MOTIVIC INTEGRATION ON SMOOTH RIGID VARIETIES AND INVARIANTS OF DEGENERATIONS

FRANÇOIS LOESER and JULIEN SEBAG

Abstract

We develop a theory of motivic integration for smooth rigid varieties. As an application we obtain a motivic analogue for rigid varieties of Serre's invariant for p-adic varieties. Our construction provides new geometric birational invariants of degenerations of algebraic varieties. For degenerations of Calabi-Yau varieties, our results take a stronger form.

1. Introduction

In the last years, motivic integration has been shown to be a quite powerful tool in producing new invariants in birational geometry of algebraic varieties over a field k, say, of characteristic zero (see [18], [1], [3], [10], [11], [12]). Let us explain the basic idea behind such results. If $h : Y \to X$ is a proper birational morphism between k-algebraic varieties, the induced morphism $\mathscr{L}(Y) \to \mathscr{L}(X)$ between arc spaces (see [10]) is a bijection outside subsets of infinite codimension. By a fundamental change of variable formula, motivic integrals on $\mathscr{L}(X)$ may be computed on $\mathscr{L}(Y)$ when Y is smooth.

In the present paper we develop similar ideas in the somewhat dual situation of degenerating families over complete discrete valuation rings with perfect residue field, for which rigid geometry appears to be a natural framework. More precisely, let R be a complete discrete valuation ring with fraction field K and perfect residue field k. We construct a theory of motivic integration for smooth* rigid K-spaces, always assumed to be quasi-compact and separated. Let X be a smooth rigid K-space of dimension d. Our construction assigns to a gauge form ω on X, that is, a nowhere vanishing differential form of degree d on X, an integral $\int_X \omega \, d\tilde{\mu}$ with value in the

*The extension to singular rigid spaces when K is of characteristic zero will be considered in a separate publication.

DUKE MATHEMATICAL JOURNAL
Vol. 119, No. 2, © 2003
Received 8 January 2002. Revision received 4 October 2002.
2000 *Mathematics Subject Classification*. Primary 14D06, 14G22, 14J17, 14J32; Secondary 11S80, 14E22, 14E30.

ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$. Here $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ is the localization with respect to the class of the affine line of the Grothendieck group of algebraic varieties over k. In concrete terms, two varieties over k define the same class in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ if they become isomorphic after cutting them into locally closed pieces and stabilizing by product with a power of the affine line. More generally, if ω is a differential form of degree d on X, we define an integral $\int_X \omega\, d\mu$ with value in the ring $\widehat{K_0(\mathrm{Var}_k)}$, which is the completion of $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ with respect to the filtration by virtual dimension (see Sec. 3.4). The construction is done by viewing X as the generic fibre of some formal R-scheme $\mathscr{X}$. To such a formal R-scheme, by means of the Greenberg functor $\mathscr{X} \mapsto \mathrm{Gr}(\mathscr{X})$, one associates a certain k-scheme $\mathrm{Gr}(\mathscr{X})$ which parametrizes unramified sections of $\mathscr{X}$. When $R = k[[t]]$ and $\mathscr{X}$ is the formal completion of $X_0 \otimes k[[t]]$ for X_0 an algebraic variety over k, $\mathrm{Gr}(\mathscr{X})$ is nothing else than the arc space $\mathscr{L}(X_0)$. We may then use the general theory of motivic integration on schemes $\mathrm{Gr}(\mathscr{X})$, which is developed in [25]. Of course, for the construction to work one needs to check that it is independent of the chosen model. This is done by using two main ingredients: the theory of weak Néron models developed in [7] and [9], and the analogue for schemes of the form $\mathrm{Gr}(\mathscr{X})$ of the change of variable formula, which is proven in [25]. In fact, the theory of weak Néron models really pervades the whole paper, and some parts of the book [7] were crying out for their use in motivic integration.*

As an application of our theory, we are able to assign in a canonical way to every smooth quasi-compact and separated rigid K-space X an element $S(X)$ in the quotient ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$, where $\mathbf{L}$ stands for the class of the affine line. When X admits a formal R-model with good reduction, $S(X)$ is just the class of the fibre of that model. More generally, if $\mathscr{U}$ is a weak Néron model of X, $S(X)$ is equal to the class of the special fibre of $\mathscr{U}$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$. In particular, it follows that this class is independent of the choice of the weak Néron model $\mathscr{U}$. This construction applies, in particular, to analytifications of smooth projective algebraic varieties over K, yielding also for such a variety X an invariant $S(X)$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

Our invariant $S(X)$ can be viewed as a rigid analogue of an invariant defined by J.-P. Serre for compact smooth locally analytic varieties over a local field. To such a variety $\tilde{X}$, Serre associates in [26], using classical p-adic integration, an invariant $s(\tilde{X})$ in the ring $\mathbf{Z}/(q-1)\mathbf{Z}$, where q denotes the cardinality of the finite field k. Counting rational points in k yields a canonical morphism $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}/(q-1)\mathbf{Z}$, and we show that the image by this morphism of our motivic invariant $S(X)$ of a smooth rigid K-space X is equal to the Serre invariant of the underlying locally analytic variety.

Unless making additional assumptions on X, one cannot hope to lift our invariant

*See the remark at the bottom of [7, p. 105].

$S(X)$ to a class in the Grothendieck ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ which would be a substitute for the class of the special fibre of *the* Néron model when such a Néron model happens to exist. In the particular situation where X is the analytification of a Calabi-Yau variety over K, that is, a smooth projective algebraic variety over K of pure dimension d with Ω^d_X trivial, the following can be achieved: one can attach to X a canonical element of $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$, which, if X admits a proper and smooth R-model $\mathscr{X}$, is equal to the class of the special fibre $\mathscr{X}_0$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$. In particular, if X admits two such models $\mathscr{X}$ and $\mathscr{X}'$, the class of the special fibres $\mathscr{X}_0$ and $\mathscr{X}'_0$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ are equal, which may be seen as an analogue of V. Batyrev's result on birational Calabi-Yau varieties [2].

The paper is organized as follows. Section 2 is devoted to preliminaries on formal schemes, the Greenberg functor, and weak Néron models. In Section 3 we review the results on motivic integration on formal schemes obtained by J. Sebag in [25] which are needed in the present work. We are then able in Section 4 to construct a motivic integration on smooth rigid varieties and to prove the main results that are mentioned in the present introduction. Finally, in Section 5, guided by the analogy with arc spaces, we formulate an analogue of the Nash problem, which is about the relation between essential (i.e., appearing in every resolution) components of resolutions of a singular variety and irreducible components of spaces of truncated arcs on the variety, for formal R-schemes with smooth generic fibre. Recently, S. Ishii and J. Kollár [17] gave an example where these two sets are not in bijection. In our setting, analogy suggests there might be some relation between essential components of weak Néron models of a given formal R-scheme $\mathscr{X}$ with smooth generic fibre and irreducible components of the truncation $\pi_n(\mathrm{Gr}(\mathscr{X}))$ of its Greenberg space for $n \gg 0$. As a very first step in that direction, we compute the dimension of the contribution of a given irreducible component to the truncation.

2. Preliminaries on formal schemes and Greenberg functor

2.1. Formal schemes

In this paper R denotes a complete discrete valuation ring with residue field k and fraction field K. We assume that k is perfect. We fix once for all a uniformizing parameter ϖ, and we set $R_n := R/(\varpi)^{n+1}$ for $n \geq 0$. In the whole paper, by a formal R-scheme, we always mean a quasi-compact, separated, locally topologically of finite type formal R-scheme, in the sense of [16, Sec 10]. A formal R-scheme is a locally ringed space $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ in topological R-algebras. It is equivalent to the data, for every $n \geq 0$, of the R_n-scheme $X_n = (\mathscr{X}, \mathscr{O}_{\mathscr{X}} \otimes_R R_n)$. The k-scheme X_0 is called the *special fibre* of $\mathscr{X}$. As a topological space, $\mathscr{X}$ is isomorphic to X_0 and $\mathscr{O}_{\mathscr{X}} = \varprojlim \mathscr{O}_{X_n}$. We have $X_n = X_{n+1} \otimes_{R_{n+1}} R_n$, and $\mathscr{X}$ is canonically isomorphic to the inductive limit of the schemes X_n in the category of formal schemes. Locally, $\mathscr{X}$ is isomorphic to an affine formal R-scheme of the form $\operatorname{Spf} A$ with A an R-algebra topologically of finite type, that is, a quotient of a restricted formal series algebra $R\{T_1, \dots, T_m\}$. If $\mathscr{Y}$ and $\mathscr{X}$ are R-formal schemes, we denote by $\operatorname{Hom}_R(\mathscr{Y}, \mathscr{X})$ the set of morphisms of formal R-schemes $\mathscr{Y} \to \mathscr{X}$, that is, morphisms between the underlying locally topologically ringed spaces over R (see [16, Sec. 10]). It follows from [16, Prop. 10.6.9] that the canonical morphism $\operatorname{Hom}_R(\mathscr{Y}, \mathscr{X}) \to \varprojlim \operatorname{Hom}_{R_n}(\mathscr{Y}_n, \mathscr{X}_n)$ is a bijection.

If k is a field, by a variety over k we mean a separated reduced scheme of finite type over k.

2.2. Extensions

Let A be a k-algebra. We set $L(A) = A$ when R is a ring of equal characteristic and $L(A) = W(A)$, the ring of Witt vectors, when R is a ring of unequal characteristic, and we denote by R_A the ring $R_A := R \widehat{\otimes}_{L(k)} L(A)$. When F is a field containing k, we denote by K_F the field of fractions of R_F. When the field F is perfect, the ring R_F is a discrete valuation ring, and, furthermore, the uniformizing parameter ϖ in R induces a uniformizing parameter in R_F. Hence, since k is assumed to be perfect, the extension $R \to R_F$ has ramification index 1 in the terminology of [7, Sec 3.6].

2.3. The Greenberg functor

We recall some material from [13] and [7, Sec. 9.6]. Let us note that, when R is a ring of equal characteristic, we can view, once a lifting of k to R is chosen, R_n as the set of k-valued points of some affine space $\mathbf{A}_k^m$ which we denote by $\mathscr{R}_n$, in a way compatible with the k-algebra structure. When R is a ring of unequal characteristic, R_n can no longer be viewed as a k-algebra. However, using Witt vectors, we may still interpret R_n as the set of k-valued points of a ring scheme $\mathscr{R}_n$, which, as a k-scheme, is isomorphic to some affine space $\mathbf{A}_k^m$. Note that we have canonical morphisms $\mathscr{R}_{n+1} \to \mathscr{R}_n$.

Now, for every $n \geq 0$, we consider the functor h_n^* which to a k-scheme T associates the locally ringed space $h_n^*(T)$ which has T as an underlying topological space and $\mathscr{H}om_k(T, \mathscr{R}_n)$ as a structure sheaf. In particular, for every perfect k-algebra A,

$$h_n^*(A) = \operatorname{Spec}\left(R_n \otimes_{L(k)} L(A)\right).$$

Taking $A = k$, we see that $h_n^* T$ is a locally ringed space over $\operatorname{Spec} R_n$.

By a fundamental result of M. Greenberg [13] (which in the equal characteristic case amounts to Weil restriction of scalars), for R_n-schemes X_n, locally of finite type, the functor

$$T \longmapsto \operatorname{Hom}_{R_n}\left(h_n^*(T), X_n\right)$$

from the category of k-schemes to the category of sets is represented by a k-scheme

$\mathrm{Gr}_n(X_n)$ which is locally of finite type. Hence, for every perfect k-algebra A,

$$\mathrm{Gr}_n(X_n)(A) = X_n\big(R_n \otimes_{L(k)} L(A)\big),$$

and, in particular, setting $A = k$, we have

$$\mathrm{Gr}_n(X_n)(k) = X_n(R_n).$$

Among basic properties, the Greenberg functor respects closed immersions, open immersions, fibred products, and smooth and étale morphisms, and it also sends affines to affines.

Now let us consider again a formal R-scheme $\mathscr{X}$. The canonical adjunction morphism $h^*_{n+1}(\mathrm{Gr}_{n+1}(X_{n+1})) \to X_{n+1}$ gives rise, by tensoring with R_n, to a canonical morphism of R_n-schemes $h^*_n(\mathrm{Gr}_{n+1}(X_{n+1})) \to X_n$, from which one derives, again by adjunction, a canonical morphism of k-schemes

$$\theta_n : \mathrm{Gr}_{n+1}(X_{n+1}) \longrightarrow \mathrm{Gr}_n(X_n).$$

In this way we attach to the formal scheme $\mathscr{X}$ a projective system $(\mathrm{Gr}_n(X_n))_{n\in\mathbf{N}}$ of k-schemes. The transition morphisms θ_n being affine, the projective limit

$$\mathrm{Gr}(\mathscr{X}) := \varprojlim \mathrm{Gr}_n(X_n)$$

exists in the category of k-schemes.

Let T be a k-scheme. We denote by $h^*(T)$ the locally ringed space that has T as an underlying topological space and $\varprojlim \mathscr{H}om_k(T, \mathscr{R}_n)$ as a structure sheaf. It is a locally ringed space over $\mathrm{Spf}\, R$ which identifies with the projective limit of the spaces $h^*_n(T)$ in the category of locally ringed spaces. Furthermore, one checks, similarly as in [16, Prop. 10.6.9], that the canonical morphism $\mathrm{Hom}_R(h^*(T), \mathscr{X}) \to \varprojlim \mathrm{Hom}_{R_n}(h^*_n(T), \mathscr{X}_n)$ is a bijection for every formal R-scheme $\mathscr{X}$.

Putting everything together, we get the following.

PROPOSITION 2.3.1

Let $\mathscr{X}$ be a quasi-compact, locally topologically of finite type formal R-scheme. The functor

$$T \longmapsto \mathrm{Hom}_R\left(h^*(T), \mathscr{X}\right)$$

from the category of k-schemes to the category of sets is represented by the k-scheme $\mathrm{Gr}(\mathscr{X})$.

In particular, for every field F containing k, there are canonical bijections

$$\mathrm{Gr}(\mathscr{X})(F) \simeq \mathrm{Hom}_R(\mathrm{Spf}\, R_F, \mathscr{X}) \simeq \mathscr{X}(R_F).$$

One should note that, in general, $\mathrm{Gr}(\mathscr{X})$ is not of finite type, even if $\mathscr{X}$ is a quasi-compact, topologically of finite type formal R-scheme.

In this paper, we always consider the schemes $\mathrm{Gr}_n(X_n)$ and $\mathrm{Gr}(\mathscr{X})$ with their reduced structure.

Sometimes, by abuse of notation, we write $\mathrm{Gr}_n(\mathscr{X})$ for $\mathrm{Gr}_n(X_n)$.

PROPOSITION 2.3.2

(1) *The functor* Gr *respects open and closed immersions and fibre products, and it sends affine formal R-schemes to affine k-schemes.*

(2) *Let $\mathscr{X}$ be a formal quasi-compact and separated R-scheme, and let $(\mathscr{O}_i)_{i\in J}$ be a finite covering by formal open subschemes. There are canonical isomorphisms $\mathrm{Gr}(\mathscr{O}_i\cap\mathscr{O}_j) \simeq \mathrm{Gr}(\mathscr{O}_i)\cap\mathrm{Gr}(\mathscr{O}_j)$, and the scheme $\mathrm{Gr}(\mathscr{X})$ is canonically isomorphic to the scheme obtained by gluing the schemes $\mathrm{Gr}(\mathscr{O}_i)$.*

Proof

Assertion (1) for the functor Gr_n is proved in [13] and [7], and it follows for Gr by taking projective limits. Assertion (2) follows from (1) and the universal property defining Gr. □

Remark 2.3.3

Assume that we are in the equal characteristic case, that is, $R = k[[\varpi]]$. For X an algebraic variety over k, we can consider the formal R-scheme $X\widehat{\otimes}R$ obtained by base change and completion. We have canonical isomorphisms $\mathrm{Gr}(X\widehat{\otimes}R) \simeq \mathscr{L}(X)$ and $\mathrm{Gr}_n(X \otimes R_n) \simeq \mathscr{L}_n(X)$, where $\mathscr{L}(X)$ and $\mathscr{L}_n(X)$ are the arc spaces considered in [10].

2.4. Smoothness

Let us recall the definition of smoothness for morphisms of formal R-schemes. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of formal R-schemes is smooth at a point x of X_0 of relative dimension d if it is flat at x and the induced morphism $f_0 : X_0 \to Y_0$ is smooth at x of relative dimension d. An equivalent condition (see [6, Lem. 1.2]) is that for every n in $\mathbf{N}$ the induced morphism $f_n : X_n \to Y_n$ is smooth at x of relative dimension d. The morphism f is smooth if it is smooth at every point of X_0. The formal R-scheme $\mathscr{X}$ is smooth at a point x of X_0 if the structural morphism is smooth at x.

Let $\mathscr{X}$ be a flat formal R-scheme of relative dimension d. We denote by $\mathscr{X}_{\mathrm{sing}}$ the closed formal subscheme of $\mathscr{X}$ defined by the radical of the Fitting ideal sheaf $\mathrm{Fitt}_d\Omega_{\mathscr{X}/R}$. The formal R-scheme $\mathscr{X}$ is smooth at a point x of X_0 (resp., is smooth) if and only if x is not in $\mathscr{X}_{\mathrm{sing}}$ (resp., $\mathscr{X}_{\mathrm{sing}}$ is empty).

2.5. Greenberg's theorem

The following statement, which is an adaptation of a result of N. Schappacher [24], is an analogue of Greenberg's theorem (see [14, Th. 1]) in the framework of formal schemes. We refer to [25] for a more detailed exposition.

THEOREM-DEFINITION 2.5.1

Let R be a complete discrete valuation ring, and let $\mathscr{X}$ be a formal R-scheme. For every $n \geq 0$, there exists an integer $n' \geq n$ such that, for every perfect field F containing k, and every x in $\mathscr{X}(R_F/\varpi^{n'+1})$, the image of x in $\mathscr{X}(R_F/\varpi^{n+1})$ may be lifted to a point in $\mathscr{X}(R_F)$. We denote by $\gamma_{\mathscr{X}}(n)$ the smallest such n'. The function $n \mapsto \gamma_{\mathscr{X}}(n)$ is called the Greenberg function *of $\mathscr{X}$.*

2.6. Rigid spaces

For $\mathscr{X}$ a flat formal R-scheme, we denote by $\mathscr{X}_K$ its generic fibre in the sense of M. Raynaud [23]. By Raynaud's theorem (see [23], [5]), the functor $\mathscr{X} \mapsto \mathscr{X}_K$ induces an equivalence between the localization of the category of quasi-compact flat formal R-schemes by admissible formal blow-ups and the category of rigid K-spaces that are quasi-compact and quasi-separated. Furthermore, $\mathscr{X}$ is separated if and only if $\mathscr{X}_K$ is separated (see [5, Prop. 4.7]). Recall that for the blow-up of an ideal sheaf $\mathscr{I}$ to be admissible means that $\mathscr{I}$ contains some power of the uniformizing parameter ϖ. In the paper all rigid K-spaces are assumed to be quasi-compact and separated.

2.7. Weak Néron models

We denote by R^{sh} a strict Henselization of R, and we denote by K^{sh} its field of fractions.

Definition 2.7.1

Let X be a smooth rigid K-variety. A weak formal Néron model* of X is a smooth formal R-scheme $\mathscr{U}$, whose generic fibre $\mathscr{U}_K$ is an open rigid subspace of X_K, and which has the property that the canonical map $\mathscr{U}(R^{\mathrm{sh}}) \to X(K^{\mathrm{sh}})$ is bijective.

The construction of weak Néron models, using Néron's smoothening process presented in [7], carries over almost literally from R-schemes to formal R-schemes, and it gives, as explained in [9], the following result.

THEOREM 2.7.2

Let $\mathscr{X}$ be a quasi-compact formal R-scheme whose generic fibre $\mathscr{X}_K$ is smooth. Then there exists a morphism of formal R-schemes $\mathscr{X}' \to \mathscr{X}$ which is the composition of

*We follow here the terminology of [9], which is somewhat different from that of [7].

a sequence of formal blow-ups with centers in the corresponding special fibres, such that every R^{sh}-valued point of $\mathscr{X}$ factors through the smooth locus of $\mathscr{X}'$.

One deduces the following omnibus statement.

PROPOSITION 2.7.3
Let X be a smooth quasi-compact and separated rigid K-space, and let $\mathscr{X}$ be a formal R-model of X, that is, a quasi-compact formal R-scheme $\mathscr{X}$ with generic fibre X. Then there exists a weak formal Néron model $\mathscr{U}$ of X which dominates $\mathscr{X}$ and which is quasi-compact. Furthermore, the canonical map $\mathscr{U}(R^{\mathrm{sh}}) \to \mathscr{X}(R^{\mathrm{sh}})$ is a bijection, and for every perfect field F containing k, the formal R_F-scheme $\mathscr{U}\widehat{\otimes}_R R_F$ is a weak Néron model of the rigid K_F-space $X\widehat{\otimes}_K K_F$. In particular, the morphism $\mathscr{U} \to \mathscr{X}$ induces a bijection between points of $\mathrm{Gr}(\mathscr{U})$ and $\mathrm{Gr}(\mathscr{X})$.

Proof
We choose a formal model $\mathscr{X}$ of X such that we are in the situation of Theorem 2.7.2. The smooth locus $\mathscr{U}$ of $\mathscr{X}'$ is quasi-compact and is a weak Néron model of $\mathscr{X}_K$ since, by [9, Lem. 2.2(ii)], every K^{sh}-valued point of $\mathscr{X}_K$ extends uniquely to a R^{sh}-valued point of $\mathscr{X}$. Also, it follows from [7, §3.6, Cor. 6] that, if $\mathscr{U}$ is a weak Néron model of the rigid K-space X, then for every perfect field F containing k, the formal R_F-scheme $\mathscr{U}\widehat{\otimes}_R R_F$ is a weak Néron model of the rigid K_F-space $X\widehat{\otimes}_K K_F$. □

Example 2.7.4
Let $\mathscr{X}$ be a regular scheme of finite presentation over Spec R, and let $\mathscr{U}$ be the open subscheme of smooth points. Then $\widehat{\mathscr{U}}$, the formal completion of $\mathscr{U}$ along the special fibre, is a weak Néron model of the rigid K-space associated to $\widehat{\mathscr{X}}$, the formal completion of $\mathscr{X}$ along the special fibre.

3. Motivic integration on formal schemes

The material in this section is borrowed from [25], to which we refer for details.

3.1. Truncation

For $\mathscr{X}$ a formal R-scheme, we denote by $\pi_{n,\mathscr{X}}$ or π_n the canonical projection $\mathrm{Gr}(\mathscr{X}) \to \mathrm{Gr}_n(X_n)$ for n in $\mathbf{N}$.

Let us first state the following corollary of Theorem 2.5.1.

PROPOSITION 3.1.1
Let $\mathscr{X}$ be a formal R-scheme. The image $\pi_n(\mathrm{Gr}(\mathscr{X}))$ of $\mathrm{Gr}(\mathscr{X})$ in $\mathrm{Gr}_n(X_n)$ is a constructible subset of $\mathrm{Gr}_n(X_n)$. More generally, if C is a constructible subset of

$\mathrm{Gr}_m(X_m)$, $\pi_n(\pi_m^{-1}(C))$ *is a constructible subset of* $\mathrm{Gr}_n(X_n)$ *for every* $n \geq 0$.

Proof

Indeed, it follows from Theorem 2.5.1 that $\pi_n(\mathrm{Gr}(\mathscr{X}))$ is equal to the image of $\mathrm{Gr}_{\gamma(n)}(X_{\gamma(n)})$ in $\mathrm{Gr}_n(X_n)$. The morphism $\mathrm{Gr}_{\gamma(n)}(X_{\gamma(n)}) \to \mathrm{Gr}_n(X_n)$ being of finite type, the first statement follows from Chevalley's theorem. For the second statement, one may assume $m = n$, and the proof proceeds as before. □

PROPOSITION 3.1.2

Let $\mathscr{X}$ *be a smooth formal separated* R*-scheme (quasi-compact, locally topologically of finite type over* R*) of relative dimension* d.

(1) *For every* n*, the morphism* $\pi_n : \mathrm{Gr}(\mathscr{X}) \to \mathrm{Gr}_n(X_n)$ *is surjective.*

(2) *For every* n *and* m *in* $\mathbf{N}$*, the canonical projection* $\mathrm{Gr}_{n+m}(X_{n+m}) \to \mathrm{Gr}_n(X_n)$ *is a locally trivial fibration for the Zariski topology with fibre* $\mathbf{A}_k^{dm}$.

We say that a map $\pi : A \to B$ is a piecewise morphism if there exists a finite partition of the domain of π into locally closed subvarieties of X such that the restriction of π to any of these subvarieties is a morphism of schemes.

3.2. Away from the singular locus

Let $\mathscr{X}$ be a formal R-scheme, and consider its singular locus $\mathscr{X}_{\mathrm{sing}}$ defined in Section 2.4. For all integers $e \geq 0$, we view $\mathrm{Gr}_e(\mathscr{X}_{\mathrm{sing},e})$ as contained in $\mathrm{Gr}_e(\mathscr{X})$, and we set

$$\mathrm{Gr}^{(e)}(\mathscr{X}) := \mathrm{Gr}(\mathscr{X}) \setminus \pi_e^{-1}\big(\mathrm{Gr}_e(\mathscr{X}_{\mathrm{sing},e})\big).$$

We say that a map $\pi : A \to B$ between k-constructible sets A and B is a piecewise trivial fibration with fibre F if there exists a finite partition of B in subsets S which are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic, as a variety over k, to $S \times F$, with π corresponding under the isomorphism to the projection $S \times F \to S$. We say that the map π is a piecewise trivial fibration over some constructible subset C of B if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration onto C.

PROPOSITION 3.2.1

Let $\mathscr{X}$ *be a flat formal* R*-scheme of relative dimension* d*. There exists an integer* $c \geq 1$ *such that, for all integers* e *and* n *in* $\mathbf{N}$ *such that* $n \geq ce$*, the projection*

$$\pi_{n+1}\big(\mathrm{Gr}(\mathscr{X})\big) \longrightarrow \pi_n\big(\mathrm{Gr}(\mathscr{X})\big)$$

is a piecewise trivial fibration over $\pi_n(\mathrm{Gr}^{(e)}(\mathscr{X}))$ *with fibre* $\mathbf{A}_k^d$.

3.3. Dimension estimates

LEMMA 3.3.1

Let $\mathscr{X}$ be a formal R-scheme whose generic fibre $\mathscr{X}_K$ is of dimension less than or equal to d. Then we have the following.

(1) *For every n in $\mathbf{N}$,*

$$\dim \pi_n\big(\mathrm{Gr}(\mathscr{X})\big) \leq (n+1)d.$$

(2) *For $m \geq n$, the fibres of the projection $\pi_m(\mathrm{Gr}(\mathscr{X})) \to \pi_n\big(\mathrm{Gr}(\mathscr{X})\big)$ are of dimension less than or equal to $(m-n)d$.*

LEMMA 3.3.2

Let $\mathscr{X}$ be a formal R-scheme whose generic fibre $\mathscr{X}_K$ is of dimension d. Let $\mathscr{S}$ be a closed formal R-subscheme of $\mathscr{X}$ such that $\mathscr{S}_K$ is of dimension less than d. Then, for all integers n, i, and ℓ such that $n \geq i \geq \gamma_{\mathscr{S}}(\ell)$, where $\gamma_{\mathscr{S}}$ is the Greenberg function of $\mathscr{S}$ defined in Theorem-Definition 2.5.1, $\pi_{n,\mathscr{X}}(\pi_{i,\mathscr{X}}^{-1}\,\mathrm{Gr}_i(\mathscr{S}))$ is of dimension less than or equal to $(n+1)\,d-\ell-1$.

3.4. Grothendieck rings

Let k be a field. We denote by $K_0(\mathrm{Var}_k)$ the abelian group generated by symbols $[S]$, for S a variety over k, with the relations $[S]=[S']$ if S and S' are isomorphic and $[S]=[S']+[S\setminus S']$ if S' is closed in S. There is a natural ring structure on $K_0(\mathrm{Var}_k)$, the product being induced by the Cartesian product of varieties,* and to any constructible set S in some variety, one naturally associates a class $[S]$ in $K_0(\mathrm{Var}_k)$. We denote by $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ the localization $K_0(\mathrm{Var}_k)_{\mathrm{loc}} := K_0(\mathrm{Var}_k)[\mathbf{L}^{-1}]$ with $\mathbf{L} := [\mathbf{A}_k^1]$. Let us note that the canonical morphism

$$K_0(\mathrm{Var}_k)/(\mathbf{L}-1)K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$$

is an isomorphism.

We denote by $F^m K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ the subgroup generated by $[S]\mathbf{L}^{-i}$ with $\dim S - i \leq -m$, and we denote by $\widehat{K_0(\mathrm{Var}_k)}$ the completion of $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ with respect to the filtration $F^{\cdot}$. (It is still unknown whether the filtration $F^{\cdot}$ is separated or not.) We also denote by $F^{\cdot}$ the filtration induced on $\widehat{K_0(\mathrm{Var}_k)}$. We denote by $\overline{K_0(\mathrm{Var}_k)_{\mathrm{loc}}}$ the image of $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ in $\widehat{K_0(\mathrm{Var}_k)}$. We put on the ring $\widehat{K_0(\mathrm{Var}_k)}$ a structure of a non-Archimedean ring by setting $\|a\| := 2^{-n}$, where n is the largest n such that $a \in F^n\widehat{K_0(\mathrm{Var}_k)}$ for $a \neq 0$ and $\|0\| = 0$.

*By the Cartesian product of two varieties S and S' over k, we mean the fibre product $S \times_k S'$ endowed with its reduced structure.

3.5. Cylinders

Let $\mathscr{X}$ be a formal R-scheme. A subset A of $\mathrm{Gr}(\mathscr{X})$ is cylindrical of level $n \geq 0$ if $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathrm{Gr}_n(\mathscr{X})$. We denote by $\mathbf{C}_{\mathscr{X}}$ the set of cylindrical subsets of $\mathrm{Gr}(\mathscr{X})$ of some level. Let us note that $\mathbf{C}_{\mathscr{X}}$ is a Boolean algebra, that is, contains $\mathrm{Gr}(\mathscr{X})$, $\emptyset$, and is stable by finite intersection, finite union, and by taking complements. It follows from Proposition 3.1.1 that if A is cylindrical of some level, then $\pi_n(A)$ is constructible for every $n \geq 0$.

A basic finiteness property of cylinders is the following.

LEMMA 3.5.1

Let A_i, $i \in I$, be a denumerable family of cylindrical subsets of $\mathrm{Gr}(\mathscr{X})$. If $A := \bigcup_{i\in I} A_i$ is also a cylinder, then there exists a finite subset J of I such that $A := \bigcup_{i\in J} A_i$.

Proof

Since $\mathrm{Gr}(\mathscr{X})$ is quasi-compact, this follows from [16, Th. 7.2.5]. □

3.6. Motivic measure for cylinders

Let $\mathscr{X}$ be a flat formal R-scheme of relative dimension d. Let A be a cylinder of $\mathrm{Gr}(\mathscr{X})$. We say that A is stable of level n if it is cylindrical of level n and if, for every $m \geq n$, the morphism

$$\pi_{m+1}\big(\mathrm{Gr}(\mathscr{X})\big) \longrightarrow \pi_m\big(\mathrm{Gr}(\mathscr{X})\big)$$

is a piecewise trivial fibration over $\pi_n(A)$ with fibre $\mathbf{A}_k^d$. We denote by $\mathbf{C}_{0,\mathscr{X}}$ the set of stable cylindrical subsets of $\mathrm{Gr}(\mathscr{X})$ of some level.

It follows from Proposition 3.1.2 that every cylinder in $\mathrm{Gr}(\mathscr{X})$ is stable when $\mathscr{X}$ is smooth. When $\mathscr{X}$ is no longer assumed to be smooth, $\mathbf{C}_{0,\mathscr{X}}$ is in general not a Boolean algebra but is an ideal of $\mathbf{C}_{\mathscr{X}}$: $\mathbf{C}_{0,\mathscr{X}}$ contains $\emptyset$, it is stable by finite union, and the intersection of an element in $\mathbf{C}_{\mathscr{X}}$ with an element of $\mathbf{C}_{0,\mathscr{X}}$ belongs to $\mathbf{C}_{0,\mathscr{X}}$. In general, $\mathrm{Gr}(\mathscr{X})$ is not stable, but it follows from Proposition 3.2.1 that $\mathrm{Gr}^{(e)}(\mathscr{X})$ is a stable cylinder of $\mathrm{Gr}(\mathscr{X})$, for every $e \geq 0$.

From first principles, one proves the following (see [10], [25]).

PROPOSITION-DEFINITION 3.6.1

There is a unique additive morphism

$$\tilde{\mu} : \mathbf{C}_{0,\mathscr{X}} \longrightarrow K_0(\mathrm{Var}_k)_{\mathrm{loc}}$$

such that $\tilde{\mu}(A) = [\pi_n(A)]\,\mathbf{L}^{-(n+1)d}$ when A is a stable cylinder of level n.

One deduces from Lemmas 3.3.1 and 3.3.2 (see [10], [25]) the following.

PROPOSITION 3.6.2

(1) *For any cylinder A in $\mathbf{C}_{\mathscr{X}}$, the limit*

$$\mu(A) := \lim_{e\to\infty} \tilde{\mu}\big(A \cap \mathrm{Gr}^{(e)}(\mathscr{X})\big)$$

exists in $\widehat{K_0(\mathrm{Var}_k)}$.

(2) *If A belongs to $\mathbf{C}_{0,\mathscr{X}}$, then $\mu(A)$ coincides with the image of $\tilde{\mu}(A)$ in $\widehat{K_0(\mathrm{Var}_k)}$.*

(3) *If A in $\mathbf{C}_{\mathscr{X}}$ is the disjoint union of a denumerable family of subsets A_i, $i \in I$, which all belong to $\mathbf{C}_{\mathscr{X}}$, then*

$$\mu(A) = \sum_{i\in I} \mu(A_i).$$

(4) *For A and B in $\mathbf{C}_{\mathscr{X}}$, $||\mu(A\cup B)|| \le \max(||\mu(A)||, ||\mu(B)||)$. If $A \subset B$, $||\mu(A)|| \le ||\mu(B)||$.*

3.7. Measurable subsets of $\mathrm{Gr}(\mathscr{X})$

For A and B subsets of the same set, we use the notation $A\triangle B$ for $(A\cup B)\setminus A\cap B$.

Definition 3.7.1

We say that a subset A of $\mathrm{Gr}(\mathscr{X})$ is *measurable* if, for every positive real number ε, there exists an ε-cylindrical approximation, that is, a sequence of cylindrical subsets $A_i(\varepsilon)$, $i \in \mathbf{N}$, such that

$$\big(A\triangle A_0(\varepsilon)\big) \subset \bigcup_{i\ge 1} A_i(\varepsilon)$$

and $||\mu(A_i(\varepsilon))|| \le \varepsilon$ for all $i \ge 1$. We say that A is *strongly measurable* if, moreover, we can take $A_0(\varepsilon) \subset A$.

THEOREM 3.7.2

If A is a measurable subset of $\mathrm{Gr}(\mathscr{X})$, *then*

$$\mu(A) := \lim_{\varepsilon\to 0} \mu\big(A_0(\varepsilon)\big)$$

exists in $\widehat{K_0(\mathrm{Var}_k)}$ and is independent of the choice of the sequences $A_i(\varepsilon)$, $i \in \mathbf{N}$.

For A a measurable subset of $\mathrm{Gr}(\mathscr{X})$, we call $\mu(A)$ the *motivic measure* of A. We denote by $\mathbf{D}_{\mathscr{X}}$ the set of measurable subsets of $\mathrm{Gr}(\mathscr{X})$.

One should note that obviously $\mathbf{C}_{\mathscr{X}}$ is contained in $\mathbf{D}_{\mathscr{X}}$.

PROPOSITION 3.7.3

(1) $\mathbf{D}_{\mathscr{X}}$ *is a Boolean algebra.*

(2) *If* A_i, $i \in \mathbf{N}$, *is a sequence of measurable subsets of* $\mathrm{Gr}(\mathscr{X})$ *with* $\lim_{i\to\infty} ||\mu(A_i)|| = 0$, *then* $\bigcup_{i\in\mathbf{N}} A_i$ *is measurable.*

(3) *Let* A_i, $i \in \mathbf{N}$, *be a family of measurable subsets of* $\mathrm{Gr}(\mathscr{X})$. *Assume that the sets* A_i *are mutually disjoint and that* $A := \bigcup_{i\in\mathbf{N}} A_i$ *is measurable. Then* $\sum_{i\in\mathbf{N}} \mu(A_i)$ *converges in* $\widehat{K_0(\mathrm{Var}_k)}$ *to* $\mu(A)$.

(4) *If* A *and* B *are measurable subsets of* $\mathrm{Gr}(\mathscr{X})$ *and if* $A \subset B$, *then* $||\mu(A)|| \leq ||\mu(B)||$.

Remark 3.7.4

In the situation of Remark 2.3.3, one can check that the notions of cylinders, stable cylinders, and measurable subsets of $\mathrm{Gr}(X\widehat{\otimes}R)$ coincide with the analogous notions introduced in [12] for subsets of $\mathscr{L}(X)$.

3.8. Order of the Jacobian ideal

Let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism of flat formal R-schemes of relative dimension d.

Let y be a point of $\mathrm{Gr}(\mathscr{Y}) \setminus \mathrm{Gr}(\mathscr{Y}_{\mathrm{sing}})$ defined over some perfect field extension F of k. We denote by $\varphi : \mathrm{Spf}\, R_F \to \mathscr{Y}$ the corresponding morphism of formal R-schemes. We define $\mathrm{ord}_\varpi(\mathrm{Jac}_h)(y)$, the order of the Jacobian ideal of h at y, as follows.

From the natural morphism $h^*\Omega_{\mathscr{X}|R} \to \Omega_{\mathscr{Y}|R}$, one deduces, by taking the dth exterior power, a morphism $h^*\Omega^d_{\mathscr{X}|R} \to \Omega^d_{\mathscr{Y}|R}$ and hence a morphism

$$(\varphi^*h^*\Omega^d_{\mathscr{X}|R})/(\mathrm{torsion}) \longrightarrow (\varphi^*\Omega^d_{\mathscr{Y}|R})/(\mathrm{torsion}).$$

Since $L := (\varphi^*\Omega^d_{\mathscr{Y}|R})/(\mathrm{torsion})$ is a free $\mathscr{O}_{R_F}$-module of rank 1, it follows from the structure theorem for finite-type modules over principal domains that the image of $M := (\varphi^*h^*\Omega^d_{\mathscr{X}|R})/(\mathrm{torsion})$ in L is either zero, in which case we set $\mathrm{ord}_\varpi(\mathrm{Jac}_h)(y) = \infty$, or $\varpi^n L$ for some $n \in \mathbf{N}$, in which case we set $\mathrm{ord}_\varpi(\mathrm{Jac}_h)(y) = n$.

3.9. The change of variable formula

If $h : \mathscr{Y} \to \mathscr{X}$ is a morphism of formal R-schemes, we still write h for the corresponding morphism $\mathrm{Gr}(\mathscr{Y}) \to \mathrm{Gr}(\mathscr{X})$.

The following lemmas are basic geometric ingredients in the proof of the change of variable formula (see Ths. 3.9.3 and 3.9.4).

LEMMA 3.9.1

Let $h : \mathscr{Y} \to \mathscr{X}$ *be a morphism between flat formal* R*-schemes of relative dimension*

d. We assume that $\mathscr{Y}$ is smooth. For e and e' in $\mathbf{N}$, we set

$$\Delta_{e,e'} := \left\{\varphi \in \mathrm{Gr}(\mathscr{Y}) \,\middle|\, \mathrm{ord}_{\varpi}(\mathrm{Jac}_h)(y) = e \text{ and } h(\varphi) \in \mathrm{Gr}^{(e')}(\mathscr{X})\right\}.$$

Then there exists c in $\mathbf{N}$ such that, for every $n \geq 2e$, $n \geq e + ce'$, for every φ in $\Delta_{e,e'}$, and for every x in $\mathrm{Gr}(\mathscr{X})$ such that $\pi_n(h(\varphi)) = \pi_n(x)$, there exists y in $\mathrm{Gr}(\mathscr{Y})$ such that $h(y) = x$ and $\pi_{n-e}(\varphi) = \pi_{n-e}(y)$.

LEMMA 3.9.2

Let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism between flat formal R-schemes of relative dimension d. We assume that $\mathscr{Y}$ is smooth. Let B be a cylinder in $\mathrm{Gr}(\mathscr{Y})$, and set $A = h(B)$. Assume that $\mathrm{ord}_{\varpi}(\mathrm{Jac}_h)$ is constant with value $e < \infty$ on B and that $A \subset \mathrm{Gr}^{(e')}(\mathscr{X})$ for some $e' \geq 0$. Then A is a cylinder. Furthermore, if the restriction of h to B is injective, then for $n \gg 0$ the following hold.

(1) *If φ and φ' belong to B and $\pi_n(h(\varphi)) = \pi_n(h(\varphi'))$, then $\pi_{n-e}(\varphi) = \pi_{n-e}(\varphi')$.*

(2) *The morphism $\pi_n(B) \to \pi_n(A)$ induced by h is a piecewise trivial fibration with fibre $\mathbf{A}_k^e$.*

For a measurable subset A of $\mathrm{Gr}(\mathscr{X})$ and a function $\alpha : A \to \mathbf{Z} \cup \{\infty\}$, we say that $\mathbf{L}^{-\alpha}$ is integrable or that α is exponentially integrable if the fibres of α are measurable and if the motivic integral

$$\int_A \mathbf{L}^{-\alpha}\, d\mu := \sum_{n \in \mathbf{Z}} \mu(\alpha^{-1}(n))\mathbf{L}^{-n}$$

converges in $\widehat{K_0(\mathrm{Var}_k)}$.

When all the fibres $\alpha^{-1}(n)$ are stable cylinders and α takes only a finite number of values on A, it is not necessary to go to the completion of $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ and one may directly define

$$\int_A \mathbf{L}^{-\alpha}\, d\tilde{\mu} := \sum_{n \in \mathbf{Z}} \tilde{\mu}\big(\alpha^{-1}(n)\big)\mathbf{L}^{-n}$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

THEOREM 3.9.3

Let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism between flat formal R-schemes of relative dimension d. We assume that $\mathscr{Y}$ is smooth. Let B be a strongly measurable subset of $\mathrm{Gr}(\mathscr{Y})$. Assume that h induces a bijection between B and $A := h(B)$. Then, for every exponentially integrable function $\alpha : A \to \mathbf{Z} \cup \infty$, the function $\alpha \circ h + \mathrm{ord}_{\varpi}(\mathrm{Jac}_h)$ is exponentially integrable on B and

$$\int_A \mathbf{L}^{-\alpha}\, d\mu = \int_B \mathbf{L}^{-\alpha \circ h - \mathrm{ord}_{\varpi}(\mathrm{Jac}_h)}\, d\mu.$$

We also need the following variant of Theorem 3.9.3.

THEOREM 3.9.4
Let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism between flat formal R-schemes of relative dimension d. We assume that $\mathscr{Y}$ and $\mathscr{X}_K$ are smooth and that the morphism $h_K : \mathscr{Y}_K \to \mathscr{X}_K$ induced by h is étale (see [8]). Let B be a cylinder in $\mathrm{Gr}(\mathscr{Y})$. Assume that h induces a bijection between B and $A := h(B)$ and that A is a stable cylinder of $\mathrm{Gr}(\mathscr{X})$. Then the fibres $B \cap \mathrm{ord}_\varpi(\mathrm{Jac}_h)^{-1}(n)$ are stable cylinders, $\mathrm{ord}_\varpi(\mathrm{Jac}_h)^{-1}(n)$ takes only a finite number of values on B, and

$$\int_A d\tilde{\mu} = \int_B \mathbf{L}^{-\mathrm{ord}_\varpi(\mathrm{Jac}_h)}\, d\tilde{\mu}$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

4. Integration on smooth rigid varieties

4.1. Order of differential forms

Let $\mathscr{X}$ be a flat formal R-scheme equidimensional of relative dimension d. Consider a differential form ω in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$. Let x be a point of $\mathrm{Gr}(\mathscr{X}) \setminus \mathrm{Gr}(\mathscr{X}_{\mathrm{sing}})$ defined over some perfect field extension F of k. We denote by $\varphi : \mathrm{Spf}\, R_F \to \mathscr{X}$ the corresponding morphism of formal R-schemes. Since $L := (\varphi^*\Omega^d_{\mathscr{X}|R})/(\mathrm{torsion})$ is a free $\mathscr{O}_{R_F}$-module of rank 1, it follows from the structure theorem for finite-type modules over principal domains that its submodule M generated by $\varphi^*\omega$ is either zero, in which case we set $\mathrm{ord}_\varpi(\omega)(x) = \infty$, or $\varpi^n L$ for some $n \in \mathbf{N}$, in which case we set $\mathrm{ord}_\varpi(\omega)(x) = n$. We may assume that $\mathscr{X}$ is affine.

Since there is a canonical isomorphism $\Omega^d_{\mathscr{X}_K}(\mathscr{X}_K) \simeq \Omega^d_{\mathscr{X}|R}(\mathscr{X}) \otimes_R K$ (see [8, Prop. 1.5]), if ω is in $\Omega^d_{\mathscr{X}_K}(\mathscr{X}_K)$, we write $\omega = \varpi^{-n}\tilde{\omega}$, with $\tilde{\omega}$ in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ and $n \in \mathbf{N}$, and we set $\mathrm{ord}_{\varpi,\mathscr{X}}(\omega) := \mathrm{ord}_\varpi(\tilde{\omega}) - n$. Clearly, this definition does not depend on the choice of $\tilde{\omega}$.

LEMMA 4.1.1
Let $h : \mathscr{Y} \to \mathscr{X}$ be a morphism between flat formal R-schemes equidimensional of relative dimension d. Let ω be in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ (resp., in $\Omega^d_{\mathscr{X}_K}(\mathscr{X}_K)$). Let y be a point in $\mathrm{Gr}(\mathscr{Y}) \setminus \mathrm{Gr}(\mathscr{Y}_{\mathrm{sing}})$, and assume that $h(y)$ belongs to $\mathrm{Gr}(\mathscr{X}) \setminus \mathrm{Gr}(\mathscr{X}_{\mathrm{sing}})$. Then

$$\mathrm{ord}_\varpi(h^*\omega)(y) = \mathrm{ord}_\varpi(\omega)\big(h(y)\big) + \mathrm{ord}_\varpi(\mathrm{Jac}_h)(y),$$

respectively,

$$\mathrm{ord}_{\varpi,\mathscr{Y}}(h_K^*\omega)(y) = \mathrm{ord}_{\varpi,\mathscr{X}}(\omega)\big(h(y)\big) + \mathrm{ord}_\varpi(\mathrm{Jac}_h)(y).$$

Proof

The proof follows directly from the definitions. □

THEOREM-DEFINITION 4.1.2

Let X be a smooth rigid variety over K of pure dimension d. Let ω be a differential form in $\Omega^d_X(X)$.

(1) *Let $\mathscr{X}$ be a formal R-model of X. Then the function $\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)$ is exponentially integrable on $\operatorname{Gr}(\mathscr{X})$ and the integral $\int_{\operatorname{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\mu$ in $\widehat{K_0(\mathrm{Var}_k)}$ does not depend on the model $\mathscr{X}$. We denote it by $\int_X \omega\, d\mu$.*

(2) *Assume furthermore that ω is a gauge form, that is, that it generates Ω^d_X at every point of X, and assume that some open dense formal subscheme $\mathscr{U}$ of $\mathscr{X}$ is a weak Néron model of X. Then the function $\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)$ takes only a finite number of values and its fibres are stable cylinders. Furthermore, the integral $\int_{\operatorname{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\tilde{\mu}$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ does not depend on the model $\mathscr{X}$. We denote it by $\int_X \omega\, d\tilde{\mu}$.*

Proof

Let us prove (2). Write $\omega = \varpi^{-n}\tilde{\omega}$ with $\tilde{\omega}$ in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ and $n \in \mathbf{N}$. Since $\mathscr{U}$ is smooth, $\Omega^d_{\mathscr{U}|R}$ is locally free of rank 1 and $\tilde{\omega}\mathscr{O}_U \otimes (\Omega^d_{\mathscr{U}|R})^{-1}$ is isomorphic to a principal ideal sheaf $(f)\mathscr{O}_U$ with f in $\mathscr{O}_U$. Furthermore, the function $\operatorname{ord}_{\varpi,\mathscr{X}}(\tilde{\omega})$ coincides with the function $\operatorname{ord}_{\varpi}(f)$ which to a point φ of $\operatorname{Gr}(\mathscr{U}) = \operatorname{Gr}(\mathscr{X})$ associates $\operatorname{ord}_{\varpi}(f(\varphi))$. The fibres of $\operatorname{ord}_{\varpi}(f)$ are stable cylinders. Since ω is a gauge form, f induces an invertible function on X; hence, by the maximum principle (see [4]), the function $\operatorname{ord}_{\varpi}(f)$ takes only a finite number of values. To prove that $\int_{\operatorname{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\tilde{\mu}$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ does not depend on the model $\mathscr{X}$, it is enough to consider the case of another model $\mathscr{X}'$ obtained from $\mathscr{X}$ by an admissible formal blow-up $h : \mathscr{X}' \to \mathscr{X}$. We may also assume that $\mathscr{X}'$ contains as an open dense formal subscheme a weak Néron model $\mathscr{U}'$ of X. The equality

$$\int_{\operatorname{Gr}(\mathscr{X}')} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}'}(\omega)}\, d\tilde{\mu} = \int_{\operatorname{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\tilde{\mu}$$

then follows from Lemma 4.1.1 and Theorem 3.9.4. Statement (1) follows similarly from Lemma 4.1.1 and Theorem 3.9.3. □

Remark 4.1.3

A situation where gauge forms naturally occur is that of reductive groups. Let G be a connected reductive group over k. B. Gross constructs in [15], using Bruhat-Tits theory, a differential form of top degree ω_G on G which is defined up to multiplication by a unit in R. One may easily check that the differential form ω_G induces a gauge form on the rigid K-group $G^{\mathrm{rig}} := (G\widehat{\otimes}R)_K$.

LEMMA 4.1.4

Let X be a smooth rigid variety over K of pure dimension d, and let ω be a gauge form on X. Let $\mathscr{O} = (O_i)_{i\in J}$ be a finite admissible covering, and set $O_I := \bigcap_{i\in I} O_i$ for $I \subset J$. Then

$$\int_X \omega\, d\tilde{\mu} = \sum_{\emptyset\neq I\subset J} (-1)^{|I|-1} \int_{O_I} \omega_{|O_I}\, d\tilde{\mu}.$$

If ω is assumed to be a differential form only in $\Omega^d_X(X)$, then

$$\int_X \omega\, d\mu = \sum_{\emptyset\neq I\subset J} (-1)^{|I|-1} \int_{O_I} \omega_{|O_I}\, d\mu.$$

Proof

Let us prove the first statement, the proof of the second one being similar. It is enough to consider the case of $|J| = 2$. Choose an R-model $\mathscr{X}$ containing a weak Néron model $\mathscr{U}$ of X as an open dense formal subscheme and such that the covering $X = O_1 \cup O_2$ is induced from a covering $\mathscr{X} = \mathscr{O}_1 \cup \mathscr{O}_2$ by open formal subschemes. It is sufficient to prove that

$$\int_{\mathrm{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\tilde{\mu} = \int_{\mathrm{Gr}(\mathscr{O}_1)} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{O}_1}(\omega_{|O_1})}\, d\tilde{\mu} + \int_{\mathrm{Gr}(\mathscr{O}_2)} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{O}_2}(\omega_{|O_2})}\, d\tilde{\mu} - \int_{\mathrm{Gr}(\mathscr{O}_1\cap\mathscr{O}_2)} \mathbf{L}^{-\operatorname{ord}_{\varpi,\mathscr{O}_1\cap\mathscr{O}_2}(\omega_{|O_1\cap O_2})}\, d\tilde{\mu},$$

which follows from the fact that for every open formal subscheme $\mathscr{O}$ of $\mathscr{X}$ the function $\operatorname{ord}_{\varpi,\mathscr{X}}(\omega)$ restricts to $\operatorname{ord}_{\varpi,\mathscr{O}}(\omega_{|\mathscr{O}_K})$ on $\mathrm{Gr}(\mathscr{O})$ and the equalities $\mathrm{Gr}(\mathscr{X}) = \mathrm{Gr}(\mathscr{O}_1) \cup \mathrm{Gr}(\mathscr{O}_2)$ and $\mathrm{Gr}(\mathscr{O}_1) \cap \mathrm{Gr}(\mathscr{O}_2) = \mathrm{Gr}(\mathscr{O}_1 \cap \mathscr{O}_2)$, which follow from Proposition 2.3.2. □

PROPOSITION 4.1.5

Let X and X' be smooth rigid K-varieties of pure dimension d and d', and let ω and ω' be gauge forms on X and X'. Then

$$\int_{X\times X'} \omega\times\omega'\, d\tilde{\mu} = \int_X \omega\, d\tilde{\mu} \times \int_{X'} \omega'\, d\tilde{\mu}.$$

If ω and ω' are assumed to be differential forms only in $\Omega^d_X(X)$, then

$$\int_{X\times X'} \omega\times\omega'\, d\mu = \int_X \omega\, d\mu \times \int_{X'} \omega'\, d\mu.$$

Proof
Let us prove the first assertion, the proof of the second one being similar. Choose R-models $\mathscr{X}$ and $\mathscr{X}'$ of X and X', respectively, containing a weak Néron model $\mathscr{U}$ of X and $\mathscr{U}'$ of X' as an open dense formal subscheme. Also, write $\omega = \varpi^{-n}\tilde{\omega}$ and $\omega' = \varpi^{-n'}\tilde{\omega}'$, with $\tilde{\omega}$ and $\tilde{\omega}'$ in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ and $\Omega^{d'}_{\mathscr{X}'|R}(\mathscr{X}')$, respectively. It is enough to check that $\tilde{\mu}(\mathrm{ord}_{\varpi,\mathscr{X}\times\mathscr{X}'}(\tilde{\omega}\times\tilde{\omega}') = m)$ is equal to $\sum_{m'+m''=m}\tilde{\mu}(\mathrm{ord}_{\varpi,\mathscr{X}}(\tilde{\omega}) = m')\times\tilde{\mu}(\mathrm{ord}_{\varpi,\mathscr{X}'}(\tilde{\omega}') = m'')$, which follows from the fact that on $\mathrm{Gr}(\mathscr{X}\times\mathscr{X}') \simeq \mathrm{Gr}(\mathscr{X})\times\mathrm{Gr}(\mathscr{X}') = \mathrm{Gr}(\mathscr{U})\times\mathrm{Gr}(\mathscr{U}')$, the functions $\mathrm{ord}_{\varpi,\mathscr{X}\times\mathscr{X}'}(\tilde{\omega}\times\tilde{\omega}')$ and $\mathrm{ord}_{\varpi,\mathscr{X}}(\tilde{\omega}) + \mathrm{ord}_{\varpi,\mathscr{X}'}(\tilde{\omega}')$ are equal. □

4.2. Invariants for gauged smooth rigid varieties
Let d be an integer greater than or equal to zero. We define $K_0(\mathrm{GSRig}^d_K)$, the Grothendieck group of gauged smooth rigid K-varieties of dimension d, as follows: as an abelian group it is the quotient of the free abelian group over symbols $[X,\omega]$ with X a smooth rigid K-variety of dimension d and ω a gauge form on X by the relations

$$[X',\omega'] = [X,\omega]$$

if there is an isomorphism $h : X' \to X$ with $h^*\omega = \omega'$, and

$$[X,\omega] = \sum_{\emptyset\neq I\subset J}(-1)^{|I|-1}[O_I,\omega_{|O_I}],$$

when $(O_i)_{i\in J}$ is a finite admissible covering of X, with the notation $O_I := \bigcap_{i\in I} O_i$ for $I \subset J$. One puts a graded ring structure on $K_0(\mathrm{GSRig}_K) := \bigoplus_d K_0(\mathrm{GSRig}^d_K)$ by requiring that

$$[X,\omega]\times[X',\omega'] := [X\times X',\omega\times\omega'].$$

Forgetting gauge forms, one defines similarly $K_0(\mathrm{SRig}^d_K)$, the Grothendieck group of smooth rigid K-varieties of dimension d, and the graded ring $K_0(\mathrm{SRig}_K) := \bigoplus_d K_0(\mathrm{SRig}^d_K)$. There are natural forgetful morphisms

$$F : K_0(\mathrm{GSRig}^d_K) \longrightarrow K_0(\mathrm{SRig}^d_K)$$

and

$$F : K_0(\mathrm{GSRig}_K) \longrightarrow K_0(\mathrm{SRig}_K).$$

PROPOSITION 4.2.1
There is a unique ring morphism

$$\tilde{\mu} : K_0(\mathrm{GSRig}_K) \to K_0(\mathrm{Var}_k)_{\mathrm{loc}}$$

which assigns to the class of a gauged smooth rigid K-variety (X,ω) the integral $\int_X \omega\, d\tilde{\mu}$.

Proof
This follows from Lemma 4.1.4 and Proposition 4.1.5. □

4.3. A formula for $\int_X \omega\, d\tilde{\mu}$

Let X be a smooth rigid variety over K of pure dimension d. Let $\mathscr{U}$ be a weak Néron model of X contained in some model $\mathscr{X}$ of X, and let ω be a form in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ inducing a gauge form on X. We denote by U_0^i, $i \in J$, the irreducible components of the special fibre of $\mathscr{U}$. By assumption, each U_0^i is smooth and $U_0^i \cap U_0^j = \emptyset$ for $i \neq j$. We denote by $\mathrm{ord}_{U_0^i}(\omega)$ the unique integer n such that $\varpi^{-n}\omega$ generates $\Omega^d_{\mathscr{X}|R}$ at the generic point of U_0^i. More generally, if ω is a gauge form in $\Omega^d_{\mathscr{X}_K}(\mathscr{X}_K)$, we write $\omega = \varpi^{-n}\tilde{\omega}$ with $\tilde{\omega}$ in $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ and $n \in \mathbf{N}$, and we set $\mathrm{ord}_{U_0^i}(\omega) := \mathrm{ord}_{U_0^i}(\tilde{\omega}) - n$.

PROPOSITION 4.3.1
Let X be a smooth rigid variety over K of pure dimension d. Let $\mathscr{U}$ be a weak Néron model of X contained in some model $\mathscr{X}$ of X, and let ω be a gauge form in $\Omega^d_{\mathscr{X}_K}(\mathscr{X}_K)$. With the above notation, we have

$$\int_X \omega\, d\tilde{\mu} = \mathbf{L}^{-d} \sum_{i \in J} [U_0^i]\, \mathbf{L}^{-\mathrm{ord}_{U_0^i}(\omega)}$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

Proof
Denote by $\mathscr{U}_0^i$ the irreducible component of $\mathscr{X}$ with special fibre U_0^i. Since $\mathrm{Gr}(\mathscr{X})$ is the disjoint union of the sets $\mathrm{Gr}(\mathscr{U}_0^i)$, we may assume that $\mathscr{X}$ is a smooth irreducible formal R-scheme of dimension d. Let ω be a section of $\Omega^d_{\mathscr{X}|R}(\mathscr{X})$ which generates $\Omega^d_{\mathscr{X}|R}$ at the generic point of $\mathscr{X}$ and induces a gauge form on the generic fibre. Let us note that the function $\mathrm{ord}_{\varpi,\mathscr{X}}(\omega)$ is identically equal to zero on $\mathrm{Gr}(\mathscr{X})$. Indeed, after shrinking $\mathscr{X}$, we may write $\omega = f\omega_0$ with ω_0 a generator $\Omega^d_{\mathscr{X}|R}$ at every point and f in $\mathscr{O}_{\mathscr{X}}(\mathscr{X})$. By hypothesis, f is a unit at the generic point of $\mathscr{X}$. Assume that, at some point x of $\mathrm{Gr}(\mathscr{X})$, $\mathrm{ord}_{\varpi} f(x) \geq 1$; it would follow that the locus of $f = 0$ is nonempty in $\mathscr{X}$, which contradicts the assumption that ω induces a gauge form on the generic fibre. Hence we get $\int_{\mathrm{Gr}(\mathscr{X})} \mathbf{L}^{-\mathrm{ord}_{\varpi,\mathscr{X}}(\omega)}\, d\tilde{\mu} = \mathbf{L}^{-d}[X_0]$, and the result follows. □

4.4. Application to Calabi-Yau varieties over K

Let X be a Calabi-Yau variety over K. By this we mean a smooth projective algebraic variety over K of pure dimension d with Ω^d_X trivial. We denote by X^{an} the rigid K-variety associated to X. Since X is proper, X^{an} is canonically isomorphic to the generic fibre of the formal completion of any proper R-model of X. In particular, X^{an} is smooth. By GAGA (see [21, Th. 2.8]), $\Omega^d_{X^{\mathrm{an}}}(X^{\mathrm{an}}) = \Omega^d_X(X) = K$.

Now we can associate to any Calabi-Yau variety over K a canonical element in the ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ which coincides with the class of the special fibre when X has a model with good reduction.

THEOREM 4.4.1

Let X be a Calabi-Yau variety over K, let $\mathscr{U}$ be a weak Néron model of X^{an}, and let ω be a gauge form on X^{an}. We denote by U_0^i, $i \in J$, the irreducible components of the special fibre of $\mathscr{U}$, and we set $\alpha(\omega) := \inf \mathrm{ord}_{U_0^i}(\omega)$. Then the virtual variety

$$[\overline{X}] := \sum_{i \in J} [U_0^i]\, \mathbf{L}^{\alpha(\omega) - \mathrm{ord}_{U_0^i}(\omega)} \tag{4.4.1}$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ depends only on X. When X has a proper smooth model with good reduction over R, $[\overline{X}]$ is equal to the class of the special fibre.

Proof

Let ω be a gauge form on X^{an}. By Proposition 4.3.1, the right-hand side of (4.4.1) is equal to $\mathbf{L}^{d+\alpha(\omega)} \int_{X^{an}} \omega \, d\tilde{\mu}$, which does not depend on ω. □

In particular, we have the following analogue of Batyrev's result on birational projective Calabi-Yau manifolds (see [2], [10]).

COROLLARY 4.4.2

Let X be a Calabi-Yau variety over K, and let $\mathscr{X}$ and $\mathscr{X}'$ be two proper and smooth R-models of X with special fibres $\mathscr{X}_0$ and $\mathscr{X}_0'$. Then

$$[\mathscr{X}_0] = [\mathscr{X}_0']$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

Remark 4.4.3

Calabi-Yau varieties over $k((t))$ with k of characteristic zero were considered in [19].

4.5. A motivic Serre invariant for smooth rigid varieties

We can now define the motivic Serre invariant for smooth rigid varieties.

THEOREM 4.5.1

There is a canonical ring morphism

$$S : K_0(\mathrm{SRig}_K) \longrightarrow K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$$

such that the diagram

$$\begin{array}{ccc} K_0(\mathrm{GSRig}_K) & \xrightarrow{\tilde{\mu}} & K_0(\mathrm{Var}_k)_{\mathrm{loc}} \\ \big\downarrow F & & \big\downarrow \\ K_0(\mathrm{SRig}_K) & \xrightarrow{S} & K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \end{array}$$

is commutative.

Proof

Since any smooth rigid K-variety of dimension d admits a finite admissible covering by affinoids $(O_i)_{i\in J}$, with $\Omega^d_{O_i}$ trivial, the morphism F is surjective. Hence it is enough to show the following statement: let $\mathscr{X}$ be a smooth formal R-scheme of relative dimension d with $\Omega^d_{\mathscr{X}|R}$ trivial, and let ω_1 and ω_2 be two global sections of $\Omega^d_{\mathscr{X}|R}$ inducing gauge forms on the generic fibre $\mathscr{X}_K$; then $\int_{\mathrm{Gr}(\mathscr{X})}(\mathbf{L}^{-\mathrm{ord}_{\varpi,\mathscr{X}}(\omega_1)} - \mathbf{L}^{-\mathrm{ord}_{\varpi,\mathscr{X}}(\omega_2)})\,d\tilde{\mu}$ belongs to $(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$. To prove this, we take ω_0 a global section of $\Omega^d_{\mathscr{X}|R}$ such that $\Omega^d_{\mathscr{X}|R} \simeq \omega_0\mathscr{O}_{\mathscr{X}}$. If ω is any global section of $\Omega^d_{\mathscr{X}|R}$, write $\omega = f\omega_0$ with f in $\mathscr{O}_{\mathscr{X}}(\mathscr{X})$. By the maximum principle, the function $\mathrm{ord}_{\varpi}(f)$ takes only a finite number of values on $\mathrm{Gr}(\mathscr{X})$. It follows that we may write $\mathrm{Gr}(\mathscr{X})$ as a disjoint union of the subsets $\mathrm{Gr}(\mathscr{X})_{\mathrm{ord}_{\varpi}(f)=n}$, where $\mathrm{ord}_{\varpi}(f)$ takes the value n. These subsets are stable cylinders, and only a finite number of them are nonempty. Hence the equality

$$\int_{\mathrm{Gr}(\mathscr{X})}(\mathbf{L}^{-\mathrm{ord}_{\varpi,\mathscr{X}}(\omega)} - \mathbf{L}^{-\mathrm{ord}_{\varpi,\mathscr{X}}(\omega_0)})\,d\tilde{\mu} = \sum_n (\mathbf{L}^{-n}-1)\tilde{\mu}\big(\mathrm{Gr}(\mathscr{X})_{\mathrm{ord}_{\varpi}(f)=n}\big)$$

holds in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$, and the statement follows. □

Remark 4.5.2

The ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ is much smaller than the ring $K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ but still quite large. Let ℓ be a prime number distinct from the characteristic of k. Then the étale ℓ-adic Euler characteristic with compact supports

$$X \longmapsto \chi_{c,\ell}(X) := \sum (-1)^i \dim H^i_{c,\text{ét}}(X, \mathbf{Q}_\ell)$$

induces a ring morphism $\chi_{c,\ell} : K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}$.

Similarly, assume that there is a natural morphism $H : K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}[u,v]$ which to the class of a variety X over k assigns its Hodge polynomial $H(X)$ for de Rham cohomology with compact support. Such a morphism is known to exist when k is of characteristic zero. Then if one sets $H_{1/2}(X)(u) := H(X)(u, u^{-1})$, one gets a morphism $H_{1/2} : K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}[u]$ since $H(\mathbf{A}^1_k) = uv$.

Now we can give the following formula for the motivic Serre invariant S in terms of a weak Néron model.

THEOREM 4.5.3
Let X be a smooth rigid variety over K of pure dimension d. Let $\mathcal{U}$ be a weak Néron model of X, and denote by U_0 its special fibre. Then

$$S([X]) = [U_0]$$

in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$.

In particular, the class of $[U_0]$ in $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}}$ does not depend on the weak Néron model $\mathcal{U}$.

Proof
By taking an appropriate admissible cover, we may assume that there exists a gauge form on X, in which case the result follows from Proposition 4.3.1 since $[U_0] = \sum_{i\in J}[U_0^i]$. (In fact, one can also prove Th. 4.5.1 that way, but we preferred to give a proof that is quite parallel to that of Serre in [26].) □

4.6. Relation with p-adic integrals on compact locally analytic varieties
Let K be a local field with residue field $k = \mathbf{F}_q$. Let us consider the Grothendieck group $K_0(\mathrm{SLocAn}_K^d)$ of compact locally analytic smooth varieties over K of pure dimension d, which is defined similarly to $K_0(\mathrm{SRig}_K^d)$, replacing smooth rigid varieties by compact locally analytic smooth varieties and finite admissible covers by finite covers. Also, a nowhere vanishing locally analytic d-form on a smooth compact locally analytic variety X of pure dimension d is called a *gauge form* on X, and one defines the Grothendieck group $K_0(\mathrm{GSLocAn}_K^d)$ of gauged compact locally analytic smooth varieties over K of pure dimension d similarly to $K_0(\mathrm{GSRig}_K^d)$. There are canonical forgetful morphisms $F : K_0(\mathrm{SRig}_K^d) \to K_0(\mathrm{SLocAn}_K^d)$ and $F : K_0(\mathrm{GSRig}_K^d) \to K_0(\mathrm{GSLocAn}_K^d)$ induced from the functor that to a rigid variety (resp., gauged variety) associates the underlying locally analytic variety (resp., gauged variety). If (X, ω) is a gauged compact locally analytic smooth variety, the p-adic integral $\int_X |\omega|$ belongs to $\mathbf{Z}[q^{-1}]$ (see [26]), and by additivity of p-adic integrals, one gets a morphism $\mathrm{int}_p : K_0(\mathrm{GSLocAn}_K^d) \to \mathbf{Z}[q^{-1}]$.

On the other hand, there is a canonical morphism $N : K_0(\mathrm{Var}_k) \to \mathbf{Z}$ which to the class of a k-variety S assigns the number of points of $S(\mathbf{F}_q)$, and which induces a morphism $N : K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}[q^{-1}]$. We also denote by N the induced morphism $K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \to \mathbf{Z}[q^{-1}]/(q-1)\mathbf{Z}[q^{-1}] \simeq \mathbf{Z}/(q-1)\mathbf{Z}$.

PROPOSITION 4.6.1
Let K be a local field with residue field $k = \mathbf{F}_q$. Then the diagram

$$\begin{array}{ccc} K_0(\mathrm{GSRig}_K^d) & \xrightarrow{\tilde{\mu}} & K_0(\mathrm{Var}_k)_{\mathrm{loc}} \\ \downarrow F & & \downarrow N \\ K_0(\mathrm{GSLocAn}_K^d) & \xrightarrow{\mathrm{int}_p} & \mathbf{Z}[q^{-1}] \end{array}$$

is commutative.

Proof
One reduces to showing the following: let $\mathscr{X}$ be a smooth formal R-scheme of dimension d, and let f be a function in $\mathscr{O}_{\mathscr{X}}(\mathscr{X})$ which induces a nonvanishing function on $\mathscr{X}_K$; then

$$N\Big(\int_{\mathrm{Gr}(\mathscr{X})} \mathbf{L}^{-\operatorname{ord}_\varpi(f)}\, d\tilde{\mu}\Big) = \int_{\mathscr{X}(R)} q^{-\operatorname{ord}_\varpi(f)}\, d\tilde{\mu}_p$$

with $d\tilde{\mu}_p$ the p-adic measure on $\mathscr{X}(R)$. It is enough to check that $N\big(\tilde{\mu}(\operatorname{ord}_\varpi(f) = n)\big)$ is equal to the p-adic measure of the set of points x of $\mathscr{X}(R)$ with $\operatorname{ord}_\varpi(f)(x) = n$, which follows from Lemma 4.6.2. □

LEMMA 4.6.2
Let K be a local field with residue field $k = \mathbf{F}_q$. Let $\mathscr{X}$ be a smooth formal R-scheme of dimension d. Let A be a (stable) cylinder in $\mathrm{Gr}(\mathscr{X})$. Then $N(\tilde{\mu}(A))$ is equal to the p-adic volume of $A \cap \mathrm{Gr}(\mathscr{X})(k)$.

Proof
Write $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathrm{Gr}_n(\mathscr{X})$. By definition, $\tilde{\mu}(A) = \mathbf{L}^{-d(n+1)}[C]$. On the other hand, $\mathscr{X}$ being smooth, the morphism $A \cap \mathrm{Gr}(\mathscr{X})(k) \to C(k)$ is surjective and its fibres are balls of radius $q^{-d(n+1)}$. It follows that the p-adic volume of $A \cap \mathrm{Gr}(\mathscr{X})(k)$ is equal to $|C(k)| q^{-d(n+1)}$. □

Let us now explain the relation with the work of Serre in [26]. Serre shows in [26] that any compact locally analytic smooth variety over K of pure dimension d is isomorphic to $r B^d$ with r an integer greater than or equal to 1 and B^d the unit ball of dimension d and that, furthermore, $r B^d$ is isomorphic to $r' B^d$ if and only if r and r' are congruent modulo $q-1$. We denote by $s(X)$ the class of r in $\mathbf{Z}/(q-1)\mathbf{Z}$. It follows from Serre's results that s induces an isomorphism $s : K_0(\mathrm{SLocAn}_K^d) \to \mathbf{Z}/(q-1)\mathbf{Z}$

and that the diagram

$$\begin{array}{ccc} K_0(\mathrm{GSLocAn}_K^d) & \xrightarrow{\mathrm{int}_p} & \mathbf{Z}[q^{-1}] \\ \downarrow & & \downarrow \\ K_0(\mathrm{SLocAn}_K^d) & \xrightarrow{s} & \mathbf{Z}/(q-1)\mathbf{Z} \end{array}$$

is commutative. The following result then follows from Proposition 4.6.1.

COROLLARY 4.6.3
Let K be a local field with residue field $k = \mathbf{F}_q$. Then the diagram

$$\begin{array}{ccc} K_0(\mathrm{SRig}_K^d) & \xrightarrow{S} & K_0(\mathrm{Var}_k)_{\mathrm{loc}}/(\mathbf{L}-1)K_0(\mathrm{Var}_k)_{\mathrm{loc}} \\ \downarrow F & & \downarrow N \\ K_0(\mathrm{SLocAn}_K^d) & \xrightarrow{s} & \mathbf{Z}/(q-1)\mathbf{Z} \end{array}$$

is commutative.

5. Essential components of weak Néron models

5.1. Essential components and the Nash problem

Since we proceed by analogy with [22], let us begin by recalling some material from that paper. We assume in this subsection that k is of characteristic zero and that $R = k[[\varpi]]$. For X an algebraic variety over k, we denote by $\mathscr{L}(X)$ its arc space as defined in [10]. In fact, in the present section, we use notation and results from [10], even when they happen to be special cases of ones in this paper. As noted in Remark 2.3.3, $\mathscr{L}(X) = \mathrm{Gr}(X \widehat{\otimes} R)$, and there are natural truncation morphisms $\pi_n : \mathscr{L}(X) \to \mathscr{L}_n(X)$ with $\mathscr{L}_n(X) = \mathrm{Gr}_n(X \otimes R_n)$.

By a desingularization of a variety X, we mean a proper and birational morphism

$$h : Y \longrightarrow X,$$

with Y a smooth variety, inducing an isomorphism between $h^{-1}(X \setminus X_{\mathrm{sing}})$ and $X \setminus X_{\mathrm{sing}}$. (Some authors omit the last condition.)

Let $h : Y \to X$ be a desingularization of X, and let D be an irreducible component of $h^{-1}(X_{\mathrm{sing}})$ of codimension 1 in Y. If $h' : Y' \to X$ is another desingularization of X, the birational map $\phi : h'^{-1} \circ h : Y \dashrightarrow Y'$ is defined at the generic point ξ of D since h' is proper; hence we can define the image of D in Y' as the closure of $\phi(\xi)$ in Y'. One says that D is an essential divisor with respect to X if, for every desingularization $h' : Y' \to X$ of X, the image of D in Y' is a divisor and that D is an essential component with respect to X if, for every desingularization $h' : Y' \to X$ of X, the image of D in Y' is an irreducible component of $h'^{-1}(X_{\text{sing}})$. In general, if D is an irreducible component of $h^{-1}(X_{\text{sing}})$, we say that D is an essential component with respect to X if there exists a proper birational morphism $p : Y' \to Y$, with Y' smooth, and a divisor D' in Y' such that D' is an essential component with respect to X and $p(D') = D$. It follows from the definitions and Hironaka's theorem that essential components of different resolutions of the same variety X are in natural bijection; hence we may denote by $\tau(X)$ the number of essential components in any resolution of X.

Let W be a constructible subset of an algebraic variety Z. We say that W is irreducible in Z if the Zariski closure $\overline{W}$ of W in Z is irreducible.

In general, let $\overline{W} = \bigcup_{1 \le i \le n} W'_i$ be the decomposition of $\overline{W}$ into irreducible components. Clearly, $W_i := W'_i \cap W$ is nonempty, irreducible in Z, and its closure in Z is equal to W'_i. We call the W_i's the *irreducible components* of W in Z.

Let E be a locally closed subset of $h^{-1}(X_{\text{sing}})$. We denote by Z_E the set of arcs in $\mathscr{L}(Y)$ whose origin lies on E but which are not contained in E. In other words, $Z_E = \pi_0^{-1}(E) \setminus \mathscr{L}(E)$. Let us note that if E is smooth and connected, $\pi_n(Z_E)$ is constructible and irreducible in $\mathscr{L}_n(Y)$. Now we set $N_E := h(Z_E)$. Since $\pi_n(N_E)$ is the image of $\pi_n(Z_E)$ under the morphism $\mathscr{L}_n(Y) \to \mathscr{L}_n(X)$ induced by h, it follows that $\pi_n(N_E)$ is constructible and irreducible in $\mathscr{L}_n(Y)$.

The following result, proved in [22], follows easily from the above remarks and Hironaka's resolution of singularities.

PROPOSITION 5.1.1 (see J. Nash [22])
Let X be an algebraic variety over k, a field of characteristic zero. Set $\mathscr{N}(X) := \pi_0^{-1}(X_{\text{sing}}) \setminus \mathscr{L}(X_{\text{sing}})$. For every $n \ge 0$, $\pi_n(\mathscr{N}(X))$ is a constructible subset of $\mathscr{L}_n(X)$. Denote by $W_n^1, \ldots, W_n^{r(n)}$ the irreducible components of $\pi_n(\mathscr{N}(X))$. The mapping $n \mapsto r(n)$ is nondecreasing and bounded by the number $\tau(X)$ of essential components occurring in a resolution of X.

Up to renumbering, we may assume that W_{n+1}^i maps to $\overline{W_n^i}$ for $n \gg 0$. Let us call the family $(W_n^i)_{n \gg 0}$ a *Nash family*. Nash shows, furthermore, that for every Nash family $(W_n^i)_{n \gg 0}$ there exists a unique essential component E in a given resolution $h : Y \to X$ of X such that $\overline{\pi_n(N_E)} = \overline{W_n^i}$ for $n \gg 0$.

Now, we can formulate the Nash problem.

PROBLEM 5.1.2 (Nash [22, p. 36])
Is there always a corresponding Nash family for an essential component? In general,

how completely do the essential components correspond to Nash families? What is the relation between $\tau(X)$ and $\lim r(n)$?

Recently, Ishii and Kollár [17] proved that the correspondence between essential components and Nash families is one-to-one for toric singularities but fails in general. In fact, they showed that the 4-dimensional hypersurface singularity $x_1^3+x_2^3+x_3^3+x_4^3+x_5^6=0$ has two essential components and only one Nash family.

Let W be a constructible subset of some variety X. We denote the supremum of the dimension of the irreducible components of the closure of W in Y by $\dim W$.

Let $h : Y \to X$ be a proper birational morphism with Y a smooth variety. Let E be a codimension 1 irreducible component of the exceptional locus of h in Y. We denote by $\nu(E)-1$ the length of $\Omega_Y^d / h^*\Omega_X^d$ at the generic point of E. Here Ω_X^d denotes the dth exterior power of the sheaf Ω_X^1 of differentials on X.

PROPOSITION 5.1.3
Let X be a variety of pure dimension d over k, a field of characteristic zero. Let $h : Y \to X$ be a proper birational morphism with Y a smooth variety, and let U be a nonempty open subset of a codimension 1 irreducible component E of the exceptional locus of h in Y. Then

$$\dim \pi_n(N_U) = (n+1)\, d - \nu(E)$$

for $n \gg 0$.

Proof
By [10, Th. 6.1], the image of $[\pi_n(N_U)]\mathbf{L}^{-(n+1)d}$ in $\widehat{K_0(\mathrm{Var}_k)}$ converges to $\mu(N_U)$ in $\widehat{K_0(\mathrm{Var}_k)}$. Since $\dim \pi_n(N_U) \leq (n+1)d$ by [10, Lem. 4.3], one deduces the fact that $\dim \pi_n(N_U) - (n+1)\, d$ has a limit. To conclude, we first note that $\overline{\pi_n(N_U)} = \overline{\pi_n(N_E)}$ for any nonempty open subset U in E. Hence we may assume that $(h^*\Omega_X^d)/\text{torsion}$ is locally free on a neighborhood of U. It then follows from [10, Prop. 6.3.2], or rather from its proof, that

$$\mu(N_U) = \mathbf{L}^{-d}[U](\mathbf{L}-1)\sum_{\ell\geq 1}\mathbf{L}^{-\ell\nu(E)}$$

in $\widehat{K_0(\mathrm{Var}_k)}$. Hence $\mu(N_U)$ belongs to $F^{\nu(E)}$ and not to $F^{\nu(E)+1}$, and the result follows. □

5.2. Essential components of weak Néron models
We return now to the setting of the present paper. We fix a flat formal R-scheme $\mathscr{X}$ of relative dimension d with smooth generic fibre $\mathscr{X}_K$. By a weak Néron model of

$\mathscr{X}$, we mean a weak Néron model $\mathscr{U}$ of $\mathscr{X}_K$ together with a morphism $h : \mathscr{U} \to \mathscr{X}$ inducing the inclusion $\mathscr{U}_K \hookrightarrow \mathscr{X}_K$. As before, we denote by U_0^i, $i \in J$, the irreducible components of the special fibre of $\mathscr{U}$. Let ξ^i denote the generic point of U_0^i. We say U_0^i is an essential component with respect to $\mathscr{X}$ if, for every weak Néron model $\mathscr{U}'$ of $\mathscr{X}$, the Zariski closure of $\pi_{0,\mathscr{U}'}(\pi_{0,\mathscr{U}}^{-1}(\xi^i))$ is an irreducible component of the special fibre of $\mathscr{U}'$. Note that being an essential component is a property relative to $\mathscr{X}$. By their very definition, essential components in different weak Néron models of $\mathscr{X}$ are in natural bijection.

We have the following analogue of Proposition 5.1.1.

PROPOSITION 5.2.1
Let $\mathscr{X}$ be a flat formal R-scheme of relative dimension d with smooth generic fibre $\mathscr{X}_K$. Denote by $W_n^1, \ldots, W_n^{r(n)}$ the irreducible components of the constructible subset $\pi_n(\mathrm{Gr}(\mathscr{X}))$ of $\mathrm{Gr}_n(\mathscr{X})$. The mapping $n \mapsto r(n)$ is nondecreasing and bounded by the number $\tau(\mathscr{X})$ of essential components occurring in a weak Néron model of $\mathscr{X}$.

Proof
Clearly, the mapping $n \mapsto r(n)$ is nondecreasing. Let $h : \mathscr{U} \to \mathscr{X}$ be a weak Néron model of $\mathscr{X}$ with irreducible components $\mathscr{U}^i$, $i \in J$. Since $\mathscr{U}^i$ is smooth and irreducible, $\pi_{n,\mathscr{U}}(\mathrm{Gr}(\mathscr{U}^i))$ is also smooth and irreducible; hence the Zariski closure of $h\big(\pi_{n,\mathscr{U}}(\mathrm{Gr}(\mathscr{U}^i))\big) = \pi_{n,\mathscr{X}}(\mathrm{Gr}(\mathscr{U}^i))$ in $\mathrm{Gr}_n(\mathscr{X})$ is irreducible. Since $\mathrm{Gr}(\mathscr{X})$ is the union of the subschemes $\mathrm{Gr}(\mathscr{U}^i)$, it follows that $r(n)$ is bounded by $|J|$. Now if $\mathscr{U}^i$ is not an essential component, there exists some weak Néron model of $\mathscr{X}$, $h' : \mathscr{U}' \to \mathscr{X}$, such that, if we denote by W^i the image of $\mathrm{Gr}(\mathscr{U}^i)$ in $\mathrm{Gr}(\mathscr{U}')$, $\pi_{n,\mathscr{U}'}(W^i)$ is contained in the Zariski closure of $\pi_{n,\mathscr{U}'}(\mathrm{Gr}(\mathscr{U}') \setminus W^i)$. It follows that $\pi_{n,\mathscr{X}}(\mathrm{Gr}(\mathscr{U}^i))$ is contained in the closure of $\pi_{n,\mathscr{X}}(\mathrm{Gr}(\mathscr{U}) \setminus \mathrm{Gr}(\mathscr{U}^i))$. The bound $r(n) \leq \tau(\mathscr{X})$ follows. □

As previously, we may, up to renumbering, assume that W_{n+1}^i maps to $\overline{W_n^i}$ for $n \gg 0$. We still call the family $(W_n^i)_{n\gg 0}$ a Nash family. Let ξ_n^i be the generic point of $\overline{W_n^i}$. By construction, ξ_{n+1}^i maps to ξ_n^i under the truncation morphism $\mathrm{Gr}_{n+1}(\mathscr{X}) \to \mathrm{Gr}_n(\mathscr{X})$; hence to the inverse system $(\xi_n^i)_{n\gg 0}$ corresponds a point ξ^i of $\mathrm{Gr}(\mathscr{X})$. Let $h : \mathscr{U} \to \mathscr{X}$ be weak Néron model of $\mathscr{X}$ with irreducible components $\mathscr{U}^j$, $j \in J$. There is a unique irreducible component $\mathscr{U}^{j(i)}$ such that the point ξ^i belongs to $\mathrm{Gr}(\mathscr{U}^{j(i)})$. Furthermore, $\overline{h\big(\pi_n(\mathrm{Gr}(\mathscr{U}^{j(i)}))\big)} = \overline{W_n^i}$ for $n \gg 0$, and it follows from the proof of Proposition 5.2.1 that $U_0^{j(i)}$ is essential.

We have the following analogue of the Nash problem.

PROBLEM 5.2.2
In general, how completely do the essential components correspond to Nash families? What is the relation between $\tau(\mathscr{X})$ *and* $\lim r(n)$*?*

For E a locally closed subset of the special fibre of $\mathscr{U}$, we set $Z_E := \pi_{0,\mathscr{U}}^{-1}(E)$ and $N_E := h(Z_E)$. Note that, for every n, $\pi_n(Z_E)$ and $\pi_n(N_E)$ are constructible subsets of $\mathrm{Gr}_n(\mathscr{U})$ and $\mathrm{Gr}_n(\mathscr{X})$, respectively. Indeed, $\pi_n(Z_E)$ is constructible since $\mathscr{U}$ is smooth; hence $\pi_n(N_E) = h(\pi_n(Z_E))$ also. We denote by $\nu(U_0^i) - 1$ the length of $\Omega^d_{\mathscr{U}|R}/h^*\Omega^d_{\mathscr{X}|R}$ at the generic point ξ^i of U_0^i.

We also have the following analogue of Proposition 5.1.3.

PROPOSITION 5.2.3
Let $\mathscr{X}$ *be a flat formal* R*-scheme of relative dimension* d *with smooth generic fibre* $\mathscr{X}_K$*. Let* $h : \mathscr{U} \to \mathscr{X}$ *be a weak Néron model of* $\mathscr{X}$*, and let* E *be an open dense subset of an irreducible component* U_0^i *of the special fibre of* $\mathscr{U}$*. Then*

$$\dim \pi_n(N_E) = (n+1)\, d - \nu(U_0^i)$$

for $n \gg 0$.

Proof
Fix an integer $e \geq 0$. By Lemma 3.9.2, for $n \gg e$,

$$\begin{aligned}\dim \pi_n(N_E \cap \mathrm{Gr}^{(e)}\, \mathscr{X}) &= \dim \pi_n\big(h^{-1}(N_E \cap \mathrm{Gr}^{(e)}\, \mathscr{X})\big) - \nu(U_0^i)\\ &= (n+1)\, d - \nu(U_0^i).\end{aligned}$$

On the other hand, it follows from Lemma 3.3.2 that

$$\dim \pi_n\big(N_E \cap (\mathscr{X} \setminus \mathrm{Gr}^{(e)}\, \mathscr{X})\big) < (n+1)\, d - \nu(U_0^i)$$

when $n \gg e \gg \nu(U_0^i)$. □

References

[1] V. V. BATYREV, “Stringy Hodge numbers of varieties with Gorenstein canonical singularities” in *Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997),* World Sci., River Edge, N.J., 1998, 1 – 32. MR 2001a:14039 315

[2] ———, “Birational Calabi-Yau n-folds have equal Betti numbers” in *New Trends in Algebraic Geometry (Warwick, U.K., 1996),* London Math. Soc. Lecture Note Ser. **264**, Cambridge Univ. Press, Cambridge, 1999, 1 – 11. MR 2000i:14059 317, 334

[3] ———, *Non-Archimedian integrals and stringy Euler numbers of log-terminal pairs,* J. Eur. Math. Soc. (JEMS) **1** (1999), 5 – 33. MR 2001j:14018 315

[4] S. BOSCH, U. GÜNTZER, and R. REMMERT, *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, Grundlehren Math. Wiss. **261**, Springer, Berlin, 1984. MR 86b:32031 330

[5] S. BOSCH and W. LÜTKEBOHMERT, *Formal and rigid geometry, I: Rigid spaces*, Math. Ann. **295** (1993), 291 – 317. MR 94a:11090 321

[6] ———, *Formal and rigid geometry, II: Flattening techniques*, Math. Ann. **296** (1993), 403 – 429. MR 94e:11070 320

[7] S. BOSCH, W. LÜTKEBOHMERT, and M. RAYNAUD, *Néron Models*, Ergeb. Math. Grenzgeb. (3) **21**, Springer, Berlin, 1990. MR 91i:14034 316, 318, 320, 321, 322

[8] ———, *Formal and rigid geometry, III: The relative maximum principle*, Math. Ann. **302** (1995), 1 – 29. MR 97e:11074 329

[9] S. BOSCH and K. SCHLÖTER, *Néron models in the setting of formal and rigid geometry*, Math. Ann. **301** (1995), 339 – 362. MR 96h:14035 316, 321, 322

[10] J. DENEF and F. LOESER, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201 – 232. MR 99k:14002 315, 320, 325, 334, 338, 340

[11] ———, "Geometry on arc spaces of algebraic varieties" in *European Congress of Mathematics (Barcelona, 2000), Vol. 1*, Progr. Math. **201**, Birkhäuser, Basel, 2001, 327 – 348. CMP 1 905 328 315

[12] ———, *Motivic integration, quotient singularities and the McKay correspondence*, Compositio Math. **131** (2002), 267 – 290. CMP 1 905 024 315, 327

[13] M. J. GREENBERG, *Schemata over local rings*, Ann. of Math. (2) **73** (1961), 624 – 648. MR 23:A3745 318, 320

[14] ———, *Rational points in Henselian discrete valuation rings*, Inst. Hautes Études Sci. Publ. Math. **31** (1966), 59 – 64. MR 34:7515 321

[15] B. H. GROSS, *On the motive of a reductive group*, Invent. Math. **130** (1997), 287 – 313. MR 98m:20060 330

[16] A. GROTHENDIECK and J. A. DIEUDONNÉ, *Éléments de géométrie algébrique, I*, Grundlehren Math. Wiss. **166**, Springer, Berlin, 1971. 317, 318, 319, 325

[17] S. ISHII and J. KOLLÁR, *The Nash problem on arc families of singularities*, preprint, arXiv:math.AG/0207171 317, 340

[18] M. KONTSEVICH, lecture at Université Paris-Sud, Orsay, December 7, 1995. 315

[19] M. KONTSEVICH and Y. SOIBELMAN, "Homological mirror symmetry and torus fibrations" in *Symplectic Geometry and Mirror Symmetry (Seoul, 2000)*, World Sci., River Edge, N.J., 2001, 203 – 263. MR 2003c:32025 334

[20] E. LOOIJENGA, *Motivic measures*, Astérisque **276** (2002), 267 – 297, Séminaire Bourbaki 1999/2000, exp. 874. CMP 1 886 763

[21] W. LÜTKEBOHMERT, *Formal-algebraic and rigid-analytic geometry*, Math. Ann. **286** (1990), 341 – 371. MR 90m:14015 334

[22] J. F. NASH JR., *Arc structure of singularities*, Duke Math. J. **81** (1995), 31 – 38. MR 98f:14011 338, 339

[23] M. RAYNAUD, "Géométrie analytique rigide d'après Tate, Kiehl, . . . " in *Table ronde d'analyse non archimédienne (Paris, 1972)*, Bull. Soc. Math. France Mém. **39** – **40**, Soc. Math. France, Paris, 1974, 319 – 327. MR 57:10012 321

[24] N. SCHAPPACHER, "Some remarks on a theorem of M. J. Greenberg" in *Proceedings of the Queen's Number Theory Conference (Kingston, Ont., 1979)*, Queen's Papers in Pure and Appl. Math. **54**, Queens Univ., Kingston, Ont., 1980, 101 – 113. MR 83f:13008 321

[25] J. SEBAG, *Intégration motivique sur les schémas formels*, to appear in Bull. Soc. Math. France, preprint, arXiv:math.AG/0112249 316, 317, 321, 322, 325

[26] J.-P. SERRE, *Classification des variétés analytiques p-adiques compactes*, Topology **3** (1965), 409 – 412. MR 31:3421 316, 336, 337

Loeser
Département de Mathématiques et Applications, Unité Mixte de Recherche 8553 du Centre National de la Recherche Scientifique, École Normale Supérieure, 45 rue d'Ulm, F-75230 Paris CEDEX 05, France; Francois.Loeser@ens.fr; http://dma.ens.fr/~loeser/

Sebag
Département de Mathématiques et Applications, Unité Mixte de Recherche 8553 du Centre National de la Recherche Scientifique, École Normale Supérieure, 45 rue d'Ulm, F-75230 Paris CEDEX 05, France; Julien.Sebag@ens.fr; http://dma.ens.fr/~sebag/