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MOTIONS OF SATELLITES AND ASTEROIDS UNDER THE INFLUENCE OF JUPITER AND THE SUN

I. Stable and Unstable Satellite Orbits

R. B. Hunter

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Summary

An attempt is made to find which initial jovicentric orbital elements give rise to stable satellite orbits about Jupiter. The orbits of a large number of hypothetical satellites of Jupiter with various values for their initial semi-major axes, inclinations and eccentricities are computed and the effect of varying these initial elements on a satellite's subsequent behaviour is demonstrated.

The orbits of the satellites are taken to be examples of the elliptical restricted three-dimensional three-body problem with the Sun and Jupiter as the two massive bodies. The equations of motion of the satellite are solved numerically by De Vogelaere's method.

1. *Introduction.* The seven outermost satellites of Jupiter have all been discovered photographically since 1904. These satellites (referred to as numbers VI to XII in order of their discovery) are very faint objects with apparent magnitudes, when Jupiter is at opposition as seen from the Earth, of between 14·7 and 19·0. The mean distances of these satellites from Jupiter are so large compared with their distances from the Sun (at least one in eighty) and the mass of the Sun is so much greater than that of Jupiter (1047 : 1) that the orbits of these satellites are very strongly perturbed by the Sun.

Table I gives mean values of the semi-major axes, periods, inclinations and

Table I

Outer satellites of Jupiter

Satellite	Semi-major axis (a.u.)	Sidereal period (years)	Inclination	Eccentricity
VI	0·0767	0·686	28°	0·158
X	0·0783	0·710	29°	0·107
VII	0·0785	0·711	28°	0·207
XII	0·142	1·728	33° R	0·169
XI	0·151	1·895	17° R	0·207
VIII	0·157	2·037	32° R	0·410
IX	0·158	2·075	23° R	0·275

From British Astronomical Association Handbook 1966.
R—retrograde orbit.

eccentricities of the seven satellites. The orbits appear to fall into three groups:

(i) Satellites VI, X and VII which are all in direct orbits with similar elements.

(ii) Satellite XII which is in a retrograde orbit slightly inside the orbits of the group (iii) satellites.

(iii) Satellites XI, VIII and IX which are in retrograde orbits with similar elements.

Roy & Ovenden (1) have discussed the occurrence of commensurabilities between the mean angular motions of these satellites and the mean angular motion of Jupiter about the Sun (see Table II). The satellites of group (i) lie close to the commensurability 1/17, while the average of the three mean motions lies even closer to the commensurability. The satellite of group (ii) is seen to lie near the commensurability 1/7 and those of group (iii) near 1/6. Again the average of the three mean motions lies even nearer the commensurability.

TABLE II

Satellites	$\frac{n_2}{n_1}$	A_2	A_1	$\frac{n_2}{n_1}-\frac{A_2}{A_1}$
Sun J VIII	0·17055	1	6	+0·00388
Sun J IX	0·17196	1	6	+0·00529
Sun J XI	0·15982	1	6	−0·00684
Sun, and mean of J VIII, J IX, J XI	0·16725	1	6	+0·00059
Sun J VI	0·05784	1	17	−0·00098
Sun J VII	0·06002	1	17	+0·00120
Sun J X	0·05867	1	17	−0·00015
Sun and mean of J VI, J VII, J X	0·05883	1	17	−0·00001
Sun J XII	0·14564	1	7	+0·00278

Note: $\frac{1}{16}-\frac{1}{17}=0{\cdot}00368$

$\frac{1}{6}-\frac{1}{7}=0{\cdot}02381$

n_1 and n_2 are mean motions of the satellite and Jupiter respectively.

Roy & Ovenden have shown that such commensurabilities between orbits in the solar system are much more common than could be merely attributed t chance and conclude that such orbits may be relatively stable due to the frequer occurrence of ' mirror configurations '.

In this paper the relative stability of various satellite orbits will be considere by computing a large number of orbits in the elliptical restricted three-dimer sional three-body problem of celestial mechanics, taking the two massive bodi to be the Sun and Jupiter and the body of negligible mass to be the satellite. Th follows on the work of Darwin (2), Strömgren (3), Goudas (4) and Chebotarev (who have computed large numbers or orbits of the massless body in the restrict three-body problem, in some cases with reference to the orbits of jovian satellit and the work of Bobone (6), Lemechova (7) and Kovalevsky (8) who have publish theories of the motions of particular outer satellites of Jupiter. The approach described in this paper lies somewhere between those of the two sets of authors, in that an attempt is made to find initial conditions that give rise to stable satellite orbits at large distances from Jupiter from the integration of a large number of satellite orbits. The orbits considered will be more like those of the real satellites of Jupiter than those computed by the first set of authors in that, for example, only Goudas considered orbits not in the plane of the Sun's apparent orbit about Jupiter and only Chebotarev took the massive bodies (the Sun and Jupiter) to move in elliptical (rather than circular) orbits about one another.

2. *Method of integration.* Let a rectangular set of axes be chosen with origin the centre of Jupiter and the x-axis pointing in the direction of the Sun's perijove, assumed to lie in a fixed direction. Let the y-axis be in the plane of the Sun's apparent orbit about Jupiter and such that the y co-ordinate of the Sun increases as it crosses the positive half of the x-axis, and let the z-axis be chosen so as to form a right-handed system.

With respect to these axes the equations of motion of a satellite of negligible mass moving under the influence of two massive bodies, the Sun and Jupiter, may be obtained in the following form by equating one of the masses to zero in equation (1) on page 9 of Smart (**9**).

$$x'' = -\frac{Gmx}{r^3} + GM\left(\frac{X-x}{\Delta^3} - \frac{X}{R^3}\right) \tag{1}$$

where x, X are the x co-ordinates of the satellite and the Sun respectively, r, Δ, R are the distances Jupiter–satellite, Sun–satellite and Jupiter–Sun respectively and G is the constant of gravitation.

The equation for y is exactly similar but, since $Z=0$, the one in z reduces to,

$$z'' = -\frac{Gmz}{r^3} - \frac{GMz}{\Delta^3}. \tag{2}$$

It was necessary to find a method of solving the three non-linear simultaneous second-order differential equations numerically. Kovalevsky (**8**) and Chebotarev (**5**) have used Cowell's method to integrate the equations while Bartlett (**10**) and Goudas (**4**) have used a modified Runga–Kutta method. In this case the fact that the first derivatives of the co-ordinates were missing from the equations suggested the use of De Vogelaere's method (**11**). It was felt that this method was preferable to the other two as it was more elegant, required less programming and has a simple starting sequence.

For a differential equation of the form,

$$x'' = f(x, t). \tag{3}$$

De Vogelaere's method consists of the cyclic use of the following equations

$$x_{r+\frac{1}{2}} = x_r + \tfrac{1}{2}hx_r' + \frac{h^2}{24}(4f_r - f_{r-\frac{1}{2}}) \tag{4}$$

$$x_{r+1} = x_r + hx_r' + \frac{h^2}{6}(f_r + 2f_{r+\frac{1}{2}}) \tag{5}$$

$$x'_{r+1} = x_r' + \frac{h}{6}(f_r + 4f_{r+\frac{1}{2}} + f_{r+1}) \tag{6}$$

where f_r denotes $f(x_r, t_r)$ and h is the interval of integration. The neglected terms are of order h^4, h^5 and h^5 respectively and the method is comparable in accuracy with the fourth-order Runga–Kutta process. The function f however is only calculated twice per step.

To begin the sequence of operations the value of $f_{-\frac{1}{2}}$ is required and is readily obtained from $x_{-\frac{1}{2}}$ which is given with sufficient accuracy by,

$$x_{-\frac{1}{2}} = x_0 - \tfrac{1}{2}hx_0' + \tfrac{1}{8}h^2 f_0. \tag{7}$$

The method may be extended to solve *simultaneous* second-order differential equations with the first derivatives missing. The equations of motion of the satellite are of the form,

$$x'' = e(x, y, z, t) \tag{8}$$

$$y'' = g(x, y, z, t) \tag{9}$$

$$z'' = k(x, y, z, t) \tag{10}$$

assuming that X and Y can be expressed as simple functions of t. Then De Vogelaere's method of solving these equations involves applying each of equations (4), (5) and (6) in turn to equations (8), (9) and (10).

Each time the functions e and g are evaluated the co-ordinates of the Sun (X and Y) have to be calculated. Since the satellite is assumed to be massless, the apparent orbit of the Sun about Jupiter is, of course, a fixed ellipse and X and Y can be determined by applying Newton's method of successive approximations to solve Kepler's equation,

$$E - e \sin E = M \tag{11}$$

for E the Sun's eccentric anomaly, knowing M its mean anomaly. Then X and Y may be obtained from,

$$X = \cos E - e \tag{12}$$

$$Y = b \sin E \tag{13}$$

where e is the eccentricity of the Sun's apparent orbit about Jupiter and b its semi-minor axis (see e.g. Smart (**12**)).

The value of the integration step used varied according to the distance of the satellite from Jupiter. The smallest value normally used was 0·7 days or $1{\cdot}6 \times 10^{-4}$ Jupiter years (one Jupiter year being equal to the sidereal period of Jupiter). Every thirty steps the program would check the length of the interval of integration to see if it should be halved or could be doubled. This was done by comparing the results obtained after a few steps with the previous interval, half the previous interval and double the previous interval respectively.

After every fourth interval check (i.e. about eight times per revolution) results were printed out including the jovicentric co-ordinates and velocity components of the satellite and the osculating values of its jovicentric orbital elements. The equations used for computing the elements from the co-ordinates and velocity components are given by Sterne (**13**).

The integrations were performed on the English Electric KDF.9 computer in the University of Glasgow. The program was written in KDF.9 Algol and with the latest compiler used (intermediate Kidsgrove) it was possible to compute one satellite revolution per minute of machine time. In all, about 2000 satellite revolutions about Jupiter were computed, the number of significant figures used in the computer being 11.

3. *Starting values and notation.* All the integrations were begun with the satellite on the semi-major axis of the Sun's apparent orbit about Jupiter when the Sun was at perijove. In addition the satellite had its velocity vector perpendicular to the line Sun–Jupiter, so that the three bodies formed a 'mirror configuration' as defined by Roy & Ovenden (**1**). According to them the past motion of the satellite would then be the mirror image about the line Sun–Jupiter of its future motion. In most cases the satellite lay between the Sun and Jupiter at the start of the integration but in a few cases it lay on the opposite side of Jupiter from the Sun.

It will be convenient to refer to a satellite orbit by the initial osculating values of its orbital semi-major axis, eccentricity and inclination to the plane of Jupiter's orbit about the Sun and the expression $\alpha/\beta/\gamma$ will refer to the satellite with semi-major axis $\alpha \cdot 10^{-2}$ Jupiter units (one Jupiter unit being the mean distance of Jupiter from the Sun i.e. 5·2028143 astronomical units), eccentricity β and inclination γ degrees. The inclination 22°·28 (the mean of the inclinations of the three outermost satellites of Jupiter) was used a great deal as it was felt to be typical of the inclinations of the real satellites. In the cases when this inclination is used the satellite orbit may be referred to as α/β where α and β have the same meanings as above. Where necessary the letter D will be used to indicate a direct orbit and R a retrograde one. The letter O will denote a satellite, the integration of whose orbit was begun with the satellite at opposition i.e. on the other side of Jupiter from the Sun.

4. *Direct and retrograde orbits.* In the next few sections the stability of various orbits of the type described above will be investigated according to the value of their initial osculating elements. An orbit will be said to be stable if the satellite is able to complete fifty revolutions about Jupiter without escaping from the planet. A satellite will be said to have escaped from Jupiter the moment its osculating jovicentric eccentricity becomes greater than unity. In practice the satellite was always found to leave the vicinity of Jupiter immediately after this had occurred and all the integrations were continued until the satellite was at least one Jupiter unit from Jupiter. A full list of the initial elements and lifetimes of the satellite orbits integrated is given in Table III.

Table IV compares the relative stability of direct and retrograde satellite orbits. Eight semi-major axes were chosen and the orbits of satellites with these initial semi-major axes and with the usual inclination (22°·28) were computed both in the direct and in the retrograde sense and with eccentricities 0 and 0·3. It should be noted that the retrograde satellites always remained in orbit about Jupiter for at least as great a number of revolutions, and usually a greater number, than the corresponding direct ones. It is of particular interest to note that a satellite with the mean elements of the three outermost retrograde satellites of Jupiter, but moving directly, escapes from Jupiter after performing only 29 revolutions about the planet. This is consistent with Moulton's (**14**) conclusion that a direct satellite with the period of Jupiter VIII would have a less stable orbit than that of Jupiter VIII, which is retrograde.

The value of 3·368 for one of the semi-major axes in Table IV corresponds to the commensurability 1/5 and the value 3·774 to the commensurability 1/4. Th

TABLE III

Starting values and lifetimes of satellites

No.	*a*	*e*	*i*	D or R	O or C	No. of revs.
1	1·49	0·158	29·9	D	C	> 100
2	2·96	0·30	22·3	D	C	29
3	2·73	0·17	32·6	D	C	> 50
4	1·49	0·158	27·9	D	C	> 100
5	3·37	0·30	22·3	D	C	9
6	3·37	0·30	22·3	R	C	> 50
7	3·37	0·00	22·3	D	C	> 50
8	3·77	0·00	22·3	D	C	3
9	3·77	0·00	22·3	R	C	> 50
10	3·91	0·00	22·3	D	C	3
11	3·91	0·00	22·3	R	C	> 50
12	3·91	0·30	22·3	D	C	2
13	3·91	0·30	22·3	R	C	> 50
14	3·77	0·30	22·3	D	C	2
15	3·77	0·30	22·3	R	C	> 50
16	1·42	0·158	27·9	D	C	> 100
17	1·49	0·168	27·9	D	C	> 100
18	5·00	0·00	22·3	D	C	5
19	5·00	0·00	22·3	R	C	> 50
20	5·00	0·30	22·3	D	C	1
21	5·00	0·30	22·3	R	C	3
22	5·00	0·30	22·3	D	O	1
23	5·00	0·30	22·3	R	O	3
24	4·50	0·00	22·3	D	C	1
25	4·50	0·00	22·3	R	C	> 50
26	4·50	0·30	22·3	D	C	3
27	4·50	0·30	22·3	R	C	4
28	4·50	0·30	22·3	D	O	3
29	4·50	0·30	22·3	R	O	5
30	2·96	0·158	27·9	D	C	> 100
31	2·96	0·30	22·3	R	C	> 100
32	2·72	0·30	22·3	D	C	> 100
33	2·72	0·168	32·6	R	C	> 100
34	6·00	0·00	22·3	D	C	0
35	6·00	0·00	22·3	R	C	4
36	1·51	0·107	28·8	D	C	> 100
37	4·50	0·15	22·3	R	C	> 50
39	5·00	0·15	22·3	R	C	> 50
40	3·37	0·30	3·0	D	C	3
41	4·50	0·25	22·3	R	C	> 50
42	3·37	0·30	15·0	D	C	3
43	5·00	0·25	22·3	R	C	12
44	3·37	0·30	28·0	D	C	9
45	5·00	0·25	5·0	R	C	> 50
46	3·37	0·30	45·0	D	C	> 50
47	5·00	0·25	60·0	R	C	3
49	5·00	0·25	15·0	R	C	> 50
50	3·37	0·30	60·0	D	C	> 50
51	5·00	0·25	45·0	R	C	3
52	3·37	0·30	75·0	D	C	> 50
53	5·00	0·25	75·0	R	C	3
54	2·96	0·30	3·0	D	C	23

D—direct, R—retrograde, O—satellite started at opposition, C—satellite started at conjunction.

a is given in terms of 10^{-2} Jupiter units.

i is in degrees.

TABLE IV

Direct and retrograde

Semi-major axis 10^{-2} J.U.	No. of revolutions $e=0$ D	No. of revolutions $e=0$ R	No. of revolutions $e=0 \cdot 3$ D	No. of revolutions $e=0 \cdot 3$ R
2·725 (XII)	—	—	>50	—
2·959 (VIII, IX and XI)	—	—	29	>50
3·368	>50	—	9	>50
3·774	3	>50	2	>50
3·908	3	>50	2	>50
4·50	1	>50	3	4
5·00	5	>50	—	3
6·00	0	4	—	—

D—direct.
R—retrograde.

value 3·908 does not correspond to a commensurability but is sufficiently close to 3·774 to provide a comparison between adjacent commensurable and non-commensurable orbits. The larger values for the semi-major axis have no particular significance. There would appear to be no evidence from Table IV to suggest that commensurable orbits are more or less stable than adjacent non-commensurable ones.

5. *Effect of eccentricity.* It may also be seen from Table IV that a satellite with initial eccentricity 0 always has at least as long a lifetime (in terms of numbers of revolutions about Jupiter) as the corresponding satellite with eccentricity 0·3. Table V shows this more clearly. The four direct orbits have eccentricities and semi-major axes corresponding to those of Jupiter XII and the mean of Jupiter VIII, IX and XI (i.e. the four orbits correspond to the four combinations of the two eccentricities and semi-major axes). It should be noted that the only satellite which escaped from Jupiter after a relatively short time was the one with both the larger semi-major axis and the greater eccentricity.

TABLE V

Effect of eccentricity

D or R	Semi-major axis	Number of revolutions $e=0$	$e=0 \cdot 15$	$e=0 \cdot 25$	$e=0 \cdot 3$
D	2·725	—	>50	—	>50
D	2·959	—	>50	—	29
R	4·5	>50	>50	>50	4
R	5·0	>50	>50	12	3

Distances in terms of 10^{-2} Jupiter units.

The eight retrograde orbits behaved similarly. For both the values of the semi-major axis used, there appears to be a cut off in stability at an eccentricity somewhere between 0·25 and 0·30. It would appear therefore that, in the case of both direct and retrograde satellites, an increase in orbital eccentricity corresponds to a decrease in stability.

6. *Effect of inclination.* To investigate the effect of altering the initial orbital inclination on the lifetime of a satellite it was decided to select one direct and one retrograde satellite, each of which had a lifetime of about ten revolutions about Jupiter, and to investigate the effect of changing the initial inclinations of these satellites on their lifetimes. It was felt that these particular satellites might be fairly sensitive to changes in their inclinations.

The retrograde satellite chosen was satellite 43, i.e. 5·0/0·25/22, which escaped from Jupiter after performing twelve revolutions about the planet. Orbits of retrograde satellites with the same semi-major axis and eccentricity as satellite 43 and with the following inclinations were computed – 5°, 15°, 45°, 60°, 75°. Their lifetimes, in terms of numbers of revolutions about Jupiter, and that of satellite 43 were as follows:

Inclination (°)	Lifetime
5	>50
15	>50
22	12
45	3
60	3
75	3

It would appear from these results that, for retrograde satellites, the smaller the inclination the greater the stability of the orbit.

The direct satellite chosen for this investigation was satellite 5, 3·37/0·3/22, and direct satellite orbits were computed with the same semi-major axis and eccentricity as satellite 5 and with the following inclinations: 3°, 15°, 28°, 45°, 60°, 75°. Their lifetimes and that of satellite 5 were as follows:

Inclination (°)	Lifetime
3	3
15	3
22	9
28	9
45	20
60	>50
75	>50

Here the pattern is exactly the opposite to the one for retrograde satellites. It would appear that, for direct satellites, the greater the orbital inclination the more stable the orbit.

As a check on this conclusion about direct orbits, the orbit of satellite 2, 2·96/0·3/22D (the mean of Jupiter VIII, IX and XI but direct) was recomputed with initial inclination 3° (satellite 54). Whereas satellite 2 escaped from Jupiter after 29 revolutions, satellite 54 escaped after 23 revolutions, which fact would appear to be consistent with the above results.

7. *Direct orbits.* In this section and the next the orbits of some of the satellites, particularly those which remained in the vicinity of Jupiter for a relatively short time, will be discussed in more detail.

Of the initially circular orbits, satellite 8, 3·77/0, was the one with the smallest semi-major axis which escaped from Jupiter after a few revolutions. Fig. 1 is the projection of the orbit on the x/y plane. The Sun was on the positive half of the x-axis at the beginning of the integration and its relative position after each revolution of the satellite is given in the bottom right hand corner of the figure. In all the figures the continuous lines represent the parts of the orbit above the x/y plane and the dotted lines the parts below the plane.

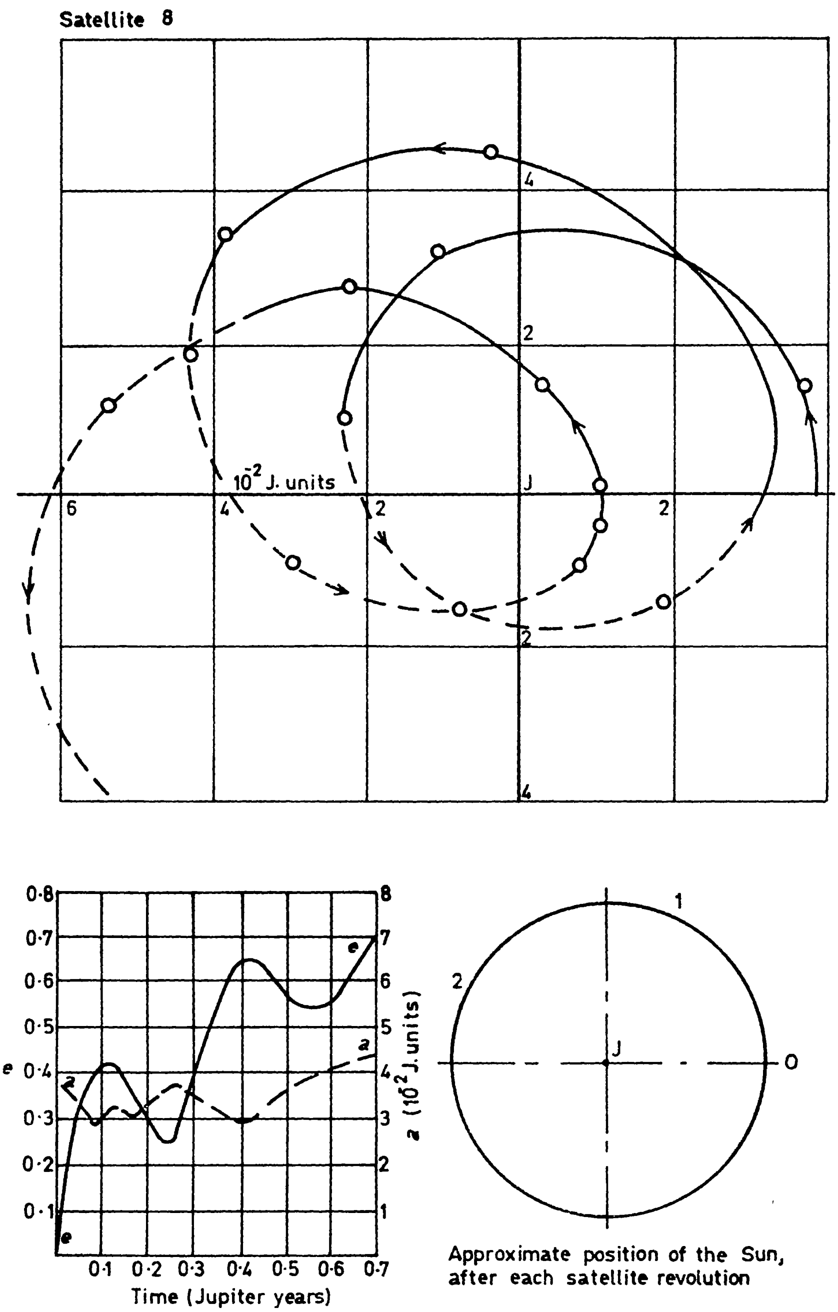

Fig. 1

It can be seen from the figure that the orbit is initially almost circular, but soon becomes more elliptical until, after a fairly close approach to Jupiter, the satellite leaves the vicinity of the planet altogether. In this respect the orbit of this satellite is typical of the escape orbits of many of the satellites.

Satellite 24, 4·5/0, (Fig. 2) makes a close approach to Jupiter after less than one revolution and then leaves the vicinity of the planet altogether.

Satellite 34, 6·0/0 (Fig. 3) does not even complete one revolution about Jupiter before escape. Clearly there is no point in considering direct initially-circular orbits with semi-major axis greater than $6{\cdot}0 \times 10^{-2}$ Jupiter units.

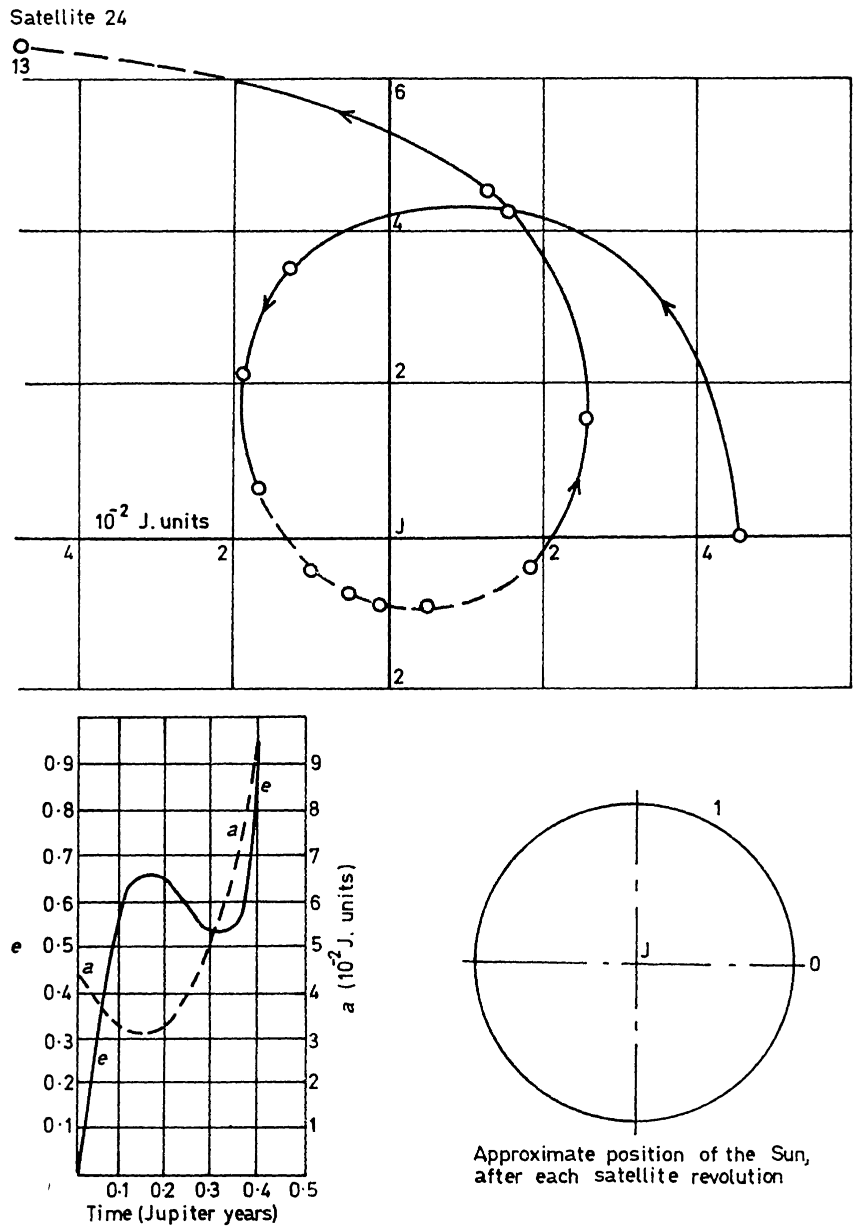

FIG. 2

Figs. 4 and 5 show the orbits of satellites 26 and 28, 4·5/0·3 and 4·5/0·3(c respectively. In each case the eccentricity again tends to increase until the satellit escapes from Jupiter.

The inclinations of these short-lived direct satellites tended to remain fair constant. For example, in the case of satellite 24, 4·5/0 with initial inclinatic 0·4873 radians, the inclination until just before the moment of escape (after or revolution about Jupiter) always lay between 0·44 and 0·49 radians.

8. *Retrograde orbits.* For short-lived retrograde satellites the pattern was again usually that of a relatively circular orbit becoming more elliptical until, after a relatively close approach to Jupiter, the satellite escaped from the planet.

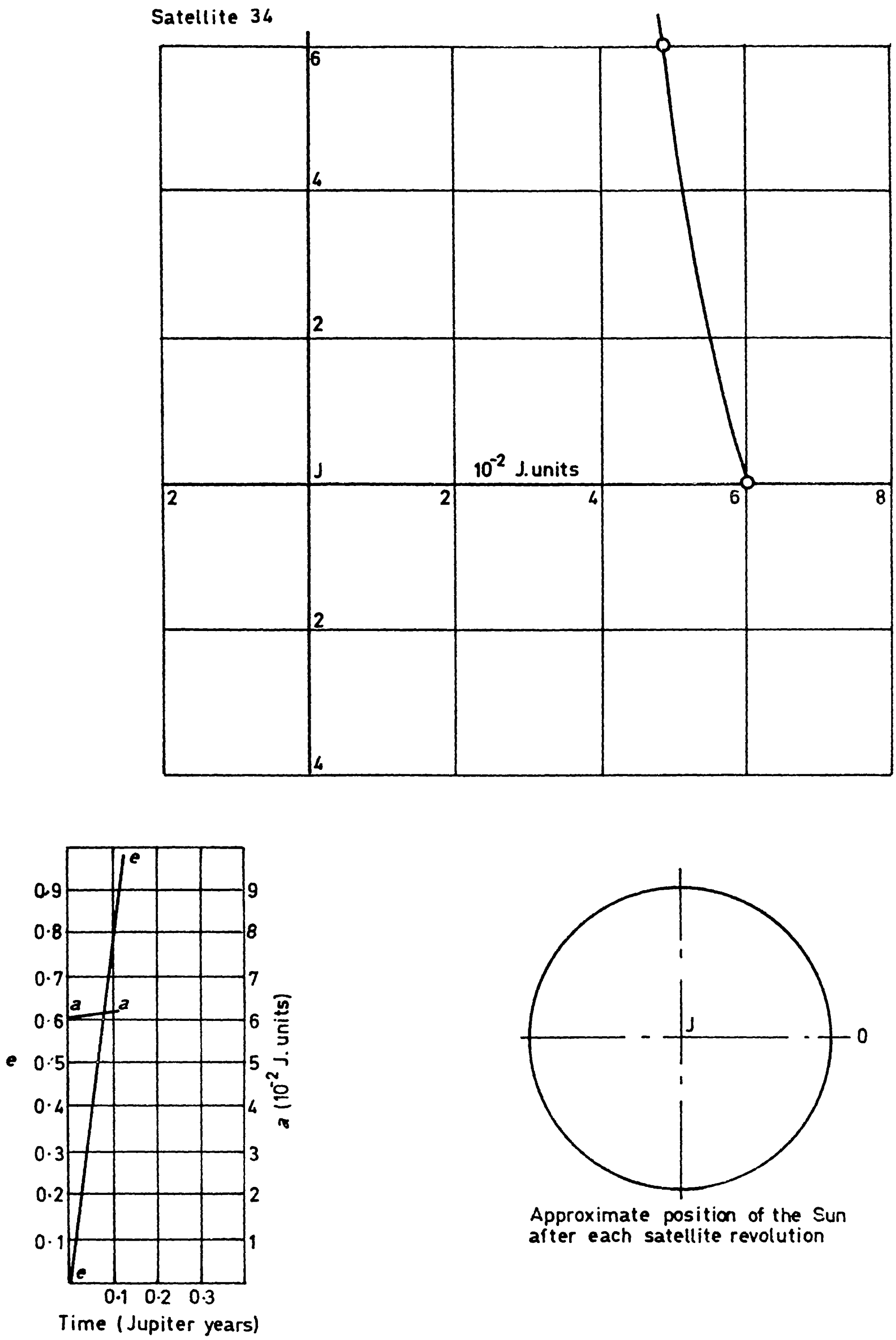

FIG. 3

Figs. 6–8 show the projections of the orbits of some of these retrograde satellites on the x/y plane. These retrograde satellites can be seen to suffer much greater variations in their orbital inclinations than the direct ones considered in the last section. For example in the case of satellite 21, 5·0/0·3, (Fig. 6) with initial inclination 0·487 radians, the inclination varies between 0·386 and 1·0 in the first three revolutions of the satellite about Jupiter.

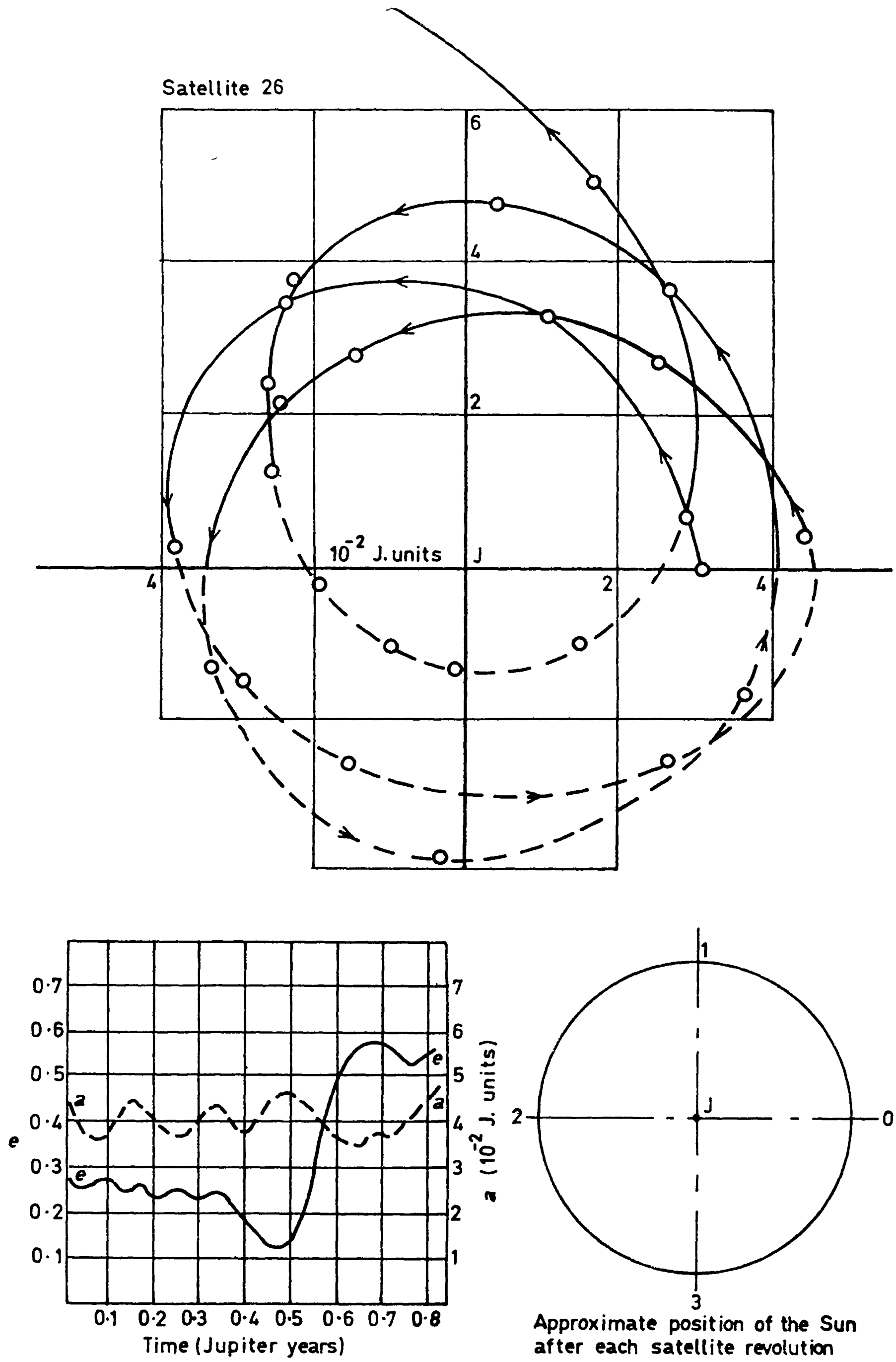

FIG. 4

Another way in which the retrograde orbits differ from the direct ones is the rate at which the lines of nodes revolve. The join of dotted line and a continuous line on any of the figures indicates the direction of one end of the line of nodes of the orbit. In the case of the retrograde orbits the line of nodes can revolve through 180° in about three revolutions of the satellite about Jupiter (see e.g. satellite 21,

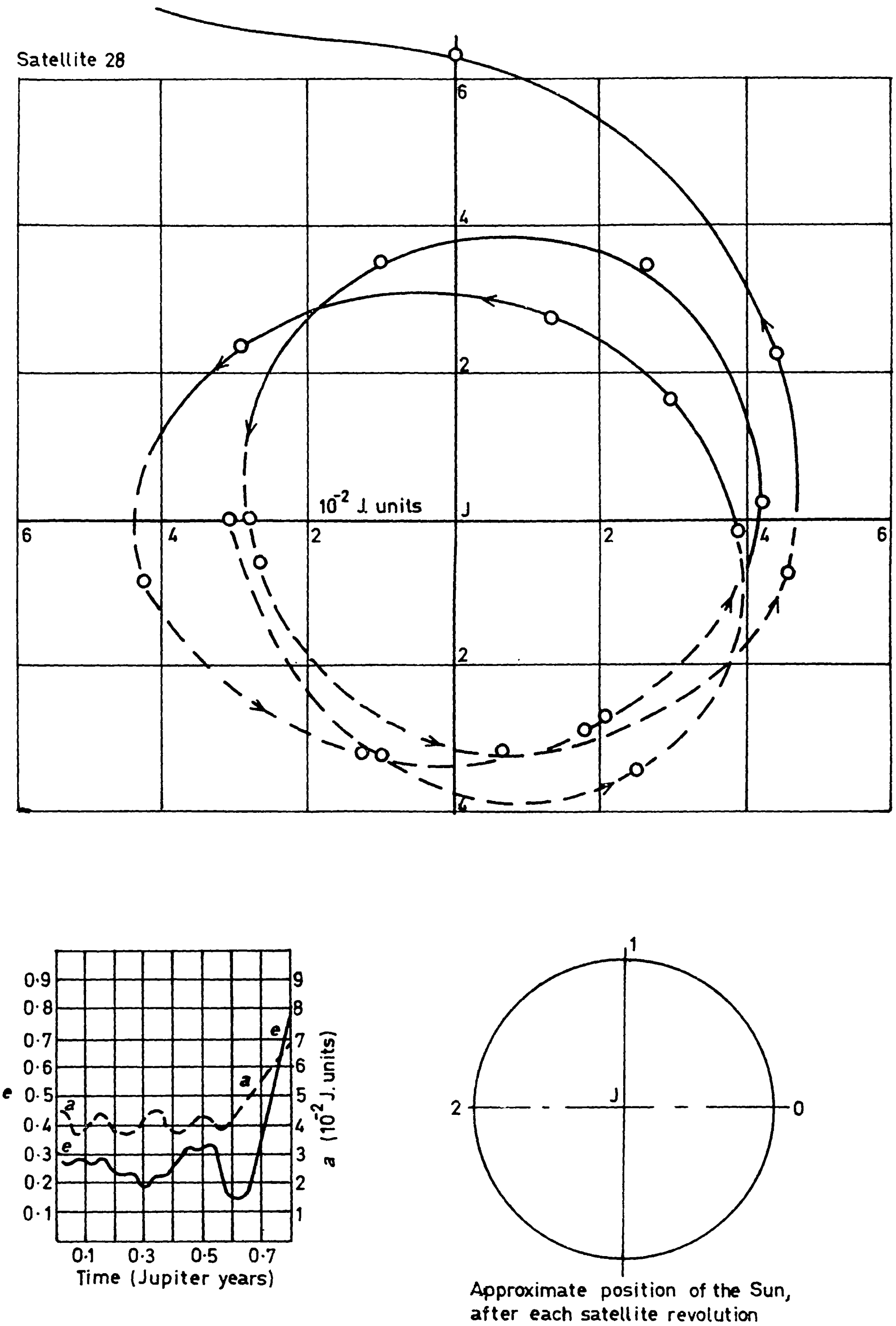

FIG. 5

Fig. 6) whereas for a direct orbit with a similar lifetime the rate is about half of this (see e.g. satellite 8, Fig. 1).

These apparent differences between the orbits of direct and retrograde satellites should not have been unexpected. What have been compared are the orbits of direct and retrograde satellites with similar lifetimes. As has already been seen, the retrograde satellites concerned are in orbits which are much further from Jupiter than the direct ones and would therefore be expected to be much more perturbed by the attraction of the Sun. As the changes in the inclinations and the movement of the nodes are caused entirely by the Sun's perturbations on the

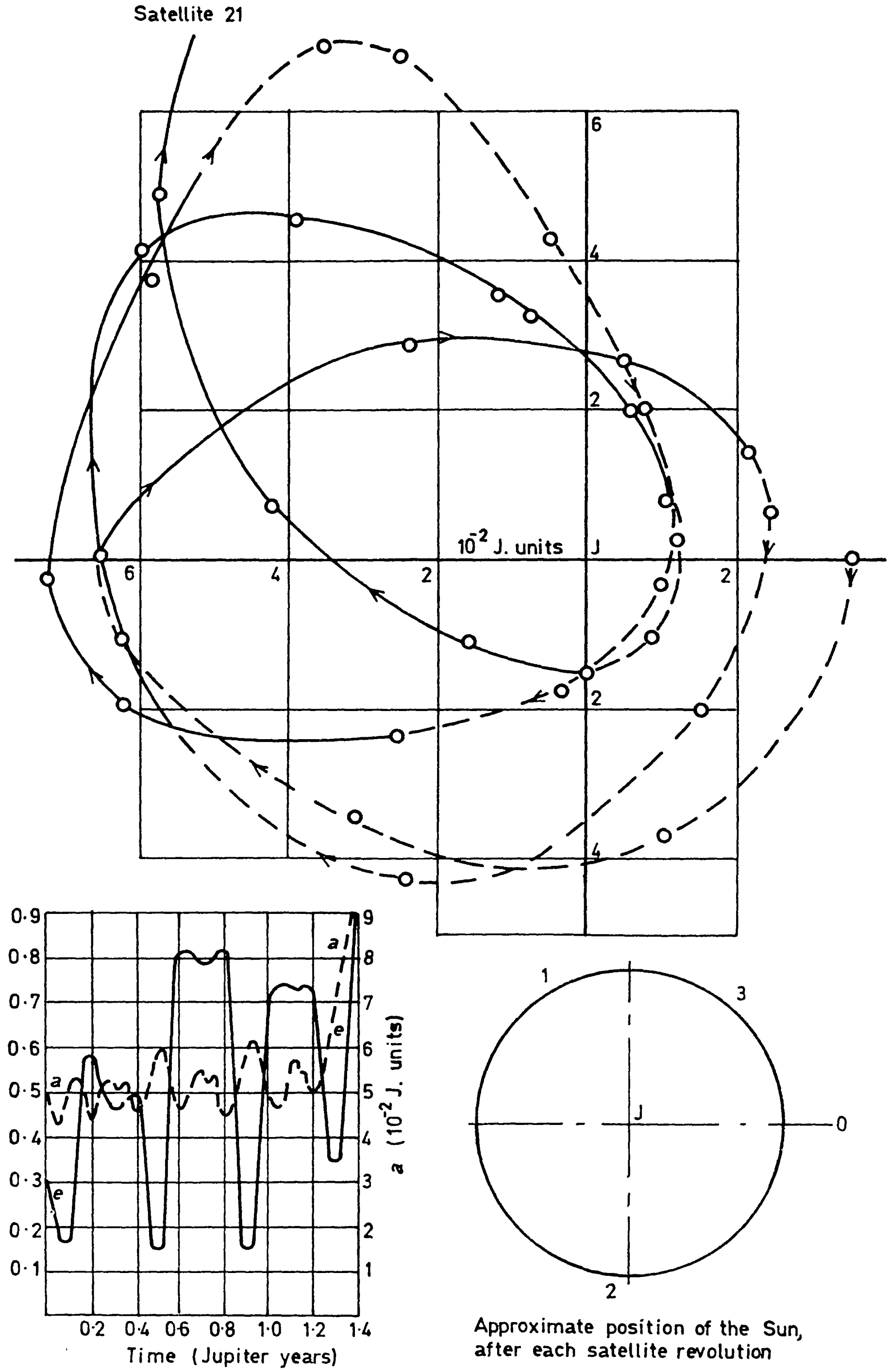

FIG. 6

orbit, it is to be expected that these effects should be greater for the retrograde satellites considered than for the direct ones. If, on the other hand, the variation in the inclination of the direct satellite 3·774/0/22 (satellite 8), with a range in inclination of 0·35 to 0·49 radians, is now compared with that of the retrograde satellite 3·774/0/22 (satellite 9) with a range in inclination of 0·39 to 0·48 radians, both taken over the first two and a half revolutions of the integration, it is seen that (at least in this case) direct and retrograde orbits at the same distance from Jupiter suffer similar changes in their inclinations due to the perturbations of the Sun.

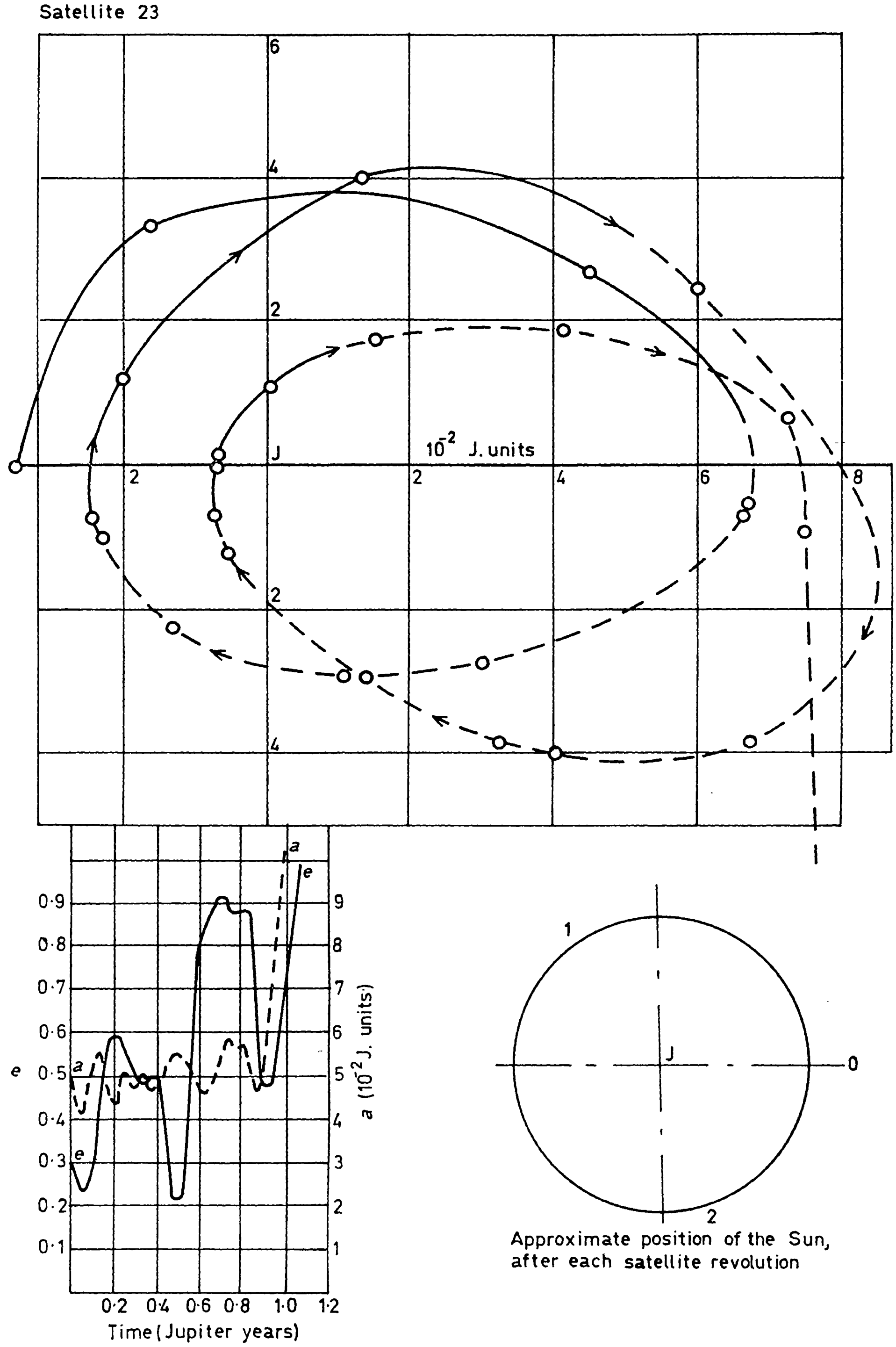

FIG. 7

9. '*Circular*' *orbits*. In the course of the integrations two almost circular retrograde orbits were discovered at distances from Jupiter well beyond that of the outermost known retrograde satellite (Jupiter IX). The orbits referred to are those of satellites 37 and 39 i.e. 4·5/0·15 and 5·0/0·15 respectively. The projection of the orbit of satellite 39 on the *x*/*y* plane is shown in Fig. 9. The integration of the orbit was continued for fifty revolutions of the satellite about Jupiter and gradually the orbit became more elliptical. For the first three revolutions of the

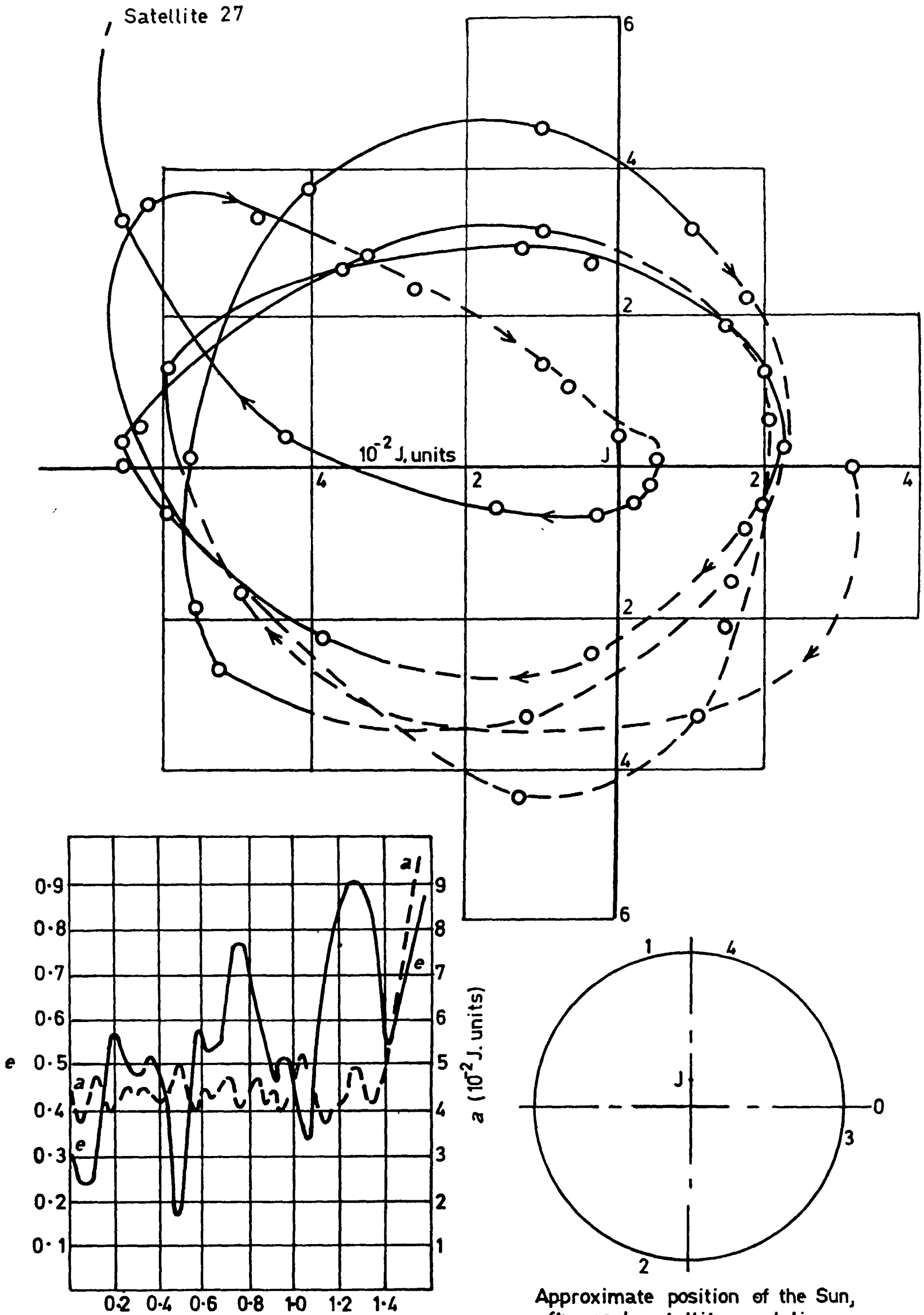

FIG. 8

satellite the radius vector always lay between $4 \cdot 21 \times 10^{-2}$ and $4 \cdot 79 \times 10^{-2}$ Jupiter units, which was much more constant than for any of the other integrations. Instead of the radius vector having about one maximum and one minimum per satellite revolution there were ten to twelve apojoves and perijoves per revolution. After a few revolutions the orbit became less circular and after ten revolutions there were about five apojoves and perijoves per revolution and the radius vector ranged between $3 \cdot 9 \times 10^{-2}$ and $5 \cdot 3 \times 10^{-2}$ Jupiter units. After forty-five revolutions of the satellite the orbit was much more typical with one apojove and one perijove per

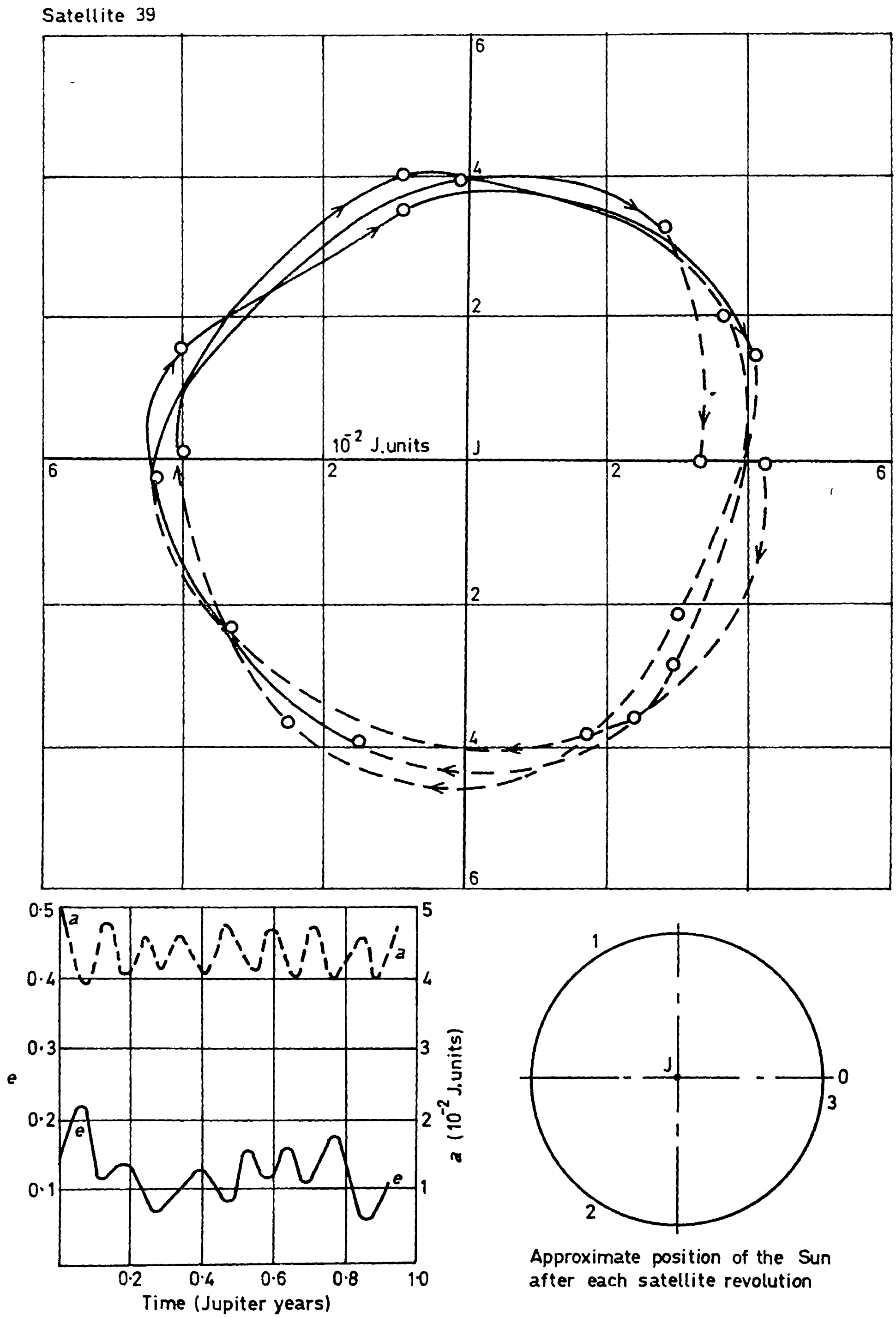

FIG. 9

revolution and a radius vector which ranged from $3{\cdot}6 \times 10^{-2}$ to $5{\cdot}7 \times 10^{-2}$ Jupiter units.

The point of interest is the near circularity of the initial part of the orbit and it is conjectured that there may be an orbit in this region which would remain almost circular for an even longer period of time.

The orbit of satellite 37 was similar to that of satellite 39 though not so markedly circular.

10. *Conclusions.* From the integrations described in this paper an attempt has been made to discover which initial osculating elements give rise to stable satellite orbits about Jupiter, when the orbits are subject to perturbations by the Sun. Various values for the initial semi-major axes, eccentricities and inclinations have been used and a fairly clear picture has emerged.

For direct satellite orbits with initial eccentricity zero and inclination about 22°, stability would appear to be possible for semi-major axes up to about 3×10^{-2} Jupiter units, while the corresponding figure retrograde orbits is about $5{\cdot}0 \times 10^{-2}$ Jupiter units. These distances may be compared with distances of about $1{\cdot}5 \times 10^{-2}$ and $3{\cdot}0 \times 10^{-2}$ Jupiter units at which the most distant direct and retrograde jovian satellites are respectively found. It seems possible therefore, that there may be more distant retrograde satellites not yet discovered but unlikely that there could be any direct ones beyond the distance of the outermost retrograde satellites of Jupiter.

Chebotarev (5) deduced from similar integrations (though always with the satellite moving in the plane of Jupiter's orbit about the Sun) that direct initially-circular orbits are stable within the sphere given by,

$$R^* = r_1 m_1^{1/2}$$

but not outside this sphere. In the equation r_1 is the distance of Jupiter from the Sun and m_1 is the ratio of the mass of Jupiter to the mass of the Sun. He also deduced that for retrograde orbits the corresponding sphere was Jupiter's sphere of influence (radius R).

The radii of these two spheres in Jupiter units are,

$$R^* = 3{\cdot}1 \times 10^{-2}, \; R = 6{\cdot}0 \times 10^{-2}.$$

Thus his conclusion regarding the radius of stability of direct orbits seems to be consistent with the results described in this paper, while his value for the radius of stability of retrograde orbits does not agree so well. The reason for the difference would appear to lie in the less stringent criterion which Chebotarev requires for the stability of an orbit, namely the ability of the satellite to perform about three revolutions about Jupiter without escaping from the planet, compared with the fifty revolutions required by the present author. It is felt that the more stringent requirements provide a more realistic assessment of the stability of a satellite orbit.

The results of this paper may also be compared with Kuiper's work (**15**). Kuiper states that the average stability limit for a satellite orbit is $0{\cdot}5 R_A$ where R_A is given by,

$$\log \frac{R_A}{a} = 0{\cdot}318 \log \mu - 0{\cdot}327 \qquad (14)$$

where a is the semi-major axis of the planet's orbit and μ is given by,

$$\mu = \frac{M_p}{M_\odot + M_p} \qquad (15)$$

where M_p and $M_\odot$ are the masses of the planet and the Sun respectively. For Jupiter,

$$0{\cdot}5 R_A = 2{\cdot}57 \times 10^{-2} \text{ Jupiter units.}$$

Kuiper adds however that, since the orbits of retrograde satellites are more stable than those of direct ones and since satellites in eccentric orbits can wander further from their parent planet than the value of their semi-major axis, it is worth looking for satellites out to a distance of $0{\cdot}75 R_A$ from a planet. For Jupiter this corresponds to a distance of $3{\cdot}85 \times 10^{-2}$ Jupiter units and the results of this paper would suggest that satellites might well exist even beyond this limit.

In Section 5 the effect of varying the initial orbital eccentricity of a satellite with given initial semi-major axis and inclination was considered. For both direct and retrograde satellites, it was found that the lifetime of a satellite was decreased as the eccentricity was increased (at least in the range of eccentricity 0 to 0·3). This is not perhaps surprising when it is realised that a satellite with a larger orbital eccentricity is able to wander further from the planet than one with a smaller one, and will therefore suffer greater perturbations due to the Sun.

Chebotarev's (5) results lead to the same conclusions regarding the greater stability of circular orbits, but it is difficult to compare the two sets of integrations as Chebotarev considered only orbits in the plane of Jupiter's orbit about the Sun and the orbits considered here are, in the main, inclined to this plane. It is tempting however, to deduce from Chebotarev's work that this result holds for eccentricities up to 0·5 (the value Chebotarev used for his elliptical orbits) and probably for higher values of the eccentricity as well.

The results of Section 6 on varying the initial inclination of an orbit are of interest but perhaps not too much inference should be drawn from them. It should be remembered that satellite orbits were chosen which, it was felt, might be fairly sensitive to changes in their inclinations. Thus the direct satellite chosen for this investigation in no way corresponded to the retrograde one chosen (the semi-major axes were $3{\cdot}37 \times 10^{-2}$ and $5{\cdot}0 \times 10^{-2}$ Jupiter units respectively) and it might be dangerous to draw conclusions regarding the effect of varying the inclinations of satellites with different values for their semi-major axes and eccentricities from the ones chosen. If however, retrograde satellites are thought of as direct satellites with inclinations between 90° and 180°, the conclusion which might be drawn from Section 6 is that, for a series of satellites with the same initial semi-major axis and eccentricity, the greater the initial inclination the greater the stability of the orbit.

Groves & Shaikh (16) have used the Jacobi integral of the circular restricted three-body problem to investigate the stability of lunar satellites perturbed by the Earth and Earth satellites perturbed by the Moon; the criterion for stability being that the zero velocity curve for the satellite concerned should be a closed curve about the parent body and that the satellite should be inside this curve. The authors point out that no long term conclusions may be drawn from the investigation due to the assumptions made in deriving the integral e.g. the assumption that the orbit of the Moon about the Earth is circular. It might have been expected however that, although the 'scale' of the systems are different, the effects of changes in the initial conditions on the stability of a lunar satellite perturbed by the Earth would be similar to those obtained for a jovian satellite perturbed by the Sun. The results of Groves and Shaikh however, are considerably different from those described in this paper. For example, Groves and Shaikh come to the conclusion that direct orbits are stable out to considerably greater distances from the Moon than retrograde ones, and that the effect of increasing the inclination of an orbit is *always* to decrease its stability.

The conclusion that direct circular lunar orbits in the plane of the Earth' motion about the Moon are more stable than retrograde ones (according to the criterion of Groves & Shaikh) may be made from an elementary consideration of the Jacobi integral in rotating co-ordinates, as given by Finlay-Freundlich (17),

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}-h \tag{16}$$

where μ, $1-\mu$ are the masses of the two large bodies, here taken to be the Earth and the Moon and r_1 and r_2 are the distances of the satellite from the Moon and the Earth respectively. For a direct and a retrograde lunar satellite, each in a circular orbit in the plane of the plane of the Moon's orbit about the Earth and with the same orbital semi-major axis, the difference in the values of h for the two satellites will depend only on their velocities with respect to the rotating co-ordinate system. The velocity will clearly be greater for the retrograde satellite than for the direct one and so the value of h will be less for the retrograde satellite than for the direct one.

For large values of h the zero-velocity curves consist of closed circles round each of the large bodies and there is a critical value of h below which the curve will be 'open' thus making it possible for a satellite to escape.

Thus for a retrograde satellite with h just below this limit and hence able to escape from the Moon, the corresponding direct satellite will have h just greater than the critical value and so will be contained within the closed zero-velocity curve. Hence it would appear that, at certain distances from the Moon direct satellites are more stable than retrograde ones and that direct satellites are stable at greater distances from the Moon than retrograde ones. The conclusions of Groves and Shaikh differ so much from the corresponding ones in this paper and from those of Chebotarev that it is felt that their criterion for the stability of satellite orbits is not a very useful one.

From the days of Darwin (2) it has been recognised that the numerical integration of orbits provides a means of gaining insight into the restricted three-body problem. It has been seen that the relatively few orbits computed in the present paper have enabled useful conclusions to be drawn regarding the stability of the orbits of jovian satellites perturbed by the Sun, and the continuation of this work should enable the region about Jupiter to be mapped out more clearly with regard to stable and unstable satellite orbits without the use of a prohibitive amount of computer time. It is hoped to publish this work in future papers.

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Computing Laboratory,
University of Newcastle upon Tyne,
1/3 *Kensington Terrace,*
Newcastle upon Tyne 2.
1967 *January*.

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