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Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups

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Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups *

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Abstract

In this paper we address the constructive controllability problem for drift-free, left-invariant systems on finite-dimensional Lie groups with fewer controls than state dimension. We consider small (ϵ) amplitude, low-frequency, periodically time-varying controls and derive average solutions for system behavior. We show how the pth-order average formula can be used to construct open-loop controls for point-to-point maneuvering of systems that require up to $(p \Leftrightarrow 1)$ iterations of Lie brackets to satisfy the Lie algebra controllability rank condition. In the cases $p = 2, 3$, we give algorithms for constructing these controls as a function of structure constants that define the control authority, i.e., the actuator capability, of the system. The algorithms are based on a geometric interpretation of the average formulas and produce sinusoidal controls that solve the constructive controllability problem with $O(\epsilon^p)$ accuracy in general (exactly if the Lie algebra is nilpotent). The methodology is applicable to a variety of control problems and is illustrated for the motion control problem of an autonomous underwater vehicle with as few as three control inputs.

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1 Introduction

Recent work in nonlinear control has drawn attention to drift-free systems with fewer controls than state variables. These arise in problems of motion planning for wheeled robots subject to nonholonomic constraints [22, 23], models of kinematic drift (or geometric phase) effects in space systems subject to appendage vibrations or articulations [12, 13], and models of self-propulsion of paramecia at low Reynolds numbers [26]. The basic state-space model takes the form,

$$\dot{x} = \sum_{i=1}^{m} F_i(x) u_i, \quad x \in \Re^n, \;\; u_i \in \Re, \;\; m < n \; . \tag{1}$$

It is well known that if the vector fields F_i satisfy a Lie algebra rank condition, then there exists a control $u = (u_1, \ldots, u_m)$ that drives the system to the origin from any initial state. However, unlike the linear setting where the controllability Grammian yields constructive controls, here the rank condition does not lead immediately to an explicit procedure for constructing controls. As a result, recent research has focused on constructing controls to achieve complete controllability [2, 14, 22, 23, 8, 20]. The success of constructive procedures based on periodically time-varying controls [22, 23, 8, 20] motivates our investigation.

Our interest in this paper is in constructive controllability using periodic forcing of drift-free, left-invariant systems of the form

$$\dot{X} = \epsilon X U, \qquad U(t) = \sum_{i=1}^{m} A_i u_i(t), \tag{2}$$

evolving on matrix Lie groups. Here, $X(t)$ is a curve in a matrix Lie group G of dimension n, $U(t)$ is a curve in the Lie algebra $\mathcal{G}$ of G, $m \leq n$ and $\{A_1, \ldots, A_n\}$ is a basis for $\mathcal{G}$. The Lie bracket $[\cdot, \cdot]$ on the matrix Lie algebra $\mathcal{G}$ is defined to be the matrix commutator $[A, B] = AB \Leftrightarrow BA$, for $A, B \in \mathcal{G}$. (For an introduction to matrix Lie groups and Lie algebras see [6]). The $u_i(\cdot)$ are assumed to be periodic functions of common period T. ϵ is a small parameter ($0 < \epsilon < 1$) such that $\epsilon u_i(\cdot)$ are interpreted as small-amplitude periodic control inputs. The set $\{A_1, \ldots, A_m\}$, where $(u_1, \ldots, u_m)$ can be actuated independently, represents the *control authority* of the system.

Our goal is to solve the complete constructive controllability problem for systems of the form (2) which can be stated formally as:

(P) Given an initial condition $X_i \in G$, a final condition $X_f \in G$ and a time $t_f > 0$, find $u(t) = (u_1(t), \ldots, u_m(t))$, $t \in [0, t_f]$, such that $X(0) = X_i$ and $X(t_f) = X_f$.

Our approach is to derive averaging theory for systems on matrix Lie groups of the form (2) and then to use the average formulas to specify open-loop controls that solve **(P)**, at least approximately. The controls are designed to drive an average system solution *exactly* thereby driving the actual system *approximately*. Open-loop controls can be used to exploit *a priori* knowledge of the system for improved system performance and reduced control effort. Intermittent feedback can then be used in conjunction with the open-loop control to reduce sensitivity to disturbances. (For related ideas see [4]). Feedback control laws, including time-varying feedback and discontinuous feedback, have been studied for nonholonomic systems ([5, 25, 9, 27]).

Equation (2) provides a general framework, or normal form, for a class of systems that includes rigid body motion control problems. For many of these problems, the system configuration space is globally described by a matrix Lie group making (2) a natural system model. The Lie group framework then leads to coordinate-free expressions for system behavior and ultimately to coordinate-free control algorithms. Further, when the systems on Lie groups are left-invariant, there is a global-ness to our solutions. That is, even if we exploit local charts to make small maneuvers, the Lie group framework allows us to move all over the configuration space without reformulating our control. This is because we can always treat the current position of the system as if it were the identity in the Lie group.

An important focus of our work is to exploit the Lie group structure to derive formulas for system response. Specifically, we show the utility of area and moment-like expressions in the controls and structure constants of the Lie algebra. The structure constants enable us to encode control authority, thus ensuring that our results naturally account for changes in control authority due to events such as actuator failures. This leads easily to constructive procedures for on-line adaptation to changes in control authority.

Averaging is used to describe an approximate solution to (2) that evolves on the matrix Lie group G, remains close to the actual solution to (2) and gives rise to straightforward procedures for specifying controls to address **(P)**. Averaging in this context is motivated by the work of Brockett [3] in which an averaging argument was used to describe the secular (linear in time) motion of the well-known two-input nilpotent system on $\Re^3$ often referred to as the Brockett system. We extend the argument to high-order averages and to systems on finite-dimensional Lie groups.

Liu and Sussmann [30, 20] also develop averaging theory to derive approximate tracking control for drift-free systems. They apply averaging theory to drift-free systems on a manifold M with highly oscillatory control inputs. Given a trajectory of a suitable "extended" system, their goal is to find a trajectory of the original system that converges to the given trajectory and use this result to derive approximate tracking controls. We, on the other hand, do not attempt to address all drift-free systems, but rather take a close look at a class of drift-free systems, i.e., those of the form (2), and exploit the Lie group framework as described above to great benefit. Additionally, while Liu and Sussmann consider high-amplitude, high-frequency control inputs, we consider small-amplitude, low-frequency control inputs. One approach is equivalent to the other by scaling time by ϵ. The result is that maneuvers in the Liu and Sussmann time scale are completed in one unit of time, while in our time scale maneuvers are completed in $O(1/\epsilon)$ units of time. However, our small-amplitude, low-frequency controls are gentler on the system and avoid significant off-course excursions.

Murray and Sastry [22, 23] and Lafferiere and Sussmann [14] derive control inputs to exactly steer drift-free systems that can be transformed into a nilpotent form, sometimes referred to as "chained form". Nilpotency refers to the fact that high-order Lie brackets of vector fields are identically zero. Our Lie group framework includes the case of nilpotent systems. For instance, certain chained-form systems can be represented in the form (2) where the Lie group G is unipotent, i.e., is upper triangular with ones along the diagonal, and the Lie algebra $\mathcal{G}$ is strictly upper triangular (nilpotent). For these nilpotent systems, our results provide exact steering controls. The fourth example below illustrates a chained-form system put in the form of (2).

There are special cases of drift-free systems that can be controlled exactly where our methods produce only an approximate solution. For instance, in [31], Walsh and Sastry describe a method to derive controls to exactly orient a spacecraft with two internal rotors configured about two of the principal axes. In this work, however, large motions are necessary to reorient the spacecraft. We emphasize that our framework is more general, allowing for a large class of systems and control input configurations and producing controls that keep the system state relatively close to any desired path. Further, as in [12, 13] our solutions give a means to compute drifts in system behavior caused by *undesirable* oscillations. Kinematic drift of a spacecraft caused by thermo-elastically induced vibrations in flexible attachments on the spacecraft is an example, c.f. [12, 13].

To further motivate the Lie group framework we give four examples.

Spacecraft Example: Equation (2) describes the kinematic spacecraft attitude control problem if we interpret $U(t)$ as the time-dependent skew symmetric matrix of spacecraft angular velocity such that X evolves on $G = SO(3)$, the special orthogonal group, where

$$SO(k) \stackrel{\triangle}{=} \{A \in \Re^{k\times k} | A^T A = I,\ \det(A) = 1\}.$$

Define $X(t) \in SO(3)$ to be the curve of rotations that maps a body-fixed orthonormal coordinate frame into an inertial coordinate frame. That is, $x_r = X(t)x_b$, where x_b is any point on the spacecraft described with respect to the body-fixed frame and x_r is the same point expressed with respect to the inertial frame. Then $X(t)$ describes the attitude of the spacecraft at time t. Define $\hat{}: \Re^3 \to so(3)$ where $so(3)$ is the space of 3×3 skew symmetric matrices and $x = (x_1, x_2, x_3)^T$ by

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{3}$$

Let $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$, and define $A_i = \hat{e}_i$, $i = 1,2,3$. Then $\{A_1, A_2, A_3\}$ is a (standard) basis for $\mathcal{G} = so(3)$ and $X(t)$ satisfies

$$\dot{X} = X\hat{\Omega}, \quad \hat{\Omega}(t) = \sum_{i=1}^{3} \Omega_i(t) A_i \tag{4}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ is the angular velocity of the spacecraft in body-fixed coordinates. Now suppose angular momentum of the spacecraft is conserved and equal to zero, i.e., there is no external torque applied to the spacecraft. Then it is possible to interpret the components of angular velocity, $\Omega_1, \Omega_2, \Omega_3$, as our small-amplitude, periodic controls, e.g., $\epsilon u_i = \Omega_i$, $i = 1, 2, 3$. For instance, the angular velocities could be effected using internal rotors. Alternatively, a point mass oscillator appended to the spacecraft could be used to control angular velocity (c.f. [16]). With this interpretation, equation (4) takes the form of (2) with $G = SO(3)$, $n = 3$, and $m \leq 3$ is the number of independent actuators. We note that any control configuration can be represented by choosing the appropriate basis for $so(3)$. For example, suppose there are only two independent control inputs defined by $\epsilon u_1 = \Omega_1 + \Omega_2$ and $\epsilon u_2 = \Omega_2 + \Omega_3$ (and $\epsilon u_3 = 0$). Then the system is described by (2) with $\{B_1, B_2, B_3\}$ as our basis for $so(3)$ where $B_1 = A_1 + A_2$, $B_2 = A_2 + A_3$, $B_3 = A_3$. Details of averaging and constructive controllability applied to the spacecraft can be found in [18].

Unicycle Example: Equation (2) describes the motion planning problem for a unicycle which rolls without slipping if we interpret $U(t)$ as the appropriate time-dependent matrix of steering velocity and translational velocity such that X evolves on $G = SE(2)$, the special Euclidean group, where

$$SE(k) \triangleq \{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \in \Re^{(k+1)\times(k+1)} | A \in SO(k), b \in \Re^k \}.$$

Here, we define $X(t) \in SE(2)$ to be the planar rigid body transformation that maps a body-fixed orthonormal frame into an inertial frame so that $X(t)$ describes the position at time t of the unicycle in the plane and its orientation at time t with respect to an inertially fixed axis. That is, for x_b a point on the unicycle described in terms of body-fixed coordinates and x_r the same point expressed in terms of inertial coordinates, $[x_r\ 1]^T = X(t)[x_b\ 1]^T$. In terms of local coordinates (x, y, θ) where (x, y) describes the unicycle's position and θ the unicycle's orientation on a plane relative to the inertial frame, X can be expressed as

$$X = \left[\begin{array}{cc|c} cos\theta & \Leftrightarrow sin\theta & x \\ sin\theta & cos\theta & y \\ \hline 0 & 0 & 1 \end{array} \right].$$

Suppose that $u_1 = \dot{\theta}$ (steering speed) and $u_2 = v$ (rolling speed) are available as controls and let

$$A_1 = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \; A_2 = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \tag{5}$$

and $A_3 = [A_1, A_2]$. Then $\{A_1, A_2, A_3\}$ defines a basis for $se(2)$, the Lie algebra associated with $SE(2)$, and $X(t)$ satisfies

$$\dot{X} = \epsilon X(A_1 u_1 + A_2 u_2) \tag{6}$$

where we have assumed small-amplitude controls. Equation (6) is of the form (2) with $G = SE(2)$, $n = 3$ and $m = 2$ and takes the same form as the spacecraft control problem with two internal rotors. Details of averaging and constructive controllability applied to the unicycle problem can be found in [18]. There it is illustrated that the controls derived to steer the unicycle are identical to those derived to control the spacecraft with two internal rotors as a result of the two systems taking the same form (6).

Underwater Vehicle Example: Equation (2) describes the kinematic motion control problem for an autonomous underwater vehicle (AUV) if we interpret $U(t)$ as the appropriate time-dependent matrix of vehicle angular and translational velocities such that X evolves on $G = SE(3)$ (see [28] for another study of an AUV on $SE(3)$). In this case, we define $X(t) \in SE(3)$ to be the rigid body transformation that maps a body-fixed orthonormal frame into an inertial frame so that $X(t)$ describes the position and orientation in three-dimensional space of the underwater vehicle at time t. Let

$$A_i = \begin{cases} \left[\begin{array}{c|c} \hat{e}_i & 0 \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 1,2,3 \\ \left[\begin{array}{c|c} 0 & e_{i-3} \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 4,5,6. \end{cases} \tag{7}$$

Then $\{A_1, \ldots, A_6\}$ defines a basis for $\mathcal{G} = se(3)$, the Lie algebra associated with $SE(3)$. Now let $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ define the angular velocity of the vehicle and $v = (v_1, v_2, v_3)^T$ the vehicle translational velocity, all with respect to the body-fixed coordinate frame. Then $X(t)$ satisfies

$$\dot{X} = X\left(\sum_{i=1}^{3} \Omega_i(t) A_i + \sum_{i=4}^{6} v_{i-3}(t) A_i\right). \tag{8}$$

We assume that we can interpret the components of $\Omega(t)$ and $v(t)$ as controls such that (8) is of the form (2), e.g., let $\epsilon u_i = \Omega_i,\ i = 1,2,3$ and $\epsilon u_i = v_{i-3},\ i = 4,5,6$. In this case

$G = SE(3)$, $n = 6$, and $m = 6$. If there are fewer than six independent actuators, i.e., $m < 6$, then some of the ϵu_i are identically zero. A different choice of basis for $se(3)$ and a different value of m reflects a different control authority.

Nilpotent System Example: As described above, systems in chained form can also typically be put in the form of drift-free, left-invariant systems on matrix Lie groups (2). As an example, consider the front-wheel drive car which can be transformed (locally about the origin) into a two-input chained-form system on $\Re^4$ [23]:

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ \dot{x}_4 &= x_3 v_1. \end{aligned} \tag{9}$$

This system can be expressed (or embedded) as evolving on the matrix Lie group consisting of elements of the form

$$X = \begin{pmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & * \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where * is arbitrary. A basis for the (nilpotent) Lie algebra of this group is given by $\{A_1, A_2, A_3, A_4\} = \{A_1, A_2, [A_2, A_1], [[A_2, A_1], A_1]\}$ where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $X(t)$ satisfies

$$\dot{X} = \epsilon X(A_1 v_1 + A_2 v_2) \tag{10}$$

which is of the form (2) with $n = 4$, $m = 2$, where we have assumed small-amplitude controls. Other two-input chained form systems, such as the kinematic car with k trailers, can similarly be described in this form (c.f. [16]).

The following is an outline of the remaining sections of this paper. In Section 2, we state some preliminaries including definitions of geometric objects that play a key role in the averaging formulas and two local representations of the solution to (2). In Section 3 we prove second and third-order averaging theorems for systems of the form (2). Our main results are an "area rule" for second-order averaging and a "moment rule" for third-order averaging. A statement of the general pth-order averaging theorem is given in Appendix A. By the *pth-order average solution* $X^{(p)}$, we mean that given a metric d on the Lie group G, $d(X(t), X^{(p)}(t)) = O(\epsilon^p)$, $\forall t \in [0, b/\epsilon]$, $b > 0$. In Section 4, we show how to use the average formulas for (approximate) constructive controllability by deriving controls that steer the average solution. The control laws become increasingly complex for increasing order of averaging, and so we seek to minimize the order of the average solution that we steer. However, a sufficiently high-order average solution is needed in order to capture the controllability of the system. We determine p_{min} where p_{min} is the smallest p such that $X^{(p)}$ can be driven from any $X_i \in G$ to any desired $X_f \in G$ and show that $(p_{min} - 1)$ is equal to the highest number of iterations of Lie brackets used to satisfy the controllability Lie algebra rank condition. The proof is constructive yielding algorithms that produce continuous, small-amplitude, low-frequency, open-loop sinusoidal controls. The algorithms are driven by the structure constants that define the control authority and controllability of the system. In Section 5 we illustrate the algorithms for two control configurations of an autonomous underwater vehicle. Conclusions are given in Section 6.

The results of this paper can be extended to the setting of abstract finite-dimensional Lie groups (c.f. [16]). To keep the notation simple, we stick to the setting of matrix Lie groups. This is sufficient for our examples.

2 Preliminaries

Our average solutions $X^{(p)}$ depend on the geometric objects described below. We make the following definitions assuming that $u(t)$ is periodic in t with period T:

$$u_{av} = (u_{av1}, \ldots, u_{avm})^T, \quad \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)^T,$$

$$u_{avi} = \frac{1}{T}\int_0^T u_i(\tau)d\tau, \quad \tilde{u}_i(t) = \int_0^t u_i(\tau)d\tau,$$

$$U_{av} = \sum_{i=1}^{m} u_{avi} A_i, \quad \tilde{U} = \sum_{i=1}^{m} \tilde{u}_i A_i.$$

So $u = \dot{\tilde{u}}$ and if $u_{av} = 0$ then $\tilde{u}$ is periodic in t with common period T.

Assume that $u_{av} = 0$ and define $Area_{ij}(T)$ to be the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period, i.e., from $t = 0$ to $t = T$. By Green's Theorem we can express this area as

$$Area_{ij}(T) = \frac{1}{2} \int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) \Leftrightarrow \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma)) d\sigma. \tag{11}$$

This area can be interpreted as the projection onto the i-j plane of the area enclosed by the curve $(\tilde{u}_1, \ldots, \tilde{u}_m)$ in one period. Define

$$\begin{aligned} a_{ij}(t) &= \frac{1}{2} \int_0^t (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) \Leftrightarrow \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma)) d\sigma. \\ &= \frac{Area_{ij}(T)t}{T} + f(t), \end{aligned} \tag{12}$$

where $f(t + T) = f(t)$, $f(0) = 0$. Define

$$m_{ijk}(T) = \frac{1}{3} \int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) \Leftrightarrow \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma)) \tilde{u}_k(\sigma) d\sigma. \tag{13}$$

Now consider the closed curve C defined by $\tilde{u}_i(t)$, $\tilde{u}_j(t)$ and $\tilde{u}_k(t)$ over one period. Let A be any oriented surface with boundary $\partial A = C$. Then by Stokes' Theorem,

$$m_{ijk}(T) = \frac{1}{3} \int_A \Leftrightarrow \tilde{u}_i d\tilde{u}_j d\tilde{u}_k \Leftrightarrow \tilde{u}_j d\tilde{u}_k d\tilde{u}_i + 2\tilde{u}_k d\tilde{u}_i d\tilde{u}_j. \tag{14}$$

So $m_{ijk}(T)$ as described by (14) can be interpreted as a first moment.

The average approximation $X^{(p)}$ will also depend on the *structure constants* $,\,^k_{ij}$ associated to a given basis for the Lie algebra $\mathcal{G}$. These are defined by

$$[A_i, A_j] = \sum_{k=1}^{n} ,\,^k_{ij} A_k, \quad i, j = 1, \ldots, n. \tag{15}$$

We define a *depth-μ Lie bracket* as μ iterated brackets, e.g., a depth-one Lie bracket is of the form $[A, B]$, a depth-two bracket is of the form $[A, [B, C]]$ or $[[A, B], C]$, a depth-three bracket is of the form $[A, [B, [C, D]]]$, etc., where $A, B, C, D \in \mathcal{G}$. A depth-zero bracket is just an element of the Lie algebra $\mathcal{G}$. We can then define structure constants associated to higher depth brackets. For example, we define *depth-two structure constants* θ^q_{ijk} associated with basis $\{A_1, \ldots, A_n\}$ according to

$$[[A_i, A_j], A_k] = [\sum_{l=1}^{n} \Gamma^l_{ij} A_l, A_k] = \sum_{l=1}^{n} \Gamma^l_{ij} [A_l, A_k] = \sum_{q=1}^{n} \sum_{l=1}^{n} \Gamma^l_{ij} \Gamma^q_{lk} A_q = \sum_{q=1}^{n} \theta^q_{ijk} A_q. \tag{16}$$

Skew symmetry of the Lie bracket on $\mathcal{G}$, $[A, B] = -[B, A]$, implies $\Gamma^k_{ij} = -\Gamma^k_{ji}$. Further,

$$Area_{ij}(T) = -Area_{ji}(T), \quad m_{ijk}(T) = -m_{jik}(T).$$

Similarly, the Jacobi identity, $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$, implies

$$\theta^p_{ijk} + \theta^p_{jki} + \theta^p_{kij} = 0. \tag{17}$$

Further,

$$m_{ijk}(T) + m_{jki}(T) + m_{kij}(T) = 0. \tag{18}$$

There are well-known controllability results for systems on Lie groups of the form (2), e.g., [1, 29, 10]. We use [24] as a convenient reference. Let

$$\mathcal{C} = \{B \mid B = [B_k, [B_{k-1}, [\cdots, [B_1, B_0] \cdots]]],\ B_i \in \{A_1, \ldots, A_m\},\ i = 0, \ldots, k\}. \tag{19}$$

By Proposition 3.15 of [24], for G a connected Lie group, if $\mathcal{G} = \text{span}\mathcal{C}$ then system (2) is *controllable*, i.e., a solution to **(P)** exists. We refer to this condition as the *Lie algebra controllability rank condition.* If this condition is satisfied using only up to depth-j brackets, i.e., $k \leq j$ in (19), then we say that system (2) is a *depth-j bracket system.*

Since, in general, there are no explicit global representations of the solution to (2) we make use of local representations: the *product of exponentials representation* given by Wei and Norman [32] and the *single exponential representation* given by Magnus [21]. We begin by defining the Wei-Norman representation.

Lemma 1 (Wei and Norman) . Let $X(t)$ be the solution to (2) with $X(0) = I$. Then $\exists t_0 > 0$ such that for $|t| < t_0$, $X(t)$ can be expressed in the form

$$X(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} \cdots e^{g_n(t)A_n} . \tag{20}$$

The Wei-Norman parameters $g = (g_1, \ldots, g_n)^T$ satisfy

$$\dot{g} = \epsilon M(g) u \ , \quad \text{for } |t| < t_0 \ , \tag{21}$$

where $g(0) = 0$ and $M(g)$ is a real analytic matrix-valued function of g. If $\mathcal{G}$ is solvable then there exists a basis of $\mathcal{G}$ and an ordering of this basis for which (21) holds globally, i.e., for all t, and in that case (21) can be integrated by quadrature. □

As shown in the work of Wei and Norman, one can express $M(g)$ of (21) in terms of the structure constants of (2). For $\|g\|$ small,

$$M(g) = I + \tilde{\xi}(g) + O(g^2), \tag{22}$$

where the ijth element of $\tilde{\xi}(g)$ is

$$\tilde{\xi}_{ij}(g) = \sum_{k=j+1}^{n} g_k, {}^{i}_{kj} \tag{23}$$

and $O(g^2)$ are higher order terms in the g_i.

It is customary to refer to components of g as the *canonical coordinates of the second kind* for G. Let W be the largest, connected open neighborhood of $0 \in \Re^n$ such that $\forall g \in W$, $M(g)$ is well-defined. Let $\Phi : \Re^n \to G$ define the mapping

$$\Phi(g) = e^{g_1 A_1} e^{g_2 A_2} \cdots e^{g_n A_n} \tag{24}$$

and define $V = \Phi(W) \subset G$. Then, the Wei-Norman formulation provides a local representation of the solution to (2) for initial condition $X(0) \in V \subset G$. Now let S be the largest neighborhood of $0 \in \Re^n$ contained in W such that $\Psi = \Phi|_S : S \to G$ is one-to-one. Let $Q = \Psi(S) \subset V$. Then $\Psi : S \to Q$ is a diffeomorphism and we can define a metric $\tilde{d} : Q \times Q \to \Re_+$ by

$$\tilde{d}(Y, Z) = d(\Psi^{-1}(Y), \Psi^{-1}(Z)) \tag{25}$$

where, for $\|\cdot\|$ a norm on $\Re^n$, $d : \Re^n \times \Re^n \to \Re_+$ is given by

$$d(\alpha, \beta) = \|\alpha \Leftrightarrow \beta\|. \tag{26}$$

As an alternative to using the Wei-Norman representation of solutions to (2), we consider Magnus' single exponential representation [21]. By Theorem III of [21] under an unspecified condition of convergence, the solution to (2) with $X(0) = I$ can be expressed as

$$X(t) = e^{Z(t)} \tag{27}$$

where $Z(t) \in \mathcal{G}$ is given by the infinite series (we show terms up to $O(\epsilon^3)$):

$$\begin{aligned} Z(t) &= \epsilon \int_0^t U(\tau) d\tau + \frac{\epsilon^2}{2} \int_0^t [\tilde{U}(\tau), U(\tau)] d\tau \\ &+ \frac{\epsilon^3}{4} \int_0^t [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)] d\sigma, U(\tau)] d\tau + \frac{\epsilon^3}{12} \int_0^t [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]] d\tau + \ldots \end{aligned} \tag{28}$$

While the convergence criterion for (28) is not given explicitly in [21], two different sufficient conditions are provided in [11] and [7], respectively. Karasev and Mosolova [11] show that (28) converges if

$$\int_0^t \| ad_{\epsilon U(\tau)} \| d\tau < \ln 2. \tag{29}$$

For G a finite-dimensional Lie group, the convergence condition (29) is equivalent to

$$\int_0^t \| \Lambda(\epsilon u(\tau)) \| d\tau < \ln 2, \tag{30}$$

where $\Lambda(\cdot)$ is an $n \times n$ matrix with ijth element $\Lambda_{ij}(\cdot)$ defined by

$$\Lambda_{ij}(v) = \sum_{k=1}^{n} v_k, \, {}^{i}_{kj}.$$

In the case that $G = SO(3)$ and $\{A_1, A_2, A_3\}$ is the standard basis for $\mathcal{G} = so(3)$, it is easy to compute that $\Lambda(\epsilon u) = \epsilon U$ and so (30) is equivalent to

$$\int_0^t \| \epsilon U(\tau) \| d\tau < \ln 2.$$

The convergence criterion given by Fomenko and Chakon [7] takes the form

$$\int_0^t \| \epsilon U(\tau) \| d\tau < \frac{\hat{b}}{M}, \tag{31}$$

where $M \geq 1$ is defined such that $\| [A, B] \| \leq M \|A\| \|B\|$ for all $A, B \in \mathcal{G}$ and the universal constant $\hat{b}$ is the radius of a disk over which a scalar differential equation, defined in [7], has an analytic solution.

Let $\hat{\Phi} : \mathcal{G} \to G$ define the mapping

$$\hat{\Phi}(Z) = e^Z, \quad Z = \sum_{i=1}^{n} z_i A_i. \tag{32}$$

Let $\hat{S}$ be the largest neighborhood of $0 \in \mathcal{G}$ such that $\hat{\Psi} = \hat{\Phi}|_{\hat{S}} : \hat{S} \to G$ is one-to-one. Let $\hat{Q} = \hat{\Psi}(\hat{S}) \subset G$. Then $\hat{\Psi} : \hat{S} \to \hat{Q}$ is a diffeomorphism and, for d given by (26), we can define a metric $\hat{d} : \hat{Q} \times \hat{Q} \to \Re_+$ by

$$\hat{d}(X, Y) = d(\hat{\Psi}^{-1}(X), \hat{\Psi}^{-1}(Y)). \tag{33}$$

Following Lazard and Tits [15] define an *admissible norm* on $\mathcal{G}$ as any norm $\| \cdot \|$ that makes $(\mathcal{G}, \| \cdot \|)$ a Banach space and satisfies

$$\|[A, B]\| \leq \|A\| \|B\|, \quad \forall A, B \in \mathcal{G}.$$

Define $B(\mathcal{G}, \rho) = \{A \in \mathcal{G} \mid \|A\| < \rho\}$. Then from Theorem 2.1 of [15], if the connected center of G, C_{G_0}, is simply connected, then the restriction of $\hat{\Phi}$ to $B(\mathcal{G}, \pi)$ is one-to-one. Consider a matrix Lie algebra $\mathcal{G} \subseteq \Re^{n \times n}$ and the induced matrix $\bar{p}$-norm $\| \cdot \|_{\bar{p}}$ on $\Re^{n \times n}$. We can always construct an admissible norm as $\| \cdot \|_L \triangleq 2 \| \cdot \|_{\bar{p}}$, since

$$\|[A, B]\|_L = 2 \|[A, B]\|_{\bar{p}} = 2 \|AB \Leftrightarrow BA\|_{\bar{p}} \leq 4 \|A\|_{\bar{p}} \|B\|_{\bar{p}} = \|A\|_L \|B\|_L.$$

In the case of simply connected C_{G_0}, we can take $\hat{S} = B(\mathcal{G}, \pi) = \{A \in \mathcal{G} \mid \|A\|_L < \pi\} = \{A \in \mathcal{G} \mid \|A\|_{\bar{p}} < \pi/2\}$. The condition on C_{G_0} holds, in particular, for finite-dimensional Lie groups with trivial centers such as $SO(3)$, $SE(2)$ and $SE(3)$. Further, for simply connected Lie groups, we can replace π by 2π, i.e., we can take $\hat{S} = B(\mathcal{G}, 2\pi)$. Thus, for all these kinds of Lie groups, we can be assured that our norm $\hat{d}$ is well-defined on a significantly sized neighborhood of the identity in G.

We note that for X in a sufficiently small neighborhood of the identity, knowing one local representation means knowing the other approximately well.

Lemma 2 Given $X \in Q \cap \hat{Q} \subset G$, let $g = \Psi^{-1}(X)$ and $Z = \hat{\Psi}^{-1}(X)$. Then $g = O(\epsilon^p)$ if and only if $Z = O(\epsilon^p)$, $p \geq 1$. In this case, $\|g_i \Leftrightarrow z_i\| = O(\epsilon^{2p})$, $i = 1, \ldots, n$.

Proof: The lemma is proved by expanding exponentials and equating the two local representations.

□

3 Averaging

Classical averaging theory is typically applied to systems evolving on $\Re^n$. To derive averaging theory for systems which evolve on Lie groups (2), we apply classical averaging theory to local representations of (2) and then transfer such estimates to the group level. The theorems in this section give formulas for the pth-order average solutions $X^{(p)}(t)$, $p = 2, 3$. For illustration we make use of the Wei-Norman product of exponentials representation for $p = 2$ and the Magnus single exponential representation for $p = 3$. The first-order average formula can be derived to be $X^{(1)}(t) = X^{(1)}(0)e^{\epsilon U_{av} t}$. This describes the effect of the DC component of $U(t)$ on the system. This is useful for control only if $m = n$. As a result, we focus on higher-order average formulas which capture Lie bracket motion of the system. A general pth-order averaging theorem is given in Appendix A. The theorems below require smooth controls; however, this requirement is relaxed in the appendix where piecewise continuous controls are sufficient.

We note that these theorems state that the formulas are valid for $X(t)$ in a neighborhood of the identity of G. However, because system (2) is left-invariant, these theorems actually give the formulas for the pth-order approximation $X^{(p)}(t)$ to the solution $X(t)$ of (2) for *any* initial condition $X(0) \in G$. Let $X_I(t)$ and $X_I^{(p)}(t)$ correspond to the actual and approximate solutions, respectively, of (2) with $X_I(0) = I \in G$. By left-invariance of (2), $X(t) = X(0)X_I(t)$ and $X^{(p)}(t) = X(0)X_I^{(p)}(t)$ is an $O(\epsilon^p)$ approximation of $X(t)$.

Theorem 3 (Second-Order Averaging: Area Rule) Consider system (2) on the Lie group G with Lie algebra $\mathcal{G}$. Assume that $U(t) \in \mathcal{G}$ is periodic in t with period T and has continuous derivatives up to third order for $t \in [0, \infty)$ and assume that $U_{av} = 0$. Let $D = \{g \in \Re^n \mid \|g\| < r\} \subset S$ (where $r > 0$ is chosen as large as possible). Suppose that $X(0) = X_0 \in Q$. Let $g(t)$ be the solution to (21) with $g(0) = g_0 = \Psi^{-1}(X_0) = O(\epsilon)$. Let

$g_0^{(2)} = ({g_1}_0^{(2)}, \ldots, {g_n}_0^{(2)})^T$ and define

$$\bar{w}_k(t) = \epsilon^2 \frac{t}{T} \sum_{i,j=1;i<j}^{m} Area_{ij}(T) \, , {}^k_{ij} + {g_k}_0^{(2)}, \tag{34}$$

$$g_k^{(2)}(t) = \epsilon \tilde{u}_k(t) + \bar{w}_k(t), \tag{35}$$

$$X^{(2)}(t) = e^{g_1^{(2)}(t)A_1} \ldots e^{g_n^{(2)}(t)A_n}, \tag{36}$$

where $, {}^k_{ij}$ and $Area_{ij}(T)$ are defined by (15) and (11), respectively. If $\|g_0 \Leftrightarrow g_0^{(2)}\| = O(\epsilon^2)$ and if $g^{(2)}(t) \in D$, $\forall t \in [0, b/\epsilon]$, $b > 0$, then,

$$\tilde{d}(X(t), X^{(2)}(t)) = O(\epsilon^2), \quad \forall t \in [0, b/\epsilon].$$

Proof: Recall by (21) and (22) that for small $\|g\|$,

$$\dot{g} = \epsilon M(g) u = \epsilon u + \epsilon \tilde{\xi}(g) u + \epsilon O(g^2) u.$$

By second-order classical averaging theory (for details see [16]),

$$\|g(t) \Leftrightarrow g^{(2)}(t)\| = O(\epsilon^2), \quad \forall t \in [0, b/\epsilon], \tag{37}$$

where

$$g^{(2)}(t) = \epsilon \tilde{u}(t) + \bar{w}(t)$$

and $\bar{w}(t)$ is the solution to

$$\dot{\bar{w}} = \epsilon^2 \frac{1}{T} \int_0^T \tilde{\xi}(\tilde{u}(\sigma)) u(\sigma) d\sigma, \quad \bar{w}(0) = g_0^{(2)}. \tag{38}$$

From the definition of $\tilde{\xi}$ (23), the kth component of the vector $\tilde{\xi}(\tilde{u})u$ is

$$\sum_{i=1}^{m} \xi_{ki}(\tilde{u}) u_i = \sum_{i=1}^{m} \sum_{j=i+1}^{m} , {}^k_{ji} \tilde{u}_j u_i. \tag{39}$$

So using integration by parts, the fact that $, {}^k_{ij} = \Leftrightarrow , {}^k_{ji}$, $\dot{\tilde{u}}_i = u_i$ and the definition of $Area_{ij}(T)$ (11) we get from substituting (39) into (38) that

$$\begin{aligned}
\bar{w}_k(t) &= \epsilon^2 \frac{t}{T} \int_0^T \sum_{i=1}^{m} \xi_{ki}(\tilde{u}(\sigma)) u_i(\sigma) d\sigma + {g_k}_0^{(2)} \\
&= \epsilon^2 \frac{t}{T} \sum_{i=1}^{m} \sum_{j=i+1}^{m} \int_0^T , {}^k_{ji} \tilde{u}_j(\sigma) \dot{\tilde{u}}_i(\sigma) d\sigma + {g_k}_0^{(2)} \\
&= \epsilon^2 \frac{t}{T} \sum_{i,j=1;i<j}^{m} \frac{1}{2} \int_0^T (\tilde{u}_j(\sigma) \dot{\tilde{u}}_i(\sigma) \Leftrightarrow \tilde{u}_i(\sigma) \dot{\tilde{u}}_j(\sigma)) d\sigma \, , {}^k_{ji} + {g_k}_0^{(2)} \\
&= \epsilon^2 \frac{t}{T} \sum_{i,j=1;i<j}^{m} Area_{ij}(T) \, , {}^k_{ij} + {g_k}_0^{(2)}.
\end{aligned}$$

For small enough ϵ, since $g^{(2)}(t) \in D \subset S$ then $g(t) \in D \subset S$, $\forall t \in [0, b/\epsilon]$. So by definition of Ψ, $X(t) = \Psi(g(t))$ and $X^{(2)}(t) = \Psi(g^{(2)}(t))$, $\forall t \in [0, b/\epsilon]$. The theorem follows by (37) and the definition of $\tilde{d}$, since $\tilde{d}(X(t), X^{(2)}(t)) = \|g(t) \Leftrightarrow g^{(2)}(t)\| = O(\epsilon^2)$, $\forall t \in [0, b/\epsilon]$. □

We show further in the next proposition that the structure constants $,_{ij}^{k}$ associated to a given basis for $\mathcal{G}$ are directly related to the Lie brackets of the vector fields defined by the columns of $M(g)$ evaluated at $g = 0$.

Lemma 4 Suppose that $\bar{w}(t)$ is defined by (34). Let $[f_1 \; f_2 \; \cdots \; f_n] = M(g)$ where f_k is the kth column of the matrix $M(g)$. Then

$$\bar{w}(t) = \frac{\epsilon^2 t}{T} \sum_{i,j=1; i<j}^{m} Area_{ij}(T)[f_i, f_j]|_{g=0} + g_0^{(2)}. \tag{40}$$

Proof: By (22) and (23) we have that

$$f_i = \begin{bmatrix} \sum_{k=i+1}^{n} g_k ,_{ki}^{1} + O(g^2) \\ \vdots \\ \sum_{k=i+1}^{n} g_k ,_{ki}^{(i-1)} + O(g^2) \\ 1 + \sum_{k=i+1}^{n} g_k ,_{ki}^{i} + O(g^2) \\ \sum_{k=i+1}^{n} g_k ,_{ki}^{i+1} + O(g^2) \\ \vdots \\ \sum_{k=i+1}^{n} g_k ,_{ki}^{n} + O(g^2) \end{bmatrix}.$$

So for $i < j$,

$$\begin{aligned} [f_i, f_j]|_{g=0} &= \frac{\partial f_j}{\partial g}|_{g=0} f_i|_{g=0} \Leftrightarrow \frac{\partial f_i}{\partial g}|_{g=0} f_j|_{g=0} \\ &= \Leftrightarrow \begin{bmatrix} ,_{ji}^{1} \\ \vdots \\ ,_{ji}^{n} \end{bmatrix} = \begin{bmatrix} ,_{ij}^{1} \\ \vdots \\ ,_{ij}^{n} \end{bmatrix}. \end{aligned}$$

which by (34) completes the proof. □

According to Theorem 3, $X^{(2)}(t)$ can be expressed as a product of exponentials where the exponents have an $O(\epsilon)$ periodic term and a secular term (a term linear in t). By (34) the secular term is proportional to the structure constants $,_{ij}^{k}$ and the projected areas $Area_{ij}(T)$ bounded by the closed curves described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period. This interpretation justifies calling Theorem 3 an area rule.

The second-order average formula derived using the single-exponential local representation for $X(t)$ (as follows from Theorem 9, Appendix A) takes the form:

$$z_k^{(2)}(t) = \epsilon\tilde{u}_k(t) + \frac{\epsilon^2 t}{T}\sum_{i,j=1;i<j}^{m} Area_{ij}(T),\,{}^{k}_{ij} + g_{k0}^{(2)}, \tag{41}$$

$$Z^{(2)}(t) = \sum_{k=1}^{n} z_k(t)A_k, \quad X^{(2)}(t) = e^{Z^{(2)}(t)}. \tag{42}$$

A comparison of the two second-order average formulas shows that $z^{(2)}(t) = g^{(2)}(t)$.

The revealing step in the proof of the single exponential area rule shows that

$$\begin{aligned} Z^{(2)}(t) &= \frac{\epsilon^2 t}{2T}\int_0^T [\tilde{U}, U](\sigma)d\sigma \\ &= \frac{\epsilon^2 t}{2T}\int_0^T [\sum_{i=1}^{m}\tilde{u}_i(\sigma)A_i, \sum_{j=1}^{m}\dot{\tilde{u}}_j(\sigma)A_j]d\sigma \\ &= \frac{\epsilon^2 t}{T}\sum_{k=1}^{n}(\sum_{i,j=1;i<j}^{m} Area_{ij}(T),\,{}^{k}_{ij})A_k, \end{aligned} \tag{43}$$

This result confirms that the formulas $X^{(2)}$ are basis independent. Additionally, (43) reveals how the secular term in the second-order approximation captures the effect of the group level version of depth-one Lie brackets. This effect is developed further in Section 4.

Theorem 5 (Third-Order Averaging: Moment Rule) Consider system (2) on the Lie group G with Lie algebra $\mathcal{G}$. Assume that $U(t) \in \mathcal{G}$ is periodic in t with period T and has continuous derivatives up to fourth order for $t \in [0, \infty)$. Further, assume that $U_{av} = 0$ and $Area_{ij}(T) = 0$, $\forall i, j$. Let $D = \{Z \in \mathcal{G} \mid \|Z\| < r\} \subset \hat{S}$ (where $r > 0$ is chosen as large as possible). b is defined according to the convergence criterion for (28). Suppose that $X(0) = X_0$. Let $Z(t)$ be given by (28) with $Z(0) = Z_0 = \hat{\Psi}^{-1}(X_0) = O(\epsilon^2)$. Let $Z_0^{(3)} = \sum_{q=1}^{n} z_{q0}^{(3)} A_q$. Define

$$z_q^{(3)}(t) = \epsilon\tilde{u}_q(t) + \sum_{i,j=1;i<j}^{m} \epsilon^2 a_{ij}(t),\,{}^{q}_{ij} \Leftrightarrow \sum_{k=1}^{m}\sum_{i,j=1;i<j}^{m} \frac{\epsilon^3 t}{T} m_{ijk}(T)\theta^q_{ijk} + z_{q0}^{(3)}, \tag{44}$$

$$Z^{(3)}(t) = \sum_{q=1}^{n} z_q^{(3)} A_q, \quad X^{(3)}(t) = e^{Z^{(3)}(t)}, \tag{45}$$

where θ^q_{ijk} and $m_{ijk}(T)$ are defined by (16) and (13), respectively. If $\|Z_0 \Leftrightarrow Z_0^{(3)}\| = O(\epsilon^3)$ and if $Z^{(3)}(t) \in D$, $\forall t \in [0, b/\epsilon]$,

$$\hat{d}(X(t), X^{(3)}(t)) = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon].$$

Proof: By classical averaging theory

$$\|Z(t) \Leftrightarrow Z^{(3)}(t)\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon] \tag{46}$$

where (compare with formulas (28) for $Z(t)$)

$$\begin{aligned} Z^{(3)}(t) &= \epsilon\tilde{U}(t) + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau + \frac{\epsilon^3 t}{4T}\int_0^T [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau \\ &+ \frac{\epsilon^3 t}{12T}\int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau + Z_0^{(3)} \\ &= \epsilon\tilde{U}(t) + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau + \frac{\epsilon^3 t}{3T}\int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau + Z_0^{(3)}. \end{aligned} \tag{47}$$

The second equality is derived by integration by parts.

By definition, we have that $\epsilon\tilde{U}(t) = \sum_{q=1}^m \epsilon\tilde{u}_q(t)A_q$. Following the steps in (43)

$$\frac{\epsilon^2}{2}\int_0^t [\tilde{U}, U](\sigma)d\sigma = \sum_{q=1}^n (\sum_{i,j=1;i<j}^m \epsilon^2 a_{ij}(t), {}^{q}_{ij})A_q. \tag{48}$$

The third term on the right side of (47) can be expanded as follows:

$$\begin{aligned} &\frac{\epsilon^3 t}{3T}\int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau \\ =\; &\frac{\epsilon^3 t}{3T}\int_0^T [\sum_{k=1}^m \tilde{u}_k(\tau)A_k, [\sum_{i=1}^m \tilde{u}_i(\tau)A_i, \sum_{j=1}^m \dot{\tilde{u}}_j(\tau)A_j]]d\tau \\ =\; &\frac{\epsilon^3 t}{T}\sum_{k=1}^m \sum_{i,j=1;i<j}^m \frac{1}{3}\int_0^T (\tilde{u}_i(\tau)\dot{\tilde{u}}_j(\tau) \Leftrightarrow \tilde{u}_j(\tau)\dot{\tilde{u}}_i(\tau))\tilde{u}_k(\tau)d\tau[A_k, [A_i, A_j]] \\ =\; &\Leftrightarrow \sum_{q=1}^n (\sum_{k=1}^m \sum_{i,j=1;i<j}^m \frac{\epsilon^3 t}{T} m_{ijk}(T)\theta_{ijk}^q)A_q, \end{aligned} \tag{49}$$

Therefore, the expression for $Z^{(3)}(t)$ given by (44)-(45) is verified. For small enough ϵ, since $Z^{(3)}(t) \in D \subset \hat{S}$ then $Z(t) \in D \subset \hat{S}$, $\forall t \in [0, b/\epsilon]$. So by (27) $X(t) = e^{Z(t)} = \hat{\Psi}(Z(t))$. The theorem follows by definition of $X^{(3)}(t)$ and $\hat{d}$. □

The third term on the right side of (44) is a purely secular term proportional to the first moments $m_{ijk}(T)$ and the depth-two structure constants θ_{ijk}^q associated with choice of basis for $\mathcal{G}$. This interpretation makes Theorem 5 a moment rule. The average formula is clearly basis independent.

Remark 6 Since by (17)-(18) $m_{ijk}(T) + m_{jki}(T) + m_{kij}(T) = 0$ and $\theta^q_{ijk} + \theta^q_{jki} + \theta^q_{kij} = 0$,

$$\sum_{q=1}^{n}\sum_{k=1}^{m}\sum_{i,j=1;i<j}^{m} m_{ijk}(T)\theta^q_{ijk}A_q = \sum_{q=1}^{n}\sum_{i,j=1;i<j}^{m}(m_{iji}(T)\theta^q_{iji} + \sum_{k=i+1}^{m}(2m_{ijk}(T) \Leftrightarrow m_{ikj}(T))\theta^q_{ijk})A_q.$$

Substituting this into the area-moment rule (44) incorporates the Jacobi identity and removes redundant terms. This is significant with regard to constructing controls to solve **(P)**.

4 Constructive Controllability

The strategy that we propose for solving (**P**) approximately can be summarized in four steps:

1. Choose intermediate target points $X_1, X_2, \ldots, X_r$ between X_i and X_f so that the "distance" between successive target points is small.

2. Specify open-loop, small-amplitude, periodic controls that drive $X(t)$ from X_i to the first target point X_1 *approximately*. To do so, specify controls that drive an $O(\epsilon^p)$ average approximation of $X(t)$ from X_i to X_1 *exactly* (p to be determined).

3. If desired, apply feedback, i.e., make appropriate modifications based on measurement of the new system state. For example, modify selection of intermediate target points.

4. Repeat steps 2 and 3 for each successive target point (letting the previous target point be the new initial position) until done.

The fact that we can make a large maneuver by repeating our technique on small steps relies on the left-invariance of our system. That is, we can always reinitialize at our current position and identify it with the identity in the Lie group.

In the case of a nilpotent system, Step 2 will drive $X(t)$ from X_i to X_1 *exactly*. This is a result of the fact that high-order Lie bracket terms are identically zero (i.e., the formula for $Z(t)$ (28) is a finite sum), and so an appropriate average provides an explicit solution to (2). The proof can be found in [16].

For Step 2, we use the average formulas of the previous section. To determine p consider the series expansion (28) for $Z(t)$, which can be thought of (locally) as the logarithm of $X(t)$. One can observe that the $O(\epsilon^p)$ term of this series is a function of a depth-$(p \Leftrightarrow 1)$ Lie bracket. Therefore, one expects that in order to be able to control $X^{(p)}$ as desired, p must be greater than or equal to p_{min} where (2) is a depth-$(p_{min} \Leftrightarrow 1)$ bracket system, i.e., the controllability rank condition is satisfied with up to depth-$(p_{min} \Leftrightarrow 1)$ brackets. We state this formally for the cases $p = 2, 3$. The general pth-order case is given in Appendix A.

Theorem 7 Suppose that system (2) on the connected Lie group G is a depth-$(p \Leftrightarrow 1)$ bracket system, $p = 2, 3$. Then the complete constructive controllability problem **(P)** can be solved with $O(\epsilon^p)$ accuracy using the formulas for $X^{(k)}(t)$, $k = 1, \ldots, p$, and p is the smallest positive integer such that this is true.

Proof: The proof is constructive and given in the form of algorithms that synthesize small-amplitude, low-frequency, continuous, sinusoidal controls. Without loss of generality we assume that $X(0) = X_i = I \in G$ and $X_f \in Q \cap \hat{Q} \subset G$ is such that $g_f = (g_{f_1}, \ldots, g_{f_n})^T = \Psi^{-1}(X_f) = O(\epsilon^{(p-1)})$ and $Z_f = \sum_{i=1}^{n} z_{f_i} A_i = \hat{\Psi}^{-1}(X_f) = O(\epsilon^{(p-1)})$. By Lemma 2 $\|z_f \Leftrightarrow g_f\| = O(2(p \Leftrightarrow 1))$. Therefore, for the order of accuracy of control that we seek, g_f and z_f can be used interchangeably.

The algorithms are designed to solve the problem $X^{(p)}(t_f) = X_f$ by solving $g^{(p)}(t_f) = g_f$ or equivalently $Z^{(p)}(t_f) = Z_f$. Multiple sub-steps are used. That is, the time interval $[0, t_f]$ is divided into subintervals, e.g., $[0, t_f] = [t_0, t_1) \cup [t_1, t_2) \cup \cdots \cup [t_{\mu-1}, t_\mu]$, $t_0 = 0$, $t_\mu = t_f$, and controls specified on each subinterval. Because as assumed above, $g_f = O(\epsilon^{(p-1)})$ and $z_f = O(\epsilon^{(p-1)})$, we can ensure that the "initial condition" for each subinterval, e.g., $g(t_0)$, $g(t_1)$, $g(t_2)$, etc., will be $O(\epsilon^{(p-1)})$, i.e., will satisfy the initial condition requirement for the averaging theorems. Thus, the appropriate averaging theorem can be applied to successive subintervals. Our controls will be specified so that the terms u_{av}, $Area_{ij}(T)$ and $m_{ijk}(T)$ will take on a single constant value on each subinterval. However, these terms may take on different values on different subintervals. Thus, for ease of notation we define the "running total" of the time-varying area terms and moment terms as $Area_{ij}(t)$ and $m_{ijk}(t)$, respectively. Let $Area_{ij}^{(r)}(T)$ and $m_{ijk}^{(r)}(T)$ be the values of the area and moment terms, respectively, during the time interval $[t_{r-1}, t_r]$ and suppose that $t \in [t_\nu, t_{\nu+1}]$, $0 \leq \nu < \mu$. Then define

$$Area_{ij}(t) = \sum_{r=1}^{\nu} \frac{(t_r \Leftrightarrow t_{r-1})}{T} Area_{ij}^{(r)}(T) + \frac{(t \Leftrightarrow t_\nu)}{T} Area_{ij}^{(\nu+1)}(T), \quad (50)$$

$$m_{ijk}(t) = \sum_{r=1}^{\nu} \frac{(t_r \Leftrightarrow t_{r-1})}{T} m_{ijk}^{(r)}(T) + \frac{(t \Leftrightarrow t_\nu)}{T} m_{ijk}^{(\nu+1)}(T). \quad (51)$$

Case (i) $p = 2$. Let

$$\mathcal{C}^{(1)} = \{C \mid C = A_k \text{ or } C = [A_i, A_j],\ i, j, k = 1, \ldots, m\}.$$

The definition of a depth-one bracket system implies $\mathcal{G} = \text{span}\mathcal{C}^{(1)}$, i.e.,

$$\begin{aligned} \mathcal{G} &= \{\sum_{k=1}^{m} c_k A_k + \sum_{i,j=1}^{m} c_{ij}[A_i, A_j],\ c_k, c_{ij} \in \Re\} \\ &= \{\sum_{k=1}^{m} c_k A_k + \sum_{i,j=1;i<j}^{m} c_{ij} \sum_{k=1}^{n} ,\, {}_{ij}^{k} A_k,\ c_k, c_{ij} \in \Re\}. \end{aligned}$$

Therefore, since $\sum_{k=1}^{n} g_{fk} A_k \in \mathcal{G}$, there exist $c_k, c_{ij} \in \Re$, $i, j, k = 1, \ldots, m$ such that

$$\sum_{k=1}^{n} g_{fk} A_k = \sum_{k=1}^{m} c_k A_k + \sum_{k=1}^{n} \sum_{i,j=1;i<j}^{m} c_{ij} ,\, {}_{ij}^{k} A_k. \quad (52)$$

By Theorem 3,

$$\sum_{k=1}^{n} g_k^{(2)}(t) A_k = \sum_{k=1}^{m} \epsilon \tilde{u}_k(t) A_k + \sum_{k=1}^{n} \sum_{i,j=1;i<j}^{m} \frac{\epsilon^2 t}{T} Area_{ij}(T) ,\, {}_{ij}^{k} A_k. \quad (53)$$

Thus, to find controls that produce $g^{(2)}(t_f) = g_f$, we equate (52) and (53) and match coefficients. That is, we choose $u_k(t)$, $t \in [0, t_f]$, $k = 1, \ldots, m$ such that

$$\epsilon \tilde{u}_k(t_f) = c_k, \quad k = 1, \ldots, m \quad \text{and} \quad (54)$$

$$\epsilon^2 Area_{ij}(t_f) = c_{ij}, \quad i, j = 1, \ldots, m,\ i < j. \quad (55)$$

Then $g^{(2)}(t_f) = g_f$ so that $X^{(2)}(t_f) = \Psi(g^{(2)}(t_f)) = \Psi(g_f) = X_f$. Thus, by Theorem 3, $\tilde{d}(X(t_f), X_f) = \tilde{d}(X(t_f), X^{(2)}(t_f)) = O(\epsilon^2)$.

Algorithm 1 below computes $u_k(t)$, $t \in [0, t_f]$, $k = 1, \ldots, m$ such that (54)-(55) are met. This is done by recognizing the geometric meaning of the terms $Area_{ij}(T)$, i.e., that

$Area_{ij}(T)$ is the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period. In particular, if we choose $\tilde{u}_i$ and $\tilde{u}_j$ to be sinusoids that are in phase then $Area_{ij}(T) = 0$. Alternatively, if they are chosen out of phase then $Area_{ij}(T) \neq 0$ is a function of the signal magnitudes and phase difference. Based on this reasoning, the final values of each of these terms can be matched independently, i.e., (54)-(55) can be met. The timing can be controlled by choosing the frequency and amplitudes of the sinusoids appropriately. In Appendix B we define two algorithm components. The first, Component 1(i) steers $\tilde{u}_k$, $k = 1, \ldots, m$ to satisfy (54) with no net change to any $Area_{ij}(t)$ term. The second, Component 1(ii) steers area terms $Area_{ij}(t), i, j, i < j$ to satisfy (55) with no net change to $\tilde{u}$. In each component the controls have an initial and final value of zero. The algorithm components are like computer subroutines that are defined once and for all and called as necessary.

ALGORITHM 1

Compute c_k, c_{ij} as follows such that (52) holds. Consider the matrix

$$\Gamma \triangleq \begin{bmatrix} \gamma_{12}^{m+1} & \gamma_{13}^{m+1} & \cdots & \gamma_{1m}^{m+1} & \gamma_{23}^{m+1} & \cdots & \gamma_{2m}^{m+1} & \cdots & \gamma_{(m-1)m}^{m+1} \\ \gamma_{12}^{m+2} & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \gamma_{12}^{n} & \gamma_{13}^{n} & \cdots & \gamma_{1m}^{n} & \gamma_{23}^{n} & \cdots & \gamma_{2m}^{n} & \cdots & \gamma_{(m-1)m}^{n} \end{bmatrix}.$$

Note that Γ has rank $n - m$. Define the generalized inverse of Γ to be $\Gamma^{\dagger} = \Gamma^T(\Gamma\Gamma^T)^{-1}$. Then let

$$\begin{bmatrix} c_{12} \\ c_{13} \\ \vdots \\ c_{(m-1)m} \end{bmatrix} = \Gamma^{\dagger} \begin{bmatrix} g_{f_{m+1}} \\ \vdots \\ g_{f_n} \end{bmatrix},$$

$$c_k = g_{f_k} - \sum_{i,j=1; i<j}^{m} c_{ij} \gamma_{ij}^{k}, \quad k = 1, \ldots, m.$$

$$\bar{S} = \{k \mid c_{kj} \neq 0, \text{ some } j > k\}, \quad r = |\bar{S}| \triangleq \text{number of elements in } \bar{S}.$$

Choose M to be a positive integer such that $M \geq 1/\pi\epsilon$. Let the period T and frequency ω of the controls be

$$T = \frac{t_f}{r(M+1)+1/2}, \qquad \omega = \frac{2\pi}{T}.$$

Then using the controls defined in Components 1(i) and 1(ii), perform the following iterations:

1. $i = 0$.

2. $i = i + 1$.

3. If $i \notin \bar{S}$ go to 5.

4. Apply Component 1(ii) for c_{ij}, $j = i + 1, \ldots, m$.

5. If $i < m \Leftrightarrow 1$ go to 2.

6. Apply Component 1(i) for c_k, $k = 1, \ldots, m$.

Then we are done and $\tilde{d}(X(t_f), X_f) = O(\epsilon^2)$ as desired.

Case (ii) $p = 3$. Let

$$\mathcal{C}^{(2)} = \{C \mid C = A_q \text{ or } C = [A_i, A_j], \text{ or } C = [[A_i, A_j], A_k], \ q, i, j, k = 1, \ldots, m\}.$$

The definition of a depth-two bracket system implies that $\mathcal{G} = \text{span}\mathcal{C}^{(2)}$, i.e.,

$$\begin{aligned}
\mathcal{G} &= \{\sum_{q=1}^{m} c_q A_q + \sum_{i,j=1}^{m} c_{ij}[A_i, A_j] + \sum_{i,j,k=1}^{m} c_{ijk}[[A_i, A_j], A_k], \ c_q, c_{ij}, c_{ijk} \in \Re\} \\
&= \{\sum_{q=1}^{m} c_q A_q + \sum_{q=1}^{n}(\sum_{i,j=1;i<j}^{m} c_{ij},{}^{q}_{ij} + \sum_{k=1}^{m}\sum_{i,j=1;i<j}^{m} c_{ijk}\theta^q_{ijk})A_q\} \\
&= \{\sum_{q=1}^{m} c_q A_q + \sum_{q=1}^{n}\sum_{i,j=1;i<j}^{m}(c_{ij},{}^{q}_{ij} + c_{iji}\theta^q_{iji} + \sum_{k=i+1}^{m} c_{ijk}\theta^q_{ijk})A_q\}
\end{aligned}$$

where the c_{ijk} in the last line are a redefinition taking into account the relations among θ^p_{ijk} induced by the Jacobi identity (17). Therefore, since $Z_f \in \mathcal{G}$, there exist $c_q, c_{ij}, c_{ijk} \in \Re$, such that

$$Z_f = \sum_{q=1}^{m} c_q A_q + \sum_{q=1}^{n}\sum_{i,j=1;i<j}^{m}(c_{ij},{}^{q}_{ij} + c_{iji}\theta^q_{iji} + \sum_{k=i+1}^{m} c_{ijk}\theta^q_{ijk})A_q. \tag{56}$$

By Theorem 5 and Remark 6

$$\begin{aligned}
Z^{(3)}(t) &= \sum_{q=1}^{m} \epsilon\tilde{u}_q(t)A_q + \sum_{q=1}^{n}\sum_{i,j=1;i<j}^{m}(\epsilon^2 a_{ij}(t),{}^{q}_{ij} \Leftrightarrow \frac{\epsilon^3 t}{T} m_{iji}(T)\theta^q_{iji} \\
&\Leftrightarrow \sum_{k=i+1}^{m} \frac{\epsilon^3 t}{T}(2m_{ijk}(T) \Leftrightarrow m_{ikj}(T))\theta^q_{ijk})A_q.
\end{aligned} \tag{57}$$

Thus, to find controls that produce $Z^{(3)}(t_f) = Z_f$, we equate (56) and (57) and match coefficients. That is, we choose $u_q(t),\ t \in [0, t_f],\ q = 1, \ldots, m$ such that

$$\epsilon \tilde{u}_q(t_f) = c_q, \quad q = 1, \ldots, m, \tag{58}$$

$$\epsilon^2 a_{ij}(t_f) = c_{ij}, \quad i, j = 1, \ldots, m,\ i < j, \tag{59}$$

$$\epsilon^3 m_{iji}(t_f) = \Leftrightarrow c_{iji}, \quad i, j = 1, \ldots, m, i < j, \quad \text{and} \tag{60}$$

$$\epsilon^3 (2m_{ijk}(t_f) \Leftrightarrow m_{ikj}(t_f)) = \Leftrightarrow c_{ijk},\ i, j, k = 1, \ldots, m,\ i < j, k. \tag{61}$$

Then $X^{(3)}(t_f) = \hat{\Psi}(Z^{(3)}(t_f)) = \hat{\Psi}(Z_f) = X_f$ and so by Theorem 5, $\hat{d}(X(t_f), X_f) = \hat{d}(X(t_f), X^{(3)}(t_f)) = O(\epsilon^3)$.

Algorithm 2 below computes $u_q(t),\ t \in [0, t_f],\ q = 1, \ldots, m$ such that (58)-(61) are met. This is done by recognizing the meaning of the geometric terms as in Case (i). In particular, the terms $m_{ijk}(T)$ and $m_{ikj}(T)$, $i < j < k$, can be controlled using sinusoids with 1-2 resonance. In Appendix B we define Component 2(ii) which addresses (59) and (60) and Component 2(iii) which addresses (61).

ALGORITHM 2 Let

$$\Xi = \{\xi \in \{m+1, \ldots, n\} \mid ,{}^{\xi}_{ij} \neq 0 \text{ some } i, j \in \{1, \ldots, m\},\ i < j\}. \tag{62}$$

Define $0 < l < (n \Leftrightarrow m)$ by $l = |\Xi|$ = number of elements in Ξ. We will assume for the purposes of the algorithm, without loss of generality, that the basis $\{A_1, \ldots, A_n\}$ is chosen and ordered such that $\{A_1, \ldots, A_{m+l}\}$ is a basis for $\Delta + [\Delta, \Delta]$, where $\Delta = \text{span}\{A_1, \ldots, A_m\}$. Compute c_q, c_{ij}, c_{ijk} such that (56) holds as follows. Consider the matrix

$$\theta \triangleq \begin{bmatrix} \theta^{m+l+1}_{121} & \theta^{m+l+1}_{122} & \cdots & \theta^{m+l+1}_{12m} & \cdots & \theta^{m+l+1}_{1mm} & \theta^{m+l+1}_{232} & \cdots & \theta^{m+l+1}_{(m-1)mm} \\ \theta^{m+l+2}_{121} & \theta^{m+l+2}_{122} & & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & & \cdot \\ \theta^{n}_{121} & \theta^{n}_{122} & & \cdots & & \cdot & \cdots & & \theta^{n}_{(m-1)mm} \end{bmatrix}$$

and the matrix

$$, \triangleq \begin{bmatrix} ,^{m+1}_{12} & ,^{m+1}_{13} & \cdots & ,^{m+1}_{1m} & ,^{m+1}_{23} & \cdots & ,^{m+1}_{2m} & \cdots & ,^{m+1}_{(m-1)m} \\ ,^{m+2}_{12} & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ ,^{m+l}_{12} & ,^{m+l}_{13} & \cdots & ,^{m+l}_{1m} & ,^{m+l}_{23} & \cdots & ,^{m+l}_{2m} & \cdots & ,^{m+l}_{(m-1)m} \end{bmatrix}.$$

Let $\alpha \triangleq m + l$. Then θ has rank $n \Leftrightarrow \alpha$ and , has rank l. Define the generalized inverse of θ to be $\theta^\dagger = \theta^T(\theta\theta^T)^{-1}$. Let

$$\begin{bmatrix} c_{121} \\ c_{122} \\ \vdots \\ c_{(m-1)mm} \end{bmatrix} = \theta^\dagger \begin{bmatrix} z_{f_{m+l+1}} \\ \vdots \\ z_{f_n} \end{bmatrix}.$$

Define the generalized inverse of , to be $,^\dagger = ,^T(,\,,^T)^{-1}$. Then let

$$\begin{bmatrix} c_{12} \\ c_{13} \\ \vdots \\ c_{(m-1)m} \end{bmatrix} = ,^\dagger \begin{bmatrix} z_{f_{m+1}} \Leftrightarrow \sum_{i,j=1;i<j}^{m} \left(c_{iji}\theta_{iji}^{m+1} + \sum_{k=i+1}^{m} c_{ijk}\theta_{ijk}^{m+1}\right) \\ \vdots \\ z_{f_{m+l}} \Leftrightarrow \sum_{i,j=1;i<j}^{m} \left(c_{iji}\theta_{iji}^{m+l} + \sum_{k=i+1}^{m} c_{ijk}\theta_{ijk}^{m+l}\right) \end{bmatrix},$$

$$c_q = z_{f_q} \Leftrightarrow \sum_{i,j=1;i<j}^{m} (c_{ij},\,{}^{q}_{ij} + c_{iji}\theta^q_{iji} + \sum_{k=i+1}^{m} c_{ijk}\theta^q_{ijk}) \quad q = 1, \ldots, m.$$

$$\bar{Y} = \{c_{ijk} \mid c_{ijk} \neq 0,\ i < j, i \leq k\},$$

$$\bar{Q} = \{c_{ijk} \in \bar{Y} \mid i < j < k\}, \quad \beta = |\bar{Q}| = \text{number of elements in } \bar{Q},$$

$$\bar{R} = \{c_{ijk} \in \bar{Y} \mid k = i\} \cup \{c_{ijk} \in \bar{Y} \mid k = j \text{ and } c_{iji} \notin \bar{Y}\},$$

$$\bar{V} = \{c_{ij} \mid c_{ij} \neq 0,\ i < j\},$$

$$\bar{W} = \{c_{ijk} \in \bar{R} \mid c_{ij} \notin \bar{V}\}, \quad \delta = |\bar{W}| + |\bar{V}|.$$

Choose M to be a positive integer such that $M \geq 1/\pi\epsilon$. Let the period T and frequency ω of the controls be

$$T = \frac{t_f}{(6\beta + 3\delta)(M+1) + 1/2}, \qquad \omega = \frac{2\pi}{T}.$$

Then use the controls defined in Components 1(i), 2(ii) and 2(iii) of Appendix B to perform the following iterations:

1. $i = 0$.
2. $i = i + 1$, $j = i$.
3. $j = j + 1$, $k = j$.
4. $k = k + 1$. If $k = m + 1$ go to 8.

5. If $c_{ijk} = c_{ikj} = 0$ go to 7.

6. Apply Component 2(iii) for c_{ijk} and c_{ikj}.

7. If $k \leq m \Leftrightarrow 1$ go to 4.

8. If $c_{ij} = c_{iji} = c_{ijj} = 0$ go to 10.

9. Apply Component 2(ii) for c_{ij}, c_{iji} and c_{ijj}.

10. If $j \leq m \Leftrightarrow 1$ go to 3.

11. If $i < m \Leftrightarrow 1$ go to 2.

12. Apply Component 1(i) for c_q for $q = 1, \ldots, m$.

Then we are done and $\hat{d}(X(t_f), X_f) = O(\epsilon^3)$ as desired.

The proof is completed by noting that for $0 < p' < p$ and $p = 2, 3$, the p'th-order average solution $X^{(p')}$ captures system behavior that includes only up to $(p' \Leftrightarrow 1)$ Lie brackets. Thus, for a depth-$(p \Leftrightarrow 1)$ bracket system, $p = 2, 3$, $X^{(p')}$ cannot be controlled as desired. □

The arguments of this section are reminiscent of [1], where Brockett works with piecewise constant controls.

5 Examples

Consider the autonomous underwater vehicle motion control problem described in Section 1. The Wei-Norman equations for $SE(3)$ with our chosen basis for $se(3)$ are

$$\begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \\ \dot{g}_4 \\ \dot{g}_5 \\ \dot{g}_6 \end{bmatrix} = \begin{bmatrix} \sec g_2 \cos g_3 & \Leftrightarrow \sec g_2 \sin g_3 & 0 & 0 & 0 & 0 \\ \sin g_3 & \cos g_3 & 0 & 0 & 0 & 0 \\ \Leftrightarrow \tan g_2 \cos g_3 & \tan g_2 \sin g_3 & 1 & 0 & 0 & 0 \\ 0 & \Leftrightarrow g_6 & g_5 & 1 & 0 & 0 \\ g_6 & 0 & \Leftrightarrow g_4 & 0 & 1 & 0 \\ \Leftrightarrow g_5 & g_4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}. \qquad (63)$$

The parameters g_1, g_2, g_3 correspond to Euler-angle type parameters and parametrize the orientation of the vehicle. The parameters g_4, g_5, g_6 parametrize the position of the vehicle.

AUV with Four Controls: First, suppose that we can control all components of angular velocity as well as one translational velocity component, i.e., $\{A_1, A_2, A_3, A_4\}$ describes our control authority. Then, $X(t)$ describes the orientation and position of the vehicle and satisfies

$$\dot{X} = \epsilon X(\sum_{i=1}^{4} u_i(t) A_i), \tag{64}$$

where $\{A_1, \ldots, A_6\}$ is the basis for $se(3)$ defined by (7). This system is a depth-one bracket system since $[A_3, A_4] = A_5$ and $[A_4, A_2] = A_6$. We have $n = 6$, $m = 4$, and the nonzero structure constants corresponding to our chosen basis for $se(3)$ can easily be computed as $\Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 = 1$, $\Gamma_{15}^6 = \Gamma_{61}^5 = 1$, $\Gamma_{42}^6 = \Gamma_{26}^4 = 1$, $\Gamma_{34}^5 = \Gamma_{53}^4 = 1$.

Following Algorithm 1, we compute

$$\Gamma = \begin{bmatrix} \Gamma_{12}^5 & \Gamma_{13}^5 & \Gamma_{14}^5 & \Gamma_{23}^5 & \Gamma_{24}^5 & \Gamma_{34}^5 \\ \Gamma_{12}^6 & \Gamma_{13}^6 & \Gamma_{14}^6 & \Gamma_{23}^6 & \Gamma_{24}^6 & \Gamma_{34}^6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

$$\Gamma^{\dagger} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}^T.$$

Therefore,

$$\begin{aligned} \begin{bmatrix} c_{24} \\ c_{34} \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} g_{f_5} \\ g_{f_6} \end{bmatrix} = \begin{bmatrix} -g_{f_6} \\ g_{f_5} \end{bmatrix}, \\ c_{12} &= c_{13} = c_{14} = c_{23} = 0, \\ c_1 &= g_{f_1} - c_{24}\Gamma_{24}^1 - c_{34}\Gamma_{34}^1 = g_{f_1}, \\ c_2 &= g_{f_2} - c_{24}\Gamma_{24}^2 - c_{34}\Gamma_{34}^2 = g_{f_2}, \\ c_3 &= g_{f_3} - c_{24}\Gamma_{24}^3 - c_{34}\Gamma_{34}^3 = g_{f_3}, \\ c_4 &= g_{f_4} - c_{24}\Gamma_{24}^4 - c_{34}\Gamma_{34}^4 = g_{f_4}, \\ \bar{S} &= \{2, 3\}, \quad r = 2. \end{aligned}$$

So we choose an integer $M \geq 1/\pi\epsilon$ and

$$T = \frac{t_f}{2(M+1) + 1/2}, \quad \omega = \frac{2\pi}{T}.$$

Then we apply Component 1(ii) for c_{24} followed by Component 1(ii) for c_{34} followed by Component 1(i) for c_1, c_2, c_3, c_4. To reduce the time and energy expended by the controls

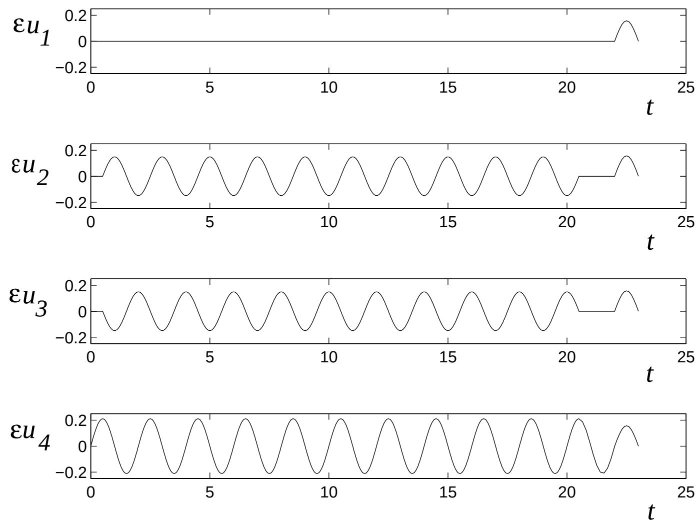

Figure 1: Control Input Signals for AUV Example with Four Controls.

we can instead apply Component 1(ii) just once. To do this let $i = 4$, $j = 2, 3$, $c_{42} = \Leftrightarrow c_{24}$, $c_{43} = \Leftrightarrow c_{34}$ and apply Component 1(ii) to match c_{42} *and* c_{43}. In this case we have $\bar{S} = \{4\}$ and $r = 1$ so we recompute

$$T = \frac{t_f}{M + 3/2}, \quad \omega = \frac{2\pi}{T}.$$

For numerical illustration, let $\epsilon = 0.1$, $g_{f_i} = 0.1$, $i = 1, \ldots, 6$, and $t_f = 23$. Choose $M = 10$, then $T = 2$, $\omega = \pi$. Figure 1 shows plots of the corresponding controls ϵu_1, ϵu_2, ϵu_3, ϵu_4 as a function of time. Figure 2 shows a simulation of the response of the Wei-Norman parameters g as a function of time. The simulation was produced by numerically solving the equations (63) using MATLAB. The horizontal dashed lines of Figure 2 represent the desired final parameter value g_f. Figure 2 shows that $g(t_f) \Leftrightarrow g_f = O(\epsilon^2)$ and equivalently $Z(t_f) \Leftrightarrow Z_f = O(\epsilon^2)$ as expected. By the results of Lazard and Tits (see Section 2) for $\mathcal{G} = se(3)$ we can let $\hat{S} = \{A \in se(3) \mid \|A\|_{\bar{p}} < \pi/2\}$ for any $\bar{p}$. From Figure 2 it is clear that $Z_f \in \hat{S}$ and $Z(t_f) \in \hat{S}$ for some choice of $\bar{p}$. Thus, $\|Z(t_f) \Leftrightarrow Z_f\| = O(\epsilon^2)$ implies $\hat{d}(X(t_f), X_f) = O(\epsilon^2)$, i.e., the AUV has been repositioned and reoriented as desired with $O(\epsilon^2)$ accuracy.

AUV with Three Controls: Now consider the case when there are only three controls available, e.g., suppose that due to an actuator failure, the third component of angular

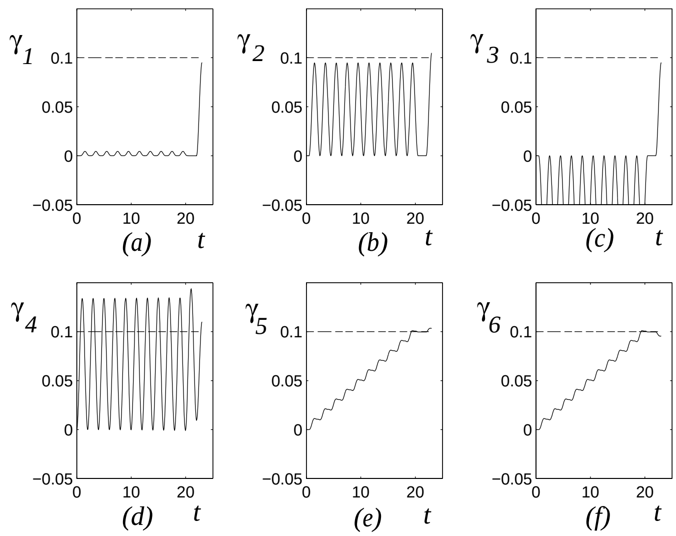

Figure 2: Response of AUV with Four Controls.

velocity can no longer be directly actuated. We can use our algorithms to adapt on-line by computing new controls based on the new relevant structure constants. Our new control authority is defined by $\{A_1, A_2, A_4\}$ and $X(t) \in SE(3)$ satisfies

$$\dot{X} = \epsilon X(u_1 A_1 + u_2 A_2 + u_4 A_4), \tag{65}$$

which is a depth-two bracket system since $[A_1, A_2] = A_3$, $[A_4, A_2] = A_6$ and $[[A_1, A_2], A_4] = A_5$. For the purposes of the algorithm, we reorder our basis for $se(3)$ such that $A_3 \Leftrightarrow A_4$ and $A_5 \Leftrightarrow A_6$. The nonzero structure constants associated with this reordered basis become $,^{4}_{12} = ,^{1}_{24} = ,^{2}_{41} = 1$, $,^{5}_{16} = ,^{6}_{51} = 1$, $,^{5}_{32} = ,^{3}_{25} = 1$, $,^{6}_{43} = ,^{3}_{64} = 1$. Further, $\theta^6_{123} = 1$. Thus, $n = 6$, $m = 3$, and so by (62), $\Xi = \{4, 5\}$ and $l = |\Xi| = 2$. Thus, we get

$$\begin{aligned} \theta &= [\theta^6_{121} \;\; \theta^6_{122} \;\; \theta^6_{123} \;\; \theta^6_{131} \;\; \theta^6_{132} \;\; \theta^6_{133} \;\; \theta^6_{231} \;\; \theta^6_{232} \;\; \theta^6_{233}] \\ &= [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; \Leftrightarrow 1 \;\; 0 \;\; 0], \\ , &= \begin{bmatrix} ,^4_{12} & ,^4_{13} & ,^4_{23} \\ ,^5_{12} & ,^5_{13} & ,^5_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \Leftrightarrow 1 \end{bmatrix}. \end{aligned}$$

$c_{123} = z_{f_5}$,
$c_{121} = c_{122} = c_{231} = c_{232} = c_{233} = c_{131} = c_{132} = c_{133} = 0.$

Note that z_{f_5} is based on the original ordering of the basis for $se(3)$ and so it is the coefficient of A_6 in the reordered basis. Further,

$$\begin{aligned}
&\begin{bmatrix} c_{12} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Leftrightarrow 1 \end{bmatrix} \begin{bmatrix} z_{f_3} \Leftrightarrow c_{123}\theta^4_{123} \\ z_{f_6} \Leftrightarrow c_{123}\theta^5_{123} \end{bmatrix} = \begin{bmatrix} z_{f_3} \\ \Leftrightarrow z_{f_6} \end{bmatrix}, \\
&c_{13} = 0, \\
&c_1 = z_{f_1} \Leftrightarrow (c_{12},\, {}^{1}_{12} + c_{23},\, {}^{1}_{23} + c_{123}\theta^1_{123}) = z_{f_1}, \\
&c_2 = z_{f_2} \Leftrightarrow (c_{12},\, {}^{2}_{12} + c_{23},\, {}^{2}_{23} + c_{123}\theta^2_{123}) = z_{f_2}, \\
&c_3 = z_{f_4} \Leftrightarrow (c_{12},\, {}^{3}_{12} + c_{23},\, {}^{3}_{23} + c_{123}\theta^3_{123}) = z_{f_4}, \\
&\bar{Y} = \{c_{123}\}, \quad \bar{Q} = \{c_{123}\}, \quad \bar{R} = \emptyset, \quad \bar{V} = \{c_{12}, c_{23}\}, \\
&\bar{W} = \emptyset, \quad \beta = 1, \quad \gamma = 2.
\end{aligned}$$

So we choose an integer $M \geq 1/\pi\epsilon$ and

$$T = \frac{t_f}{12(M+1) + 1/2}, \qquad \omega = \frac{2\pi}{T}.$$

Then we apply Component 2(iii) for c_{123}, followed by Component 2(ii) for c_{12}, followed by Component 2(ii) for c_{23}, followed by Component 2(i) for c_1, c_2, c_3. These components will specify controls u_1, u_2 and u_3. However, u_3 is really our original control u_4 since it is the coefficient of the original A_4.

For this particular system we note that the execution of the algorithm is longer than necessary, i.e., there are steps which have zero net effect on the system. Thus, to save time and energy we eliminate the unnecessary steps of the control algorithm defined above. The total time duration of the parts left out is $9(M+1)T$ so we recompute

$$T = \frac{t_f}{3(M+1) + 1/2}, \quad \omega = \frac{2\pi}{T}.$$

For numerical illustration, let $\epsilon = 0.2$, $t_f = 37$ and $g_{f_1} = 0.05$, $g_{f_2} = 0.05$, $g_{f_3} = 0.04$, $g_{f_4} = 0.06$, $g_{f_5} = 0.05$, $g_{f_6} = 0.05$ (recalling from Lemma 2 that for the algorithm that we can set $z_f = g_f$). Choose $M = 5$, then $T = 2$, $\omega = \pi$. Figure 3 shows plots of the corresponding controls ϵu_1, ϵu_2 and ϵu_4 as a function of time. Figure 4 shows a simulation of the response of the Wei-Norman parameters g as a function of time. The horizontal dashed lines of Figure 4 represent the desired final parameter values g_f. Figure 4 shows that $g(t_f) \Leftrightarrow g_f = O(\epsilon^3)$ as expected. We conclude that the AUV has been repositioned and reoriented as desired with $O(\epsilon^3)$ accuracy. Reorientation with only roll and pitch actuators was demonstrated experimentally using this algorithm on an underwater vehicle in the neutral buoyancy facility of the Space Systems Laboratory at the University of Maryland. For details see [17].

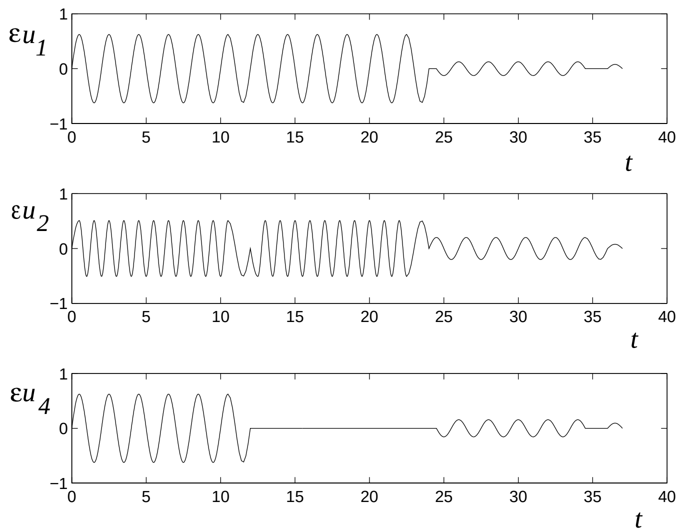

Figure 3: Control Input Signals for AUV Example with Three Controls.

6 Conclusions

We have derived average formulas for the solution to (2) and used them to specify small-amplitude periodic controls that solve the complete constructive controllability problem **(P)** approximately (exactly if the system is nilpotent). We have shown that the smallest order of the average formula sufficient to solve **(P)** is one more than the number of Lie bracket iterations needed for the system to satisfy the Lie algebra controllability rank condition. The results were developed for the pth-order average approximation where $p = 2, 3$; however, the general pth-order average theorem is stated in Appendix A. The proof of the controllability result is constructive and was given in the form of algorithms for generating open-loop controls. Structure constants, which define the control authority of the system, drive the algorithms. A change in control authority such as an actuator failure may be described by a change in structure constants and, thus, can be accommodated on-line using the algorithms. One might consider the algorithms of this paper to be a "motion script" generator. Thus, a change in structure results in a change in script.

Averaging theory for systems on Lie groups also holds promise for understanding and controlling systems with drift. That is, while the algorithms derived in this paper are valid

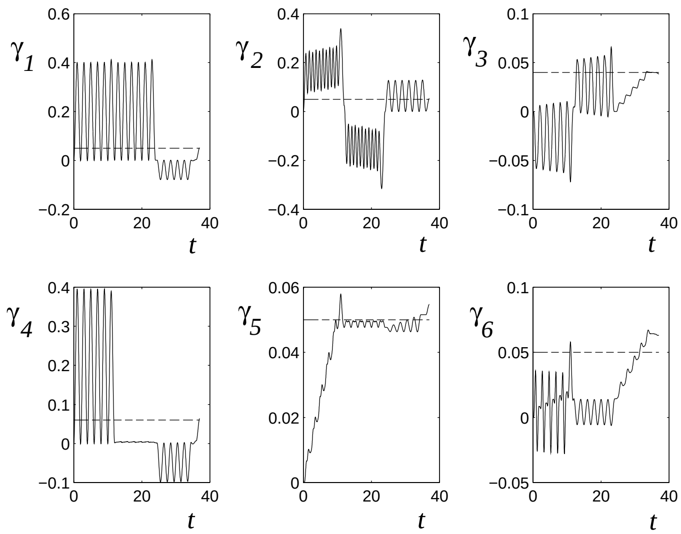

Figure 4: Response of AUV with Three Controls.

only for drift-free systems of the form (2), the averaging theory here does not rely on the drift-free assumption. We have already applied our averaging theory on Lie groups to the problem of controlling a class of switched electrical networks which can be modelled as systems on Lie groups with drift [19].

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Appendix A

In this appendix we state the general pth-order averaging theorem and constructive controllability theorem. The proofs can be found in [16]. The theorems make use of the following recursive formula given by Fomenko and Chakon [7] for the terms in the infinite series expression for $Z(t)$ where $X(t) = e^{Z(t)}$

Theorem 8 (Fomenko and Chakon) Let $\delta = \hat{b}/M$, where $M \geq 1$ is a constant such that $\|[A, B]\| \leq M\|A\|\|B\|$, $\forall A, B \in \mathcal{G}$ and $\hat{b}$ is a universal constant. Suppose that $U(t)$ is a piecewise continuous curve in $\mathcal{G}$ and $\int_0^t \|\epsilon U(\tau)\| d\tau < \delta$. Then $Z(t) = \sum_{i=1}^{\infty} \epsilon^i Z_i(t)$ is a convergent series. The terms $Z_i(t)$ are uniquely defined by

$$\begin{aligned} Z_1(t) = T_0(t) &= \int_0^t U(\tau) d\tau, \\ (i+1) Z_{i+1}(t) &= T_i + \sum_{r=1}^{i} \{ \frac{1}{2} [Z_r, T_{i-r}] \\ &\quad + \sum_{q \geq 1,\, 2q \leq r} k_{2q} \sum_{\sum_{j=1}^{2q} m_j = r,\, m_j > 0} [Z_{m_1}, [Z_{m_2}, \ldots, [Z_{m_{2q}}, T_{i-r}] \ldots]\}, \\ T_k(t) &= \int_0^t [U(\tau_1), \int_0^{\tau_1} [U(\tau_2), \ldots, \int_0^{\tau_k} U(\tau_{k+1}) d\tau_{k+1}] d\tau_k] \ldots] d\tau_1. \end{aligned} \tag{66}$$

We note that each term Z_i is composed of depth-$(i \Leftrightarrow 1)$ brackets.

Theorem 9 (pth-Order Averaging: Area-Moment Rule) Consider system (2) on the Lie group G with Lie algebra $\mathcal{G}$. Assume that $U(t)$ is a piecewise continuous, bounded curve in $\mathcal{G}$. Let $b > 0$ be such that $\int_0^t \|U(\tau)\| d\tau < \delta$, $\forall t \in [0, b]$, where δ is as defined in Theorem 8. Further, assume that $U(t)$ is periodic in t of period T, $\forall t \in [0, \infty)$. Let $p \geq 1$ be an integer. For $p > 1$ assume that $T_k(T) = 0$, $k = 0, \ldots, p \Leftrightarrow 2$, where $T_k(t)$ is defined by (66). Suppose that $X(0) = X_0 \in \hat{Q} \subset G$ is such that $Z_0 = \hat{\Psi}^{-1}(X_0) = O(\epsilon^{p-1})$ if $p > 1$ and $Z_0 = O(\epsilon)$ if $p = 1$. Define

$$Z^{(p)}(t) = \sum_{i=1}^{p-1} (\Leftrightarrow 1)^{i+1} \epsilon^i Z_i(t) + (\Leftrightarrow 1)^{p+1} \epsilon^p \frac{t}{T} Z_p(T) + Z_0^{(p)}, \tag{67}$$

$$X^{(p)}(t) = e^{Z^{(p)}(t)}, \tag{68}$$

where $Z_i(t)$ are defined by (66). If $\|Z_0 \Leftrightarrow Z_0^{(p)}\| = O(\epsilon^p)$ and $Z^{(p)}(t) \in \hat{S}$, $\forall t \in [0, b/\epsilon]$, then

$$\hat{d}(X(t), X^{(p)}(t)) = O(\epsilon^p), \quad \forall t \in [0, b/\epsilon]. \tag{69}$$

Further, for $t = NT$, N an integer,

$$Z^{(p)}(NT) = (\Leftrightarrow 1)^{p+1}\epsilon^p N Z_p(T) + Z_0^{(p)}. \tag{70}$$

Remark 10 Further explicit decomposition of $Z^{(p)}$ into terms like areas and structure constants can be found in [16] imitating the argument for $Z^{(2)}$ in (43) and $Z^{(3)}$ in (49).

Theorem 11 Suppose that system (2) on the connected Lie group G with Lie algebra $\mathcal{G}$ is a depth-p' bracket system. Let $p = p' + 1$. Then the complete constructive controllability problem $(\mathbf{P})$ can be solved with $O(\epsilon^p)$ accuracy using the formulas $X^{(r)}(t)$ given by (68) for $r = 1, \ldots, p$. Further, p is the minimum positive integer such that this is true.

Appendix B

In this appendix, we present the components used in the algorithms of this paper. The sinusoidal controls are typically sub-optimal. However, given the chosen sinusoidal structure of the controls, the amplitudes are selected to minimize energy (integral of sum of squares of inputs). In the following control laws, if a control component is not explicitly prescribed it should be set equal to zero.

Component 1(i)

Given: c_k, $k = 1, \ldots, m$, T, ω and current time t_0.

Goal: Let $t_1 = t_0 + T/2$. We define $\epsilon u_k(t)$, $k = 1, \ldots, m$, $t \in [t_0, t_1]$, continuous, such that $\epsilon \tilde{u}_k(t_1) = c_k$ and $\epsilon u_k(t_1) = \epsilon u_k(t_0) = 0$.

Controls:

$$\epsilon u_k(t) = \frac{1}{2} c_k \omega sin(\omega(t \Leftrightarrow t_0)), \qquad t_0 \le t \le t_1.$$

Component 1(ii)

Given: c_{ij}, for i and $j = i + 1, \ldots, m$, T, ω, M and current time t_0.

Goal: Let $t_1 = t_0 + (M+1)T$. Let

$$\begin{aligned} \alpha_i &= \left(\sum_{j=i+1}^{m} \frac{c_{ij}^2}{\pi^2 M^2} \right)^{1/4}, \\ \alpha_j &= \frac{c_{ij}}{\alpha_i \pi M}, \qquad j = i+1, \ldots, m. \end{aligned}$$

Specify continuous, zero-mean controls $u_i(t)$, $u_j(t)$, $j = i+1, \ldots, m$, $t \in [t_0, t_1]$ such that $\epsilon^2 Area_{ij}(t_1) = c_{ij}$, $\epsilon u_i(t_0) = \epsilon u_i(t_1) = \epsilon u_j(t_0) = \epsilon u_j(t_1) = 0$.

Controls:

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_i \omega \sin(\omega(t \Leftrightarrow t_0)) \\ \epsilon u_j(t) = 0 \end{array}\right\} \; t_0 \le t \le t_0 + \tfrac{T}{4} = s_1$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_i \omega \cos(\omega(t \Leftrightarrow s_1)) \\ \epsilon u_j(t) = \alpha_j \omega \sin(\omega(t \Leftrightarrow s_1)) \end{array}\right\} \; s_1 \le t \le s_1 + MT = s_2$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_i \omega \cos(\omega(t \Leftrightarrow s_2)) \\ \epsilon u_j(t) = 0 \end{array}\right\} \; s_2 \le t \le s_2 + \tfrac{3T}{4} = t_1$$

Note that $\epsilon^2 Area_{ij}(t_1) = \alpha_i \alpha_j \pi M = c_{ij}$ and the goal is met.

<u>Component 2(ii)</u>

Given: $i < j$, c_{ij}, c_{iji}, c_{ijj}, T, ω, M and current time t_0.

Goal: Let $t_1 = t_0 + 3(M+1)T$ and

$$\begin{array}{lll} \alpha_{i1} = \sqrt{\frac{c_{ij}}{\pi M}}, & \alpha_{i2} = \left(\frac{32 c_{iji}^2}{\pi^2 M^2} \right)^{1/6}, & \alpha_{i3} = \left(\frac{32 c_{ijj}^2}{\pi^2 M^2} \right)^{1/6}, \\ \alpha_{j_1} = \frac{c_{ij}}{\alpha_{i1} \pi M}, & \alpha_{j_2} = \frac{2 c_{iji}}{\alpha_{i2}^2 \pi M}, & \alpha_{j3} = \frac{-2 c_{ijj}}{\alpha_{i3}^2 \pi M}. \end{array}$$

Specify continuous, zero-mean controls $u_i(t)$ and $u_j(t)$, $t \in [t_0, t_1]$, such that $\epsilon^3 Area_{ij}(t_1) = c_{ij}$, $\epsilon^3 m_{iji}(t_1) = \Leftrightarrow c_{iji}$, $\epsilon^3 m_{ijj}(t_1) = \Leftrightarrow c_{ijj}$ and $\epsilon u_i(t_1) = \epsilon u_i(t_0) = \epsilon u_j(t_1) = \epsilon u_j(t_0) = 0$.

Controls:

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i1} \omega \sin(\omega(t \Leftrightarrow t_0)) \\ \epsilon u_j(t) = 0 \end{array}\right\} \; t_0 \le t \le t_0 + \frac{T}{4} = s_1$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i1} \omega \cos(\omega(t \Leftrightarrow s_1)) \\ \epsilon u_j(t) = \alpha_{j_1} \omega \sin(\omega(t \Leftrightarrow s_1)) \end{array}\right\} \; s_1 \le t \le s_1 + MT = s_2$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i1} \omega \cos(\omega(t \Leftrightarrow s_2)) \\ \epsilon u_j(t) = 0 \end{array}\right\} \; s_2 \le t \le s_2 + \frac{3T}{4} = s_3$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i2}\omega \sin(\omega(t \Leftrightarrow s_3)) \\ \epsilon u_j(t) = 2\alpha_{j_2}\omega \sin(\omega(t \Leftrightarrow s_3)) \end{array}\right\} \quad s_3 \le t \le s_3 + \frac{T}{4} = s_4$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i2}\omega \cos(\omega(t \Leftrightarrow s_4)) \\ \epsilon u_j(t) = 2\alpha_{j_2}\omega \cos(2\omega(t \Leftrightarrow s_4)) \end{array}\right\} \quad s_4 \le t \le s_4 + MT = s_5$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \alpha_{i2}\omega \cos(\omega(t \Leftrightarrow s_5)) \\ \epsilon u_j(t) = 2\alpha_{j_2}\omega \cos(\omega(t \Leftrightarrow s_5)) \end{array}\right\} \quad s_5 \le t \le s_5 + \frac{3T}{4} = s_6$$

$$\left.\begin{array}{l} \epsilon u_i(t) = 2\alpha_{j_3}\omega \sin(\omega(t \Leftrightarrow s_6)) \\ \epsilon u_j(t) = \alpha_{i3}\omega \sin(\omega(t \Leftrightarrow s_6)) \end{array}\right\} \quad s_6 \le t \le s_6 + \frac{T}{4} = s_7$$

$$\left.\begin{array}{l} \epsilon u_i(t) = 2\alpha_{j_3}\omega \cos(2\omega(t \Leftrightarrow s_7)) \\ \epsilon u_j(t) = \alpha_{i3}\omega \cos(\omega(t \Leftrightarrow s_7)) \end{array}\right\} \quad s_7 \le t \le s_7 + MT = s_8$$

$$\left.\begin{array}{l} \epsilon u_i(t) = 2\alpha_{j_3}\omega \cos(\omega(t \Leftrightarrow s_8)) \\ \epsilon u_j(t) = \alpha_{i3}\omega \cos(\omega(t \Leftrightarrow s_8)) \end{array}\right\} \quad s_8 \le t \le s_8 + \frac{3T}{4} = s_9$$

The condition on $Area_{ij}(t)$ is met during the time interval $[0, s_3]$, the condition on $m_{iji}(t)$ is met during $[s_3, s_6]$ and the condition on $m_{ijj}(t)$ is met during $[s_6, s_9]$.

Component 2(iii)

Given: $i < j < k$, c_{ijk}, c_{ikj}, T, ω, M and current time t_0.

Goal: Let $t_1 = t_0 + 6(M+1)T$. Let $d_1 = 2(\frac{2}{3}c_{ijk} + \frac{1}{3}c_{ikj})/\pi M$ and $d_2 = 2(\frac{1}{3}c_{ijk} + \frac{2}{3}c_{ikj})/\pi M$.

Select

$$\begin{array}{lll} \rho_{j_1} = \left(\frac{d_1}{6}\right)^{1/3}, & \rho_{i_1} = \left(\left|\frac{d_1}{\rho_{j_1}}\right|\right)^{1/2}, & \rho_{k_1} = \frac{d_1}{\rho_{i_1}\rho_{j_1}}, \\ \rho_{j_2} = \left(\frac{d_2}{6}\right)^{1/3}, & \rho_{i_2} = \left(\left|\frac{d_2}{\rho_{j_2}}\right|\right)^{1/2}, & \rho_{k_2} = \frac{d_2}{\rho_{i_2}\rho_{j_2}}. \end{array}$$

We specify continuous, zero-mean controls $u_i(t)$, $u_j(t)$ and $u_k(t)$, $t \in [t_0, t_1]$, such that $\epsilon^3(2m_{ijk}(t_1) \Leftrightarrow m_{ikj}(t_1)) = \Leftrightarrow c_{ijk}$, $\epsilon^3(2m_{ikj}(t_1) \Leftrightarrow m_{ijk}(t_1)) = \Leftrightarrow c_{ikj}$, $\epsilon u_i(t_1) = \epsilon u_i(t_0) = \epsilon u_j(t_1) = \epsilon u_j(t_0) = \epsilon u_k(t_1) = \epsilon u_k(t_0) = 0$. Further, $Area_{ij}(t_1) = Area_{ik}(t_1) = Area_{jk}(t_1) = 0$.

Controls:

$$\left.\begin{array}{l} \epsilon u_i(t) = \rho_{i_1}\omega \sin(\omega(t \Leftrightarrow t_0)) \\ \epsilon u_j(t) = 2\rho_{j_1}\omega \sin(\omega(t \Leftrightarrow t_0)) \\ \epsilon u_k(t) = \rho_{k_1}\omega \sin(\omega(t \Leftrightarrow t_0)) \end{array}\right\} \quad t_0 \le t \le t_0 + \frac{T}{4} = s_1$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \rho_{i_1}\omega \cos(\omega(t \Leftrightarrow s_1)) \\ \epsilon u_j(t) = 2\rho_{j_1}\omega \cos(2\omega(t \Leftrightarrow s_1)) \\ \epsilon u_k(t) = \rho_{k_1}\omega \cos(\omega(t \Leftrightarrow s_1)) \end{array}\right\} \quad s_1 \le t \le s_1 + MT = s_2$$

$$\left.\begin{array}{l} \epsilon u_i(t) = \rho_{i1}\omega\cos(\omega(t \Leftrightarrow s_2)) \\ \epsilon u_j(t) = 2\rho_{j1}\omega\cos(\omega(t \Leftrightarrow s_2)) \\ \epsilon u_k(t) = \rho_{k1}\omega\cos(\omega(t \Leftrightarrow s_2)) \end{array}\right\} \quad s_2 \leq t \leq s_2 + \frac{3T}{4} = s_3$$

The condition on $m_{ijk}(t)$ is met during the time interval $[0, s_3]$. However, the values of $m_{iji}(t)$ and $m_{jkk}(t)$ at $t = s_3$ may be different from their initial condition.

So repeat the controls above replacing t_0 with s_3, s_1 with s_4, s_2 with s_5 and s_3 with s_6. Also, replace ρ_{j_1} by $\Leftrightarrow\rho_{j_1}$ and set $\epsilon u_k(t) = 0,\ t \in [s_3, s_6]$. During $[s_3, s_6]$, the original value of $m_{iji}(t)$ is restored. Repeat the controls above again, this time replacing t_0 with s_6, s_1 with s_7, s_2 with s_8 and s_3 with s_9. Also, replace ρ_{j_1} by $\Leftrightarrow\rho_{j_1}$ and set $\epsilon u_i(t) = 0,\ t \in [s_6, s_9]$. Then, during $[s_6, s_9]$ the original value of $m_{jkk}(t)$ is restored.

Finally, rerun the entire set of controls for $t \in [t_0, s_9]$, exchanging the roles of j and k, augmenting the indices of the time intervals appropriately. Also, replace ρ_{i1} by ρ_{i2}, ρ_{j_1} by ρ_{j_2} and ρ_{k1} by ρ_{k2}. Then $t_1 = s_{18}$ and $\epsilon^3(2m_{ijk}(t_1) \Leftrightarrow m_{ikj}(t_1)) = \Leftrightarrow c_{ijk}$, $\epsilon^3(2m_{ikj}(t_1) \Leftrightarrow m_{ijk}(t_1)) = \Leftrightarrow c_{ikj}$. Thus, the goal is met.