

Motion of Solids in Fluids when the Flow is not Irrotational.

By G. I. TAYLOR, M.A.

(Communicated by Prof. H. Lamb, F.R.S. Received April 13, 1916.)

The chief interest in the results obtained in the following pages lies in the fact that a mathematical result has been obtained concerning the motion of solids in fluids which is verified accurately when recourse is had to experiment, with real solids moving in real fluids. This is so exceptional a circumstance that it is hoped that the interest which it gives to the mathematical work will serve to extenuate, to a certain extent, the clumsiness of the methods employed.

The problem solved is two-dimensional. An infinite cylindrical body of any cross-section moves in a uniformly rotating fluid with its generators parallel to the axis of rotation. The stream lines and the reaction between the solid and the fluid are found.

Suppose that a stream function ψ' has been found which represents the irrotational motion of an incompressible fluid when a cylindrical solid (or several cylindrical solids) of the required cross-section is moved in an assigned manner starting from rest in a fluid which has a given boundary or has a given irrotational motion at infinity. ψ' is a function of x and y , the co-ordinates of a point in a plane perpendicular to the axis of rotation, and t , the time.

Since the motion is irrotational ψ' satisfies the relation $\nabla^2\psi' = 0$ everywhere, and $-\partial\psi'/\partial s = V_n'$ at the solid boundaries, where V_n' represents the velocity normal to the boundary of a point on the surface of a cylindrical solid moving in the fluid, and $\partial\psi'/\partial s$ represents the rate of change in ψ' measured in a direction along the solid boundary. These, together with the conditions at infinity, if the fluid is unenclosed, are the necessary and sufficient conditions for determining ψ' . The components of velocity of the fluid are then

$$u' = -\partial\psi'/\partial y \quad \text{and} \quad v' = \partial\psi'/\partial x.$$

Now consider the function

$$\psi = \psi' + \frac{1}{2}\omega(x^2 + y^2), \quad (1)$$

where ω is a constant both in regard to space and to time. It satisfies the dynamical equations of motion, $D/Dt(\nabla^2\psi) = 0$ for $\nabla^2\psi = 2\omega$, which is constant; and it is the stream function of the fluid motion obtained when the whole system represented by ψ' is rotated with uniform angular velocity ω

about the origin. The boundary conditions of the rotating system are evidently satisfied if the cylindrical solids move relative to the rotating system in the same way that they moved relative to fixed axes in the case of the motion represented by ψ' . Hence it appears that the system consisting of the cylindrical solids and the fluid in which they move may be rotated uniformly without affecting the motion of the fluid relative to the rotating system, provided the cylinders are constrained to move, relative to the rotating system, in the same way that they moved, relative to fixed axes, when the system was not rotating.* If, however, the solids are free to move under the action of their own inertia and of the pressure of the fluid, the rotation will make a considerable difference to the relative motion of the solids and the fluid. It therefore becomes important to find the pressure at any point.

Let p' be the pressure at the point (x, y) in the irrotational case, and let p be the pressure when the whole system is rotated.

The equations for p and p' are

$$\left. \begin{aligned} -\frac{1}{\rho} \frac{\partial p'}{\partial x} &= \frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} \\ -\frac{1}{\rho} \frac{\partial p'}{\partial y} &= \frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} -\frac{1}{\rho} \frac{\partial p}{\partial x} &= \frac{\delta u}{\delta t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\ -\frac{1}{\rho} \frac{\partial p}{\partial y} &= \frac{\delta v}{\delta t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \end{aligned} \right\} \quad (3)$$

where u and v are the components of velocity in the rotational motion.

The symbol $\delta u/\delta t$ has been used to represent the rate of change, at a point fixed in space, in the component of velocity parallel to a fixed direction which momentarily coincides with the axis of x .

This is not the same thing as $\partial u/\partial t$. Since u may be regarded as being known in terms of the co-ordinates $(x$ and $y)$, referred to rotating axes, and t the time, $\partial u/\partial t$ represents the rate of change in the component of the velocity of the fluid which is parallel to the rotating axis of x at a point which moves with the axes. It is evident that $\partial u/\partial t = \partial u'/\partial t$ and $\partial v/\partial t = \partial v'/\partial t$.

To find the value of $\delta u/\delta t$ and $\delta v/\delta t$, consider the rate of change, at a point fixed in space, in the component of velocity parallel to a fixed direction which momentarily makes an angle β with the axis of x .

* It will be shown later that this proposition cannot be extended to the case of the three-dimensional motion.

The component of the velocity of the fluid parallel to this direction is $u \cos \beta + v \sin \beta$.

After a short interval of time, δt , the co-ordinates of the fixed point relative to the moving axes are

$$x + \omega y \delta t \quad \text{and} \quad y - \omega x \delta t.$$

The components of velocity parallel to the rotating axes (which now make an angle $\omega \delta t$ with their previous positions) are

$$u + \left(\frac{\partial u}{\partial t} + \omega y \frac{\partial u}{\partial x} - \omega x \frac{\partial u}{\partial y} \right) \delta t \quad \text{and} \quad v + \left(\frac{\partial v}{\partial t} + \omega y \frac{\partial v}{\partial x} - \omega x \frac{\partial v}{\partial y} \right) \delta t.$$

The component of velocity parallel to the fixed direction is therefore

$$\begin{aligned} & \left[u + \left(\frac{\partial u}{\partial t} + \omega y \frac{\partial u}{\partial x} - \omega x \frac{\partial u}{\partial y} \right) \delta t \right] \cos (\beta - \omega \delta t) \\ & \quad + \left[v + \left(\frac{\partial v}{\partial t} + \omega y \frac{\partial v}{\partial x} - \omega x \frac{\partial v}{\partial y} \right) \delta t \right] \sin (\beta - \omega \delta t). \end{aligned}$$

The rate of change in velocity parallel to the fixed direction is therefore

$$\left(\frac{\partial u}{\partial t} + \omega y \frac{\partial u}{\partial x} - \omega x \frac{\partial u}{\partial y} - \omega v \right) \cos \beta + \left(\frac{\partial v}{\partial t} + \omega y \frac{\partial v}{\partial x} - \omega x \frac{\partial v}{\partial y} + \omega u \right) \sin \beta.$$

Putting $\beta = 0$ we find

$$\frac{\delta u}{\delta t} = \frac{\partial u}{\partial t} + \omega y \frac{\partial u}{\partial x} - \omega x \frac{\partial u}{\partial y} - \omega v$$

and putting $\beta = \frac{1}{2} \pi$,

$$\frac{\delta v}{\delta t} = \frac{\partial v}{\partial t} + \omega y \frac{\partial v}{\partial x} - \omega x \frac{\partial v}{\partial y} + \omega u.$$

Substituting these values in (3), subtracting equations (2) and substituting for u and v , it will be found that

$$-\frac{1}{\rho} \frac{\partial}{\partial x} (p - p') = -\omega^2 x - 2\omega v' \quad \text{and} \quad -\frac{1}{\rho} \frac{\partial}{\partial y} (p - p') = -\omega^2 y + 2\omega u'.$$

These equations may be integrated in the form

$$(p - p') / \rho = \frac{1}{2} \omega^2 (x^2 + y^2) + 2\omega \psi'. \tag{4}$$

At this stage it is easy to prove that the proposition proved on p. 100 cannot be extended to the case of three-dimensional motion.

Let u', v', w' be the components of the velocity of a fluid in irrotational motion. Suppose that the motion defined by

$$u = u' - \omega y, \quad v = v' + \omega x, \quad w = w',$$

is possible.

Downloaded from https://royalsocietypublishing.org/ on 09 August 2022

Proceeding as before, it will be found that the pressure equations can be reduced to

$$-\frac{1}{\rho} \frac{\partial}{\partial x} (p-p') = -\omega^2 x - 2\omega v',$$

$$-\frac{1}{\rho} \frac{\partial}{\partial y} (p-p') = -\omega^2 y + 2\omega u',$$

$$-\frac{1}{\rho} \frac{\partial}{\partial z} (p-p') = 0.$$

These are not consistent unless u' and v' are independent of z ; that is, unless the motion is two-dimensional.

Let us now apply (4) to find the resultant force and couple which the fluid pressure exerts on a solid moving in a rotating fluid.

Let F_x' , F_y' , and G' be the resultant forces and couple due to fluid pressure on the solid in the case when the system is not rotating. F_x' and F_y' are supposed to act at the centre of gravity C of the area of the cross-section of the solid. Let F_x , F_y , and G be the corresponding quantities in the case when the system is rotating.

If χ represents the angle between the normal to the surface of the solid and the axis of x , then

$$F_x - F_x' = - \int (p-p') \cos \chi ds,$$

$$F_y - F_y' = - \int (p-p') \sin \chi ds,$$

$$G - G' = \int (p-p') (\eta \cos \chi - \xi \sin \chi) ds,$$

where ξ and η are the co-ordinates of a point on the surface referred to axes parallel to the axes of x and y , and passing through C ; and the integrals are

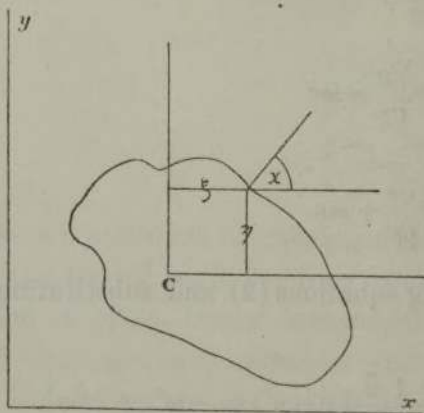
taken round the surface of the solid. Substituting the value of $p-p'$ given by (4) these may be integrated.

Thus

$$-\frac{F_x - F_x'}{\rho} = \frac{1}{\rho} \int (p-p') \cos \chi ds = \frac{\omega^2}{2} \int (x^2 + y^2) \cos \chi ds + 2\omega \int \psi' \cos \chi ds. \quad (5)$$

Now $\int y^2 \cos \chi ds$ vanishes since $\cos \chi ds = dy$. Also $\frac{\omega^2}{2} \int x^2 \cos \chi ds = \omega^2 A x_0$,

where A is the area of cross-section of the solid, and (x_0, y_0) are the co-ordinates of its centroid.



$2\omega \int_s \psi' \cos \chi ds$ may be integrated by parts. It then becomes

$$-2\omega \int_s y \frac{d\psi'}{ds} ds = -2\omega \int_s (y_0 + \eta) \frac{d\psi'}{ds} ds = -2\omega \int_s \eta \frac{d\psi'}{ds} ds, \quad (6)$$

since $\int_s y_0 \frac{d\psi'}{ds} ds$ evidently vanishes.

Now $-\partial\psi'/\partial s$ represents the velocity of the fluid normal to the surface of the solid. The boundary condition which must be satisfied by ψ' is

$$-\partial\psi'/\partial s = (\dot{x}_0 - \Omega\eta) \cos \chi + (\dot{y}_0 + \Omega\xi) \sin \chi,$$

where Ω is the angular velocity of the body.

Substituting in (6) and remembering that $\cos \chi ds = d\eta$ and $\sin \chi ds = -d\xi$, it will be found that

$$2\omega \int_s \psi' \cos \chi ds = 2\omega \int_s \eta (\dot{x}_0 - \Omega\eta) d\eta - 2\omega \int_s (\dot{y}_0 + \Omega\xi) \eta d\xi.$$

The first of these integrals vanishes and the second may be written

$$-2\omega\dot{y}_0 \int_s \eta d\xi - 2\omega\Omega \int_s \eta\xi d\xi,$$

Now $\int_s \eta d\xi = -A$ and $\int_s \eta\xi d\xi = 0$, since C is the centroid of the area of cross-section.

Hence from (5) $-(F_x - F'_x)/\rho = \omega^2 Ax_0 + 2\omega A\dot{y}_0.$

Similarly it will be found that

$$-(F_y - F'_y)/\rho = \omega^2 Ay_0 - 2\omega A\dot{x}_0.$$

It will be noticed that $\omega^2 Ax_0$ and $\omega^2 Ay_0$ are the components of a force $\omega^2 AR$ acting radially, R being the distance of C from the centre of rotation. Also $2\omega A\dot{y}_0$ and $-2\omega A\dot{x}_0$ are the components of a force $2\omega AQ$ acting at right angles to the direction of motion of C relative to the rotating axes, Q being the relative velocity of C .

Now consider the couple $G - G'$ due to the rotation

$$\frac{G - G'}{\rho} = \int_s \frac{p - p'}{\rho} (\eta d\eta + \xi d\xi).$$

Substituting from (4),

$$G - G' = \frac{\omega^2}{2} \int_s \{ (x_0 + \xi)^2 + (y_0 + \eta)^2 \} (\eta d\eta + \xi d\xi) + 2\omega \int_s \psi' (\eta d\eta + \xi d\xi).$$

Neglecting all terms which contain only powers of ξ or of η^* and integrating the second integral by parts, this becomes

$$\frac{G-G'}{\rho} = \frac{\omega^2}{2} \left[2x_0 \int_s \xi \eta d\eta + 2y_0 \int_s \eta \xi d\xi + \int_s \xi^2 \eta d\eta + \int_s \eta^2 \xi d\xi \right] - 2\omega \int \frac{\xi^2 + \eta^2}{2} \frac{d\psi'}{ds} ds.$$

Now
$$\int_s \xi \eta d\eta = \int_s \eta \xi d\xi = 0,$$

and
$$\int_s \xi^2 \eta d\eta + \int_s \eta^2 \xi d\xi = \frac{1}{2} \int_s d(\xi^2 \eta^2) = 0;$$

also since
$$-(\partial\psi'/\partial s) ds = (\dot{x}_0 - \Omega\eta) d\eta + (\dot{y}_0 + \Omega\xi) d\xi,$$

$(G-G')/\rho$ reduces to

$$2\omega \left[\dot{x}_0 \int_s \frac{1}{2} \xi^2 d\eta - \dot{y}_0 \int_s \frac{1}{2} \eta^2 d\xi - \Omega \int_s \xi^2 \eta d\eta - \Omega \int_s \eta^2 \xi d\xi \right] = 0.$$

The forces due to fluid pressure, which act on a body moving in an assigned manner in a rotating fluid, may therefore be regarded as being made up as follows:—

(1) The forces F_x' , F_y' , and the couple G' which would act on the body if it moved in the same way relatively to the fluid at rest.

(2) A force equivalent to $\rho\omega^2 AR$ acting towards the centre of rotation through C.

(3) A force $2\rho\omega AQ$ acting at C in a direction perpendicular to the relative motion of C and the rotating axes, and directed to the left if the rotation of the fluid is anti-clockwise.

We can therefore solve any problem on the motion of cylindrical solids in a rotating fluid if we can obtain a solution of a similar problem respecting the motion of the solids in a fluid at rest.

Now, consider the forces and the couple which it is necessary to apply to a solid body of mass M , in order that it may move in an assigned manner relatively to rotating axes. Suppose that a force F' and a couple G' must be applied at its centre of gravity, in order that it may move in the assigned manner relatively to fixed axes. The additional force which it is necessary to apply when the system is rotating uniformly with angular velocity ω may be shown to consist of a force $2M\omega Q$ perpendicular to the direction of the velocity Q of the centre of gravity relative to the rotating system, together with a force $M\omega^2 R$ acting through the centre of gravity towards the centre of rotation.

* For they vanish when integrated round a closed contour.

It will be noticed that, if the position of the centre of gravity of the solid coincides with the centroid of its cross-section, and if the mass per unit length of the solid is equal to ρA , that is to say, if the mass and the centre of gravity of the solid are the same as those of the fluid displaced, then these forces are the same as those which act on the solid, owing to the additional pressures in the fluid due to its rotation.

These considerations lead to the conclusion that, if a solid of the same density as the fluid be moved along a certain path by certain assigned external forces, then a uniform rotation of the whole system, including the external force, makes no difference to the path which the solid pursues relative to the system.

This theorem applies only to the case of two-dimensional motion. In the case of a finite cylinder, for instance, it seems almost obvious that the pressures due to the rotation must fall off towards its ends. It is natural to suppose, therefore, that the reaction of the fluid would not be sufficient to hold a finite cylinder in its path when the whole system is rotated.

The case of a sphere moving in a rotating fluid presents considerable mathematical difficulties, but the initial motion has been investigated by Mr. J. Proudman, who has kindly consented to allow the author to make use of his results, though they are not yet published.* He finds that, if a sphere of volume V starts from rest in the rotating fluid and moves with uniform velocity along a straight line relative to the rotating system, it is acted on initially by a force $V\rho\omega^2R$ directed towards the centre of the rotation (which is at a distance R from the centre of the sphere) and by a force $\frac{1}{2}V\rho Q\omega$ acting in a direction perpendicular to its path. But in order that a sphere of the same density as the fluid, that is, one whose mass is $V\rho$, may move along a straight path relative to the rotating system, it must be acted on by a force $V\rho\omega^2R$ directed towards the centre of rotation and by a force $2V\rho\omega Q$ perpendicular to its path.

The forces due to fluid pressure are not sufficient to supply the second of these. If, therefore, the sphere were drawn through the rotating fluid by means of a string, it would not move in the direction the string was pulling it, but would be deflected to the left if the fluid were rotating clockwise, and to the right if were rotating anticlockwise. On the other hand, if a cylinder of the same density were drawn through the rotating fluid, the force necessary to hold it in its straight path would be supplied by the fluid pressure. The cylinder would therefore move straight through the fluid in the direction the string was pulling it.

* Since the above was written Mr. Proudman has published his results. They appeared in 'Roy. Soc. Proc.,' A, vol. 92, pp. 408-424 (1916).

These conclusions have been tested and completely verified by means of experiments made by the author in the Cavendish Laboratory with water in a rotating tank.

Experiments made with a Rotating Tank of Water.

A glass tank full of water was mounted so that it could be rotated about a vertical axis at various speeds by means of an electric motor. The speeds varied from 2 to 6 seconds per revolution. Two bodies were prepared, one cylindrical and the other spherical. The former consisted of a piece of thin-walled brass tube about 6 in. \times $\frac{3}{4}$ in. stopped at the end with waxed cork, while the other was a spherical glass bulb. They were weighted until they would fall very slowly through water, and the positions of the weights were adjusted till they would stay almost at rest in any position in the water. The centres of gravity of the bodies were then coincident with the centres of gravity of the water displaced by them.

A simple mechanism was next devised to tow them through the tank from one end to the other. It consisted of a wood pulley about 4 inches in diameter, mounted on a vertical spindle which was driven into a wood bridge, fixed to the tank over the middle of it. This spindle coincided with the axis of rotation of the tank. Cotton was then wound round the pulley, passed through some small rings screwed into a board fixed to one end of the tank, and led horizontally along the tank to the cylinder or sphere, which was fixed at the other end.

The body was held in a holder while the tank and water were being brought to a state of uniform rotation. A device was arranged so that the holder could release the body and at the same moment the wood pulley on which the cotton was wound could be fixed in space. As the tank was then rotating round the pulley the cotton wound up round it, and pulled the bodies along the middle of the tank from one end to the other.

Result.—It was found that the cylinder moved straight through the middle of the tank. Even when the tank was rotating very rapidly the cylinder always passed over the central line. The sphere, however, was violently deviated to the left (the tank was rotating clockwise). When the tank was rotated quite slowly, about once in 6 seconds, the sphere would not quite touch the side, though it never came up to the stop at the other end from a direction less than 45° away from the central line. When the tank rotated more rapidly the complete path could never be seen, because the sphere always hit the side of the tank before it had gone more than a few inches in the direction along which the cotton was trying to pull it. After striking the side of the tank the sphere would follow the side along, touching all the time,

till it got to a position close to the other end where the string was pulling in a direction making an angle of about 50° with the side of the tank. It would leave the side and approach the point towards which the cotton was pulling it along a curved path.

The accuracy with which the experiments just described verify the hydrodynamical theory of rotating fluids is at first sight most surprising. Besides the fact that there is apparently no other case in which experiments made with real solids moving in real fluids agree with the predictions of hydrodynamics, it is known that the stream lines of a real fluid round a circular cylinder in particular bear no resemblance to the stream lines used in the ordinary hydrodynamical theory. It will be noticed, however, that in order that there may be agreement between theory and experiment in the particular respect to which attention has been drawn, it is unnecessary that the actual flow pattern shall be the same as the flow pattern contemplated in the ordinary hydrodynamical theory. All that is necessary is that the flow pattern in the case of the cylinder shall be two-dimensional, while that in the case of the sphere shall be three-dimensional.

Experiments with Vortex Rings in a Rotating Fluid.

The theory explained on p. 105 leads to the conclusion that if a homogeneous solid, which is not cylindrical, be projected in a rotating fluid of the same density as itself it will be deviated, to the left if the rotation is clockwise, and to the right if the rotation is anti-clockwise, of the path it would pursue through the fluid if the whole system were not rotating. Now, a vortex ring affects the fluid round it in much the same way as a solid ring of the same dimensions as the cyclic portion of the flow system. If it is projected through a fluid at rest it travels along a straight line. We should expect, therefore, that if a vortex ring were projected through a rotating fluid it would follow a curved path relative to the fluid, being deviated to the left if the fluid were rotating clockwise.

This conclusion was tested experimentally and found to be correct. A small vortex box with a rubber top and a circular hole in the side was made. This was filled with a solution of fluorescein and placed in one end of the tank, which was filled with water and held fixed. On striking the rubber lightly a vortex ring was produced which travelled straight down the tank and struck the middle of the opposite end.

The same experiment was repeated when the tank and vortex box were rotating. On tapping the box, rings started out in the same direction as before, but were deflected in a curved path, so that they hit the side instead of the end of the tank. By tapping the box quite lightly and rotating the

tank fairly rapidly the rings could be made to turn in such small circles that they came round and struck the vortex box again without touching the side of the tank on the way. They would, in fact, turn in a circle whose diameter was only about four times the diameter of the rings.

It was pointed out by Dr. F. W. Aston, to whom the writer was showing this experiment, that the rings appeared to remain parallel to a plane fixed in space, while the rest of the fluid rotated. He suggested that the gyroscopic action prevented the ring from being deviated from this plane, and that in order that the ring might move relative to the fluid in a direction perpendicular to its plane it would have to move through the fluid along a curved path.

Motion of a Circular Cylinder in a Fluid which has a Steady Rotational Motion at Infinity but does not Necessarily Rotate as a Whole.

The results given in the rest of this paper have no immediate practical interest. The author entered on the investigation with a view to getting an idea of how the instability which is known to exist in a uniformly shearing laminar flow would be likely to manifest itself, and to find out whether the characteristics of the motion of solids in rotating fluids, which have been discussed in the first part of this paper, have any counterpart in the case of solids moving a fluid whose undisturbed motion is a uniform laminar flow.

The problem of finding the motion of a circular cylinder in a rotationally moving fluid divides itself naturally into two parts, that of finding the stream function for a given motion of the cylinder, and that of finding the force which the pressure associated with that stream function exerts on the cylinder. The stream function for a certain type of rotational flow in which the vorticity is uniform will now be found.

Let (r, θ) be the polar co-ordinates of a point referred to axes through the centre of the cylinder, and let (x_0, y_0) be the co-ordinates of the centre of the cylinder referred to fixed axes, so that the equation $\theta = 0$ represents a line parallel to the axis of x at a distance y_0 from it.

Consider the stream function

$$\psi = \frac{1}{2}\zeta r^2 + (Ar + B/r) \cos \theta + (Cr + D/r) \sin \theta + (Er^2 + F/r^2) \cos 2\theta + (Gr^2 + H/r^2) \sin 2\theta. \quad (7)$$

It satisfies the equation $\nabla^2\psi = 2\zeta$ everywhere.

If, therefore, the constants A, B, C, etc., be so chosen that the boundary condition

$$\partial\psi/r\partial\theta + \dot{x}_0 \cos \theta + \dot{y}_0 \sin \theta = 0 \quad (8)$$

is satisfied where $r = a$, a being the radius of the cylinder, then ψ is the

stream function which represents the motion of a fluid which, if the cylinder were removed, would be moving in accordance with the velocities given by the stream function.

$$\psi_1 = \zeta r^2/2 + Ar \cos \theta + Cr \sin \theta + Er^2 \cos 2\theta + Gr^2 \sin 2\theta. \quad (9)$$

Now (8) must be satisfied for all values of θ ; hence we may equate coefficients of $\cos \theta$, $\sin \theta$, $\cos 2\theta$, and $\sin 2\theta$, separately to zero. In this way the following relations between the constants are determined:—

$$A + B/a^2 - \dot{y}_0 = 0, \quad C + D/a^2 + \dot{x}_0 = 0, \quad Ea + F/a^3 = 0, \quad Ga + H/a^3 = 0. \quad (10)$$

It will be noticed that ψ_1 , the stream function of the fluid before the introduction of the cylinder, is expressed in terms of co-ordinates referred to moving axes. In order to find the motion of a cylinder in a fluid whose undisturbed motion before the introduction of the cylinder is known with reference to fixed axes, we must transform (9) so as to give ψ_1 in terms of co-ordinates x and y referred to the fixed axes used to fix the position of the cylinder. The transformation is performed by putting

$$r \cos \theta = x - x_0, \quad r \sin \theta = y - y_0.$$

ψ_1 then becomes

$$\frac{1}{2} \zeta \{ (x - x_0)^2 + (y - y_0)^2 \} + A(x - x_0) + C(y - y_0) + E \{ (x - x_0)^2 - (y - y_0)^2 \} + 2G(x - x_0)(y - y_0). \quad (11)$$

If the motion of the fluid before the introduction of the cylinder be given by the function

$$\psi_1 = \frac{1}{2} \zeta (x^2 + y^2) + A'x + C'y + E'(x^2 - y^2) + 2G'xy, \quad (12)$$

where A' , C' , E' , G' are given constants, we find, by equating coefficients of x , y , x^2 , xy , y^2 in (11) and (12), the following relations determining A , C , E , G , in terms of A' , C' , E' , G' , x_0 and y_0 ,

$$\left. \begin{aligned} -\zeta x_0 + A - 2E x_0 - 2G y_0 &= A', & -\zeta y_0 + C + 2E y_0 - 2G x_0 &= C', \\ E &= E', & G &= G'. \end{aligned} \right\} \quad (13)$$

Solving (10) and (13) we obtain the following values of A , B , C , D , E , F , G , H ,

$$\left. \begin{aligned} A &= A' + \zeta x_0 + 2E' x_0 + 2G' y_0, \\ B &= -a^2 (-\dot{y}_0 + A' + \zeta x_0 + 2E' x_0 + 2G' y_0), \\ C &= C' + \zeta y_0 - 2E' y_0 + 2G' x_0, \\ D &= -a^2 (\dot{x}_0 + C' + \zeta y_0 - 2E' y_0 + 2G' x_0), \\ E &= E', & F &= -E' a^4, \\ G &= G', & H &= -G' a^4. \end{aligned} \right\} \quad (14)$$

Hence the stream function is obtained for the motion of a cylinder in a fluid whose undisturbed motion may be expressed by a stream function of the form ψ_1 . The two particular cases which are of the greatest interest are those of uniform rotation, for which $\psi_1 = \frac{1}{2} \omega (x^2 + y^2)$, and uniformly shearing laminar flow, for which $\psi_1 = -\frac{1}{2} \alpha y^2$, α being the rate of shear. Before discussing these cases, however, it is necessary to find an expression in terms of ψ for the force on the cylinder.

In general there does not appear to be a simple pressure integral like Bernoulli's for the case of irrotational motion, or the expression given in equation (4) for the pressure in a rotating fluid. It is necessary to go back to the original equations of motion of the fluid.

If the rate of change in pressure along a direction which makes an angle χ with the axis of x be represented by the symbol dp/ds_χ , ds_χ representing an element of length in the direction χ , then the equation of motion is

$$-\frac{1}{\rho} \frac{dp}{ds_\chi} = \frac{D}{Dt} (v_\chi),$$

where v_χ represents the component of velocity of the fluid in the direction χ . Its value may be found in terms of ψ by the equation

$$v_\chi = \partial\psi/\partial r \cdot \sin(\chi - \theta) - \partial\psi/r\partial\theta \cdot \cos(\chi - \theta). \quad (15)$$

Now $\frac{D}{Dt}(v_\chi)$ may be written

$$\frac{\delta v_\chi}{\delta t} - \frac{\partial\psi}{r\partial\theta} \frac{\partial}{\partial r} (v_\chi) + \frac{\partial\psi}{\partial r} \frac{\partial}{r\partial\theta} (v_\chi), \quad (16)$$

where $\delta v_\chi/\delta t$ represents, as before, the rate of change in v_χ at a point fixed in space.

If δr , $\delta\theta$ are the changes in the co-ordinates of a fixed point in time δt ,

$$\frac{\delta v_\chi}{\delta t} = \frac{\partial v_\chi}{\partial t} + \frac{\partial v_\chi}{\partial\theta} \frac{\delta\theta}{\delta t} + \frac{\partial v_\chi}{\partial r} \frac{\delta r}{\delta t}, \quad (17)$$

where $\partial v_\chi/\partial t$ represents the rate of change in v_χ at a point fixed relative to the moving axes. The value of $\partial v_\chi/\partial t$ may be obtained by differentiating the expression (15) with respect to time, which occurs in all terms which contain x_0 , y_0 , \dot{x}_0 or \dot{y}_0 .

The values of δr and $\delta\theta$ may be found by resolving the velocity of c , the centre of the cylinder, along and perpendicular to r .

Thus

$$\delta r = -(\dot{x}_0 \cos \theta + \dot{y}_0 \sin \theta) \delta t, \quad r \delta\theta = (\dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta) \delta t;$$

substituting in (17),

$$\frac{\delta v_\chi}{\delta t} = \frac{\partial v_\chi}{\partial t} + (\dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta) \frac{\partial v_\chi}{r\partial\theta} - (\dot{x}_0 \cos \theta + \dot{y}_0 \sin \theta) \frac{\partial v_\chi}{\partial r};$$

substituting this in (16),

$$-\frac{1}{\rho} \frac{\partial p}{\partial s_x} = \frac{D}{Dt} (v_x) = \frac{\partial v_x}{\partial t} - \left(\frac{\partial \psi}{r \partial \theta} + \dot{x}_0 \cos \theta + \dot{y}_0 \sin \theta \right) \frac{\partial v_x}{\partial r} + \left(\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right) \frac{\partial v_x}{r \partial \theta}.$$

Now $-\partial \psi / r \partial \theta - \dot{x}_0 \cos \theta - \dot{y}_0 \sin \theta$ represents the component, normal to the surface, of the relative velocity of the fluid and the cylinder. It must therefore vanish.

Hence
$$-\frac{1}{\rho} \frac{\partial p}{\partial s_x} = \frac{\partial v_x}{\partial t} + \left(\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right) \frac{\partial v_x}{r \partial \theta}$$

and substituting for v_x from (15),

$$-\frac{1}{\rho} \frac{\partial p}{\partial s} = \sin(\chi - \theta) \left[\frac{\partial^2 \psi}{\partial r \partial t} + \left(\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right) \left(\frac{\partial^2 \psi}{r \partial r \partial \theta} - \frac{\partial \psi}{r^2 \partial \theta} \right) \right] + \cos(\chi - \theta) \left[-\frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial t} + \left(\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right) \left\{ -\frac{\partial \psi}{r \partial \theta} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{r \partial \theta} \right) \right\} \right].$$

If χ be put equal to $\frac{1}{2}\pi + \theta$, we obtain the variation in pressure round the cylinder in the form

$$-\frac{1}{\rho} \left[\frac{\partial p}{r \partial \theta} \right]_{r=a} = \left[\frac{\partial^2 \psi}{\partial r \partial t} + \left(\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right) \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) \right]_{r=a}. \quad (18)$$

If F_x and F_y represent the components of the resultant force acting on the cylinder due to fluid pressure,

$$F_x = - \int_0^{2\pi} p \cos \theta a d\theta, \quad F_y = - \int_0^{2\pi} p \sin \theta a d\theta.$$

These may be integrated by parts.

$$F_x \text{ then becomes } a^2 \int_0^{2\pi} \left[\frac{\partial p}{r \partial \theta} \right]_{r=a} \sin \theta d\theta \quad \text{and} \quad F_y = -a^2 \int_0^{2\pi} \left[\frac{\partial p}{r \partial \theta} \right]_{r=a} \cos \theta d\theta. \quad (19)$$

By substituting the value obtained for $\left[\frac{\partial p}{r \partial \theta} \right]_{r=a}$ in (18) we can find the force exerted by fluid pressure when the cylinder has any assigned motion for which a stream function can be found.

This method will be applied to two particular cases: In Case (1) the general motion of the fluid is one of uniform rotation. This problem has been solved already in the first part of this paper, but it seems worth while to verify the calculation. In Case (2) the general motion of the fluid is one of uniform shearing.

Case 1.—The stream function of the general motion of the fluid is

$\psi_1 = \frac{1}{2} \omega (x^2 + y^2)$. In this case, then, $\zeta = \omega$ and $A' = C' = E' = G' = 0$. The stream function of the motion round the cylinder is

$$\psi = \omega \frac{r^2}{2} + \left\{ \omega x_0 r - \frac{a^2}{r} (-\dot{y}_0 + \omega x_0) \right\} \cos \theta + \left\{ \omega y_0 r - \frac{a^2}{r} (\dot{x}_0 + \omega y_0) \right\} \sin \theta.$$

Substituting in (18) the value of $-\frac{1}{\rho} \left[\frac{\partial p}{r \partial \theta} \right]_{r=a}$ may be found, and substituting this value in (19) it will be found that

$$-F_x = \pi \rho a^2 (\ddot{x}_0 + 4\omega \dot{y}_0 - 2\omega^2 x_0), \quad F_y = \pi \rho a^2 (\dot{y}_0 + 4\omega \dot{x}_0 + 2\omega^2 y_0).$$

If these expressions be transformed by the transformation

$$x_0 = R \cos(\phi + \omega t), \quad y_0 = R \sin(\phi + \omega t),$$

so that R, ϕ , are the polar co-ordinates of a point referred to axes which rotate with the fluid, it will be found that the forces F_x, F_y , may be resolved into components F_R along R , and F_ϕ perpendicular to it where

$$F_R = \pi \rho a^2 \{ -\ddot{R} + R\dot{\phi}^2 - 2R\omega\dot{\phi} - R\omega^2 \},$$

$$F_\phi = \pi \rho a^2 \{ -R\ddot{\phi} - 2\dot{R}\dot{\phi} + 2\omega\dot{R} \}.$$

This agrees with the results obtained on p. 104, for the force whose components are F_R and F_ϕ may be regarded as being made up in the following way: (1) a force $\pi \rho a^2 \times$ (acceleration of the cylinder relative to the rotating axes); (2) a force $\pi \rho a^2 \omega^2 R$ acting towards the centre of rotation; and (3) a force $2\pi \rho a^2 \omega \times$ (velocity of the cylinder relative to the rotating axes) acting at right angles to the direction of relative motion. These are evidently the same as the three forces discussed on p. 104.

Case 2.—The general motion of the fluid is one of uniform shearing. The fluid moves parallel to the axis of x with velocity αy , which increases at a uniform rate as y increases. In this case $\psi_1 = -\frac{1}{2} \alpha y^2$, which may be written

$$\psi_1 = -\frac{1}{4} \alpha (x^2 + y^2) + \frac{1}{4} \alpha (x^2 - y^2).$$

Comparing this with (12) it appears that $\zeta = -\frac{1}{2} \alpha$ and $E' = \frac{1}{4} \alpha$, while $A' = C' = G' = 0$. Hence from (7) and (14)

$$\psi = -\alpha \frac{r^2}{4} + \frac{a^2 \dot{y}_0}{r} \cos \theta - \left\{ \alpha y_0 r + \frac{a^2}{r} (\dot{x}_0 - \alpha y_0) \right\} \sin \theta + \frac{\alpha}{4} \left(r^2 - \frac{a^4}{r^2} \right) \cos 2\theta.$$

Hence differentiating and putting $r = a$,

$$\left[\frac{\partial^2 \psi}{\partial r \partial t} \right]_{r=a} = -\dot{y}_0 \cos \theta + \dot{x}_0 \sin \theta - 2\alpha \dot{y}_0 \sin \theta,$$

$$\left[\frac{\partial \psi}{\partial r} + \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta \right]_{r=a} = \alpha (\cos 2\theta - \frac{1}{2}) - 2\dot{y}_0 \cos \theta + 2(\dot{x}_0 - \alpha y_0) \sin \theta,$$

$$\left[\frac{\partial}{\partial r} \left(\frac{\partial \psi}{r \partial \theta} \right) \right]_{r=a} = \frac{1}{a} \{ 2\dot{y}_0 \sin \theta + (\dot{x}_0 - \alpha y_0) \cos \theta - 2\alpha a \sin 2\theta \}.$$

Hence from (18)

$$\begin{aligned}
 -\frac{1}{\rho} \left[\frac{\partial p}{r \partial \theta} \right]_{r=a} &= \dot{x}_0 \sin \theta - \dot{y}_0 \cos \theta - 2\alpha \dot{y}_0 \sin \theta \\
 &+ \frac{2}{a} \{ \dot{y}_0 \sin \theta + (\dot{x}_0 - \alpha \dot{y}_0) \cos \theta - \alpha a \sin 2\theta \} \\
 &\times \{ 2(\dot{x}_0 - \alpha \dot{y}_0) \sin \theta - 2\dot{y}_0 \cos \theta + a\alpha (\cos 2\theta - \frac{1}{2}) \}.
 \end{aligned}$$

Hence from (19)

$$F_x = -\pi \rho a^2 \{ \ddot{x}_0 - \alpha \dot{y}_0 \}, \quad F_y = -\pi \rho a^2 \{ \ddot{y}_0 + 2\alpha (\dot{x}_0 - \alpha \dot{y}_0) \}. \quad (20)$$

This result will now be applied to find the motion of a cylinder of the same density as the fluid when it is projected from the origin with velocity whose components are U and V.

The equations of motion of the cylinder are

$$\begin{aligned}
 \pi \rho a^2 \ddot{x}_0 &= -\pi \rho a^2 \{ \ddot{x}_0 - 2\alpha \dot{y}_0 \}, & \pi \rho a^2 \ddot{y}_0 &= -\pi \rho a^2 \{ \ddot{y}_0 + 2\alpha (\dot{x}_0 - \alpha \dot{y}_0) \}, \\
 \ddot{x}_0 - \alpha \dot{y}_0 &= 0, & \ddot{y}_0 &= -\alpha (\dot{x}_0 - \alpha \dot{y}_0).
 \end{aligned}$$

The first of these may be integrated in the form

$$\dot{x}_0 - \alpha \dot{y}_0 = \text{constant}.$$

That is to say, the component parallel to the axis of x of the relative velocity of the cylinder and the fluid is constant and equal to U. The acceleration of the cylinder in the direction of the axis of x is constant and equal to $-\alpha V$. If $U = 0$, that is to say, if the cylinder is shot off in a direction perpendicular to the direction of shear, then the component of velocity parallel to the axis of y is constant, and the fluid pressure is just sufficient to give the cylinder the acceleration αV , which is necessary in order that the velocity of the cylinder relative to the fluid round it may remain constant. This property of uniformly shearing fluids appears to be analogous to a certain extent to the property of rotating fluids discussed on p. 105.