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MOTIVIC EXPONENTIAL INTEGRALS AND A MOTIVIC THOM-SEBASTIANI THEOREM

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1. Introduction

1.1. Let f and f' be germs of analytic functions on smooth complex analytic varieties X and X' and consider the function $f \oplus f'$ on $X \times X'$ given by $f \oplus f'(x, x') = f(x) + f'(x')$. The Thom-Sebastiani theorem classically states that the monodromy of $f \oplus f'$ on the nearby cycles is isomorphic to the product of the monodromy of f and the monodromy of f'. (In the original form of the theorem in [16], the functions were assumed to have isolated singularities.) It is now a common idea that the Thom-Sebastiani theorem is best understood by using Fourier transformation and exponential integrals because of the formula

$$\int \exp\big(t\big(f \oplus f'\big)\big) = \int \exp(tf) \cdot \int \exp\big(tf'\big). \tag{1.1}$$

Indeed, by using asymptotic expansions of such integrals for $t \to \infty$, A. Varchenko proved a Thom-Sebastiani theorem for the Hodge spectrum in the isolated singularity case [20] (see also [14]), the general case being due to M. Saito (see [19], [11], and [13]).

The aim of the present paper is to give a motivic meaning to equation (1.1) and to deduce a motivic Thom-Sebastiani theorem. To explain our approach, we begin by reviewing some known results on p-adic exponential integrals.

1.2. Let K be a finite extension of $\mathbf{Q}_p$. Let us denote by R the valuation ring of K, by P the maximal ideal of R, and by k the residue field of K. The cardinality of k is denoted by q, so $k \simeq \mathbf{F}_q$. For z in K, $\operatorname{ord} z \in \mathbf{Z} \cup \{+\infty\}$ denotes the valuation of z, $|z| = q^{-\operatorname{ord} z}$, and $\operatorname{ac}(z) = z\pi^{-\operatorname{ord} z}$, where π is a fixed uniformizing parameter for R.

Let $f \in R[x_1, \dots, x_m]$ be a nonconstant polynomial. Let $\Phi : R^m \to \mathbf{C}$ be a locally constant function with compact support. Let α be a character of $R^\times$, that is, a morphism $R^\times \to \mathbf{C}^\times$ with finite image. For i in $\mathbf{N}$, set

$$Z_{\Phi,f,i}(\alpha) = \int_{\{x \in R^m \mid \operatorname{ord} f(x) = i\}} \Phi(x)\alpha\big(\operatorname{ac} f(x)\big)|dx|,$$

where $|dx|$ denotes the Haar measure on K^m, normalized so that R^m is of measure 1.

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We denote by Ψ the standard additive character on K, defined by

$$z \longmapsto \Psi(z) = \exp\bigl(2i\pi \operatorname{Tr}_{K/\mathbf{Q}_p} z\bigr).$$

For i in $\mathbf{N}$, we consider the exponential integral

$$E_{\Phi,f,i} = \int_{R^m} \Phi(x)\Psi\bigl(\pi^{-(i+1)} f(x)\bigr)\,|dx|.$$

Let α be a character of $R^\times$. The conductor of α, $c(\alpha)$, is defined as the smallest $c \geq 1$ such that α is trivial on $1+P^c$, and one associates to α the Gauss sum

$$g(\alpha) = q^{1-c(\alpha)} \sum_{v \in (R/P^{c(\alpha)})^\times} \alpha(v)\Psi\Bigl(v/\pi^{c(\alpha)}\Bigr).$$

Proposition 1.2.1 (See §1 of [3]). *For any i in $\mathbf{N}$,*

$$E_{\Phi,f,i} = \int_{\{x\in R^m \mid \operatorname{ord} f(x) > i\}} \Phi(x)\,|dx| + (q-1)^{-1} \sum_{\alpha} g\bigl(\alpha^{-1}\bigr) Z_{\Phi,f,i-c(\alpha)+1}(\alpha).$$

Here $i - c(\alpha) + 1 \geq 0$. Moreover, if the critical locus of f in Supp Φ *is contained in $f^{-1}(0)$, then, for all except a finite number of characters α, the integrals $Z_{\Phi,f,j}(\alpha)$ are zero for all j.*

Corollary 1.2.2 (Using Theorem 3.3 of [3]). *Assume that Φ is residual (i.e., that* Supp Φ *is contained in R^m, and that $\Phi(x)$ depends only on x modulo P), and that the critical locus of f in* Supp Φ *is contained in $f^{-1}(0)$. Assume furthermore that the divisor $f = 0$ has good reduction (in the sense that the conditions in Theorem 3.3 of* [3] *are satisfied). Then*

$$E_{\Phi,f,i} = \int_{\{x\in R^m \mid \operatorname{ord} f(x) > i\}} \Phi(x)\,|dx| + (q-1)^{-1} \sum_{\substack{\alpha \\ c(\alpha)=1}} g\bigl(\alpha^{-1}\bigr) Z_{\Phi,f,i}(\alpha).$$

So we see that p-adic exponential integrals may be expressed as linear combinations of p-adic integrals involving multiplicative characters with Gauss sums as coefficients.

When k is a field of characteristic zero, there is a $k((t))$-analogue of p-adic integration, motivic integration, introduced by M. Kontsevich. In particular, by the results of [4] and [5], it is possible to define motivic analogues of the p-adic integrals $Z_{\Phi,f,i}(\alpha)$ in 1.2.2 as elements of a Grothendieck group of Chow motives. Since in this analogy the $k((t))$-case always has good reduction, it becomes quite natural to use the equality in Corollary 1.2.2 as a candidate for the definition of motivic exponential integrals, and this is indeed what we do in this paper. To achieve this aim, we enlarge slightly our virtual motives by attaching virtual motives to Gauss sums in a way very similar to Anderson's construction of ulterior motives [1]. Now equation (1.1) is no longer trivial, and the main result of the paper is that it still holds true for our motivic exponential integrals (Theorem 4.2.4). We deduce from this result a motivic analogue of the Thom-Sebastiani theorem (Theorem 5.2.2). Passing to Hodge realization, this gives a proof of the Thom-Sebastiani theorem for the Hodge spectrum (Corollary 6.2.4).

2. Adding ulterior motives to the Grothendieck group

2.1. We fix a base field k, which we assume throughout the paper to be of characteristic zero, and we denote by $\mathcal{V}_k$ the category of smooth and projective k-schemes. For an object X in $\mathcal{V}_k$ and an integer d, we denote by $A^d(X)$ the Chow group of codimension-d cycles with rational coefficients modulo rational equivalence. Objects of the category $\mathcal{M}_k$ of (rational) k-motives are triples (X, p, n) where X is in $\mathcal{V}_k$, p is an idempotent (i.e., $p^2 = p$) in the ring of correspondences $\operatorname{Corr}^0(X, X)$ $(= A^d(X \times X)$ when X is of pure dimension d), and n is an integer. If (X, p, n) and (Y, q, m) are motives, then

$$\operatorname{Hom}_{\mathcal{M}_k}\big((X, p, n), (Y, q, m)\big) = q \operatorname{Corr}^{m-n}(X, Y) p.$$

Composition of morphisms is given by composition of correspondences. The category $\mathcal{M}_k$ is additive, $\mathbf{Q}$-linear, and pseudo-Abelian, and there is a natural tensor product on $\mathcal{M}_k$. We denote by h the functor $h : \mathcal{V}_k^\circ \to \mathcal{M}_k$, which sends an object X to $h(X) = (X, \mathrm{id}, 0)$ and a morphism $f : Y \to X$ to its graph in $\operatorname{Corr}^0(X, Y)$. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\operatorname{Spec} k, \mathrm{id}, -1)$. There is a canonical isomorphism $h(\mathbf{P}^1_k) \simeq 1 \oplus \mathbf{L}$.

When E is a field of characteristic zero, one defines similarly the category $\mathcal{M}_{k,E}$ of k-motives with coefficients in E, by replacing the Chow groups $A^\cdot$ by $A^\cdot \otimes_{\mathbf{Q}} E$. Let $K_0(\mathcal{M}_{k,E})$ be the Grothendieck group of the pseudo-Abelian category $\mathcal{M}_{k,E}$. It is also the abelian group associated to the monoid of motives with coefficients in E. The tensor product on $\mathcal{M}_{k,E}$ induces a natural ring structure on $K_0(\mathcal{M}_{k,E})$. For m in $\mathbf{Z}$, let $F^m K_0(\mathcal{M}_{k,E})$ denote the subgroup of $K_0(\mathcal{M}_{k,E})$ generated by $h(S, f, i)$, with $i - \dim S \geq m$. This gives a filtration of the ring $K_0(\mathcal{M}_{k,E})$, and we denote by $\widehat{K}_0(\mathcal{M}_{k,E})$ the completion of $K_0(\mathcal{M}_{k,E})$ with respect to this filtration.

Remark 2.1.1. We expect, but do not know how to prove, that the filtration $F^\cdot$ on $K_0(\mathcal{M}_{k,E})$ is separated. This assertion is clearly implied by the conjectural existence (cf. [15, page 185]) of additive functors $h^{\leq j} : \mathcal{M}_{k,E} \to \mathcal{M}_{k,E}$, $j \in \mathbf{Z}$, such that, for any $X = h(S, f, i)$ in $\mathcal{M}_{k,E}$, the $h^{\leq j}(X)$ form a filtration of X with $h^{\leq -2i-1}(X) = 0$, $h^{\leq 2\dim S - 2i}(X) = X$, and $h^{\leq j}(\mathbf{L}X) = \mathbf{L}h^{\leq j-2}(X)$ for all j. In particular, using the existence of weight filtrations, one obtains, without using any conjecture, that the kernel of the canonical morphism

$$K_0\big(\mathcal{M}_{k,E}\big) \longrightarrow \widehat{K}_0\big(\mathcal{M}_{k,E}\big)$$

is killed by étale and Hodge realizations.

2.2. We begin by recalling some material from [4]. Let G be a finite abelian group and let $\hat{G}$ be its complex character group. We denote by $\mathcal{V}_{k,G}$ the category of smooth and projective k-schemes with G-action. Let E be a subfield of $\mathbf{C}$ containing all the roots of unity of order dividing $|G|$. For X in $\mathcal{V}_{k,G}$ and g in G, we denote by $[g]$ the correspondence given by the graph of multiplication by g. For α in $\hat{G}$, we consider the idempotent

$$f_\alpha := |G|^{-1} \sum_{g \in G} \alpha^{-1}(g)[g]$$

in $\mathrm{Corr}^0(X,X) \otimes E$, and we denote by $h(X,\alpha)$ the motive $(X, f_\alpha, 0)$ in $\mathcal{M}_{k,E}$. We denote by $\mathrm{Sch}_{k,G}$ the category of separated schemes of finite type over k with G-action satisfying the following condition: the G-orbit of any closed point of X is contained in an affine open subscheme. This condition is clearly satisfied for X quasiprojective and ensures the existence of X/G as a scheme. Objects of $\mathrm{Sch}_{k,G}$ are called G-schemes, and in this paper all schemes with G-action are assumed to be G-schemes.

The following result, proved in [4] as a consequence of results in [7] and [21], generalizes to the G-action case, a result that has been proved by Gillet and Soulé [6] and Guillén and Navarro Aznar [7].

Theorem 2.2.1. *Let k be a field of characteristic zero. There exists a unique map*

$$\chi_c : \mathrm{Ob}\,\mathrm{Sch}_{k,G} \times \hat{G} \longrightarrow K_0\big(\mathcal{M}_{k,E}\big)$$

such that

(1) *if X is smooth and projective with G-action, for any character α,*

$$\chi_c(X,\alpha) = [h(X,\alpha)];$$

(2) *if Y is a closed G-stable subscheme in a scheme X with G-action, for any character α,*

$$\chi_c(X \setminus Y,\alpha) = \chi_c(X,\alpha) - \chi_c(Y,\alpha);$$

(3) *if X is a scheme with G-action and if U and V are G-invariant open subschemes of X, for any character α,*

$$\chi_c(U \cup V,\alpha) = \chi_c(U,\alpha) + \chi_c(V,\alpha) - \chi_c(U \cap V,\alpha).$$

Furthermore, χ_c is determined by conditions (1) and (2).

2.3. In this subsection, we gather some elementary statements we need.

Proposition 2.3.1. *Let k be a field of characteristic zero.*

(1) *For any X in* $\mathrm{Ob}\,\mathrm{Sch}_{k,G}$,

$$\chi_c(X) = \sum_{\alpha \in \hat{G}} \chi_c(X,\alpha).$$

(2) *Let X be in* $\mathrm{Ob}\,\mathrm{Sch}_{k,G}$. *Assume the G-action factors through a morphism*

of finite abelian groups $G \to H$. *If* α *is not in the image of* $\hat{H} \to \hat{G}$, *then* $\chi_c(X, \alpha) = 0$.

(3) *Let* X *and* Y *be in* $\mathrm{Ob}\,\mathrm{Sch}_{k,G}$ *and let* G *act diagonally on* $X \times Y$. *Then*

$$\chi_c(X \times Y, \alpha) = \sum_{\beta \in \hat{G}} \chi_c(X, \beta) \cdot \chi_c\big(Y, \alpha\beta^{-1}\big).$$

(4) *Let* X *be in* $\mathrm{Ob}\,\mathrm{Sch}_{k,H}$ *and let* $f : G \to H$ *be a morphism of finite abelian groups. Then, for any* α *in* $\hat{G}$,

$$\chi_c(X, \alpha) = \sum_{\hat{f}(\beta)=\alpha} \chi_c(X, \beta),$$

with $\hat{f} : \hat{H} \to \hat{G}$ *the dual morphism between character groups.*

Proof. Statements (1)–(3) are proven in Proposition 1.3.3 of [4]. They are all consequence of (4), whose proof is similar: The statement is obvious when X is smooth projective, and the general case follows by additivity of χ_c. □

LEMMA 2.3.2. *Let* a *be an integer and let* $\mu_d(k)$ *act on* $\mathbf{G}_{m,k}$ *by multiplication by* ξ^a, $\xi \in \mu_d(k)$. *For any nontrivial character* α *of* $\mu_d(k)$,

$$\chi_c\big(\mathbf{G}_{m,k}, \alpha\big) = 0.$$

Proof. This is Lemma 1.4.3 of [4]. □

We now discuss motivic Euler characteristics of quotients. The following lemma is well known.

LEMMA 2.3.3. *Let* X *be a smooth projective scheme with* G*-action, let* H *be a subgroup of* G, *and let* α *be a character of* G/H. *Assume the quotient* X/H *is smooth. Then* $h(X/H, \alpha) \simeq h(X, \alpha \circ \varrho)$, *where* ϱ *is the projection* $G \to G/H$.

Proof. This is Lemma 1.5.1 of [4]. □

In general, one has the following result, which was conjectured in [4] and proved in [2] by del Baño Rollin and Navarro Aznar.

THEOREM 2.3.4. *If* X *is a scheme with* G*-action,* H *a subgroup of* G, *and* α *a character of* G/H, *then* $\chi_c(X/H, \alpha) = \chi_c(X, \alpha \circ \varrho)$, *where* ϱ *is the projection* $G \to G/H$.

Remark 2.3.5. Theorem 2.3.4 is a direct consequence of Lemma 2.3.3 when X is a smooth curve or when X is smooth and may be embedded in a smooth projective scheme Y with G-action such that the quotient Y/H is smooth, and such that $Y \setminus X$ is the union of finitely many smooth closed G-stable subvarieties intersecting transversally and having smooth images in Y/H that intersect transversally.

2.4. Jacobi motives. We fix an integer $d \geq 1$. We denote by $\mu_d(k)$ the group of d-roots of 1 in k and by ζ_d a fixed primitive dth root of unity in $\mathbf{C}$. We assume from now on that k contains all d-roots of unity.

We set

$$A_d := K_0\big(\mathcal{M}_{k,\mathbf{Q}[\zeta_d]}\big).$$

For $n \geq 1$, we consider the affine Fermat variety F_d^n, defined by the equation $x_1^d + \cdots + x_n^d = 1$ in $\mathbf{A}_k^n$, and its closure in $\mathbf{P}_k^n$, which we denote by W_d^n. Hence, W_d^n is defined in $\mathbf{P}_k^n$ by the equation $-X_0^d + \cdots + X_n^d = 0$, with $x_i = X_i/X_0$, $i \geq 1$.

The action of $\mu_d(k)$ on each coordinate induces a natural action of the group $\mu_d(k)^n$ on F_d^n. Hence, for $\alpha_1, \ldots, \alpha_n$ characters of $\mu_d(k)$, we define the Jacobi motive $J(\alpha_1, \ldots, \alpha_n)$ as the element

$$J(\alpha_1, \ldots, \alpha_n) := \chi_c\big(F_d^n, (\alpha_1, \ldots, \alpha_n)\big)$$

in A_d. It is clear that $J(\alpha_1, \ldots, \alpha_n)$ is symmetric in the α_i's. We also define $[\alpha(-1)] := \chi_c(x^d = -1, \alpha)$. Remark that $[\alpha(-1)] = 1$, if k contains a dth root of -1. We need the following proposition, which is classical in other contexts.

PROPOSITION 2.4.1. *The following relations hold in A_d.*
(1) *We have $J(1,1) = \mathbf{L}$.*
(2) *We have $J(1,\alpha) = 0$, if $\alpha \neq 1$.*
(3) *If $\alpha \neq 1$, $J(\alpha, \alpha^{-1}) = -[\alpha(-1)]$.*
(4) *We have*

$$J\big(\alpha_1, \alpha_2\big)\big[J\big(\alpha_1\alpha_2, \alpha_3\big) - \varepsilon\big] = J\big(\alpha_1, \alpha_2, \alpha_3\big) - \delta,$$

with $\varepsilon = \delta = 0$, if $\alpha_1\alpha_2 \neq 1$; $\varepsilon = 1$, $\delta = [\alpha_1(-1)](\mathbf{L} - 1)$, if $\alpha_1\alpha_2 = 1$ and $\alpha_1 \neq 1$; and $\varepsilon = 1$, $\delta = \mathbf{L}$, if $\alpha_1 = \alpha_2 = 1$.

Proof. Relation (1) follows directly from Remark 2.3.5 because the quotient of the curve F_d^2 by $\mu_d(k) \times \mu_d(k)$ is the affine line $\mathbf{A}_k^1$.

To prove (2), observe that the quotient of the curve F_d^2 by $\mu_d(k) \times \{1\}$ is the Kummer cover $x_2^d = 1 - x_1$ of the affine line. Hence, by Remark 2.3.5, $J(1,\alpha) = \chi_c(\mathbf{A}_k^1, \alpha)$, with the natural action of $\mu_d(k)$ on $\mathbf{A}_k^1$, and the assertion follows from Lemma 2.3.2 and Proposition 2.3.1.

Let us now prove (3). The character α being nontrivial, observe that we have

$$\chi_c\Big(F_d^2 \cap \{x_2 = 0\}, \big(\alpha, \alpha^{-1}\big)\Big) = 0.$$

For instance, this follows from Proposition 2.3.1. Hence,

$$J\big(\alpha, \alpha^{-1}\big) = \chi_c\Big(F_d^2 \setminus \{x_2 = 0\}, \big(\alpha, \alpha^{-1}\big)\Big).$$

Now we may identify $F_d^2 \setminus \{x_2 = 0\}$ with the affine curve $u^d - v^d = -1; v \neq 0$, via the change of variable $u = x_1 x_2^{-1}$, $v = x_2^{-1}$. Taking into account the $\mu_d(k) \times \mu_d(k)$-action, we get

$$\chi_c\Big(F_d^2 \setminus \{x_2 = 0\}, \big(\alpha, \alpha^{-1}\big)\Big) = \chi_c\Big(\big(u^d - v^d = -1; v \neq 0\big), (\alpha, 1)\Big).$$

By Remark 2.3.5,

$$\begin{aligned}\chi_c\big(\big(u^d - v^d = -1; v \neq 0\big), (\alpha, 1)\big) &= \chi_c\big(\big(u^d = v - 1; v \neq 0\big), \alpha\big)\\ &= \chi_c\big(\mathbf{A}_k^1, \alpha\big) - \chi_c\big(u^d = -1, \alpha\big)\\ &= -\big[\alpha(-1)\big],\end{aligned}$$

because $\chi_c(\mathbf{A}_k^1, \alpha) = 0$ by Lemma 2.3.2 and Proposition 2.3.1.

To prove (4), we consider the morphism

$$f : F_d^2 \times \big(F_d^2 \setminus \{y_1 = 0\}\big) \longrightarrow F_d^3 \setminus \big\{z_1^d + z_2^d = 0\big\}$$

given by

$$\big((x_1, x_2), (y_1, y_2)\big) \longmapsto (z_1 = x_1 y_1, z_2 = x_2 y_1, z_3 = y_2).$$

This morphism identifies $F_d^3 \setminus \{z_1^d + z_2^d = 0\}$ with the quotient of $F_d^2 \times (F_d^2 \setminus \{y_1 = 0\})$ by the kernel Γ of the morphism $\mu_d(k)^4 \to \mu_d(k)^3$ given by $(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_1\xi_3, \xi_2\xi_3, \xi_4)$. It follows from Remark 2.3.5 and Proposition 2.3.1(3) that

$$\begin{aligned}&\chi_c\big(F_d^2, (\alpha_1, \alpha_2)\big) \cdot \chi_c\big(F_d^2 \setminus \{y_1 = 0\}, (\alpha_1\alpha_2, \alpha_3)\big)\\ &\qquad\qquad = \chi_c\big(F_d^3 \setminus \big\{z_1^d + z_2^d = 0\big\}, (\alpha_1, \alpha_2, \alpha_3)\big).\end{aligned}$$

Indeed, f extends to the rational map

$$W_d^2 \times W_d^2 \longrightarrow W_d^3$$

given by

$$\begin{aligned}&\big(\big[X_0, X_1, X_2\big], \big[Y_0, Y_1, Y_2\big]\big)\\ &\qquad\qquad \longmapsto \big[Z_0 = X_0Y_0, Z_1 = X_1Y_1, Z_2 = X_2Y_1, Z_3 = X_0Y_2\big].\end{aligned}$$

Now let Z and Z' denote, respectively, the blow-up of $W_d^2 \times W_d^2$ along $\{X_0 = 0\} \times \{Y_1 = 0\}$ and the blow-up of W_d^3 along $\{Z_0 = Z_3 = 0\} \cup \{Z_1 = Z_2 = 0\}$. It is classical (see [17]) that f extends to a morphism $\bar{f} : Z \to Z'$, that the actions of $\mu_d(k)^4$ and $\mu_d(k)^3$ extend to actions on Z and Z', respectively, and that $\bar{f}$ identifies Z' with the quotient of Z by Γ. (Of course we could also use here Theorem 2.3.4 directly instead of Remark 2.3.5.) To complete the proof of (4), we only have to prove that

$\chi_c(F_d^2 \cap \{y_1 = 0\}, (\alpha_1\alpha_2, \alpha_3)) = \varepsilon$ and that $\chi_c(F_d^3 \cap \{z_1^d + z_2^d = 0\}, (\alpha_1, \alpha_2, \alpha_3)) = \delta$. The first equality is clear since, by Proposition 2.3.1,

$$\chi_c\big(F_d^2 \cap \{y_1 = 0\}, (\alpha_1\alpha_2, \alpha_3)\big) = \chi_c(y_1 = 0, \alpha_1\alpha_2) \cdot \chi_c\big(y_2^d = 1, \alpha_3\big).$$

To prove the second one, we remark that

$$\chi_c\big(F_d^3 \cap \big\{z_1^d + z_2^d = 0\big\} \cap \{z_1 \text{ or } z_2 \neq 0\}, (\alpha_1, \alpha_2, \alpha_3)\big) \\ = \chi_c\big(u^d = -1, \alpha_1\big) \cdot \chi_c\big(\mathbf{G}_{m,k}, \alpha_1\alpha_2\big) \cdot \chi_c\big(w^d = 1, \alpha_3\big).$$

This follows from Proposition 2.3.1, by using the change of variable $u = z_1 z_2^{-1}$, $v = z_2$, and $w = z_3$. The result is now a consequence of Lemma 2.3.2, since $\chi_c(F_d^3 \cap \{z_1^d + z_2^d = 0\} \cap \{z_1 = z_2 = 0\}, (\alpha_1, \alpha_2, \alpha_3))$ is equal to 1 or 0, according to whether α_1 and α_2 are both trivial or not. □

2.5. We assume from now on that k contains all the roots of unity. When d divides d' we have a canonical surjective morphism of groups $\mu_{d'}(k) \to \mu_d(k)$ given by $x \mapsto x^{d'/d}$ that dualizes to an injective morphism of character groups $\widehat{\mu}_d(k) \to \widehat{\mu}_{d'}(k)$. We set $\widehat{\mu}(k) := \varinjlim \widehat{\mu}_d(k)$. We identify $\widehat{\mu}_d(k)$ with the subgroup of elements of order dividing d in $\widehat{\mu}(k)$.

We denote by F the subfield of $\mathbf{C}$ generated by the roots of unity, and we set

$$A := K_0\big(\mathcal{M}_{k,F}\big)$$

and

$$\widehat{A} := \widehat{K}_0\big(\mathcal{M}_{k,F}\big).$$

We have a natural ring morphism $A_d \to A$. When no confusion occurs, we still denote by the same symbol the image in A of an element in A_d. In particular, if α_1 and α_2 are elements of $\widehat{\mu}(k)$, which are images of elements $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of $\widehat{\mu}_d(k)$, we denote by $J(\alpha_1, \alpha_2)$ the image of $J(\tilde{\alpha}_1, \tilde{\alpha}_2)$ in A, which is independent from the choice of d, by Remark 2.3.5. For α in $\widehat{\mu}(k)$, we define similarly $[\alpha(-1)]$ in A.

We now consider the ring U obtained from the ring A by adding all the Gauss sums motives associated to $\widehat{\mu}(k)$. This construction is strongly reminiscent of Anderson's construction of "ulterior motives" [1].

We define U as the free A-module with basis G_α, α in $\widehat{\mu}(k)$. We define an A-algebra structure on U by stating the following relations:

(2.5.1) $\quad G_1 = -1$

(2.5.2) $\quad G_\alpha G_{\alpha^{-1}} = [\alpha(-1)]\mathbf{L}, \quad$ for $\alpha \neq 1$

(2.5.3) $\quad G_{\alpha_1} G_{\alpha_2} = J(\alpha_1, \alpha_2) G_{\alpha_1\alpha_2}, \quad$ for $\alpha_1, \alpha_2, \alpha_1\alpha_2 \neq 1$.

PROPOSITION 2.5.1. *The algebra U is associative and commutative.*

Proof. The commutativity is clear and the associativity follows directly from Proposition 2.4.1. □

3. Motivic integrals of multiplicative characters

3.1. Let X be a k-variety, that is, a separated and reduced k-scheme of finite type. We denote by $\mathscr{L}(X)$ the scheme of germs of arcs on X. It is a scheme over k, and for any field extension $k \subset K$ there is a natural bijection

$$\mathscr{L}(X)(K) \simeq \mathrm{Mor}_{k\text{-schemes}}\left(\operatorname{Spec} K[[t]], X\right)$$

between the set of K-rational points of $\mathscr{L}(X)$ and the set of germs of arcs with coefficients in K on X. We call K-rational points of $\mathscr{L}(X)$, for K a field extension of k, arcs on X, and $\varphi(0)$ is called the origin of the arc φ. More precisely, the scheme $\mathscr{L}(X)$ is defined as the projective limit

$$\mathscr{L}(X) := \varprojlim \mathscr{L}_n(X)$$

in the category of k-schemes of the schemes $\mathscr{L}_n(X)$ representing the functor

$$R \mapsto \mathrm{Mor}_{k\text{-schemes}}\left(\operatorname{Spec} R[t]/t^{n+1}R[t], X\right)$$

defined on the category of k-algebras. (The existence of $\mathscr{L}_n(X)$ is well known—see [5]—and the projective limit exists, since the transition morphisms are affine.) We denote by π_n the canonical morphism, corresponding to truncation of arcs,

$$\pi_n : \mathscr{L}(X) \longrightarrow \mathscr{L}_n(X).$$

The schemes $\mathscr{L}(X)$ and $\mathscr{L}_n(X)$ are always considered with their reduced structure.

3.2. In [5], we defined the boolean algebra $\mathbf{B}_X$ of semi-algebraic subsets of $\mathscr{L}(X)$. We refer to [5] for the precise definition, but we recall that if A is a semi-algebraic subset of $\mathscr{L}(X)$, the image $\pi_n(A)$ is constructible in $\mathscr{L}_n(X)$, and that for $f : X \to Y$, a morphism of k-varieties, the image by f of any semi-algebraic subset of $\mathscr{L}(X)$ is a semi-algebraic subset of $\mathscr{L}(Y)$. Both statements are consequences of results by Pas [8].

Let A be a semi-algebraic subset of $\mathscr{L}(X)$. We call A *weakly stable at level* $n \in \mathbf{N}$, if A is a union of fibers of $\pi_n : \mathscr{L}(X) \to \mathscr{L}_n(X)$. We call A *weakly stable*, if it is stable at some level n. Note that weakly stable semi-algebraic subsets form a boolean algebra.

Let X, Y, and F be algebraic varieties over k, and let A (resp., B) be a constructible subset of X (resp., Y). We say that a map $\pi : A \to B$ is a *piecewise morphism*, if there exists a finite partition of B into subsets S that are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and such that the restriction of π to $\pi^{-1}(S)$ is a morphism of k-varieties. We say that a map $\pi : A \to B$ is a *piecewise trivial fibration with fiber* F, if there exists a finite partition of B into subsets S that are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic, as a variety over k, to $S \times F$, with π corresponding under the isomorphism to the projection $S \times F \to S$.

We say that the map π is a *piecewise trivial fibration over* some constructible subset C of B, if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration onto C. Let X be an algebraic variety over k of pure dimension m, and let A be a semi-algebraic subset of $\mathcal{L}(X)$. We call A *stable at level* $n \in \mathbf{N}$, if A is weakly stable at level n and $\pi_{i+1}(\mathcal{L}(X)) \to \pi_i(\mathcal{L}(X))$ is a piecewise trivial fibration over $\pi_i(A)$ with fiber $\mathbf{A}_k^m$ for all $i \geq n$.

We call A *stable*, if it is stable at some level n. Note that the family of stable semi-algebraic subsets of $\mathcal{L}(X)$ is closed under taking finite intersections and finite unions. If X is smooth, then A is stable at level n, if it is weakly stable at level n.

3.3. We fix an integer $d \geq 1$ and assume k contains all d-roots of unity. Let $f : X \to \mathbf{G}_{m,k}$ be a morphism of k-varieties. For any character α of order d of $\mu_d(k)$, we may define an element $[X, f^*\mathcal{L}_\alpha]$ of A_d as follows.

The morphism $[d] : \mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$, given by $x \mapsto x^d$, is a Galois covering with Galois group $\mu_d(k)$. We consider the fiber product

$$\begin{tikzcd}
\widetilde{X}_{f,d} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathbf{G}_{m,k} \arrow[r, "{[d]}"] & \mathbf{G}_{m,k}.
\end{tikzcd}$$

The scheme $\widetilde{X}_{f,d}$ is endowed with an action of $\mu_d(k)$, so we can define

$$\big[X, f^*\mathcal{L}_\alpha\big] := \chi_c\big(\widetilde{X}_{f,d}, \alpha\big).$$

More generally, if X is constructible in some k-variety and if $f : X \to \mathbf{G}_{m,k}$ is a piecewise morphism, we may define $[X, f^*\mathcal{L}_\alpha] = \sum_{S\in\mathcal{S}}[S, f_{|S}^*\mathcal{L}_\alpha]$ by taking an appropriate partition $\mathcal{S}$ of X into locally closed subvarieties, using the additivity of χ_c.

The next statement follows directly from the additivity of χ_c.

LEMMA 3.3.1. *Let X and Y be constructible in some k-varieties, and let $f : X \to \mathbf{G}_{m,k}$ and $g : Y \to X$ be piecewise morphisms. Assume that g is a piecewise trivial fibration with fiber F. For any character α of order d of $\mu_d(k)$, the following holds:*

$$[Y, (f \circ g)^*\mathcal{L}_\alpha] = \chi_c(F)[X, f^*\mathcal{L}_\alpha].$$

3.4. Let X be an algebraic variety over k of pure dimension m, and let $f : X \to \mathbf{A}_k^1$ be a morphism of k-varieties. By the very definition of semi-algebraic subsets, the set

$$\{\mathrm{ord}_t\, f = i\} := \{\varphi \in \mathcal{L}(X) \mid \mathrm{ord}_t\, f \circ \varphi = i\}$$

is a semi-algebraic subset of $\mathcal{L}(X)$, for any integer $i \geq 0$. We define similarly the semi-algebraic subsets $\{\mathrm{ord}_t\, f > i\}$. We denote by $\bar{f}_i$ the mapping $\{\mathrm{ord}_t\, f = i\} \to \mathbf{G}_{m,k}$, which, to a point φ in $\{\mathrm{ord}_t\, f = i\}$, associates the constant term of the series $t^{-i}(f \circ \varphi)$. (Sometimes we use the same notation $\bar{f}_i$ to denote the natural extension

$\{\mathrm{ord}_t f \geq i\} \to \mathbf{A}^1_k$.) Now let W be a stable semi-algebraic subset of $\mathscr{L}(X)$ that is contained in $\{\mathrm{ord}_t f = i\}$ for some i. Choose an integer $n \geq i$ such that W is stable at level n. The mapping $\bar{f}_i$ factors to a piecewise morphism

$$\bar{f}_{i|\pi_n(W)} : \pi_n(W) \to \mathbf{G}_{m,k}.$$

Let α be a character of order d of $\mu_d(k)$, which we also view as an element of $\widehat{\mu}(k)$. By Lemma 3.3.1, the virtual motive $[\pi_n(W), \bar{f}^*_{i|\pi_n(W)}\mathscr{L}_\alpha]\mathbf{L}^{-(n+1)m}$ is independent of n. So we may define

$$\int_W^{\sim} \alpha(\mathrm{ac}\, f)\, d\mu$$

as the image of

$$\big[\pi_n(W), \bar{f}^*_{i|\pi_n(W)}\mathscr{L}_\alpha\big]\mathbf{L}^{-(n+1)m}$$

in A.

Proposition 3.4.1. *Let X be an algebraic variety over k of pure dimension m, and let $f : X \to \mathbf{A}^1_k$ be a morphism of k-varieties. For any α in $\widehat{\mu}(k)$, there exists a unique map*

$$W \longmapsto \int_W \alpha(\mathrm{ac}\, f)\, d\mu$$

from $\mathbf{B}_X$ to $\widehat{A}$ satisfying the following three properties.

(3.4.2) *If $W \in \mathbf{B}_X$ is stable and contained in $\{\mathrm{ord}_t f = i\}$ for some i, then $\int_W \alpha(\mathrm{ac}\, f)\, d\mu$ coincides with the image of $\int_W^{\sim} \alpha(\mathrm{ac}\, f)\, d\mu$ in $\widehat{A}$.*

(3.4.3) *If $W \in \mathbf{B}$ is contained in $\mathscr{L}(S)$ with S a closed subvariety of X with* $\dim S < \dim X$, *then $\int_W \alpha(\mathrm{ac}\, f) d\mu = 0$.*

(3.4.4) *Let W_i be in $\mathbf{B}_X$ for each i in $\mathbf{N}$. Assume that the W_i's are mutually disjoint and that $W := \bigcup_{i\in\mathbf{N}} W_i$ is semi-algebraic. Then the series*

$$\sum_{i\in\mathbf{N}} \int_{W_i} \alpha(\mathrm{ac}\, f)\, d\mu$$

is convergent in $\widehat{A}$ and converges to $\int_W \alpha(\mathrm{ac}\, f)\, d\mu$.

Moreover, we have

(3.4.5) *If W and W' are in $\mathbf{B}_X$, $W \subset W'$, and if $\int_{W'} \alpha(\mathrm{ac}\, f)\, d\mu$ belongs to the closure $F^m\widehat{A}$ of $F^m A$ in $\widehat{A}$, then $\int_W \alpha(\mathrm{ac}\, f)\, d\mu$ belongs to $F^m\widehat{A}$.*

Proof. This is completely similar to the proof of Proposition 3.2 of [5]. □

In [5], we also defined simple functions $W \to \mathbf{Z}$ for W in $\mathbf{B}_X$. A typical example of a simple function is the following. Consider a coherent sheaf of ideals $\mathscr{I}$ on X and denote by $\mathrm{ord}_t \mathscr{I}$ the function $\mathrm{ord}_t\mathscr{I} : \mathscr{L}(X) \to \mathbf{N} \cup \{+\infty\}$ given by $\varphi \mapsto \min_g \mathrm{ord}_t g(\varphi)$, where the minimum is taken over all g in the stalk $\mathscr{I}_{\pi_0(\varphi)}$ of $\mathscr{I}$ at $\pi_0(\varphi)$. The function $\mathrm{ord}_t \mathscr{I}$ is a simple function.

Hence, for W in $\mathbf{B}_X$ and $\lambda : W \to \mathbf{Z} \cup \{+\infty\}$ a simple function, we can define

$$\int_W \alpha(\mathrm{ac}\, f)\mathbf{L}^{-\lambda}\, d\mu := \sum_{n\in\mathbf{Z}} \int_{W\cap\lambda^{-1}(n)} \alpha(\mathrm{ac}\, f)\, d\mu \mathbf{L}^{-n}$$

in $\widehat{A}$, whenever the right-hand side converges, in which case we say $\alpha(\mathrm{ac}\, f)\mathbf{L}^{-\lambda}$ is integrable on W. If the function λ is bounded from below, then $\alpha(\mathrm{ac}\, f)\mathbf{L}^{-\lambda}$ is integrable on W, because of (3.4.5).

For Y a k-variety, we denote by Ω^1_Y the sheaf of differentials on Y and by Ω^d_Y the dth exterior power of Ω^1_Y. If Y is smooth and $\mathcal{F}$ is a coherent sheaf on Y, together with a natural morphism $\iota : \mathcal{F} \to \Omega^d_Y$, we denote by $\mathcal{I}(\mathcal{F})$ the sheaf of ideals on Y that is locally generated by functions $\iota(\omega)/dy$ with ω a local section of $\mathcal{F}$ and dy a local volume form on Y, and by $\mathrm{ord}_t\, \mathcal{F}$ the simple function $\mathrm{ord}_t\, \mathcal{I}(\mathcal{F})$.

Proposition 3.4.6. *Let $h : Y \to X$ be a proper birational morphism, with Y smooth, let W be a semi-algebraic subset of $\mathcal{L}(X)$, and let $\lambda : \mathcal{L}(X) \to \mathbf{N}$ be a simple function. Then,*

$$\int_W \alpha(\mathrm{ac}\, f)\mathbf{L}^{-\lambda}\, d\mu = \int_{h^{-1}(W)} \alpha\big(\mathrm{ac}(f\circ h)\big)\mathbf{L}^{-\lambda\circ h - \mathrm{ord}_t\, h^*\left(\Omega^d_X\right)}\, d\mu.$$

Proof. This follows directly from Lemma 3.4 of [5]. □

4. Motivic exponential integrals and the main result

4.1. We set $\widehat{U} := U \otimes_A \widehat{A}$. We also consider the subring A_{loc} of $\widehat{A}$ generated by the image of A in $\widehat{A}$ and the series $(1-\mathbf{L}^{-n})^{-1}$, $n \in \mathbf{N}\setminus\{0\}$. We denote by U_{loc} the tensor product $U \otimes_A A_{\mathrm{loc}}$, which is naturally a subring of $\widehat{U}$.

4.2. Let X be an irreducible algebraic variety over k and let $f : X \to \mathbf{A}^1_k$ be a morphism of k-varieties. Let D be the divisor defined by $f = 0$ in X. By a *very good* resolution of (X, D), we mean a couple (Y, h), with Y a smooth and connected k-scheme of finite type, $h : Y \to X$ a proper morphism, such that the restriction $h : Y \setminus h^{-1}(D \cup \mathrm{Sing}\, X) \to X \setminus (D \cup \mathrm{Sing}\, X)$ is an isomorphism, such that the ideal sheaf $\mathcal{I}(h^*(\Omega^d_X))$ is invertible (this last condition is of course irrelevant when X is smooth), and such that the union of $(h^{-1}(D))_{\mathrm{red}}$ and the support of the divisor associated to $\mathcal{I}(h^*(\Omega^d_X))$ has only normal crossings as a subscheme of Y. Such resolutions always exist by Hironaka's theorem.

Let E_i, $i \in J$, be the irreducible (smooth) components of $\big(h^{-1}(D \cup \mathrm{Sing}\, X)\big)_{\mathrm{red}}$. Let W be a reduced subscheme of $f^{-1}(0)$. We call a very good resolution (Y, h) of (X, D) a very good resolution of (X, D, W), if $(h^{-1}(W))_{\mathrm{red}}$ is the union of some E_i's.

Let d be a positive integer greater than or equal to 1. We say d is big with respect to (f, g, W), if

$$\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t f = i\}} \alpha(\mathrm{ac}\, f)\mathbf{L}^{-\mathrm{ord}_t g}\, d\mu = 0$$

for every integer i, whenever the order of α does not divide d.

The following proposition is proven together with Theorem 4.2.4.

PROPOSITION 4.2.1. *Let X be an irreducible algebraic variety over k; let $f : X \to \mathbf{A}_k^1$ and $g : X \to \mathbf{A}_k^1$ be morphisms of k-varieties. Let W be a reduced subscheme of $f^{-1}(0)$. Let d be a positive integer greater than or equal to 1. Assume there exists a very good resolution (Y, h) of $(X, fg = 0, W)$ such that d is a multiple of the multiplicities of the E_i's in the divisor of $f \circ h$ on Y. Then d is big with respect to (f, g, W).*

Definition 4.2.2. Let X be an irreducible algebraic variety over k; let $f : X \to \mathbf{A}_k^1$ and $g : X \to \mathbf{A}_k^1$ be morphisms of k-varieties. Let W be a reduced subscheme of $f^{-1}(0)$. For integers $i \geq 0$, we set

$$\begin{aligned}(4.2.2)\quad & \int_{\pi_0^{-1}(W)} \exp\left(t^{-(i+1)} f\right) \mathbf{L}^{-\operatorname{ord}_t g}\, d\mu \\ & := \int_{\pi_0^{-1}(W) \cap \{\operatorname{ord}_t f > i\}} \mathbf{L}^{-\operatorname{ord}_t g}\, d\mu + \sum_{\alpha \in \widehat{\mu}(k)} \frac{1}{\mathbf{L}-1} G_{\alpha^{-1}} \int_{\pi_0^{-1}(W) \cap \{\operatorname{ord}_t f = i\}} \alpha(\operatorname{ac} f) \mathbf{L}^{-\operatorname{ord}_t g}\, d\mu\end{aligned}$$

in $\widehat{U}$. The sum is finite since, by Proposition 4.2.1, there exists an integer d that is big with respect to (f, g, W).

Remark 4.2.3. The integral (4.2.2) belongs to U_{loc} because of Proposition 5.1 in [5] and the definitions in 3.4.

If $f : X \to \mathbf{A}_k^1$ and $f' : X' \to \mathbf{A}_k^1$ are morphisms of varieties, we denote by $f \oplus f' : X \times X' \to \mathbf{A}_k^1$ the morphism given by composition of the morphism (f, f') with the addition morphism $\oplus : \mathbf{A}_k^1 \times \mathbf{A}_k^1 \to \mathbf{A}_k^1$.

We can now state the main result of the paper.

MAIN THEOREM 4.2.4. *Let X and X' be irreducible algebraic varieties over k. Let $f : X \to \mathbf{A}_k^1$, $g : X \to \mathbf{A}_k^1$, $f' : X' \to \mathbf{A}_k^1$, and $g' : X' \to \mathbf{A}_k^1$ be morphisms of k-varieties. Let W (resp., W') be a reduced subscheme of $f^{-1}(0)$ (resp., $f'^{-1}(0)$). For every $i \geq 0$,*

$$\begin{aligned}(4.2.4)\quad & \int_{\pi_0^{-1}(W \times W')} \exp\left(t^{-(i+1)}(f \oplus f')\right) \mathbf{L}^{-\operatorname{ord}_t gg'}\, d\mu \\ & = \left(\int_{\pi_0^{-1}(W)} \exp\left(t^{-(i+1)} f\right) \mathbf{L}^{-\operatorname{ord}_t g}\, d\mu\right) \cdot \left(\int_{\pi_0^{-1}(W')} \exp\left(t^{-(i+1)} f'\right) \mathbf{L}^{-\operatorname{ord}_t g'}\, d\mu\right).\end{aligned}$$

Remark 4.2.5. Of course, by relations (2.5.1), (2.5.2), and (2.5.3), (4.2.4) is equivalent to the following relations in A_{loc}, where, for notational convenience, we write $[\alpha; f; g]$ for the integrand $\alpha(\mathrm{ac}\, f)\mathbf{L}^{-\mathrm{ord}_t g}\, d\mu$,

(4.2.6)

$$\begin{aligned}
&\int_{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=i\}} \big[\alpha; f\oplus f'; gg'\big] \\
&\quad = \frac{1}{\mathbf{L}-1} \sum_{\alpha_1\neq 1\,\alpha_2\neq 1\,\alpha_1\alpha_2=\alpha} J\big(\alpha_1^{-1}, \alpha_2^{-1}\big)\cdot\bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}}[\alpha_1; f; g]\bigg) \\
&\qquad \cdot\bigg(\int_{\pi_0^{-1}(W')\cap\{\mathrm{ord}_t\, f'=i\}}\big[\alpha_2; f'; g'\big]\bigg) - \frac{1}{\mathbf{L}-1}\bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}}[1; f; g]\bigg) \\
&\qquad \cdot\bigg(\int_{\pi_0^{-1}(W')\cap\{\mathrm{ord}_t\, f'=i\}}\big[\alpha; f'; g'\big]\bigg) + \cdots + \bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f>i\}}[1; f; g]\bigg) \\
&\qquad \cdot\bigg(\int_{\pi_0^{-1}(W')\cap\{\mathrm{ord}_t\, f'=i\}}\big[\alpha; f'; g'\big]\bigg) + \cdots,
\end{aligned}$$

for $\alpha \neq 1$ (here $\cdots$ means "same term as before with (W, f, g) and (W', f', g') interchanged") and

(4.2.7)

$$\begin{aligned}
&\int_{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'>i\}} \big[1; f\oplus f'; gg'\big] \\
&\qquad - \frac{1}{\mathbf{L}-1}\int_{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=i\}} \big[1; f\oplus f'; gg'\big] \\
&\quad = \sum_{\alpha\neq 1} \frac{[\alpha(-1)]\mathbf{L}}{(\mathbf{L}-1)^2}\bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}}[\alpha; f; g]\bigg)\cdot\bigg(\int_{\pi_0^{-1}(W')\cap\{\mathrm{ord}_t\, f'=i\}}\big[\alpha^{-1}; f'; g'\big]\bigg) \\
&\qquad + \bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f>i\}}[1; f; g] - \frac{1}{\mathbf{L}-1}\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}}[1; f; g]\bigg)\cdot\bigg(\cdots\bigg).
\end{aligned}$$

Again, the sums are finite by Proposition 4.2.1.

4.3. Proof of Proposition 4.2.1 and Theorem 4.2.4. Applying Proposition 3.4.6 to very good resolutions of $(X, fg = 0, W)$ and $(X', f'g' = 0, W')$ and using additivity of χ_c, one reduces to the case when X is smooth of dimension m, $x_1, \ldots, x_m$ are regular functions on X inducing an étale map $X \to \mathbf{A}_k^m$, $f = u\prod_{i=1}^r x_i^{n_i}$, with $n_i > 0$, $g = v\prod_{i=1}^m x_i^{m_i}$, with $m_i \geq 0$, u and v are units, W is the union of the hypersurfaces $x_i = 0$, for i in $I \subset \{1, \ldots, r\}$, and similarly with $'$ for X', f', g', and W'. We call this situation the DNC (divisor with normal crossings) case.

We denote by $\tilde{F}_d^n$ the open subvariety of F_d^n defined by $x_i \neq 0$, $i = 1, \ldots n$. It is stable under the $\mu_d(k)^n$-action.

Proposition 4.3.1. *Assume d is big with respect to (f, g, W) and (f', g', W'). For any α in $\widehat{\mu}(k)$ of order dividing d and any integer $i \geq 0$, the following relation holds in A:*

(4.3.1)

$$\int_{\substack{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=i\}\\ \cap\{\mathrm{ord}_t\, f=i\}\cap\{\mathrm{ord}_t\, f'=i\}}} \big[\alpha; f\oplus f'; gg'\big] = \frac{1}{\mathbf{L}-1}\sum_{\alpha_1\alpha_2=\alpha} \chi_c\Big(\tilde{F}_d^2, \big(\alpha_1^{-1}, \alpha_2^{-1}\big)\Big)\cdot\bigg(\int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}} [\alpha_1; f; g]\bigg) \cdot\bigg(\int_{\pi_0^{-1}(W')\cap\{\mathrm{ord}_t\, f'=i\}} \big[\alpha_2; f'; g'\big]\bigg).$$

Here we keep the same notation for $\chi_c(\tilde{F}_d^2, (\alpha_1^{-1}, \alpha_2^{-1}))$ and its image in A, and α_1 and α_2 are assumed to be of order dividing d.

Proof. By Proposition 3.4.6, one reduces to the DNC case. For $i = 0$, the result is clear, the domain of integration being empty, so we may assume $i \geq 1$. Now we remark that $\pi_0^{-1}(W) \cap \{\mathrm{ord}_t\, f = i\}$ is the disjoint union of semi-algebraic sets

$$W_\gamma := \Big\{\varphi \mid \mathrm{ord}_t x_j(\varphi) = \gamma_j\Big\},$$

for $\gamma = (\gamma_1, \ldots, \gamma_r)$, with $\sum_{j=1}^r \gamma_j n_j = i$ and $\gamma_j > 0$ when $j \in I$. On W_γ, we may consider the function $\bar{x}_j : W_\gamma \to \mathbf{G}_{m,k}$, which, to a point φ, associates the constant term of the series $t^{-\gamma_j} x_j(\varphi)$, for $1 \leq j \leq r$. Set

$$K = \Big\{j \in \{1, \ldots, r\} \mid \gamma_j > 0\Big\}.$$

The assumptions made imply that K is not empty. Now we remark that on W_γ,

$$\bar{f}_i = u \prod_{j\in K} \bar{x}_j^{n_j}$$

with u a unit that is a function of the $\bar{x}_j$'s for $j \notin K$ only. Hence, there exists $\ell > 0$ such that

$$W_\gamma = \pi_\ell^{-1}\big(Z \times \mathbf{G}_{m,k}^K\big)$$

with Z a smooth variety, and such that the restriction of $\bar{f}_i$ to W_γ is equal to

$$\Bigg(u \prod_{j\in K} t_j^{n_j}\Bigg) \circ \pi_\ell$$

with u a unit on Z and t_j the canonical coordinate on the corresponding $\mathbf{G}_{m,k}$ factor. Let a be the gcd of the n_i's. By using an appropriate torus isomorphism of $\mathbf{G}_{m,k}^K$ and changing Z, we deduce that

$$W_\gamma = \pi_\ell^{-1}\bigl(Z \times \mathbf{G}_{m,k}\bigr),$$

with Z a smooth variety, and that the restriction of $\bar{f}_i$ to W_γ is equal to $(ut^a)\circ\pi_\ell$, with u a unit on Z and t the canonical coordinate on the $\mathbf{G}_{m,k}$ factor. Now the result is a direct consequence of the next lemma and the following proposition. □

Lemma 4.3.2. *With the above notation and assumptions, if the order of α_1 does not divide a, then*

$$\int_{W_\gamma} [\alpha_1; f; g] = 0.$$

Proof. This is a direct consequence of Lemma 1.4.4 of [4] (compare with the proof of Proposition 2.2.2(2) of [4]). □

We remark that Proposition 4.2.1 follows now directly from Lemma 4.3.2.

Proposition 4.3.3. *Let $u_1 : X_1 \to \mathbf{G}_{m,k}$ and $u_2 : X_2 \to \mathbf{G}_{m,k}$ be morphisms of algebraic varieties over k. Let a, b, and d be in $\mathbf{N}\setminus\{0\}$. Assume a and b divide d. Denote by $(X_1 \times \mathbf{G}_{m,k} \times X_2 \times \mathbf{G}_{m,k})^0$ the complement in $X_1 \times \mathbf{G}_{m,k} \times X_2 \times \mathbf{G}_{m,k}$ of the divisor of $u_1t_1^a + u_2t_2^b$. For any character α of $\mu_d(k)$,*

$$\begin{aligned}\text{(4.3.3)}\quad &\bigl[(X_1 \times \mathbf{G}_{m,k} \times X_2 \times \mathbf{G}_{m,k})^0, \bigl(u_1t_1^a + u_2t_2^b\bigr)^*\mathcal{L}_\alpha\bigr] \\ &= (\mathbf{L}-1) \sum_{\substack{\alpha_1\alpha_2=\alpha \\ \alpha_1^a=1,\ \alpha_2^b=1}} \chi_c\Bigl(\tilde{F}_d^2, \bigl(\alpha_1^{-1}, \alpha_2^{-1}\bigr)\Bigr)\bigl[X_1, u_1^*\mathcal{L}_{\alpha_1}\bigr]\bigl[X_2, u_2^*\mathcal{L}_{\alpha_2}\bigr].\end{aligned}$$

Proof. By definition, the left-hand side (LHS) of (4.3.3) is equal to

$$\chi_c\bigl(u_1(x_1)t_1^a + u_2(x_2)t_2^b = w^d \mid (x_1, x_2, t_1, t_2, w) \in X_1 \times X_2 \times \mathbf{G}_{m,k}^3, \alpha\bigr).$$

Here the action of $\mu_d(k)$ is the standard one on the last $\mathbf{G}_{m,k}$ and trivial on the other factors. Hence, by Theorem 2.3.4,

$$\begin{aligned}\text{LHS} = \chi_c\bigl(&u_1(x_1) = v_1^a, u_2(x_2) = v_2^b, v_1^at_1^a + v_2^bt_2^b \\ &= w^d \mid (x_1, x_2, t_1, t_2, v_1, v_2, w) \in X_1 \times X_2 \times \mathbf{G}_{m,k}^5, (1, 1, \alpha)\bigr).\end{aligned}$$

Here the group action is the action of $\mu_a(k) \times \mu_b(k) \times \mu_d(k)$, which is componentwise on the last three $\mathbf{G}_{m,k}$ factors and trivial on the others. Making the toric change of variable

$$(t_1, t_2, v_1, v_2, w) \longmapsto \bigl(T_1 = v_1t_1w^{-d/a}, T_2 = v_2t_2w^{-d/b}, v_1, v_2, w\bigr),$$

this may be rewritten as

$$\begin{aligned}\text{LHS} = \chi_c\bigl(&u_1(x_1) = v_1^a, u_2(x_2) = v_2^b, T_1^a + T_2^b \\ &= 1 \mid (x_1, x_2, T_1, T_2, v_1, v_2, w) \in X_1 \times X_2 \times \mathbf{G}_{m,k}^5, (1, 1, \alpha)\bigr).\end{aligned}$$

By Proposition 2.3.1(3) and Lemma 2.3.2, we deduce

$$\mathrm{LHS}=(\mathbf{L}-1)\sum_{\substack{\alpha_1\in\widehat{\mu}_a(k)\\ \alpha_2\in\widehat{\mu}_b(k)}}\big[X_1,u_1^*\mathscr{L}_{\alpha_1}\big]\big[X_2,u_2^*\mathscr{L}_{\alpha_2}\big]\chi_c\big(\tilde{F}_d^2,\big(\alpha_1^{-1},\alpha_2^{-1},\alpha\big)\big).$$

Here the $\mu_a(k)\times\mu_b(k)\times\mu_d(k)$-action on F_d^2 is given by

$$(\xi_1,\xi_2,\xi_3):(T_1,T_2)\longmapsto\big(\xi_1\xi_3^{-d/a}T_1,\xi_2\xi_3^{-d/b}T_2\big).$$

The result follows because, by Proposition 2.3.1(4), $\chi_c(\tilde{F}_d^2,(\alpha_1^{-1},\alpha_2^{-1},\alpha))$ is equal to $\chi_c(\tilde{F}_d^2,(\alpha_1^{-1},\alpha_2^{-1}))$ (with the standard $\mu_a(k)\times\mu_b(k)$-action), if $\alpha_1\alpha_2=\alpha$ and to zero otherwise. □

Proposition 4.3.4. *Let X and X' be irreducible algebraic varieties over k. Let $f:X\to\mathbf{A}_k^1$, $g:X\to\mathbf{A}_k^1$, $f':X'\to\mathbf{A}_k^1$, and $g':X'\to\mathbf{A}_k^1$ be morphisms of k-varieties. Let W (resp., W') be a reduced subscheme of $f^{-1}(0)$ (resp., $f'^{-1}(0)$). Let $i\geq 0$ be an integer.*

(1) *For any $\alpha\neq 1$ in $\widehat{\mu}(k)$ and any integer $j>i$,*

$$\int_{\substack{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=j\}\\ \cap\{\mathrm{ord}_t\, f=i\}\cap\{\mathrm{ord}_t\, f'=i\}}}\big[\alpha;f\oplus f';gg'\big]=0.$$

(2) *For any integer $j>i$,*

$$\int_{\substack{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=j+1\}\\ \cap\{\mathrm{ord}_t\, f=i\}\cap\{\mathrm{ord}_t\, f'=i\}}}\big[1;f\oplus f';gg'\big] =\mathbf{L}^{-1}\int_{\substack{\pi_0^{-1}(W\times W')\cap\{\mathrm{ord}_t\, f\oplus f'=j\}\\ \cap\{\mathrm{ord}_t\, f=i\}\cap\{\mathrm{ord}_t\, f'=i\}}}\big[1;f\oplus f';gg'\big].$$

Proof. As before, we reduce to the DNC case and we use the fact that

$$\pi_0^{-1}(W)\cap\{\mathrm{ord}_t\, f=i\}$$

is the disjoint union of semi-algebraic sets

$$W_\gamma:=\big\{\varphi\mid \mathrm{ord}_t x_j(\varphi)=\gamma_j\big\},$$

for $\gamma=(\gamma_1,\ldots,\gamma_r)$, with $\sum_{j=1}^r\gamma_j n_j=i$. We consider again the functions $\bar{x}_j:W_\gamma\to\mathbf{G}_{m,k}$. On W_γ we may write $\bar{f}_i=v\prod_{1\leq j\leq r}\bar{x}_j^{n_j}$ with v a unit. Let φ be a point in W_γ. We write

$$x_j\big(\varphi(t)\big)=t^{\gamma_j}\bar{x}_j\left(1+\sum_{k\geq 1}a_{k,j}t^k\right)$$

for $1 \leq j \leq r$, and

$$x_j\bigl(\varphi(t)\bigr) = \sum_{k\geq 0} a_{k,j} t^k$$

for $r < j \leq m$. Similar notation is used for f' and for φ' in $W'_{\gamma'}$. We may assume that $\gamma_1 \geq 1$. For $\ell > i$, the coefficient of t^ℓ in $f(\varphi(t)) + f'(\varphi'(t))$ is equal to

$$\sum_{j=1}^{r} n_j a_{\ell-i,j} \bar{f}_i + P + \sum_{j'=1}^{r'} n'_{j'} a'_{\ell-i,j'} \bar{f}'_i + P' \tag{3}$$

where P (resp., P') is a polynomial in the variables $a_{k,j}$ and $\bar{x}_j$ (resp., $a'_{k,j'}$ and $\bar{x}'_{j'}$), with $k \leq \ell - i$, having as coefficients regular functions in $\pi_0(\varphi)$ (resp., $\pi_0(\varphi')$). Moreover, since $\gamma_1 \geq 1$, the polynomial P does not involve the variable $a_{\ell-1,1}$ (but might contain the variable $a_{\ell-1,2}$ when $\gamma_2 = 0$).

Let Γ_ℓ be the locus of $\operatorname{ord}_t(f \oplus f') \geq \ell$ in $\pi_\ell\bigl(W_\gamma \times W'_{\gamma'}\bigr) \subset \mathcal{L}_\ell(X \times X')$, and let Γ_ℓ^+ be the locus of $\operatorname{ord}_t(f \oplus f') > \ell$ in Γ_ℓ. From (3), for ℓ replaced by $i+1, \ldots, \ell$, it follows that $a_{\ell-i,1}$ does not appear in the equations defining the variety Γ_ℓ, and that Γ_ℓ^+ is the hypersurface of Γ_ℓ defined by equating (3) to zero. Taking the function (3) as a new coordinate on Γ_ℓ, instead of $a_{\ell-1,1}$, we see that $\Gamma_\ell \simeq \Gamma_\ell^+ \times \mathbf{A}_k^1$, with the mapping $\overline{(f \oplus f')_\ell} : \Gamma_\ell \setminus \Gamma_\ell^+ \to \mathbf{G}_{m,k}$ (which is given by (3)) corresponding to the projection of $\Gamma_\ell^+ \times \mathbf{G}_{m,k}$ onto the last factor. Assertion (2) follows directly, and assertion (1) is now a consequence from Proposition 2.3.1 and Lemma 2.3.2. $\square$

We are now able to conclude the proof of Theorem 4.2.4. We remark first that

$$\chi_c\bigl(\tilde{F}_d^2, (\alpha_1, \alpha_2)\bigr) = \begin{cases} J(\alpha_1, \alpha_2), & \text{if } \alpha_1 \neq 1 \text{ and } \alpha_2 \neq 1, \\ -1, & \text{if } \alpha_1 \neq 1 \text{ and } \alpha_2 = 1 \text{ or } \alpha_1 = 1 \text{ and } \alpha_2 \neq 1, \\ \mathbf{L} - 2, & \text{if } \alpha_1 = 1 \text{ and } \alpha_2 = 1. \end{cases}$$

If $\alpha \neq 1$, relation (4.2.6) follows directly from Proposition 4.3.1 and Proposition 4.3.4(1). Assume now that $\alpha = 1$. We set

$$a_i := \int_{\pi_0^{-1}(W) \cap \{\operatorname{ord}_t f = i\}} [1; f; g] \quad \text{and} \quad A_i := \int_{\pi_0^{-1}(W) \cap \{\operatorname{ord}_t f > i\}} [1; f; g]$$

and define similarly a'_i and A'_i. For $k \geq i$, we also set

$$a_{i,k} := \int_{\substack{\pi_0^{-1}(W \times W') \cap \{\operatorname{ord}_t f \oplus f' = k\} \\ \cap \{\operatorname{ord}_t f = i\} \cap \{\operatorname{ord}_t f' = i\}}} \bigl[1; f \oplus f'; gg'\bigr].$$

Let us denote by RHS the right-hand side of (4.2.7). Since, by Proposition 2.4.1, $\chi_c(\tilde{F}_d^2, (\alpha, \alpha^{-1}))$ is equal to $-[\alpha(-1)]$ if $\alpha \neq 1$ and to $\mathbf{L} - 2$ if $\alpha = 1$, we deduce from Proposition 4.3.1 the relation

$$\begin{aligned}\mathrm{RHS} &= -\frac{\mathbf{L}}{\mathbf{L}-1}a_{i,i} + a_i a_i' + A_i A_i' - \frac{1}{\mathbf{L}-1}a_i A_i' - \frac{1}{\mathbf{L}-1}a_i' A_i \\ &= -\frac{\mathbf{L}}{\mathbf{L}-1}\Big(a_i a_i' - \sum_{i<\ell} a_{i,\ell}\Big) + a_i a_i' + A_i A_i' - \frac{1}{\mathbf{L}-1}a_i A_i' - \frac{1}{\mathbf{L}-1}a_i' A_i.\end{aligned}$$

The left-hand side (LHS) of (4.2.7) is equal to

$$A_i A_i' + \sum_{k\le i<\ell} a_{k,\ell} - \frac{1}{\mathbf{L}-1}\Big(a_i A_i' + a_i' A_i + a_i a_i' + \sum_{k<i} a_{k,i} - \sum_{i<\ell} a_{i,\ell}\Big).$$

Hence, we obtain

$$\mathrm{LHS} \;-\; \mathrm{RHS} \;=\sum_{k<i<\ell} a_{k,\ell} - \frac{1}{\mathbf{L}-1}\sum_{k<i} a_{k,i} = \sum_{k<i<\ell} a_{k,\ell} - \frac{\mathbf{L}^{-1}}{1-\mathbf{L}^{-1}}\sum_{k<i} a_{k,i}.$$

The result now follows, since one deduces from Proposition 4.3.4(2) that, for fixed k and i, with $k<i$,

$$\sum_{i<\ell} a_{k,\ell} = \frac{\mathbf{L}^{-1}}{1-\mathbf{L}^{-1}}a_{k,i}. \qquad \square$$

5. Motivic Thom-Sebastiani theorem

5.1. Let B be any of the rings A_{loc}, $\widehat{A}$, U_{loc}, $\widehat{U}$. We consider the ring of Laurent polynomials $B[T,T^{-1}]$ and its localisation $B[T,T^{-1}]_{\mathrm{rat}}$ obtained by inverting the multiplicative family generated by the polynomials $1-\mathbf{L}^a T^b$, a, b in $\mathbf{Z}$, $b \neq 0$. Notice in this definition we could restrict to $b>0$ or to $b<0$. Hence, by expanding denominators into formal series, there are canonical embeddings of rings

$$\exp_T : B\big[T,T^{-1}\big]_{\mathrm{rat}} \hookrightarrow B\big[T^{-1},T\big]\big]$$

and

$$\exp_{T^{-1}} : B\big[T,T^{-1}\big]_{\mathrm{rat}} \hookrightarrow B\big[\big[T^{-1},T\big].$$

Here $B[T^{-1},T]]$ (resp., $B[[T^{-1},T]$) denotes the ring of series $\sum_{i\in\mathbf{Z}} a_i T^i$ with $a_i=0$ for $i \ll 0$ (resp., $i \gg 0$). By taking the difference $\exp_T - \exp_{T^{-1}}$ of the two expansions, we obtain an embedding

$$\tau : B\big[T,T^{-1}\big]_{\mathrm{rat}}/B\big[T,T^{-1}\big] \hookrightarrow B\big[\big[T^{-1},T\big]\big]$$

where $B[[T^{-1},T]]$ is the group of formal Laurent series with coefficients in B. Let $\varphi = \sum_{i\in\mathbf{Z}} a_i T^i$ and $\psi = \sum_{i\in\mathbf{Z}} b_i T^i$ be series in $B[[T^{-1},T]]$. We define their Hadamard product as the series

$$\varphi * \psi := \sum_{i\in\mathbf{Z}} a_i b_i T^i.$$

PROPOSITION 5.1.1. *Let φ and ψ be series in $B[[T^{-1},T]]$. If they belong to the image of τ, then their Hadamard product $\varphi * \psi$ is also in the image of τ.*

Proof. Let P_1 and P_2 be in $B[T,T^{-1}]_{\mathrm{rat}}$. From the formula

$$\frac{1}{(1-\mathbf{L}^a T^d)(1-\mathbf{L}^b T^d)} = (\mathbf{L}^a - \mathbf{L}^b)^{-1}\left(\frac{\mathbf{L}^a}{1-\mathbf{L}^a T^d} - \frac{\mathbf{L}^b}{1-\mathbf{L}^b T^d}\right),$$

it follows that there exists a d in $\mathbf{N}\setminus\{0\}$ such that, modulo $B[T,T^{-1}]$, both P_1 and P_2 are B-linear combinations of elements $T^r(1-\mathbf{L}^a T^d)^{-k}$, with $r \in \mathbf{N}$, $a \in \mathbf{Z}$, $k \in \mathbf{N}\setminus\{0\}$. Thus, modulo $B[T,T^{-1}]$, both $\exp_T(P_1)$ and $\exp_T(P_2)$ are B-linear combinations of elements of the form

$$\varrho := \sum_{n\in\mathbf{N}} f(n)\mathbf{L}^{na}T^{nd+r}$$

in $B[T,T^{-1}]_{\mathrm{rat}}$, with $a \in \mathbf{Z}$, $r \in \mathbf{N}$, $r < d$, and f a polynomial with coefficients in $\mathbf{Q}$ such that $f(\mathbf{Z}) \subset \mathbf{Z}$. We claim that

$$\tau(\varrho) = \sum_{n\in\mathbf{Z}} f(n)\mathbf{L}^{na}T^{nd+r}.$$

Indeed, if $f(n) = \binom{k+n-1}{k-1}$, then $\varrho = T^r(1-\mathbf{L}^a T^d)^{-k}$, and an explicit calculation, using the relation

$$\binom{k-m-1}{k-1} = (-1)^{k-1}\binom{m-1}{k-1},$$

proves the claim in this special case. Hence the claim holds for any f, considering f as a linear combination of such special f's. The Hadamard product of elements of the form ϱ has the same form. Thus the claim implies that the Hadamard product commutes with τ for elements of the form ϱ, which implies the result by the previous considerations. □

Let us denote by $B[[T]]_{\mathrm{rat}}$ the intersection of $B[[T]]$ with the image of $\exp_T$. It follows from the above proposition that $B[[T]]_{\mathrm{rat}}$ is stable by Hadamard product. Let $\varphi = \exp_T(P)$ be in $B[[T]]_{\mathrm{rat}}$. We denote by $\lambda(\varphi)$ the constant term in the expansion of $\exp_{T^{-1}}(P)$.

PROPOSITION 5.1.2. *Let φ and ψ be series in $TB[[T]]_{\mathrm{rat}}$. Then,*

$$\lambda(\varphi * \psi) = -\lambda(\varphi)\cdot\lambda(\psi).$$

Proof. Let us remark that $\lambda(\varphi)$ depends only upon the class of φ modulo additive translation by $TB[T]$. Hence, we may assume there exist P and Q in $B[T,T^{-1}]_{\mathrm{rat}}$ such that $\exp_T(P) = \varphi$, $\exp_T(Q) = \psi$, and $\exp_{T^{-1}}(P)$ and $\exp_{T^{-1}}(Q)$ belong to $B[[T^{-1}]]$. By Proposition 5.1.1, there exists R in $B[T,T^{-1}]_{\mathrm{rat}}$ such that $\exp_T(R) = \varphi * \psi$ and $\exp_{T^{-1}}(R) = -\exp_{T^{-1}}(P)\exp_{T^{-1}}(Q)$. The result follows. □

5.2. Let X be an irreducible algebraic variety over k of pure dimension m. Let $f : X \to \mathbf{A}^1_k$ be a morphism. Let W be a reduced subscheme of $f^{-1}(0)$. We set

$$E_{W,f}(T) = \sum_{i>0} \left[\int_{\pi_0^{-1}(W)} \exp\left(t^{-(i+1)} f\right) d\mu \right] T^i$$

in $U_{\mathrm{loc}}[[T]]$. For any α in $\widehat{\mu}(k)$, we set

$$Z_{W,f,\alpha}(T) = \sum_{i>0} \left[\int_{\pi_0^{-1}(W) \cap \mathrm{ord}_t f = i} \alpha(\mathrm{ac}\, f)\, d\mu \right] T^i$$

in $A_{\mathrm{loc}}[[T]]$. When X is smooth, $Z_{W,f,\alpha}(T)$ is equal to the natural image in $A_{\mathrm{loc}}[[T]]$ of $\int_W (f^s, \alpha)$, with the notation of [4], setting $T = \mathbf{L}^{-s}$. Hence, it follows from Theorem 2.2.1 of [4] that $Z_{W,f,\alpha}(T)$ belongs to $A_{\mathrm{loc}}[[T]]_{\mathrm{rat}}$. This still holds when X is no longer smooth, by resolution of singularities and Proposition 3.4.6 (adapting the proof of Theorem 2.2.1 of [4] in a straightforward way).

We set

$$\mathcal{S}^{\psi}_{\alpha,W,f} := \frac{\mathbf{L}^m}{1-\mathbf{L}} \lambda\left(Z_{W,f,\alpha}(T)\right)$$

in $\widehat{A}$. When X is smooth and α is of order d, we defined in Definition 4.1.2 of [4] an element $S_{\alpha,x}$ of A_d, well-defined modulo $(\mathbf{L}-1)$-torsion, for x a closed point in $f^{-1}(0)$, which corresponds to the α-equivariant part of the motivic Euler characteristic of nearby cycles. Its image in $\widehat{A}$ is just what we call now $\mathcal{S}^{\psi}_{\alpha,\{x\},f}$. We remark that there is no $(\mathbf{L}-1)$-torsion in $\widehat{A}$ (see also Remark 2.1.1). By Theorem 2.2.1 and Lemma 4.1.1 of [4], $\mathcal{S}^{\psi}_{\alpha,W,f}$ belongs to the image of A, even when X is no longer smooth, by resolution of singularities and Proposition 3.4.6.

To deal with vanishing cycles, we set $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f}$ for $\alpha \neq 1$, and $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f} - \chi_c(W)$ for $\alpha = 1$.

PROPOSITION 5.2.1. *The series* $E_{W,f}(T)$ *belongs to* $U_{\mathrm{loc}}[[T]]_{\mathrm{rat}}$ *and*

$$\lambda\left(E_{W,f}(T)\right) = -\mathbf{L}^{-m} \sum_{\alpha \in \widehat{\mu}(k)} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W,f}.$$

Proof. By the very definitions,

$$E_{W,f}(T) = \frac{1}{\mathbf{L}-1} \sum_{\alpha \in \widehat{\mu}(k)} G_{\alpha^{-1}} Z_{W,f,\alpha}(T) + P(T)$$

with

$$P(T) = \sum_{i \geq 0} \int_{\pi_0^{-1}(W) \cap \{\mathrm{ord}_t f > i\}} d\mu\, T^i - \int_{\pi_0^{-1}(W) \cap \{\mathrm{ord}_t f > 0\}} d\mu.$$

One may write $P(T)$ as

$$P(T) = (T-1)^{-1}\Big(Z_{W,f,1}(T) - \int_{\pi_0^{-1}(W)\cap\{\mathrm{ord}_t f>0\}} d\mu\Big) - \int_{\pi_0^{-1}(W)} d\mu.$$

Since it follows from Theorem 2.2.1 of [4] that $Z_{W,f,1}(T)$ belongs to $A_{\mathrm{loc}}[[T]]_{\mathrm{rat}}$ and that $\exp_{T^{-1}}(Z_{W,f,1}(T))$ belongs to $A_{\mathrm{loc}}[[T^{-1}]]$, we deduce that $\lambda(P(T)) = -\mathbf{L}^{-m}\chi_c(W)$. □

Motivic Thom-Sebastiani Theorem 5.2.2. *Let X and X' be irreducible algebraic varieties over k of pure dimension m and m'. Let $f : X \to \mathbf{A}^1_k$ and $f' : X' \to \mathbf{A}^1_k$ be morphisms of k-varieties. Let W(resp.,W') be a reduced subscheme of $f^{-1}(0)$ (resp., $f'^{-1}(0)$). Then*

$$\sum_\alpha G_{\alpha^{-1}}\mathcal{S}^{\phi}_{\alpha,W\times W',f\oplus f'} = \Big(\sum_\alpha G_{\alpha^{-1}}\mathcal{S}^{\phi}_{\alpha,W,f}\Big)\cdot\Big(\sum_\alpha G_{\alpha^{-1}}\mathcal{S}^{\phi}_{\alpha,W',f'}\Big).$$

Proof. By Theorem 4.2.4, $E_{W\times W',f\oplus f'} = E_{W,f} * E_{W',f'}$. The series $E_{W,f}$ and $E_{W',f'}$ having no constant term, the result follows from Proposition 5.1.2 and Proposition 5.2.1. □

6. Hodge realization and the Hodge spectrum

6.1. In this section, we assume $k = \mathbf{C}$. For $d \geq 1$, there is an embedding of $\widehat{\mu}_d(\mathbf{C})$ in $\mathbf{Q}/\mathbf{Z}$ given by $\alpha \mapsto a$ with $\alpha(e^{2\pi i/d}) = e^{2\pi i a}$. This gives an isomorphism $\widehat{\mu}(\mathbf{C}) \simeq \mathbf{Q}/\mathbf{Z}$. We denote by γ the section $\mathbf{Q}/\mathbf{Z} \to [0,1)$.

A $\mathbf{C}$-Hodge structure of weight n is just a finite-dimensional bigraded vector space $V = \bigoplus_{p+q=n} V^{p,q}$ or, equivalently, a finite-dimensional vector space V with decreasing filtrations $F^{\cdot}$ and $\overline{F}^{\cdot}$ such that $V = F^p \oplus \overline{F}^q$ when $p+q = n+1$. We define similarly a *rational* $\mathbf{C}$-Hodge structure of weight n, by allowing p and q to belong to $\mathbf{Q}$ but still requiring $p+q \in \mathbf{Z}$.

We denote by $K_0(\mathrm{MHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of $\mathbf{C}$-Hodge structures (it is also the Grothendieck group of the abelian category of complex mixed Hodge structures), and we denote by $K_0(\mathrm{RMHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of rational $\mathbf{C}$-Hodge structures.

The Hodge realization functor induces a morphism $H : A \to K_0(\mathrm{MHS}_{\mathbf{C}})$ that is zero on the kernel of the morphism $A \to \widehat{A}$ by Remark 2.1.1. So we extend H to the image of A in $\widehat{A}$.

This morphism may be extended to a morphism $H : U \to K_0(\mathrm{RMHS}_{\mathbf{C}})$ as follows. For p and q in $\mathbf{Q}$ with $p+q$ in $\mathbf{Z}$, we denote by $H^{p,q}$ the class of the rank-1 vector space with bigrading (p,q). We set $H(G_1) = -1$ and $H(G_\alpha) = -H^{1-\gamma(\alpha),\gamma(\alpha)}$ for $\alpha \neq 1$. This is compatible with the relations (2.5.1), (2.5.2), and (2.5.3) since, by a standard calculation (see [17]),

$$H(J_{\alpha_1,\alpha_2}) = -H^{1-(\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)),\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)}$$

when $\alpha_1 \neq 1$, $\alpha_2 \neq 1$, and $\alpha_1\alpha_2 \neq 1$. Similarly, as before, since H vanishes on the kernel of the morphism $U \to \widehat{U}$, we extend it to the image of this morphism.

6.2. For X, a complex algebraic variety, we denote by $\mathrm{MHM}(X)$ the abelian category of mixed modules on X constructed by M. Saito in [9] and [10]. In the definition of mixed Hodge modules, it is required that the underlying perverse sheaf is defined over $\mathbf{Q}$. To allow more flexibility, we also use the category $\mathrm{MHM}'(X)$ of bifiltred $\mathcal{D}$-modules on X, which are direct factors of objects of $\mathrm{MHM}(X)$ as bifiltred $\mathcal{D}$-modules. We denote by $D^b(\mathrm{MHM}(X))$ and $D^b(\mathrm{MHM}'(X))$ the corresponding derived categories.

Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ be a morphism. We denote by ψ_f^H and ϕ_f^H the nearby and vanishing cycle functors for mixed Hodge modules as defined in [10], and we denote by T_s the semisimple part of the monodromy operator. Note that ψ_f^H and ϕ_f^H on mixed Hodge modules correspond to $\psi_f[-1]$ and $\phi_f[-1]$ on the underlying perverse sheaves. If M is a mixed Hodge module on X, we denote by $\psi_{f,\alpha}^H M$ the object of $\mathrm{MHM}'(X)$ that corresponds to the eigenspace of T_s for the eigenvalue $\exp(2\pi i\gamma(\alpha))$. These definitions extend to the Grothendieck group of the abelian category $\mathrm{MHM}'(X)$.

For any object K of $D^b(\mathrm{MHM}(X))$, we denote by $\chi_c(X, K)$ the class of $Rp_!(K)$ in $K_0(\mathrm{MHS}_{\mathbf{C}})$, where p is the projection onto $\operatorname{Spec}\mathbf{C}$. Clearly, this definition may be extended to $D^b(\mathrm{MHM}'(X))$.

THEOREM 6.2.1. *Let X be a smooth and connected complex algebraic variety of dimension m, let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ be a morphism, and let $i_W : W \hookrightarrow f^{-1}(0)$ be a reduced subscheme of $f^{-1}(0)$. The following equalities hold:*

$$H\big(\mathcal{S}^{\psi}_{\alpha,W,f}\big) = (-1)^{m-1}\chi_c\big(W, i_W^*\psi^H_{f,\alpha}\mathbf{C}^H_X[m]\big),$$

$$H\big(\mathcal{S}^{\phi}_{\alpha,W,f}\big) = (-1)^{m-1}\chi_c\big(W, i_W^*\phi^H_{f,\alpha}\mathbf{C}^H_X[m]\big).$$

Proof. When W is a point, the first equality is Theorem 4.2.1 of [4]. The proof of the general case is completely similar. The second equality follows directly from the first one. □

Definition 6.2.2. Let X be a smooth and connected complex algebraic variety of dimension m, let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ be a morphism, and let $i_W : W \hookrightarrow f^{-1}(0)$ be a reduced subscheme of $f^{-1}(0)$. We set

$$\tilde{\chi}_c\Big(W, i_W^*\psi^H_f\mathbf{C}^H_X[m]\Big) = \sum_{\alpha} H\big(G_{\alpha^{-1}}\big)\chi_c\Big(W, i_W^*\psi^H_{f,\alpha}\mathbf{C}^H_X[m]\Big)$$

and

$$\tilde{\chi}_c\Big(W, i_W^*\phi^H_f\mathbf{C}^H_X[m]\Big) = \sum_{\alpha} H\big(G_{\alpha^{-1}}\big)\chi_c\Big(W, i_W^*\phi^H_{f,\alpha}\mathbf{C}^H_X[m]\Big).$$

We deduce from Theorem 5.2.2 and Theorem 6.2.1 the following corollary.

COROLLARY 6.2.3. *Let X and X' be smooth and connected complex algebraic varieties of pure dimension m and m'. Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ and $f' : X' \to \mathbf{A}^1_{\mathbf{C}}$ be morphisms of algebraic varieties. Let W (resp., W') be a reduced subscheme of $f^{-1}(0)$ (resp., $f'^{-1}(0)$). Then*

$$\tilde{\chi}_c\Big(W \times W', i^*_{W\times W'}\phi^H_{f\oplus f'}\mathbf{C}^H_{X\times X'}\big[m+m'\big]\Big) = -\tilde{\chi}_c\Big(W, i^*_W\phi^H_f\mathbf{C}^H_X[m]\Big)\cdot\tilde{\chi}_c\Big(W', i^*_{W'}\phi^H_{f'}\mathbf{C}^H_{X'}\big[m'\big]\Big).$$

Let X be a smooth, complex, algebraic variety of pure dimension m. Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ be a morphism of algebraic varieties, and let x be a closed point of $f^{-1}(0)$. Let us recall the definition of the spectrum $\mathrm{Sp}(f,x)$ given in [18] and [12] (which differs from that of [19] by multiplication by t). Let H be a complex mixed Hodge structure with an automorphism T of order dividing d. We define the Hodge spectrum of (H,T) as $\mathrm{HSp}(H,T) = \sum_{\alpha\in 1/d\mathbf{Z}} n_\alpha t^\alpha \in \mathbf{Z}[t^{-(1/d)}, t^{1/d}]$, with $n_\alpha = \dim\mathrm{Gr}^p_F H_\lambda$, for $\lambda = \exp(2\pi i\alpha)$ and $p = [\alpha]$, where H_λ is the eigenspace of T with eigenvalue λ, and F is the Hodge filtration. This definition extends to the Grothendieck group of the abelian category of complex mixed Hodge structures with an automorphism T of order dividing d. Note that $\mathrm{HSp}(H(k),T) = t^{-k}\mathrm{HSp}(H,T)$, where (k) is the Tate twist. We denote by ι the $\mathbf{Z}$-algebra automorphism of $\mathbf{Z}[t^{-(1/d)}, t^{1/d}]$ defined by $\iota(t^{1/d}) = t^{-(1/d)}$. Now we define $\mathrm{Sp}(f,x)$ as

$$\mathrm{Sp}(f,x) := t^m\iota\left(\sum_{j\in\mathbf{Z}}(-1)^j\mathrm{HSp}\Big(H^j i^*_x\phi^H_f\mathbf{C}^H_X[m], T_s\Big)\right).$$

COROLLARY 6.2.4. *Let X and X' be smooth and connected complex algebraic varieties of pure dimension m and m'. Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ and $f' : X' \to \mathbf{A}^1_{\mathbf{C}}$ be morphisms of algebraic varieties. Let x and x' be closed points in $f^{-1}(0)$ and $f'^{-1}(0)$. Then*

$$\mathrm{Sp}\big(f\oplus f', (x,x')\big) = \mathrm{Sp}(f,x)\cdot\mathrm{Sp}\big(f',x'\big).$$

Corollary 6.2.4 was first proved by A. Varchenko in [20] when f and f' have isolated singularities (see also [14]). The general case is due to M. Saito (see [19], [11], and [13]).

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