Motion detection, the Wigner distribution function, and the optical fractional Fourier transform

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It is shown that both surface tilting and translational motion can be independently estimated by use of the speckle photographic technique by capturing consecutive images in two different fractional Fourier domains. A geometric interpretation, based on use of the Wigner distribution function, is presented to describe this application of the optical fractional Fourier transform when little prior information is known about the motion.

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Speckle photography is a practical means of extracting in-plane translation and tilting motion information by use of captured intensity information.[1,2] Tilt measurement, for example, involves the capture of the optical Fourier transform of the reflected surface field. Adding or subtracting two sequential images and numerically calculating the Fourier transform (FT) of the result produces a set of interference fringes whose period is inversely proportional to any constant shift in field spatial frequency and so in surface tilt. In this case no information is known about surface translation. Angular velocity and acceleration can be found based on a series of sequentially captured images. The resolution and the dynamical range of detectable movements are fixed and depend on the wavelength used, the reflected field speckle size (including decorrelation effects), and the measurement system used, e.g., the system's point-spread function and the CCD's pixel size and sensitivity. The time resolution depends on the speed of the camera used.

The fractional Fourier transform (FRT), which is a generalization of the FT, has received a great deal of attention in the optics literature.[3–7] The combination of the holographic interferometric principle and an optical implementation of the FRT was shown previously to permit simultaneous tilt and in-plane translation detection.[8] It was also shown that using an optical FRT system permits speckle photography to be extended to allow for simultaneous measurement of mixed translation and tilt movement.[9] Furthermore, it has been experimentally demonstrated[10] that the extra degree of freedom made available by the use of an optical implementation of the FRT permits controlled variation of the minimum resolution and dynamical range of measurement of tilting (rotational) motion. The experimental results were achieved with an optical FRT system called a fake-zoom lens.[11] Varying the distances between the lenses permits the generation of several fractional order planes with constant magnification (scale factor).

The Wigner distribution function[12,13] (WDF) has been shown to be of practical value for the understanding of optical signal-processing systems. Wigner's representation uses both spatial frequency and position simultaneously to describe the optical field. The Wigner representation of a field $u(x)$ can be defined as

$$W(x,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y \exp(-jky)u^{*}(x-y/2)u(x+y/2). \tag{1}$$

It is a pseudo distribution function, and $W(x,k)$ can have negative values. To find the intensity, $I(x) = |u(x)|^2$, we integrate $W(x,k)$ over k. Similarly, to find the spatial frequency distribution, $\tilde{I}(k) = |\mathrm{FT}[u(x)]|^2 = |\tilde{U}(k)|^2$, we integrate $W(x,k)$ over x.

Previous publications[8–10] have reported that both tilt and translation could be simultaneously determined because it was assumed either (i) that a fixed linear relationship existed between the tilting motion and the translation motion such that measurement in a single fractional domain could be specified in the speckle photography setup or (ii) that a reference field was available (from, for example, a hologram) that permitted the retrieval of phase information. In these ways the translation and tilting motions could both be completely determined.

In this Letter we show that, even if no fixed relationship between tilt and translation is known and no reference field is available, the motion of a surface can still be found by use of a variable FRT-based speckle photographic system. For simplicity in our analysis we assume one-dimensional fields and we use the WDF to provide a geometrical interpretation of the results.

In Fig. 1 a WDF, $W(x,k)$, which we use to designate a field reflected from a surface, is designated by a contour centered at the origin $(0,0)$. If the surface moves slightly, the corresponding WDF may be presented as simply a shifted version of the initial WDF to coordinates (ξ,κ) in phase space, becoming $W(x-\xi, k-\kappa)$, to correspond to a translation of magnitude ξ combined with a shift in spatial frequency of size κ. This motion can be described in many exactly equivalent

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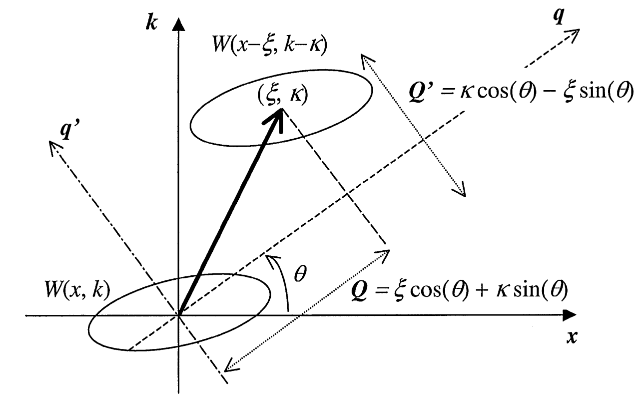

Fig. 1. Shift of a WDF in phase space and the F_θ plane.

ways: For example, in the spatial domain,

$$u(x) \rightarrow u(x - \xi)\exp(+j\kappa x). \tag{2a}$$

In the spatial frequency domain,

$$U(k) = F[u(x)] \rightarrow U(k - \kappa)\exp(+j\xi k), \tag{2b}$$

whereas, as already indicated, in phase space we can write that

$$W(x, k) \rightarrow W(x - \xi, k - \kappa). \tag{2c}$$

In the FRT domain we can describe the motion by using the normalized parameter definition of the FRT of angle θ.[14,15] Initially the field in this FRT domain is given by

$$F_\theta[u(x)](q) = U_\theta(q) = \frac{1}{(2\pi|\sin\theta|)^{1/2}} \times \exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2} + J\left(\frac{\theta}{\pi}\right)\right] + \frac{j}{2}q^2\cot\theta\right\}\int_{-\infty}^{+\infty} u(x) \times \exp\left(+\frac{j}{2}x^2\cot\theta - jqx\csc\theta\right)\mathrm{d}x. \tag{3}$$

Following motion, the projection on the fractional Fourier axis is given by

$$F_\theta[u(x-\xi)\exp(j\kappa x)](q) = \frac{1}{(2\pi|\sin\theta|)^{1/2}} \times \exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2} + J\left(\frac{\theta}{\pi}\right)\right] + \frac{j}{2}\cot\theta\left[q^2 + \xi^2 - 2\frac{q\xi}{\cos\theta} + 2\kappa\xi\tan\theta\right]\right\} \times \int_{-\infty}^{+\infty} u(y)\exp\left[+\frac{j}{2}y^2\cot\theta - j(q-Q)y\csc\theta\right]\mathrm{d}y, \tag{4}$$

where we have used the substitution $y = x - \xi$.

Comparing Eqs. (3) and (4), we can see that the FRT of the original field, $U_\theta(q)$, has been multiplied by a phase factor and been shifted by an amount $Q = \xi\cos\theta + \kappa\sin\theta$. From Fig. 1 we identify this shift distance in q as the projection onto q of the actual shift distance $\sqrt{\xi^2 + \kappa^2}$. We can now write in addition to Eqs. (2) that

$$U_\theta(q) \rightarrow U(q - Q)\exp[+j\Phi(q)], \tag{5a}$$

where

$$\Phi(q) = q\cot\theta\left(Q - \frac{\xi}{\cos\theta}\right) + \frac{\cot\theta}{2}\left(\xi^2 - Q^2\right) + \kappa\xi = qQ' - \frac{QQ'}{2} + \frac{\kappa\xi}{2}. \tag{5b}$$

We note that $Q' = \kappa\cos\theta - \xi\sin\theta$; see Fig. 1. We further note that $Q' = \mathrm{d}Q/\mathrm{d}\theta$ and that $\sqrt{Q^2 + Q'^2} = \sqrt{\xi^2 + \kappa^2}$. In the special case when the shift in the WDF is parallel to q, $\tan\theta = \xi/\kappa$, $Q = \sqrt{\xi^2 + \kappa^2}$, and $Q' = 0$. In this case $\Phi(q) = \kappa\xi/2$, which was previously identified as significant in fractional-Fourier-based holographic interferometry.[8]

Following the usual speckle photographic procedure, we subtract the resultant intensities, i.e., the absolute values of Eqs. (2) and (4) captured in the FRT plane, and take the FT of the result, which yields

$$|\mathrm{FT}\{|U_\theta(q)|^2 + |U_\theta(q-Q)\exp[j\Phi(q)]|^2\}| = 2\,\mathrm{FT}[I_\theta(q)](q')\cos(Qq'/2), \tag{6}$$

where $I_\theta(q) = |U_\theta(q)|^2$ and is equal to the integration of $W(x,k)$ over q', the axis that is perpendicular to q; see Fig. 1.

Examining Eq. (6), we see that the value of shift Q along q can be found from the resultant speckle fringe pattern. Inasmuch as θ is also known, the magnitude of the total shift in phase space can be estimated as $\sqrt{\xi^2 + \kappa^2} = Q/\cos\theta$. However, the values of ξ and κ are still not independently known. In general, we require two projections to be able to completely determine the two components of the shift vector. Clearly these projections do not have to be on orthonormal axes.

To acquire two projections we assume that we can vary our FRT angle by an amount $\Delta\theta$ between measurements. In this case $\theta \rightarrow \theta + \Delta\theta$; i.e., we are now projecting the same WDFs onto a different FRT domain. In this case there will be a change in value of Q as the FRT order changes, i.e., $Q \rightarrow Q + \Delta Q = \xi\cos(\theta + \Delta\theta) + \kappa\sin(\theta + \Delta\theta)$. By use of the speckle photographic technique, both Q and $Q + \Delta Q$ can be determined as described above, yielding two simultaneous equations in two unknowns. Solving these, we get that

$$\xi = \frac{Q\sin(\theta + \Delta\theta) - (Q + \Delta Q)\sin(\theta)}{\sin(\Delta\theta)}, \tag{7a}$$

$$\kappa = \frac{-Q\cos(\theta + \Delta\theta) + (Q + \Delta Q)\cos(\theta)}{\sin(\Delta\theta)}. \tag{7b}$$

If we assume that $\Delta\theta$ is small, carrying out Taylor series expansions will yield

$$\xi \approx -\frac{\Delta Q}{\Delta\theta}\sin(\theta) + Q\cos(\theta) - \frac{\Delta\theta}{6}(3Q + \Delta Q)\sin(\theta) + O(\Delta\theta)^3 \approx -\frac{\mathrm{d}Q}{\mathrm{d}\theta}\sin(\theta) + Q\cos(\theta)\,, \tag{8a}$$

$$\kappa \approx \frac{\Delta Q}{\Delta\theta}\cos(\theta) + Q\sin(\theta) + \frac{\Delta\theta}{6}(3Q + \Delta Q)\cos(\theta) + O(\Delta\theta)^3 \approx \frac{\mathrm{d}Q}{\mathrm{d}\theta}\cos(\theta) + Q\sin(\theta)\,. \tag{8b}$$

Clearly we are free to choose a value of θ that simplifies the implementation of our system. If we choose $\theta = \pi/2$, which corresponds to an optical FT, then $\kappa = Q$ and $\xi = -(\mathrm{d}Q/\mathrm{d}\theta)$.

In conclusion, it has been shown that, by optically generating FRT planes of suitable order within a speckle photographic system, one can estimate both the tilting and the translation of the surface. The method requires the capture of four speckle images, two in one fractional domain and two in a second, i.e., $I_\theta(q_1;t_1)$, $I_{\theta+\Delta\theta}(q_2;t_2)$, $I_\theta(q_1;t_3)$, and $I_{\theta+\Delta\theta}(q_2;t_4)$, where for example the time sequence may be of the form $t_1 < t_2 \ll t_3 < t_4$. No discussion of possible techniques to implement a variable-order FRT has been presented here. However, clearly a practical system would require access to an accurate, fast electronically controlled method of FRT order variation. Furthermore, no discussion of speckle size or decorrelation or of the effect of the optics used on the operation of the system, e.g., noise introduction by the optical FRT itself,[16] has been presented. These parameters will be critically important in determining the capabilities of any such system.

We believe that the geometrical method presented here provides a new way to describe and analyze optical metrology systems. Furthermore, it provides physical insights that have allowed us to propose new metrology systems. Initial experimental results have already been presented in the literature,[9] and the practicality of these systems is currently being examined.

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