MOTION PLANNING IN THE PRESENCE OF MOVING OBSTACLES

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ABSTRACT

This paper investigates the computational complexity of planning the motion of a body B in 2-D or 3-D space, so as to avoid collision with moving obstacles of known, easily computed, trajectories. Dynamic movement problems are of fundamental importance to robotics, but their computational complexity has not previously been investigated.

We provide evidence that the 3-D dynamic movement problem is intractable even if B has only a constant number of degrees of freedom of movement. In particular, we prove the problem is *PSPACE*-hard if B is given a velocity modulus bound on its movements and is *NP* hard even if B has no velocity modulus bound, where in both cases B has 6 degrees of freedom. To prove these results we use a unique method of simulation of a Turing machine which uses time to encode configurations (whereas previous lower bound proofs in robotics used the system position to encode configurations and so required unbounded number of degrees of freedom).

We also investigate a natural class of dynamic problems which we call *asteroid avoidance problems* : B, the object we wish to move, is a convex polyhedron which is free to move by translation with bounded velocity modulus, and the polyhedral obstacles have known translational trajectories but cannot rotate. This problem has many applications to robot, automobile, and aircraft collision avoidance. Our main positive results are polynomial time algorithms for the 2-D asteroid avoidance problem with bounded number of obstacles as well as single exponential time and $n^{O(\log n)}$ space algorithms for the 3-D asteroid avoidance problem with an unbounded number of obstacles. Our techniques for solving these asteroid avoidance problems are novel in the sense that they are completely unrelated to previous algorithms for planning movement in the case of static obstacles.

We also give some additional positive results for various other dynamic movers problems, and in particular give polynomial time algorithms for the case in which B has no velocity bounds and the movements of obstacles are algebraic in space-time.

Motion Planning in the Presence of Moving Obstacles

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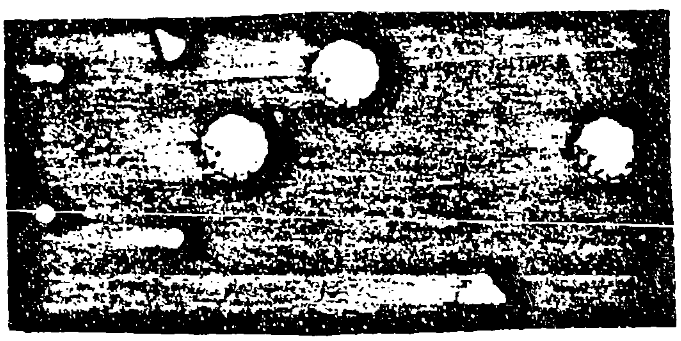

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We provide evidence that the 3-D dynamic movement problem is intractable even if B has only a constant number of degrees of freedom of movement. In particular, we prove the problem is *PSPACE*-hard if B is given a velocity modulus bound on its movements and is *NP* hard even if B has no velocity modulus bound, where in both cases B has 6 degrees of freedom. To prove these results we use a unique method of simulation of a Turing machine which uses time to encode configurations (whereas previous lower bound proofs in robotics used the system position to encode configurations and so required unbounded number of degrees of freedom).

We also investigate a natural class of dynamic problems which we call *asteroid avoidance problems*. B, the object we wish to move, is a convex polyhedron which is free to move by translation with bounded velocity modulus, and the polyhedral obstacles have known translational trajectories but cannot rotate. This problem has many applications to robot, automobile, and aircraft collision avoidance. Our main positive results are polynomial time algorithms for the 2-D asteroid avoidance problem with bounded number of obstacles as well as single exponential time and $n^{O(\log n)}$ space algorithms for the 3-D asteroid avoidance problem with an unbounded number of obstacles. Our techniques for solving these asteroid avoidance problems are novel in the sense that they are completely unrelated to previous algorithms for planning movement in the case of static obstacles.

We also give some additional positive results for various other dynamic movers problems, and in particular give polynomial time algorithms for the case in which B has no velocity bounds and the movements of obstacles are algebraic in space-time.

1. INTRODUCTION

1.1 Static Movers Problems

The *static movers problem* is to plan a collision-free motion of a body B in 2-D or 3-D space avoiding a set of obstacles stationary in space. For example, B may be a sofa which we wish to move through a room crowded with furniture, or B may be an articulated robot arm which we wish to move in a fixed workspace.

[Reif, 79] first showed that a generalized 3-D static movers problem is *PSPACE*-hard, where B consists of n linked polyhedra. [Hopcroft, Joseph and Whitesides, 84] and [Hopcroft, Schwartz and Sharir, 84] later proved *PSPACE*-lower bounds for 2-D static movers problems. If the number of degrees of freedom of motion is kept constant then the problem has polynomial-time solutions provided that the geometric constraints on the motion can be stated algebraically [Schwartz and Sharir, 83b]. More efficient polynomial time algorithms for various specific cases of static movers problems are given in [Lozano-Perez and Wesley, 79, Reif, 79, Schwartz and Sharir, 83a,c, 84, Hopcroft, Joseph and Whitesides, 85, Sharir and Ariel-Sheffi, 83, O'Dunlaing, Sharir and Yap, 83, O'Dunlaing and Yap, 85]

1.2 Dynamic Movers Problems

In this paper, we consider the problem of planning a collision-free motion of a body B which is free to move within some 2-D or 3-D space S, containing several obstacles which move in S along known trajectories. We require that the obstacle trajectories be easily computable functions of time, and not be at all dependent on any movement of B. Some applications are:

(1) Robotic Collision Avoidance. B might be a robot arm which must be moved through a workspace such as an assembly line in which various machine parts make predictable movements.

(2) Automobile Collision Avoidance. B is an automobile with an automatic steering system which must avoid collision with other automobiles with known trajectories on a highway.

(3) Aircraft Collision Avoidance. B is an aircraft which we wish to automatic-pilot through an airspace containing a number of aircraft and other obstacles with known flight paths

(4) Spacecraft Navigation. B might be a spacecraft which we wish to automatically maneuver among a field of moving obstacles, such as asteroids.

Although the dynamic movers problem is fundamental to robotics, we know of no previous work which has considered the computational complexity of such problems

We can formally define a *dynamic movers* problem as follows. Let B be an arbitrary fixed system of moving bodies (each of which can translate and rotate, and some of which may be hinged), having overall d degrees of freedom B is allowed to move within a space S which contains a collection of obstacles moving in an arbitrary (but known) manner. To cope with the time-varying environment we represent the time as an additional parameter of the configuration of B. More precisely, we define the *free configuration space FP* of B to consist of all pairs $[X,t] \in E^{(d+1)}$, where $X \in E^d$ represents a configuration of B, and such that if at time t the system B is at configuration X then B does not meet any obstacle at that time. In this representation of FP a continuous motion of B is represented by a continuous arc $[x_t,t]=p(t)$, which is *monotone* in t. Note that the slope of this arc (relative to the t-axis) represents the "velocity" (i.e. the rate of change of the parameters of the motion) of B If we impose no restrictions on this velocity, any such t-monotone path corresponds to a possible motion of B. However, the dynamic version of the problem is usually further complicated by imposing certain constraints on the allowed motions of B One such constraint is that the velocity of B has a bounded modulus (the modulus is the Euclidean norm of the velocity vector) Such a constraint of a "uniform" bound on velocity of B is particularly appropriate if B is a single rigid body free only to translate. most of the versions of the problem (e.g. the asteroid avoidance problem) studied in this paper will be of this kind

Using the above terminology, the problem that we wish to solve is Given an initial free configuration $[X_0,0]$ and a final free configuration $[X_1,T]$, plan a continuous motion of B (if one exists) between these configurations which will avoid collision with the obstacles, or else report that no such motion is possible (Note that we also specify the time T at which we want to be at the final configurations X_1, as will be seen below, a variant of our techniques can be used to obtain minimal time movement of B) In other words, we wish to find a monotone path in FP between the two configurations $[X_0,0]$ and $[X_1,T]$, where the path satisfies the velocity modulus bounds constraint (if imposed)

The goal of this paper is to systematically investigate the complexity of various fundamental classes of dynamic movement planning problems

1.3 Summary of Our Results

In summary, the *main results* of this paper are:

(1) *PSPACE lower bounds* of 3-D dynamic movement planning of a single disk with bounded velocity and rotating obstacles.

(2) *decision algorithms* for 1-D, 2-D or 3-D dynamic movement planning of a polyhedron with bounded velocity and purely translating obstacles.

We also have additional results for some dynamic movement planning problems with unbounded velocity.

1.4 Our Lower Bound Results for Rotating Obstacles.

In the case the obstacles rotate, they may generate non-algebraic trajectories in space-time which appear to make movement planning intractable. Our *main negative result*, given in Section 2, is a proof that 3-D dynamic movement planning with rotating obstacles is *PSPACE*-hard, even in the case the object to be moved is a disc with bounded velocity. (We also have a related *NP*-hardness result, described below, in the case B has no velocity bounds.)

Remark. All previously known lower bound results for movers problems utilize the *position* of B for encoding n bits, and thus require that B have $\Omega(n)$ degrees of freedom. We use substantially different techniques for our lower bound results. In particular, we use *time* to encode the configuration of a Turing machine that we wish to simulate (therefore we call our construction a "time-machine"). In our lower bound construction it suffices that B have only $O(1)$ degrees of freedom. (In contrast, static movement planning is polynomial time decidable in case B has only $O(1)$ degrees of freedom.) The key to our *PSPACE*-hardness proof is a "delay box" construction, which by use of rotating obstacles generates an exponential number of disconnected components in the free configuration space.

1.5 Efficient Algorithms for Asteroid Avoidance Problems

In Section 3 we investigate an interesting class of tractable dynamic movement problems where the obstacles do not rotate. An *asteroid avoidance problem* is the dynamic movement problem where each of the obstacles is a polyhedron with a fixed (possibly distinct) translational velocity and direction, and B is a convex polyhedron which may make arbitrary translational movements but with a bounded velocity modulus. Neither B nor the obstacles may rotate. (This problem is named after the well known ASTEROID video game where a spacecraft of limited velocity modulus must be maneuvered to avoid swiftly moving asteroids.) The problem is efficiently solved in the 1-D case by line scanning techniques but is quite difficult in the 2-D and 3-D cases.

The assumptions of the asteroid avoidance problem are applicable in many of the above mentioned practical problems, such as robot, automobile, airplane and spacecraft collision avoidance problems, where both B and the obstacles are approximated by convex polyhedra.

The *major positive results* of this paper are a polynomial time algorithm for the 2-D asteroid avoidance problem with a bounded number of convex obstacles as well as $2^{n^{O(1)}}$ time and $n^{O(\log n)}$ space decision algorithms for the 3-D asteroid avoidance problem with an unbounded number of obstacles. The methods we develop (such as the use of normal movements) to solve asteroid avoidance problems are quite different from those previously used to solve static movement problems and seem likely to be fundamental to the efficient solution of other problems in planning dynamic movement.

We also have a simple polynomial time algorithm for the 3-D asteroid avoidance problem with an unbounded number of obstacles where B has maximum velocity bound greater than any obstacle, and must only avoid collision.

1.6 Dynamic Movers Problems with no Velocity Bounds on B

In the final Section 4 of this paper, we consider the complexity of dynamic movement planning in the case B, the object to be moved, has no velocity modulus bounds. We first show that the 3-D dynamic movement problem for a cylinder B with unrestricted velocity is NP-hard

We then consider algorithms for dynamic movement planning in the case in which no velocity bounds are imposed on the motion of B, and the geometric constraints on the possible positions of B can be specified by algebraic equalities and inequalities (in the parameters describing the possible degrees of freedom of B and in *time*). We show that this problem is solvable in polynomial time for any ***fixed*** moving system B (which may consist of several independent hinged translating and rotating bodies in 2-D or 3-D).

2. A TIME MACHINE SIMULATION OF PSPACE

We show here that

THEOREM 2.1. *Dynamic movement planning in the case of bounded velocity is PSPACE-hard, even in the case where the body B to be moved is a disc.*

Proof. Let M be a deterministic Turing machine with space bound $S(n) = n^{O(1)}$. We can assume M has tape alphabet $\{0,1\}$, state set $Q = \{0, \ldots, |Q|-1\}$ with initial state 0 and accepting state 1. A *configuration* of M consists of a tuple $C = (u, q, h)$ where $u \in \{0,1\}^{S(n)}$ is the current tape contents, $q \in Q$ is the current state, and $h \in \{1, \ldots, S(n)\}$ is the position of the tape head. Let $next(C)$ be the configuration immediately succeeding C. Given input string $w \in \{0,1\}^n$ considered to be a binary number, the initial configuration is $C_0 = (w0^{S(n)-n}, 0, 0)$. We can assume $(0^{S(n)}, 1, 0)$ is the accepting configuration. We can also assume that if M accepts, then it does so in exactly $T = 2^{cS(n)}$ steps for some constant $c > 0$. Thus M accepts iff C_T is accepting, where $C_0, C_1, \ldots, C_T$ is the sequence of configurations of M satisfying $C_i = next(C_{i-1})$ for $i = 1, \ldots, T$.

To simulate the computation of M on input w, we will construct a 3-D instance of the dynamic movers problem where the body B to be moved is a disc of radius 1, and where we bound the velocity modulus of B by $v = 100\ |Q|S(n)$. The basic idea of our simulation is to use *time* to encode the current configuration of M.

Let

$$N = S(n) + \lceil \log |Q| \rceil + \lceil \log S(n) \rceil .$$

We shall encode each configuration $C = (u, q, h)$ as an N bit binary number

$$\#(C) = u + q 2^{S(n)} + h 2^{S(n) + \lceil \log |Q| \rceil} ,$$

so $0 \le \#(C) < 2^N$. A *surface configuration* of M is a triple $<u_h, q, h>$ where $u_h \in \{0,1\}$ is the value of the tape cell currently scanned, q is the current state and h is the head position. Note that there are only a polynomial number of surface configurations. For each $q \in Q$, $h \in \{1, \ldots, S(n)\}$, and $u_h \in \{0,1\}$, we associate a distinguished position $P_{<u_h, q, h>}$ of B in 3-dimensional space corresponding to surface configuration $<u_h, q, h>$ of M.

We will fix a distinguished initial position p_0 of B in 3-dimensional space B is located at p_0 at the initial time $t_0 = w$. The dynamic movers problem will be to move B so that it is at position p_0 also at time $t_T = 2^{S(n)} + T2^N$ We will construct a collection of moving obstacles which will force B to move to position p_0 exactly at each time $t_i \geq w$ such that $\lfloor t_i \rfloor = \#C_i + i2^N$, and $t_i < \lfloor t_i \rfloor + \frac{2}{v}$ Thus the lower N bits of t_i encode the configuration C_i and the higher bits encode the step number. (Note that t_0 encodes the initial configuration, at step 0, and t_T encodes the final configuration at step T.)

To simulate M, we need two kinds of devices, one to test that M is at a particular surface configuration, and the other to simulate one step of M at a specific surface configuration The first kind of device is constructed as follows To test a given bit b_j, in position j of t_i, we require that B be forced (by a semidisc rotating once every $\frac{2}{v}$ time units) to enter a cylindrical box (which we call a "test box") with two exit slots: $exit_0$ and $exit_1$. We design the box so that it is swept by a semidisc once every $\frac{1}{v}$ time units, and furthermore $b_j = 1$ iff $exit_1$ is open iff $exit_0$ is closed at time $\lfloor t_i \rfloor + \Delta_j$, where Δ_j is the time required by B to reach the entry slot of this test box. A semidisc rotating once every 2^j time units can be used to open and close these exits at the appropriate times Thus we can design the test box so that in $\frac{6}{v}$ time units, B is forced to move through $exit_{b_j}$.

Hence (by using a balanced tree of such test boxes plus some additional sweeping semidiscs), we can force B to be moved from p_0 to arrive in distinguished position $P_{<u_h,q,h>}$ in time at least $\lfloor t_i \rfloor + 1$ and less than $\lfloor t_i \rfloor + 1 + \frac{2}{v}$ Let $C_{i+1} = next(C_i) = (u',q',h')$ be the configuration of M immediately following C_i Since $\#C_{i+1} - \#C_i$ depends only on $<u_h,q,h>$, there is a function $g(u_h,q,h)$ such that $\#C_{i+1} = \#C_i + g(u_h,q,h)$ and $g(u_h,q,h) < 2^N$ Hence we will require an additional gadget, to be described below, to force B to move from position $P_{<u_h,q,h>}$ back again to position p_0 at time t_{i+1} such that

$$\lfloor t_{i+1} \rfloor = \#C_{i+1} + (i+1)2^N = \lfloor t_i \rfloor + g(u_h,q,h) + 2^N ,$$

and $t_{i+1} \leq \lfloor t_{i+1} \rfloor + \frac{2}{v}$ The total time delay for this move must be $g(u_h,q,h) + 2^N - 1$.

Thus our key remaining construction still required is a "delay Δ box" (where Δ is a number less than 2^{2N}) If B enters the delay box at any time $t \geq 0$ such that $t < \lfloor t \rfloor + \frac{2}{v}$, then B must be made to exit the delay box at a time at least $\lfloor t \rfloor + \Delta$, and at most $\lfloor t \rfloor + \Delta + \frac{2}{v}$ Note we can assume Δ is greater than a constant, say 10, or else the construction is trivial (Our construction is not trivial, however, in the general case where Δ is exponential in N since it is based on an explicit construction of an exponential number of disconnected components in the free configuration space, using only a small number of moving (essentially rotating) obstacles having polynomially describable velocities)

Our delay box consists of a fixed torus-shaped obstacle, plus some additional moving obstacles (see Figure 1). We can precisely define this torus as the surface generated by the revolution of an (imaginary) circle of radius 3 around the x axis, so that its center is always at distance $\frac{\Delta v}{2\pi}$ from the x axis, and so that the circle is always coplanar with the x-axis. Let Θ be the angular position of a point with respect to rotation around the x-axis.

The torus will have open *entrance* and *exit* slots at $\Theta = 0$ and $\Theta = \pi$, respectively, just sufficiently wide for entrance and exit of disc B from the torus. The idea of our delay box construction will be to create various disconnected "free spaces" within the torus in which B must be located. These free spaces will be constructed so that they move within the torus π radians of Θ (i.e., make $1/2$ a revolution) in Δ time units. Once B enters the torus via the entrance slot, our construction will force B to be located in exactly one such free space, and revolve with it around the torus until B leaves the interior of the torus at the exit slot after the required delay of Δ time units.

We now show precisely how to create these moving "free spaces". A moving obstacle D moves through the interior of the torus with angular velocity (with respect to Θ) of $16 + \frac{1}{2\Delta}$ revolutions per time unit. D consists of three discs D_0, D_1, D_2 placed face to face so that their centers are nearly in contact and so that they are each coplanar with the x-axis. Discs D_0, D_1, D_2 are of radius almost 3. D_0 has a $1/4$ section removed, D_1 has a $3/4$ section removed, and D_2 has a $1/2$ section removed. D_1 and D_2 each rotate around their center, but D_0 does not. Let ψ_i be the angular displacement of D_i as it rotates around its center, for $i=1,2$. We set the angular velocity of D_1 with respect to ψ_1 to be the same as the angular velocity of D_1 with respect to Θ. We set the angular velocity of D_2 with respect to ψ_2 to be 32Δ revolutions per time unit (see Figure 2).

We assume that when D has angular displacement $\Theta = \frac{3\pi}{2}$, D_1 is positioned so that the remaining solid quarter section of D_1 completely overlaps the removed quarter section of D_0. This creates an immobile "dead space" at $\Theta = \frac{3\pi}{2}$ every roughly $\frac{1}{16}$ time units, which B cannot cross (because its velocity is too small), and will force B (if it is to avoid collision) to exit the torus via the exit slot at $\Theta = \pi$. However, while D has angular displacement Θ, $0 \le \Theta \le \pi$, the removed $3/4$ section of D_1 completely overlaps the removed quarter section of D_0 which therefore remains completely unobscured (see Figure 3).

Observe that for $0 \le \Theta \le \pi$ a "free space" is created during about $1/2$ revolution of D_2 around its center (i.e. when the removed quarter section of D_0 and the removed half-section of D_2 sufficiently overlap to accommodate B between them), and B can be located in this free space without contacting an obstacle. On the next roughly $1/2$ revolution of D_2 around its center, a "dead space" is created (i.e., when the removed sections of discs D_0, D_2 do not sufficiently overlap), and B cannot be located in this dead space. Since D_2 rotates around its center at most 2Δ times every time interval in which D rotates through the torus, at most 2Δ such free spaces are created during one revolution of D around the torus (see Figure 4).

Since D makes an integal number plus $\frac{1}{2\Delta}$ revolutions of Θ every time unit, each free space moves $\frac{1}{2\Delta}$ revolutions of Θ every time unit, and thus each free space moves $1/2$ a revolution of Θ in Δ time units as required in our construction. Moreover we have chosen the size of the torus and $\Delta \ge 10$, so that it is easy to verify that the maximum velocity v of B is sufficient for B to enter the torus, to move along within a free space and to finally exit the torus. Finally, we claim that B cannot move between any two distinct free spaces while in the interior of the torus. If this was possible, then B could move across a dead space without colliding with D. But D makes a revolution of ϑ at least every $\frac{1}{16}$ time units. In this time, B (which has maximum velocity v) can move at most distance $\frac{v}{16}$, which is less than the minimum distance $\frac{1}{4\Delta} \frac{\Delta v}{2\pi} 2\pi = \frac{v}{2}$ between any two free spaces, a contradiction.

Finally, to complete our construction, we observe that it is easy (by use of a binary tree of cylindrical tube obstacles and sweeping semidiscs) to force B to move from the exit of each such torus to p_0, so that the overall delay in reaching p_0 is Δ, as required. A description of the above construction can easily be computed by an $O(\log n)$ space bounded deterministic Turing Machine. $\square$

Remark: This "time-machine" construction can be simplified further, to the case involving dynamic movement planning in 2-D space in the presence of a single moving obstacle which is a single point. Giving this obstacle a rather irregular (but still polynomially describable) motion, we can simulate both testing devices and delay devices and any additional obstacles needed to force B to move from and back to the starting position p_0. Nevertheless, we prefer the construction given here since it uses more natural and regular kinds of motion. (We are grateful to Jack Schwartz for making this observation.)

3. EFFICIENT ALGORITHMS FOR THE ASTEROID AVOIDANCE PROBLEM

Our *PSPACE*-hardness result of the previous section indicates that it may be inherently difficult to solve dynamic movers problems where the obstacles rotate. Therefore we confine our attention to the following case, which we call the *asteroid avoidance problem*.

Assume that B is an arbitrary convex polyhedron in d-space which can move only by translating with maximum velocity modulus v but without rotating (so that its motion has only d translational degrees of freedom). We also assume that each of the obstacles is a convex polyhedron which moves (without rotating) at a fixed and known velocity (which may vary from one obstacle to another). Finally, we assume the obstacles never collide with each other. The free configuration space FP (including time as an extra degree of freedom, as above) is $(d + 1)$-dimensional. While the case $d = 1$ is easy to solve, the cases $d = 2,3$ of the asteroid avoidance problem are quite challenging, and require some interesting algorithmic techniques.

We have efficient algorithms for various asteroid avoidance problems. These results utilize some basic facts described in the next two subsections, of which the most important is that normal movement suffice.

3.1 Reduction to the Movement of a Point

Before continuing, we use the following simple device (see [Lozano-Perez and Wesley, 79]) to reduce the problem to the case in which B is a single moving point. Let B_0 denote the set of points occupied by B at time $t = 0$. Replace each moving obstacle C by the set $C - B_0$ (which consists of pointwise differences of points of C and points of B_0). Suppose that we wish to plan an admissible motion of B from the initial position B_0 to a final position B_1, and let X_1 denote the relative displacement of B_1 from B_0. Then such a motion exists if and only if there exists an admissible motion of a single point from the origin to X_1 which avoids collision with the moving displaced bodies $C - B_0$ (each such body moving with the same velocity as the obstacle body C). Since the displaced bodies are also convex polyhedra we have reduced the problem to a similar one in which B can be assumed to be a single moving point. Hereafter in this section, we assume this.

3.2 Normal Movements

We will require some special notation for various types of movement of B over a given time interval. In all the following types of movement of B, we allow (the point) B to touch an obstacle boundary, but do not allow it to move to the interior of any obstacle, and require that B not exceed a maximum velocity modulus v.

(1) A *static movement* is one in which B does not move (i.e., has 0 velocity)

(2) A *direct movement* is a movement of B with a constant velocity vector (with modulus $\le v$). During a direct movement, B may touch an obstacle only at the endpoints of that movement.

(3) A *contact movement* is a movement of B in which B moves on the boundary of an obstacle C (i.e., the boundary of the region of FP induced by the movement of C). In the $2-D$ asteroid avoidance problem, we also require that any such (maximal) contact movement begin and end at (contact with) vertices of an obstacle. In the $3-D$ asteroid avoidance problem, we require that each contact movement begin and end only at (contact with) edges or vertices of an obstacle.

(4) A *fundamental movement* of B is a direct movement followed possibly by a contact movement.

(5) A *normal movement* of B is a (possibly empty) sequence of fundamental movements of B where the movements must satisfy the following restrictions:

R1: Between any two distinct direct movements, there must be a contact movement, and

R2: No two distinct (maximal) contact movements are allowed to visit (the boundary of) the same obstacle.

Note that R1 requires that a normal movement does not change its direction except while in contact with an obstacle. R2 ensures that a normal movement consists of $\le k+1$ fundamental movements, where k is the number of obstacles.

LEMMA 3.1. *B has a collision-free movement $p(t) = [X_t, t]$ from $[X_0, 0]$ to $[X_T, T]$ iff B has a sequence of fundamental movements from $[X_0, 0]$ to $[X_T, T]$ satisfying R1.*

Proof. If B has a sequence of fundamental movements from $[X_0, 0]$ to $[X_T, T]$ where B may possibly touch some of the obstacles, then since the obstacles are assumed never to collide, and the initial and final positions of B are free, it is easily seen that by a perturbation of the given sequence of movements, one can obtain a collision-free movement from $[X_0, 0]$ to $[X_T, T]$.

For the converse part, consider the class K of all paths $p(t) = [X_t, t]$ in space-time from $[X_0, 0]$ to $[X_T, T]$, whose slope at any given time is of modulus at most v, and which avoids penetration into the interior of any obstacle. By assumption K is not empty. Let $\pi_0 \in K$ be the shortest path in K (where the length of a path in K is its Euclidean length in E^{d+1}).

Observe that if $\pi \in K$ and if $[X_1, t_1], [X_2, t_2] \in \pi$, then the path π', obtained by replacing the portion of π between these two points by the straight segment joining them (in space-time), is such that its slope at any given time is $\le v$. Since the space-time trajectory of each obstacle is a convex polyhedron, it follows, using standard shortest-path arguments, that π_0 must be a polygonal path.

which consists of an alternating sequence of free straight segments and of polygonal subpaths in which B is in contact with an obstacle. Moreover, the vertices of π_C must lie along edges of the space-time trajectories of the moving obstacles, so they correspond to contacts of B with obstacle vertices. Thus π_C is a sequence of fundamental movements satisfying R1. □

Lemma 3.2. *B has a collision-free movement from $[X_0,0]$ to $[X_T,T]$ iff B has a normal movement from $[X_0,0]$ to $[X_T,T]$.*

Proof. By Lemma 3.1, we can assume B has a movement $[X_t,t]$ defined for $0 \le t \le T$, consisting of a sequence of fundamental movements beginning at times $0 \le t_1, t_2, \ldots, t_m \le T$ satisfying R1. If restriction R2 is violated then there must be times t_i, t_j such $[X_t,t]$ is in contact with the same obstacle C during times t_i and t_j. But since C is convex, its trajectory C^* in space-time is also convex. It is then easy to construct a single contact path $[X_t',t]$ along C^* for $t_i \le t \le t_j$, such that $X_{t_1}' = X_{t_1}$ and $X_{t_2}' = X_{t_2}$, and such that the slope of this path at any given time is of modulus $\le v$. Repeating this process as required, we get a normal movement satisfying both R1 and R2.

The other direction follows from Lemma 3.1. □

3.3 The Asteroid Avoidance Problem with One Degree of Freedom of Movement

We will first consider the case of a 1-D asteroid avoidance problem, where we assume B is constrained to move along a fixed line (in the presence of 2-D convex polygonal obstacles which can pass through that line). The problem is not difficult in this case since B has only one degree of freedom movement. (Nevertheless, a brief investigation of this case will aid the reader to understand better the techniques which we use for the more difficult cases of $d = 2,3$ degrees of freedom.) Let n be the total number of obstacle edges. Let k be the number of obstacles. By the reduction of Section 3.1, we can assume B is a single point.

THEOREM 3.1. *The asteroid avoidance problem can be solved in time $O(n \log n)$ If B is constrained to move only along a 1-dimensional line L.*

Proof. The key observation is that the (space-time) configuration space FP in this case is a 2-dimensional space bounded by polygonal barriers generated by the uniform motions along L of the intersections of obstacle edges with L. We explicitly construct FP using a scan-line technique. We first sort in time $O(n \log n)$ all obstacle edges and vertices in the order of times in which they first intersect L. Let this sorted sequence be $t_1, \ldots, t_m$ (where $m = O(n)$). As we sweep the scan line across time, we maintain for each time t the set FP_t of all accessible free configurations at time t, and also a sorted list Q of the intersections of obstacle edges with L at time t. Suppose $[X_0,0]$ is the initial configuration of B. Initially FP_0 consists of the single point $[X_0,0]$ in space-time, and the initial value of Q is easily calculated in time $O(n \log n)$. Inductively, suppose for for some $t_i \ge 0$ we have constructed FP_{t_i}. We represent FP_{t_i} as an ordered, finite sequence of disjoint intervals $I_1, \ldots, I_{s_i}$ of L, whose union is the set of all points X such that there is a collision-free movement of B, whose velocity modulus never exceeds v, from $[X_0,0]$ to $[X,t_i]$. Let t_{i+1} be the next time following t_i that an obstacle vertex intersects L. $FP_{t_{i+1}}$ can easily be constructed from FP_{t_i} by observing that (1) the boundaries of each of the intervals I_j expand with velocity v, and (2) an obstacle deletes any portion of an interval I_j that it intersects on the line L, moreover, this intersection is itself an interval (since the object is convex) whose endpoints move with constant velocities for t in the interval $[t_i,t_{i+1}]$

From (2) it follows that the total number of intervals ever inwerted by the algorithm into L is at most $2k \leq O(n)$. Each step in the construction of $FP_{t_{i+1}}$ from FP_{t_i} can thus create a new interval in L, change the velocity of an end-point of such an interval already in L, and optionally also merge pairs of such adjacent intervals into single intervals. But the overall number of such merges cannot exceed the number of intervals ever inserted into L, i.e. is at most $O(n)$. The total time of the algorithm is therefore $O(n \log n)$. □

3.4 A Polynomial Time Algorithm for the 2-D Asteroid Avoidance Problem for a Bounded Number of Obstacles

In this subsection, we consider the 2-D asteroid avoidance problem. The configuration space FP in this case is 3-dimensional. We can assume, by the reduction of Section 3.1, that B is a single point. We wish to move B from $[X_0, 0]$ to $[X_T, T]$. The obstacles $C_1, \ldots, C_k$ are k convex polygons. Let n, the *size* of the problem, be the total number of vertices and edges of the obstacles. We will show that if k is a constant, then we can solve the problem in $n^{O(1)}$ time.

Our basic technique will be to first consider the problem of computing the time intervals in which single direct and contact movements can be made, and then use a recursive method to determine the time intervals in which it is possible to do normal movements.

For technical reasons, we consider the initial and final positions of B to be additional immobile "obstacles" $C_0 = X_0$, $C_{k+1} = X_T$, each consisting of a single vertex. Let $V(C_j)$ be the set of vertices of obstacle C_j for $j = 1, \ldots, k$ and let $V(C_0) = \{X_0\}$ and $V(C_{k+1}) = \{X_T\}$. Let $V = \bigcup_{j=0}^{k+1} V(C_j)$ be the set of all vertices. Note that for each $j = 0, \ldots, k+1$, all vertices $a \in V(C_j)$ undergo a translational motion with the same fixed velocity vector.

We will use I to denote the set of times a certain event will occur. Let $|I|$ denote the minimum number of disjoint intervals into which the points of I can be partitioned. Clearly, I can be written using $O(|I|)$ inequalities. We will store the intervals of I in sorted order using a balanced binary tree of size $O(|I|)$, in which we can do insertions and deletions in time $O(\log |I|)$.

For each $a, a' \in V(C_j)$, let $CM_{a,a'}(I)$ be the set of all times $t' \geq 0$ for which vertex a' can be reached at time t' by a contact movement of B on the boundary of C_j starting at vertex a at some time $t \in I$.

Lemma 3.3. *$CM_{a,a'}(I)$ can be computed in time $O(|I| + |V(C_j)|)$ and furthermore $|CM_{a,a'}(I)| \leq |I|$.*

Proof. There are fixed reals $0 \leq \Delta_1 \leq \Delta_2$ (possibly infinite) such that vertex a' can be reached from vertex a by a contact movement within minimum delay Δ_1 and maximum delay Δ_2. These delay parameters Δ_1, Δ_2 can be easily computed (by computing the sum of the delay bounds required for near-contact movement of each of the edges of C_j from a to a') in time $O(|V(C_j)|)$. Then

$$CM_{a,a'}(I) = \{t' \mid \Delta_1 + t \leq t' \leq \Delta_2 + t,\ t \in I\}.$$

so $|CM_{a,a'}(I)| \leq |I|$, and can be computed within time $O(|I| + |V(C_j)|)$. □

For each $a, a' \in V$, let $DM_{a,a'}(I)$ be the set of all times $t' \geq 0$ such that vertex a' can be reached at time t' by a single direct movement of B starting at vertex a at some time $t \in I$.

To calculate $DM_{a,a'}(I)$, we consider the following subproblem. Find the set $F_{a,a'}$ of all pairs of times t, t' such that the position $a'(t')$ of a' at time t' can be reached from the position $a(t)$ of a at time t by a single direct movement

Fix a time t and let $A(t)$ denote the set of all times t' such that the slope of the motion from $[a(t),t]$ to $[a'(t'),t']$ has modulus $\le v$. Plainly $A(t)$ is a closed interval $[t_1,t_2]$. Consider the triangle Δ whose corners are $w = [a(t),t]$, $w_1 = [a'(t_1),t_1]$, $w_2 = [a'(t_2),t_2]$. For each obstacle C_j, its space-time trajectory C_j^* intersects Δ at a convex set Δ_j. The two tangents from w to Δ_j cut an interval $A_j(t)$ off the segment w_1w_2. $A_j(t)$ is exactly the set of positions $[a'(t'),t']$ of a' which are not reachable from $[a(t),t]$ by a single direct movement, due to the interference of C_j. Let $I_j(t)$ denote the projection of $A_j(t)$ onto the t-axis. The set $F_{a,a'}$ is then

$$\{(t,t') : t' \not\in \bigcup_{j=1}^{k} I_j(t)\} .$$

Suppose $F_{a,a'}$ has been calculated. Then

$$DM_{a,a'} = \{t' : \exists t \in I, (t,t') \in F_{a,a'}\} .$$

To calculate the two-dimensional set $F_{a,a'}$ we can use a standard technique of sweeping a line $t = const$ across the (t,t')-plane. Note that for each t and j, each endpoint of $I_j(t)$ is determined by a specific vertex of C_j, and that, given such a vertex v the corresponding endpoint $e_v(t)$ of $I_j(t)$ is an algebraic function in t of constant degree. Hence the structure of $F_{a,a'} \cap \{t = const\}$ can change during the sweeping only at points t where two functions $e_v(t)$, $e_{v'}(t)$ overlap, or where one such function has a vertical tangent, i.e. at $O(n^2)$ points at most. This readily implies

Lemma 3.4: *$F_{a,a'}$ can be calculated in time $O(n^2\log n)$, and stored in $O(n^2)$ space. Furthermore, for each I, $|DM_{a,a'}(I)| \le (|I| + n^2)k$, and $DM_{a,a'}(I)$ can be calculated in time $(|I| + n^2)k$.*

Proof: The first part follows by the sweeping technique mentioned above. The second part follows from the fact that, as a result of the sweeping, the t-axis is split into $O(n^2)$ intervals, over each one of which the combinatorial structure of $F_{a,a'}$ remains constant, and consists of at most $k + 1$ disjoint intervals. Hence, merging these intervals with the intervals of I, we can calculate $DM_{a,a'}(I)$ in a straightforward manner within the asserted time bound, and also obtain the required bound on the complexity of that set. □

THEOREM 3.2. *The asteroid avoidance problem can be solved in time $O(n^{2(k+2)}k)$ and hence in time $n^{O(1)}$ in the case of $k = O(1)$ obstacles.*

Proof. Initially let $I_{X_0}^{(0)} = \{t, 0 \le t \le T\}$ and let $I_a^{(0)} = \phi$ for each $a \in v - \{X_0\}$. Inductively for some $i \ge 0$, suppose for each $a \in v$, $I_a^{(i)}$ is the set of times t that vertex a is reachable from $[X_0,0]$ by a (collision-free) movement of B consisting of $\le i$ fundamental movements. Then for each $a' \in V$,

$$J_{a'}^{(i)} = \bigcup_{a \in V} DM_{a,a'}(I_a^{(i)})$$

is the set of times that vertex a' is reachable from $[X_0,0]$ by a movement of B consisting of $\le i$ fundamental movements followed by a direct movement. Hence if $a'' \in V(C_j)$ then

$$I_{a''}^{(i+1)} = \bigcup_{a' \in V(C_j)} CM_{a',a''}(J_{a'}^{(i)})$$

is the set of times that vertex a'' is reachable from $[X_0,0]$ by a movement of B consisting of $\le i+1$ fundamental movements. Thus for each $a \in V$, $I_a^{(k+1)}$ is the set of times vertex a is reachable by a normal movement of B from $[X_0,0]$. By Lemma 3.2 such a normal movement suffices. So $T \in I_{X_T}^{(k+1)}$ iff there exists a collision-free movement of B from $[X_0,0]$ to $[X_T,T]$.

Lemmas 3.3 and 3.4 imply $|I_a^{(i)}| \leq O(n^{2i+2}k)$ and so the $i-th$ step takes time $O(n^2(n^{2i+2}k+k\log(n^{2i+3}k)))$. Therefore, the total time is $O(n^{2(k+2)}k)$. □

3.5. A Decision Algorithm for the 3-D Asteroid Avoidance Problem with an Unbounded Number of Obstacles

We next consider the 3-D asteroid avoidance problem. The configuration space FP is in this case 4-dimensional. By the results of Section 3.1, we can assume we wish to move a point B from $[X_0,0]$ to $[X_T,T]$, avoiding k convex polyhedral obstacles $C_1, \ldots, C_k$. In this case the *size* n of the problem is the total number of edges of the polyhedra. We will show that the problem is decidable.

Recall that each contact movement is required to begin and end at an obstacle edge or vertex. We will consider each obstacle edge $e = (u,v)$ to be directed from u to v. If e has length L we will for $0 \leq y \leq 1$, let $e(y)$ denote the point on e at distance $y \cdot L$ from vertex u, so $e(0) = u$ and $e(1) = v$. Let $E = \{e_1, \ldots, e_n\}$ be the set of all obstacle edges. Let $E(C_j) \subseteq E$ be the set of (directed) edges of obstacle C_j for $j = 1, \ldots, k$.

For technical reasons we again consider the initial and final positions of B to be immobile obstacles $C_0 = X_0$ and $C_{k+1} = X_T$. We consider $E(C_0)$ to contain a single edge of length 0 at point X_0 and $E(C_{k+1})$ to contain a single edge of length 0 at point X_T.

An *open formula* $F(y_1, \ldots, y_r)$ in the theory of real closed fields consists of a logical expression containing conjunctions, disjunctions, and negations of atomic formulas, where each atomic formula is an equality or inequality involving rational polynomials in the variables $y_1, \ldots, y_r$. A (partially quantified) formula in this theory is a formula of the form $Q_1 y_1 \cdots Q_a y_a\, F(y_1 \cdots y_r)$ where $a \leq r$, and where each Q_i is an existential or a universal quantifier. Such a formula will be called an *algebraic predicate*; its *degree* is the maximum degree of any polynomial within the formula, and its *size* is the number of atomic formulas it contains. We will use the following results:

LEMMA 3.5 [Collins, 75]. *A given formula of the theory of real closed fields of size n, constant degree, with r variables can be decided in deterministic time $n^{2^{O(r)}}$.*

LEMMA 3.6 [Ben-Or, Kozen, and Reif, 84]. *A given formula of the theory of real closed fields of size n constant degree and r variables can be decided in $n^{O(r)}$ space*

We will first show that we can describe by algebraic predicates the time intervals for which fundamental movements can be made, and then use the existential theory of real closed fields to decide the feasibility of movements consisting of finite sequences of these fundamental movements. Below we fix $e_i, e_{i'} \in E(C_j)$ and $0 \leq y, y' \leq 1$. Let $cm(i,i',y,y',\Delta)$ be the predicate which holds just if B has a contact movement along a single face of C_j from $e_i(y)$ to $e_{i'}(y')$ with delay Δ (i.e. the motion takes Δ time units)

LEMMA 3.7 *$cm(i,i',y,y',\Delta)$ can be constructed in polynomial time as an algebraic predicate of size $n^{O(1)}$ with constant degree and no quantified variables.*

Proof. Let $face(i,i')$ be the predicate that holds iff e_i and $e_{i'}$ are both on the same face of an obstacle. Let (w_x,w_y,w_z) be the velocity vector of obstacle C_j. Let (u_x,u_y,u_z) be distance vector from $e_i(y)$ to $e_{i'}(y')$, where u_x,u_y,u_z are all linear functions of y,y'. B will move in contact with C_j with velocity vector (v_x,v_y,v_z) with modulus

$$\sqrt{v_x^2+v_y^2+z_z^2} \le v \quad .$$

If B moves from $e_i(y)$ to $e_{i'}(y')$ with delay Δ then we must have $v_x\Delta = w_x\Delta + u_x$, $v_y\Delta = w_y\Delta + u_y$, and $v_z\Delta = w_z\Delta + u_z$.

Solving for v_x, v_y, v_z and substituting into $v_x^2 + v_y^2 + v_z^2$, we derive the formula

$$cm(i,i',y,y',\Delta) \equiv$$
$$\left[\left[\left(\frac{w_x\Delta+u_x}{\Delta}\right)^2 + \left(\frac{w_y\Delta+u_y}{\Delta}\right)^2 + \left(\frac{w_z\Delta+u_z}{\Delta}\right)^2\right] \le v^2\right] \wedge face\ (i,i') .$$

Let $dm(i,i',y,y',t,t')$ be the predicate which holds just if B has a (collision-free) direct movement from $e_i(y)$ at time t, to $e_{i'}(y')$ at time t'. The following is proved using arguments similar to those used in Lemma 3.4.

LEMMA 3.8 *$dm(i,i',y,y',t,t')$ can be constructed in polynomial time as an algebraic predicate of size and degree $n^{O(1)}$ with no quantified variables.*

Proof. For a given set of obstacles $C \subseteq \{C_1, \ldots, C_k\}$ let $dm_C(i,i',y,y',t,t')$ be defined as above, except that we allow possible collisions of B with obstacles in $\{C_1, \ldots, C_k\} - C$. Then $dm_\phi(i,i',y,y',t,t')$ can easily be given as an algebraic predicate of size $n^{O(1)}$ bounding the time t' to a single (possibly empty) interval, whose bounds vary algebraically with t.

Inductively we can write $dm_{\{C_1, \ldots, C_l\}}(i,i',y,y',t,t')$ as the conjunction of $dm_{\{C_1, \ldots, C_{l-1}\}}(i,i',y,y',t,t')$ and an algebraic predicate of size $n^{O(1)}$, restricting t' outside a single (possibly empty) interval of time. Thus $dm(i,i',y,y',t,t') = dm_{\{C_1, \ldots, C_k\}}(i,i',y,y',t,t')$ is an algebraic predicate of size $n^{O(1)}$.

Let $fm(i,i',y,y',t,t')$ hold iff there is a fundamental movement of B from $e_i(y)$ at time t to $e_{i'}(y')$ at time t'. Lemmas 3.7 and 3.8 imply

$$fm(i,i',y,y',t,t') \equiv dm(i,i',y,y',t,t') \vee cm(i,i',y,y',\ t'-t)$$

$$\vee \exists i'',y'',\Delta \quad dm(i,i'',y,y'',t,t'-\Delta) \wedge cm(i'',i',y'',y,\Delta)$$

LEMMA 3.9. *$fm(i,i',y,y',t,t')$ can be constructed in polynomial time as an algebraic predicate of size $n^{O(1)}$, constant degree and $O(1)$ quantified variables.*

Let $m(i,i',y,y',t,t')$ be the predicate that holds iff B has a collision-free movement from $e_i(y)$ at time t to $e_{i'}(y')$ at time t'. Note that the formula for $m(i,i',y,y',t,t')$ appears to require $\Omega(n)$ quantified variables. We now show

LEMMA 3.10. *$m(i,i',y,y,t,t')$ can be constructed in polynomial time as an algebraic predicate of size $n^{O(1)}$ with constant degree using $O(\log n)$ quantified variables.*

Proof. For each $\ell = 0,1, \ldots, \log n$, we define $m^{(\ell)}(i,i',y,y',t,t')$ to be predicate that holds iff B has a movement from $e_i(y)$ at time t to $e_{i'}(y')$ at time t' consisting of a sequence of $\le 2^\ell$ fundamental movements. Clearly, $m^{(0)}(i,i',y,y',t,t') = fm(i,i',y,y',t,t')$. We can then define

$$m^{(\ell+1)}(i,i',y,y',t,t') \equiv \exists i'',y'',t''$$

$$m^{(l)}(i,i'',y,y'',t,t'') \wedge m^{(l)}(i'',i',y'',y',t'',t') \ .$$

However, this definition, when applied recursively yields a formula of size $\geq 2^n$. A more compact definition is gotten by

$$m^{(l+1)}(i,i',y,y',t,t') \equiv \exists i'',y'',t'' \ \forall a_1,a_2,a_3,a_4,a_5,a_6$$

$$[(a_1 = i \wedge a_2 = i'' \wedge a_3 = y \wedge a_4 = y'' \wedge a_5 = t \wedge a_6 = t'') \vee$$

$$(a_1 = i'' \wedge a_2 = i' \wedge a_3 = y'' \wedge a_4 = y' \wedge a_5 = t'' \wedge a_6 = t')]$$

$$\supset m^{(l)}(a_1,a_2,a_3,a_4,a_5,a_6)$$

The formula $m^{(\log n)}(i,t',y,y',t,t')$ is of size $n^{O(1)} \log n \leq n^{O(1)}$ and requires only $O(\log n)$ quantified variables. By Lemmas 3.1 and 3.2 we have $m(i,i',y,y',t,t') \equiv m^{(\log n)}(i,i',y,y',t,t')$. □

THEOREM 3.3. *The 3-D asteroid avoidance problem can be solved in time $2^{n^{O(1)}}$, or alternatively in space $n^{O(\log n)}$.*

Proof. We assume immobile obstacle edges e_1, e_2 such that $e_1(0) = X_0$ and $e_2(0) = X_T$. Hence by Lemma 3.10, B has a collision-free movement from $[X_0,0]$ to $[X_T,T]$ iff $m(1,2,0,0,0,T)$ holds.

Since $m(i,2,0,0,0,T)$ has $n^{O(1)}$ size and $O(\log n)$ variables, we can test satisfiability of $m(1,2,0,0,0,T)$ by Lemma 3.5 in time $2^{n^{O(1)}}$, or alternatively by Lemma 3.6 in space $n^{O(\log n)}$. □

3.6 The 3-D Asteroid Avoidance Problem with Slow Obstacles

Let w_{max} be the maximum velocity modulus of any obstacle, and let v denote as above the maximum velocity modulus of B. The *slow asteroid avoidance problem* is a restricted 3-D asteroid avoidance problem where we require $w_{max} \leq v$, and we wish to plan an admissible motion of B which will avoid collision with any of the moving obstacles within a given interval of time. That is, we do not specify a final desired position of B but are only interested in the ability of B to avoid collision with slowly moving obstacles.

Let C be one of the moving obstacles whose velocity is w. Then the space-time volume swept by C is

$$C^* \equiv \{[Y+wt,t] : Y \in C_0, t \geq 0\}$$

where C_0 is the volume occupied by C at time $t = 0$. Hence C^* is also a convex body. Next define the *shadow* of C to be

$$s(C) \equiv \{[X,t] : \forall |u| \leq v \ \exists \tau \geq 0, [X+\tau u, t+\tau] \in C^*\} \ .$$

In other words, $s(C)$ consists of points $[X,t]$ such that if we proceed from them at any fixed admissible velocity, we will encounter a point on C^*. The intuition behind this definition is that $s(C)$ contains space-time positions from which it is impossible to escape the moving C, using any fixed admissible velocity. The following lemma shows that we cannot escape from any of these points under any choice of varying (but admissible) velocity:

LEMMA 3.11. *For each $Z \in s(C)$, any path starting at Z whose pointwise velocity remains bounded by v will hit a point in C^*.*

Proof. Suppose the contrary, and let p be such a path which misses C^*. Without loss of generality we can assume that p is a polygonal path (so that along each segment of this path the velocity is constant). Consider the first segment of p connecting a point $[X_1, t_1]$ in $s(C)$ to a point $[X_2, t_2]$ outside $s(C)$, and suppose that the velocity along this segment is u. In particular

$$X_2 = X_1 + (t_2 - t_1)u \ .$$

Since $[X_1, t_1] \in s(C)$, and since u is an admissible velocity, there exists a time τ (necessarily larger than t_2) such that

$$W_1 = [X_1 + (\tau - t_1)u, \tau] \in C^* \ .$$

On the other hand, $[X_2, t_2]$ is not in $s(C)$ so that, by definition, there exists another admissible velocity u_0 so that the straight path p_0 going from $[X_2, t_2]$ at velocity u_0 never meets C^*. By the same argument used above, there also exists a time $\tau_0 > t_1$ such that

$$W_2 = [X_1 + (\tau_0 - t_1)u_0, \tau_0] \in C^* \ .$$

But C^* is convex, so that the whole segment connecting W_1 with W_2 must be contained in C^*. This however, is impossible, because this segment intersects p_0, a contradiction which proves our claim. □

LEMMA 3.12. *Let C_1 and C_2 be two distinct moving obstacles with velocity modulus $\le v$. Then $s(C_1) \cap s(C_2) = \phi$.*

Proof. Suppose the contrary, and let $Z = [X, t]$ be a point in $s(C_1) \cap s(C_2)$ Choose any admissible velocity u, and proceed from Z at velocity u Since Z belongs to both shadows, it follows that there exist times τ_1, τ_2, both larger than t, such that $Z_1 = [X + u(\tau_1 - t), \tau_1] \in C_1^*$ and $Z_2 = [X + u(\tau_2 - t), \tau_2] \in C_2^*$ Without loss of generality assume $\tau_1 < \tau_2$. Then by the proof of Lemma 3.11 we have $Z_1 \in s(C_2)$. But Z_1 is a point on the moving C_1, which implies that a point on C_1 will eventually meet C_2, which contradicts our assumption that the moving obstacles do not collide. □

THEOREM 3.4. *If $w_{max} \le v$ and the initial position of B is not in any shadow $s(C)$ then it is always possible for B to avoid collision with the moving obstacles. Furthermore, if the obstacles are k polyhedra with a total of n edges, then the required motion of B can be computed in time $(n+k)^{O(1)} = n^{O(1)}$.*

Proof. Let $[X_0, 0]$ be the initial configuration at time $t_0 = 0$.

It suffices for B to remain immobile for $t \ge 0$, as long as $[X_0, t]$ contacts no obstacle shadow. Let $t' \ge 0$ be the first time (if ever) that $[X_0, t']$ contacts an obstacle shadow $s(C)$. Let w be the velocity vector of C. During times t, $t' \le t \le \infty$, we give B near-contact motion which remains on the external boundary of $s(C)$ using only translation of velocity $|w|$ in the direction of w. Since $w_{max} \le v$, the velocity modulus of these movements do not exceed v. Thus we have established the existence of a collision-free movement $[X, t]$ for $0 \le t \le \infty$. Hence B can always avoid collision with any obstacle Since the shadows of 3-D polyhedral obstacles can be easily computed in polynomial time, the required collision-free motion can also be computed in polynomial time. □

4. DYNAMIC MOVEMENT PROBLEMS WITH UNRESTRICTED VELOCITY

Throughout the last two sections we have assumed B had a given velocity modulus bound. Here we will allow B to have unrestricted motion, and in particular we will impose no velocity bounds.

This case appears still intractable, as we show that the 3-D dynamic movement problem for the case B is a cylinder with unrestricted motion, is NP-hard. Again this proof requires that B has only $O(1)$ degrees of freedom and we make critical use of the presence of rotating obstacles to encode time.

We will next show, in contrast with what has just been stated, that the problem is polynomial time if all the obstacle motions are algebraic in space-time, that is the movement of B is constrained by algebraic inequalities of bounded degrees (for example B consists of a bounded number of 3-D linked polyhedra), and there is no bound on the velocity modulus of B.

4.1 The Case of Unrestricted Motions in the Presence of Rotating Obstacles is NP-hard

We will reduce the 3-satisfiability problems to that of planning the motion of a cylindrical body B in 3-space in the presence of several rotating obstacles. A *semidisc* is a disc with half its interior removed so that it is bounded by a semicircle and a line segment. Suppose that we are given an instance of 3-satisfiability involving n Boolean variables $x_1, \ldots, x_n$. With each variable x_i we associate several semidiscs $D_{i,k}$ of radius 1, each rotating in some plane lying parallel to the $x-y$ plane at some height $h_{i,k}$ with its center at some point $u_{i,k}$. For each $i = 1, \ldots, n$, all the semidiscs $D_{i,k}$ rotate with the same angular velocity $v_i = \frac{\pi}{2^{i-1}}$. Thus the first set of semidiscs complete half a revolution in 1 time unit, the second set in 2 time units, and so forth. Hence, if U is a sufficiently small disc contained in the interior of the 2-D unit disc near its perimeter, and if the initial positions of the rotating semidiscs are chosen appropriately, then after t whole time units each semidisc $D_{i,k}$ will cover the set $(U+u_{i,k}) \times \{h_{i,k}\}$ if and only if the i-th binary digit of t is 1. We assume that the cross section of B has an area smaller than that of U.

Suppose that the given instance of 3-satisfiability involves p clauses, where the m-th clause has the form $z_{m_1} \vee z_{m_2} \vee z_{m_3}$, where each z_i is either x_i or the complement of x_i. We represent this clause by three semidiscs $D_{m_1,m}, D_{m_2,m}, D_{m_3,m}$, all placed on a plane at some height h_m (without touching or intersecting each other), such that their centers all lie on the y axis of this plane, and such that the empty half of $D_{m_i,m}$ is placed initially to the right of the y-axis if $z_{m_i} = x_{m_i}$; otherwise the semidisc is placed initially with its empty half to the left of the y axis. We then construct three narrow tunnels, all connecting some point C_m lying between the $(m-1)$-th plane and the m-th plane just introduced, to a point C_{m+1} lying above the new plane. Each tunnel is circular, and its intersection with the plane is a sufficiently small disc lying within the right half of the corresponding disc D near its highest (in y) point. This construction implies that at time t the body B that we wish to move can quickly go from C_m to C_{m+1} iff the assignment of the i-th binary digit of t to the variable x_i, for each $i = 1, \ldots, n$, satisfies the m-th clause. It follows that we can move B from an initial position C_1 to a final C_{m+1} iff there exists a time t for which the above assignment satisfies the given instance of the 3-satisfiability problem. (It is easy to add more rotating discs that would enforce B to traverse the whole system of tunnels in a very short time that begins at an integral number of time units.) This proves that

THEOREM 4.1. *In the presence of rotating obstacles, dynamic motion planning of a body B with no velocity modulus bound is NP-hard, even in the case where the body B is a rigid cylinder.*

Remark: As in the case of the time-machine construction in Section 2, this construction can also be simplified to a two-dimensional dynamic movement planning with a single moving point obstacle, at the cost of using an irregular and more complex motion of that obstacle.

4.2 The Case of Unrestricted Algebraic Motions

Let B be an arbitrary fixed system of moving bodies with a total of d degrees of freedom. Let S be a space bounded by an arbitrary collection of moving obstacles. Let the (space-time) free configuration space FP of B be defined as in Section 1. We will assume that the problem is algebraic in the sense that the geometric constraints on the possible free configurations of B (i.e. the constraints defining FP) can be expressed as algebraic (over the rationals) equalities and inequalities in the $d+1$ parameters $[X,t]$.

Remark. Some of the motions used in the preceding lower bound proofs are not algebraic in the above sense. The simplest such motion is rotation of a two-dimensional body about a fixed center. Indeed, suppose for simplicity that the rotating body is a single point at distance r from the center of rotation (which we assume to be the origin). Then the curve in space-time traced by the rotating point is a helix, parametrized as $(x,y,t) = (r\cos\omega t, r\sin\omega t, t)$, which is certainly not algebraic.

To obtain a polynomial-time solution to this problem, we decompose E^{d+1} into a *cylindrical algebraic decomposition* as proposed by Collins [Collins, 75] (or Collins' decomposition in short; cf. [Cooke and Finney, 67] for a basic description of cell complexes) relative to the set P of polynomials appearing in the definition of FP. Roughly speaking, this technique partitions E^{d+1} into finitely many connected cells, such that on each of these cells each polynomial of P has a constant sign (zero, positive, or negative). Thus FP is the union of a subset of these cells, and it is a simple matter to identify those cells which are contained in FP (we refer to such cells as *free* Collins cells). Moreover, by using the modified decomposition technique presented in [Schwartz and Sharir, 83b], one can also compute the *adjacency* relationships between Collins cells (i.e., find pairs $[c_1,c_2]$ of Collins cells such that one of these cells is contained in the boundary of the other). Thus any continuous path in FP can be mapped to the sequence of free Collins cells through which it passes, and conversely, for any such sequence of free adjacent Collins cells we can construct a continuous path in FP passing through these cells in order. This observation has been used in [Schwartz and Sharir, 83b] to reduce the continuous (static) motion planning problem to the discrete problem of searching for an appropriate path in an associated *connectivity graph* whose nodes are the free Collins cells, and whose edges connect pairs of adjacent such cells.

We would like to apply the same ideas to the dynamic problem that we wish to solve, but we face here the additional problem that we are allowed to consider only t-monotone paths in FP. To overcome this difficulty, we note that the Collins decomposition procedure is recursive, proceeding through one dimension at a time. When it comes to decompose the subspace E^{i+1}, it has already decomposed E^i into "base" cells, and the decomposition of E^{i+1} will be such that for each base cell b of E^i there will be constructed several "layered" cells of E^{i+1} all projecting into b. Hence if we apply the Collins decomposition technique in such a way that the time axis t is decomposed in the innermost recursive step, it follows that each final cell c (in E^{d+1}) consists of points $[X,t]$ whose t either lies between two boundary times $t_0(c) < t_1(c)$ or is constant. Moreover, If c is a Collins cell of the first type, then it is easy to show, using induction on the dimension, that for any point $[X_0, t_0(c)]$ lying on the "lower" boundary of c, and for any point $[X_1, t_1(c)]$ on its "upper" boundary, there exists a continuous monotone path through c connecting these two points. In fact, the preceding property also holds if one or both of these points are interior to c.

These observations suggest the following procedure.

(1) Apply the Collins decomposition technique to E^{d+1} relative to the set of polynomials defining FP, so that t is the innermost dimension to be processed. Also find the adjacency relationship between the Collins cells, using the technique described in [Schwartz and Sharir, 83b].

(2) Construct a *connectivity graph CG*, which is a *directed* graph defined as follows: The nodes of CG are the free Collins cells. A directed edge $[c, c']$ connects two free cells c and c' provided that (a) c and c' are adjacent; (b) either c and c' both project onto the same base segment on the t axis, or c projects onto an open t segment $(t_0(c), t_1(c))$ and c' projects onto its upper endpoint $t_1(c)$, or c' projects onto an open t segment $(t_0(c'), t_1(c'))$ and c projects onto its lower endpoint $t_0(ctag)$. Intuitively, each edge of CG represents a crossing between two adjacent cells which is either stationary in time (crossing in a direction orthogonal to t), or else progresses forward in time.

(3) Find the cells c_0, c_1 containing respectively the initial and final configurations $[X_0, 0]$, $[X_1, T]$. Then search for a directed path in CG from c_0 to c_1. If there exists such a path then there also exists a motion in FP between the two given configurations (and the latter motion can be effectively constructed from the path in CG), otherwise no such motion exists.

To see that the procedure just described is correct, note first that if p is a continuous motion through FP between the initial and final configurations (which we assume to cross between Collins cells only finitely many times), then it is easily seen that the sequence of free cells through which p passes constitutes a directed path in CG. Conversely, if p' is a directed path in CG between c_0 and c_1, then p' can be transformed into a continuous (monotone) motion through FP as follows. First choose for each free Collins cell c a representative interior point $[X_c, t_c]$, such that the representative points of all the cells that project onto the same base segment on the t axis have the same t value. Then transform each edge $[c, c']$ of p' into a monotone path in FP as follows. If $t_c = t_{c'}$ (i.e., if the crossing from c to c' is orthogonal to the time axis), then connect $[X_c, t_c]$ to $[X_{c'}, t_c]$ by any path which is contained in the union $c \cup c'$ and on which t is held constant, the existence of such a path is guaranteed by the property of Collins cells noted above. If $t_c < t_{c'}$ then connect $[X_c, t_c]$ to $[X_{c'}, t_{c'}]$ by a monotone path contained in $c \cup c'$, again the existence of such a path is guaranteed by the structure of Collins cells. The resulting path p is plainly continuous, is contained in FP and is weakly monotone in t. (Note that the crossings of the first type in which t remains constant represents extreme situations where the velocity of B is infinite. However, since p is continuous and FP is open, one can easily modify p slightly so as to make it strictly monotone in time.) This establishes the correctness of our procedure.

Since the Collins decomposition is of size polynomial in the number of given polynomials and in their maximal degree (albeit doubly exponential in the number of degrees of freedom d), and can be computed within time of similar polynomial complexity, it follows that

THEOREM 4.2. *The dynamic unrestricted version of the movers problem for a fixed moving B can be solved in the general (space-time) algebraic case in time polynomial in the number of obstacles, the number of parts of B, and their maximal algebraic degree*

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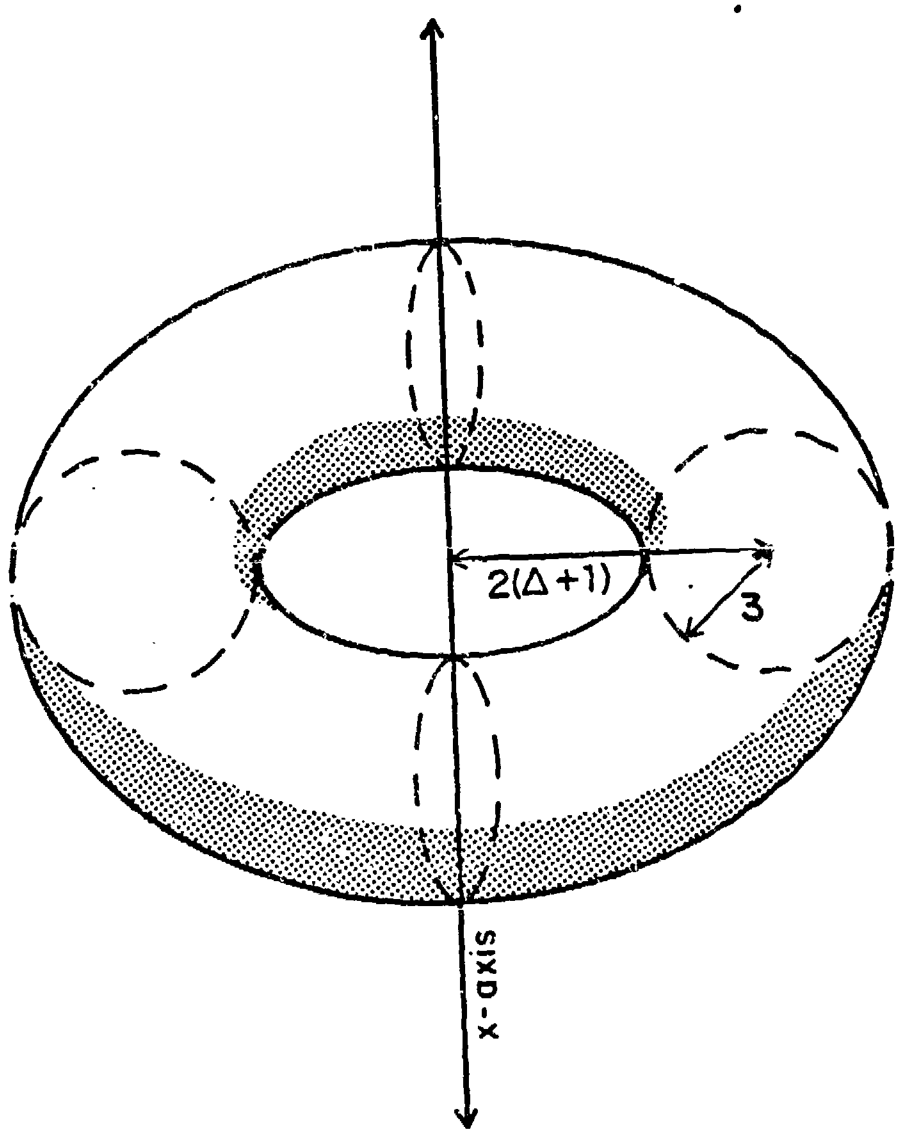

Figure 1. The construction of a torus by the movement of a circle of radius 3 around the x-axis.

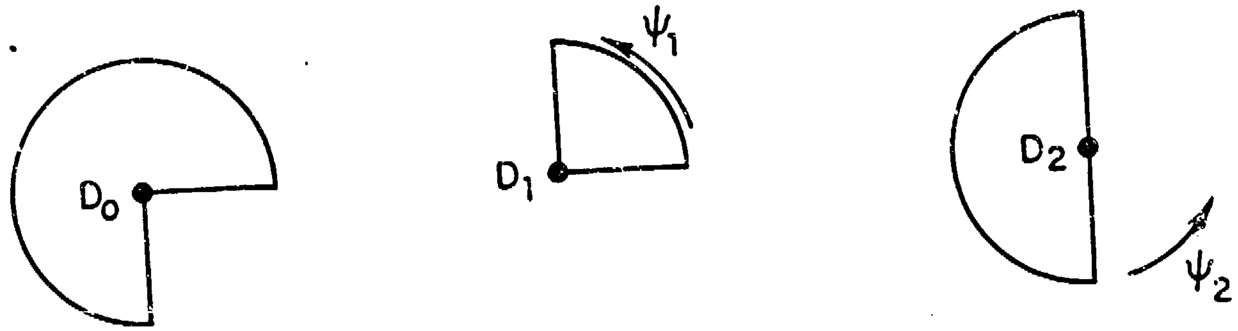

Figure 2. The disks D_0, D_1, D_2.

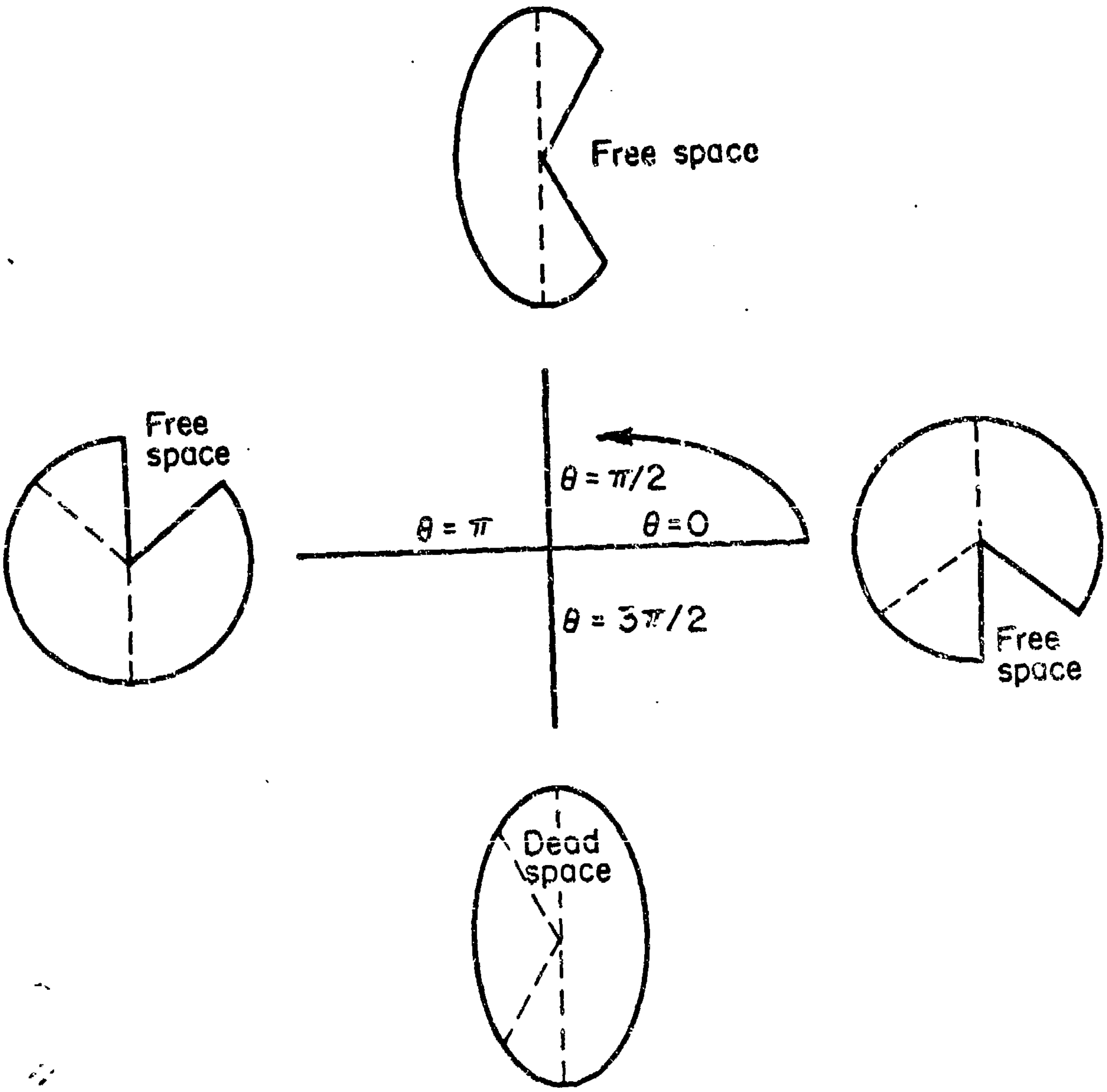

Figure 3. Snapshots of $D_0 \cup D_1$ at angular displacements of $\theta = 0, \pi/2, \pi, 3\pi/4$.

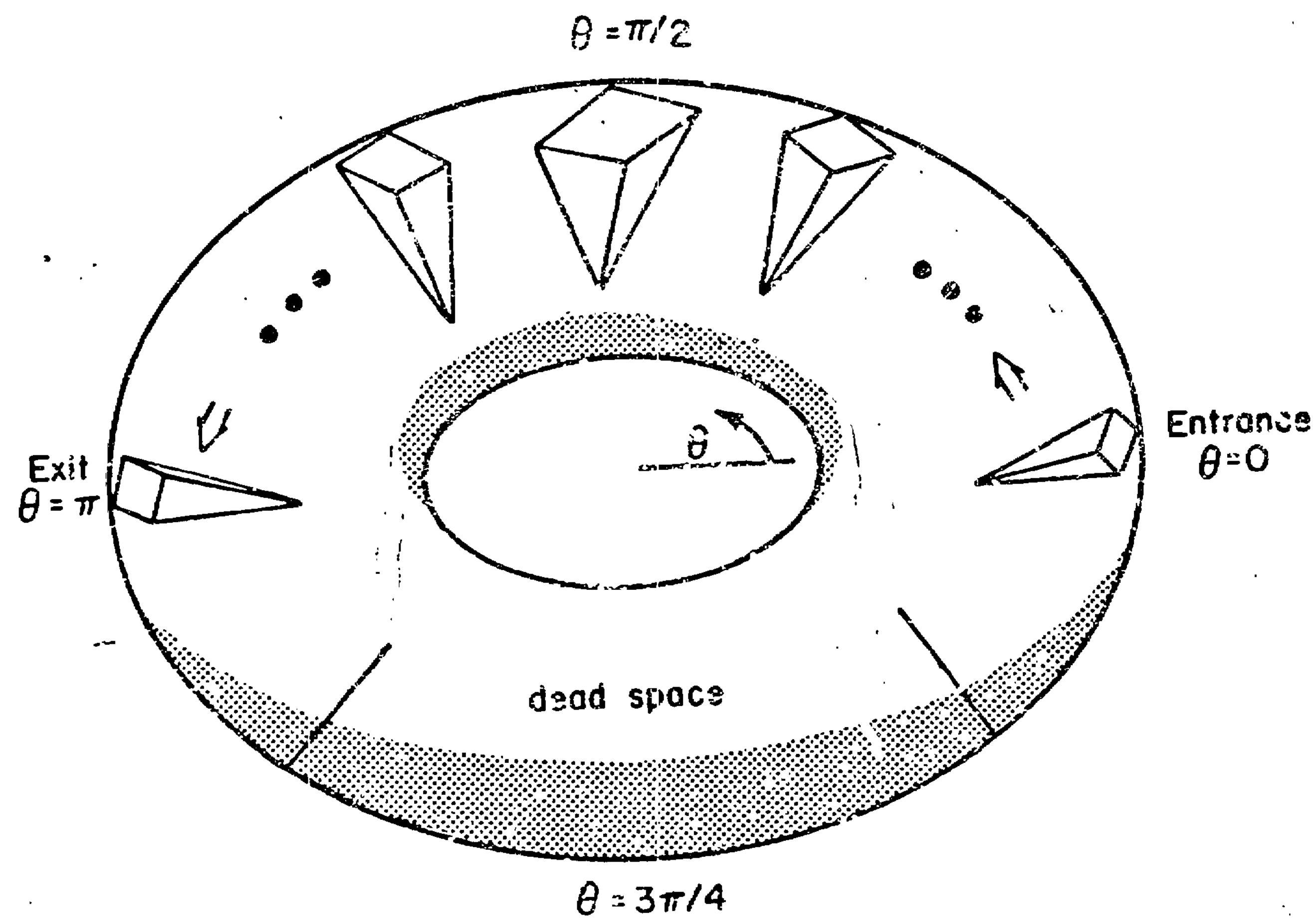

Figure 4. The free spaces generated by the movement of D.