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Motivic Multiple Zeta Values and Superstring Amplitudes

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Abstract

The structure of tree–level open and closed superstring amplitudes is analyzed. For the open superstring amplitude we find a striking and elegant form, which allows to disentangle its α'–expansion into several contributions accounting for different classes of multiple zeta values. This form is bolstered by the decomposition of motivic multiple zeta values, i.e. the latter encapsulate the α'–expansion of the superstring amplitude. Moreover, a morphism induced by the coproduct maps the α'–expansion onto a non–commutative Hopf algebra. This map represents a generalization of the symbol of a transcendental function. In terms of elements of this Hopf algebra the α'–expansion assumes a very simple and symmetric form, which carries all the relevant information. Equipped with these results we can also cast the closed superstring amplitude into a very elegant form.

Contents

1. Introduction

One important question in quantum field theory is finding a simple principle to easily compute physical quantities such as Feynman integrals describing higher–order quantum corrections. Analytic results for Feynman integrals are encoded by transcendental functions such as multiple polylogarithms or elliptic functions [1]. These functions, which depend on the kinematic invariants, have a rich algebraic structure and obey a variety of different classes of relations among each other. Although these equations may allow to obtain a short and simple answer in practice it is not straightforward how to concretely apply and disentangle these relations to arrive at this simple answer. Hence, a guiding principle to get a grip on these relations is important.

A recent step towards an implicit application of these relations, which also leads to quite remarkable simplifications [2], is the concept of the symbol of a transcendental function, which maps the combinatorics of relations among different multiple polylogarithms to the combinatorics of a tensor algebra [3]. All the functional identities between the polylogarithms are mapped to simple algebraic relations in the tensor algebra over the group of rational functions. A generalization of the symbol approach is the coproduct structure of multiple polylogarithms [4,5]. The advantage of the coproduct structure is, that it also keeps track of multiple zeta values (MZVs) in contrast to the symbol S, for which we have $S(\pi), S(\zeta) = 0$. Recently, in Ref. [6] the coproduct structure has been applied for a concrete physical amplitude.

The properties of scattering amplitudes in both gauge and gravity theories suggest a deeper understanding from string theory, *cf.* Ref. [7] for a recent review. Many field theory objects and relations such as Bern–Carrasco–Johansson (BCJ) [8] or Kawai–Lewellen–Tye (KLT) [9] relations can be easily derived from and understood in string theory by tracing these identities back to the monodromy properties of the string world–sheet [10,11]. In this context we also like to mention the concept of transcendentality of Feynman integrals [12], which has a natural explanation from superstring amplitudes given by generalized Euler integrals [13]. Moreover, the concept of symbols and coproduct structure for Feynman integrals might have a natural appearance in string theory. In fact, in this work we shall demonstrate, that the aforementioned coproduct structure allows to cast the α'–expansion of the tree–level open and closed superstring amplitude into a short and symmetric form.

Generically, the string amplitudes are given by integrals over vertex operator positions on the Riemann surface describing the interacting string world–sheet. At higher loops there is also an integral over the moduli space of this manifold. At tree–level such integrals over positions boil down to generalized Euler integrals [14]. Expanding the latter w.r.t. to powers in the string tension α' yields higher–order string corrections to Yang–Mills (YM) theory. Their expansion coefficients are given by MZVs multiplying some polynomials in the kinematic invariants: at each order in α' only a set of MZVs of a fixed transcendentality degree (transcendentality level [12]) appears. In practice extracting these orders from the integrals [14,15], which boils down to computing generalized Euler–Zagier sums, is both cumbersome and provides quite complicated expressions: the appearance of various MZVs of different depth seems to lack any sorted structure. Furthermore, there is no selection principle to choose the right basis of MZVs in the α'–expansion. Just as computing amplitudes in field theory a lot of their simplicity and symmetry structure is lost by using not the most appropriate approach. In other words, though the final result may have a simple structure, the actual computation might not be able to reproduce this simplicity and yield a difficult answer.

In fact, by passing from the MZVs to their motivic versions [4,5] and then mapping the latter to elements of a Hopf algebra endows the superstring amplitude with its motivic structure. More precisely, the isomorphism ϕ, which is induced by the coproduct, maps the α'–expansion of the open superstring amplitude $\mathcal{A}$ into the very short and intriguing form in terms of elements f_i of a non–commutative Hopf algebra:

$$\mathcal{A} \xrightarrow{\phi} \left(\sum_{k=1}^{\infty} f_2^k \ P_{2k}\right) \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbf{N}^+-1}} f_{i_1} f_{i_2} \ldots f_{i_p} \ M_{i_p} \ldots M_{i_2} M_{i_1} \right\} A \ . \qquad (1.1)$$

In Eq. (1.1) the vector A encompasses a basis of YM subamplitudes, the matrices P_{2k} and M_{2n+1} encode polynomials of degree $2k$ and $2n+1$, respectively in α' and the kinematic invariants. As the vector A the string amplitude $\mathcal{A}$ represents a vector of the same dimension, *cf.* section 3 for further notational details. All the relevant information of the α'–expansion of the open superstring amplitude is encapsulated in (1.1) without further specifying the latter explicitly in terms of MZVs. This way all relations between MZVs are automatically built in as simple algebraic relations following from the coalgebra structure. Furthermore, the result is independent on any particular selection of a basis of MZVs. Finally, in contrast to the symbol the map ϕ, which is invertible, does not lose any information on the amplitude.

The organization of the present work is as follows. In section 2 we review those aspects of MZVs, which will be needed in the sequel. In section 3 we present our findings for the α'–expansion of the N–point open superstring amplitude. After some short exhibition on the work [5] of F. Brown on motivic MZVs in section 4 we compute the decompositions of motivic MZVs from weight 11 until weight 16 and compare the result with the structure of the open superstring amplitude. Equipped with these results in section 5 we investigate the motivic structure of the open superstring amplitude and derive (1.1). In section 6 we use our open superstring results to also cast the closed string amplitude into a compact form. In Appendix A we present some more results on the decomposition of motivic MZVs.

2. Aspects of multiple zeta values

One prime object in both quantum field theory and string theory are multiple zeta values (MZVs):

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l} \quad , \quad n_l \in \mathbf{N}^+ \ , \ n_r \geq 2 \ . \tag{2.1}$$

In this section we review some of their aspects. They can be written as special cases [16]

$$\zeta_{n_1,\ldots,n_r} = (-1)^r \ G(\underbrace{0,\ldots,0}_{n_r-1},1\ldots,\underbrace{0,\ldots,0}_{n_1-1},1;1) \tag{2.2}$$

of multiple polylogarithms [16,17]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} \ G(a_2,\ldots,a_n;t) \ , \tag{2.3}$$

with $G(z)=1$ and $a_i, z \in \mathbf{C}$. In (2.1) the sum $w=\sum_{l=1}^r n_l$ is called the transcendentality degree or weight of (2.1) and r its depth. The integral representation (2.2) is useful to establish various properties and relations of (2.1). The set of integral linear combinations of MZVs (2.1) is a ring, since the product of any two values can be expressed by a (positive) integer linear combination of the other MZVs [18], *e.g.*

$$\zeta_m \ \zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n} \ . \tag{2.4}$$

This relation is known as quasi–shuffle or stuffle relation. There are many relations over $\mathbf{Q}$ among MZVs, *e.g.* $\zeta_{1,4} = 2\zeta_5 - \zeta_2\zeta_3$. We define the (commutative) $\mathbf{Q}$–algebra $\mathcal{Z}$ spanned by all MZVs over $\mathbf{Q}$. The latter is the (conjecturally direct) sum over the $\mathbf{Q}$–vector spaces $\mathcal{Z}_N$ spanned by the set of MZVs (2.1) of total weight $w=N$, with $n_r \geq 2$, *i.e.* $\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k$. For a given weight $w \in \mathbf{N}$ the dimension $\dim_{\mathbf{Q}}(\mathcal{Z}_N)$ of the space $\mathcal{Z}_N$ is conjecturally given by $\dim_{\mathbf{Q}}(\mathcal{Z}_N) = d_N$, with $d_N = d_{N-2} + d_{N-3}, \ N \geq 3$ and $d_0 = 1, \ d_1 = 0, \ d_2 = 1$ [18]. Starting at weight $w=8$ MZVs of depth greater than one $r>1$ appear in the basis. By applying stuffle, shuffle, doubling, generalized doubling relations and duality it is possible to reduce the MZVs of a given weight to a minimal set [19]. For $D_{w,r}$ being the number of independent MZVs at weight $w>2$ and depth r, which cannot be reduced to primitive MZVs of smaller depth and their products, it is believed, that $D_{8,2}=1, D_{10,2}=1, \ D_{11,3}=1, \ D_{12,2}=1$ and $D_{12,4}=1$ [20]. For $Z=\frac{\mathcal{Z}_{>0}}{\mathcal{Z}_{>0}\mathcal{Z}_{>0}}$ the space of irreducible MZVs we have: $dim(Z_w) \equiv \sum_r D_{w,r} = 1,0,1,0,1,1,1,1,2,2,3,3,4,5$ for $w=3,\ldots,16$, respectively [20,19].

The selection of a basis of MZVs can be performed by following some principles. For instance a minimal depth representation may be preferable. In addition, one may write as many elements of the basis as possible with positive odd indices n_l only. However, it is not possible to achieve this for the whole basis, *i.e.* a number of basis elements needs at least two even entries [19]. Up to weight $w = 16$, one can choose the following basis elements, displayed in the following three tables, *cf.* Tables 1–3.

w	2	3	4	5	6	7	8	9	10	11		12	
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5	ζ_3^2	ζ_7	$\zeta_{3,5}$	ζ_9	$\zeta_{3,7}$	$\zeta_{3,3,5}$	$\zeta_2\ \zeta_3^3$	$\zeta_{1,1,4,6}$	$\zeta_2\ \zeta_{3,7}$
				$\zeta_2\ \zeta_3$	ζ_2^3	$\zeta_2\ \zeta_5$	$\zeta_3\ \zeta_5$	ζ_3^3	$\zeta_3\ \zeta_7$	$\zeta_{3,5}\ \zeta_3$	$\zeta_2\ \zeta_9$	$\zeta_{3,9}$	$\zeta_2^2\ \zeta_{3,5}$
						$\zeta_2^2\ \zeta_3$	$\zeta_2\ \zeta_3^2$	$\zeta_2\ \zeta_7$	ζ_5^2	ζ_{11}	$\zeta_2^2\ \zeta_7$	$\zeta_3\ \zeta_9$	$\zeta_2\ \zeta_5^2$
							ζ_2^4	$\zeta_2^2\ \zeta_5$	$\zeta_2\ \zeta_{3,5}$	$\zeta_3^2\ \zeta_5$	$\zeta_2^3\ \zeta_5$	$\zeta_5\ \zeta_7$	$\zeta_2\ \zeta_3\ \zeta_7$
								$\zeta_2^3\ \zeta_3$	$\zeta_2\ \zeta_3\ \zeta_5$		$\zeta_2^4\ \zeta_3$	ζ_3^4	$\zeta_2^2\ \zeta_3\ \zeta_5$
									$\zeta_2^2\ \zeta_3^2$				$\zeta_2^3\ \zeta_3^2$
									ζ_2^5				ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9		12	

Table 1: *Basis elements for $\mathcal{Z}_w$, with $2 \leq w \leq 10$.*

w	13		14		15		
$\mathcal{Z}_w$	$\zeta_{3,3,7}$	$\zeta_2\ \zeta_{3,3,5}$	$\zeta_{3,3,3,5}$	$\zeta_2\ \zeta_{1,1,4,6}$	$\zeta_{1,1,3,4,6}$	$\zeta_2\ \zeta_{3,3,7}$	$\zeta_2^2\ \zeta_{3,3,5}$
	$\zeta_{3,5,5}$	$\zeta_2\ \zeta_3\ \zeta_{3,5}$	$\zeta_{3,11}$	$\zeta_2\ \zeta_{3,9}$	$\zeta_{3,3,9}$	$\zeta_2\ \zeta_{3,5,5}$	$\zeta_2^2\ \zeta_3\ \zeta_{3,5}$
	ζ_{13}	$\zeta_2\ \zeta_{11}$	$\zeta_{5,9}$	$\zeta_2\ \zeta_3\ \zeta_9$	$\zeta_{5,3,7}$	$\zeta_2\ \zeta_{13}$	$\zeta_2^2\ \zeta_{11}$
	$\zeta_{3,7}\ \zeta_3$	$\zeta_2\ \zeta_3^2\ \zeta_5$	$\zeta_{3,3,5}\ \zeta_3$	$\zeta_2\ \zeta_5\ \zeta_7$	ζ_{15}	$\zeta_2\ \zeta_3\ \zeta_{3,7}$	$\zeta_2^2\ \zeta_3^2\ \zeta_5$
	$\zeta_{3,5}\ \zeta_5$	$\zeta_2^2\ \zeta_3^3$	$\zeta_{3,5}\ \zeta_3^2$	$\zeta_2\ \zeta_3^4$	$\zeta_{1,1,4,6}\ \zeta_3$	$\zeta_2\ \zeta_5\ \zeta_{3,5}$	$\zeta_2^3\ \zeta_3^3$
	$\zeta_3^2\ \zeta_7$	$\zeta_2^2\ \zeta_9$	$\zeta_3\ \zeta_{11}$	$\zeta_2^2\ \zeta_{3,7}$	$\zeta_{3,9}\ \zeta_3$	$\zeta_2\ \zeta_3^2\ \zeta_7$	$\zeta_2^3\ \zeta_9$
	$\zeta_3\ \zeta_5^2$	$\zeta_2^3\ \zeta_7$	$\zeta_3^3\ \zeta_5$	$\zeta_2^3\ \zeta_{3,5}$	$\zeta_9\ \zeta_3^2$	$\zeta_2\ \zeta_3\ \zeta_5^2$	$\zeta_2^4\ \zeta_7$
		$\zeta_2^4\ \zeta_5$	$\zeta_5\ \zeta_9$	$\zeta_2^2\ \zeta_5^2$	$\zeta_3\ \zeta_5\ \zeta_7$		$\zeta_2^5\ \zeta_5$
		$\zeta_2^5\ \zeta_3$	ζ_7^2	$\zeta_2^2\ \zeta_3\ \zeta_7$	ζ_3^5		$\zeta_2^6\ \zeta_3$
				$\zeta_2^3\ \zeta_3\ \zeta_5$	$\zeta_{3,7}\ \zeta_5$		
				$\zeta_2^4\ \zeta_3^2$	ζ_5^3		
				ζ_2^7	$\zeta_{3,5}\ \zeta_7$		
d_w	16		21		28		

Table 2: *Basis elements for $\mathcal{Z}_w$, with $13 \leq w \leq 15$.*

w	16			
$\mathcal{Z}_w$	$\zeta_{1,1,6,8}$	$\zeta_2\ \zeta_3\ \zeta_{3,3,5}$	$\zeta_2\ \zeta_{3,3,3,5}$	$\zeta_2^2\ \zeta_{1,1,4,6}$
	$\zeta_{3,3,3,7}$	$\zeta_2\ \zeta_3^2\ \zeta_{3,5}$	$\zeta_2\ \zeta_{3,11}$	$\zeta_2^2\ \zeta_{3,9}$
	$\zeta_{3,3,5,5}$	$\zeta_2\ \zeta_3\ \zeta_{11}$	$\zeta_2\ \zeta_{5,9}$	$\zeta_2^2\ \zeta_5\ \zeta_7$
	$\zeta_{3,13}$	$\zeta_2\ \zeta_3^3\ \zeta_5$	$\zeta_2\ \zeta_5\ \zeta_9$	$\zeta_2^3\ \zeta_{3,7}$
	$\zeta_{5,11}$	$\zeta_2^2\ \zeta_3^4$	$\zeta_2\ \zeta_7^2$	$\zeta_2^4\ \zeta_{3,5}$
	$\zeta_3\ \zeta_{3,3,7}$	$\zeta_2^2\ \zeta_3\ \zeta_9$		$\zeta_2^3\ \zeta_5^2$
	$\zeta_3\ \zeta_{3,5,5}$	$\zeta_2^3\ \zeta_3\ \zeta_7$		ζ_2^8
	$\zeta_3\ \zeta_{13}$	$\zeta_2^4\ \zeta_3\ \zeta_5$		
	$\zeta_{3,7}\ \zeta_3^2$	$\zeta_2^5\ \zeta_3^2$		
	$\zeta_{3,5}\ \zeta_3\ \zeta_5$			
	$\zeta_3^3\ \zeta_7$			
	$\zeta_3^2\ \zeta_5^2$			
	$\zeta_7\ \zeta_9$			
	$\zeta_{3,5}^2$			
	$\zeta_5\ \zeta_{11}$			
	$\zeta_{3,3,5}\ \zeta_5$			
d_w	37			

Table 3: *Basis elements for* $\mathcal{Z}_{16}$.

A slight generalization of (2.3) represents the integral I_γ over a product of closed one–forms [16]

$$I_\gamma(a_0;a_1,\ldots,a_n;a_{n+1})=\int_\gamma \frac{dz}{z-a_1}\ldots\frac{dz}{z-a_n}\ , \tag{2.5}$$

with γ a path in $M=\mathbf{C}/\{a_1,\ldots,a_n\}$ with endpoints $\gamma(0)=a_0\in M,\ \gamma(1)=a_{n+1}\in M$. For the map

$$\rho(n_1,\ldots,n_r)=10^{n_1-1}\ldots10^{n_r-1}\ , \tag{2.6}$$

with $n_r\geq 2$ Kontsevich observed that:

$$\zeta_{n_1,\ldots,n_r}=(-1)^r\ I_\gamma(0;\rho(n_1\ldots n_r);1)\ . \tag{2.7}$$

This defines an element in the category $MT(\mathbf{Z})$ of mixed Tate motives over $\mathbf{Z}$. It is an Abelian tensor category, whose simple objects are the Tate motives $\mathbf{Q}(n)$. The periods of $MT(\mathbf{Z})$ are $\mathbf{Q}$–linear combinations of $\zeta_{n_1,\ldots,n_r}$ [21].

3. Open superstring amplitude

The string S–matrix, which describes string scattering processes involving on–shell string states as external states, comprises a perturbative expansion in the string tension α'

and the string coupling constant $g_{\rm string}$. From this expansion one may extract for a given order in α' and $g_{\rm string}$ the relevant interaction terms of the low–energy effective action.

Open superstring theory contains a massless vector identified as a gauge boson. Its interactions are studied by gluon scattering amplitudes. Geometrically, at tree–level the latter are described by a disk with (integrated) insertions of gluon vertex operators. Due to the extended nature of strings the amplitudes generically represent non–trivial functions of the string tension α'. In the effective field theory description this α'–dependence gives rise to a series of infinitely many higher order gauge operators governed by positive integer powers in α'. The classical YM term is reproduced in the zero–slope limit $\alpha'\to 0$, while its modification can be derived by studying the higher orders in α' of the tree–level gluon scattering amplitudes.

At string tree–level the complete open string N–point superstring amplitude has been computed in [22,23]. The main result is written in a strikingly compact form[1]

$$\mathcal{A}(1,\ldots,N)=\sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^\sigma_{(1,\ldots,N)}(\alpha')\ , \tag{3.1}$$

where A_{YM} represent $(N-3)!$ color ordered Yang–Mills (YM) subamplitudes, $F^\sigma(\alpha')$ are generalized Euler integrals encoding the full α'–dependence of the string amplitude and $i_\sigma=\sigma(i)$. The labels $(1,\ldots,N)$ in $F^\sigma_{(1,\ldots,N)}$ are related to the integration region of the integrals: choosing an ordering of the vertex operator positions z_i along the boundary of the disk determines the color–ordering of the superstring subamplitude. The system of $(N-3)!$ multiple hypergeometric functions F^σ appearing in (3.1) are given as generalized Euler integrals

$$\begin{aligned}F^{(23\ldots N-2)}_{(1,\ldots,N)}(s_{ij})&=(-1)^{N-3}\int\limits_{z_i<z_{i+1}}\prod_{j=2}^{N-2}dz_j\ \left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\ \left\{\ \prod_{k=2}^{N-2}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\ \right\}\ ,\\ &=(-1)^{N-3}\int\limits_{z_i<z_{i+1}}\prod_{j=2}^{N-2}dz_j\ \left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\\ &\quad\times\left\{\left(\prod_{k=2}^{[N/2]}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right)\ \left(\prod_{k=[N/2]+1}^{N-2}\ \sum_{n=k+1}^{N-1}\frac{s_{kn}}{z_{kn}}\right)\right\}\ ,\end{aligned}\tag{3.2}$$

with permutations $\sigma\in S_{N-3}$ acting on all indices within the curly brace. Above, $[\ldots]$ denotes the Gauss bracket $[x]=\max_{n\in\mathbf{Z},n\leq x}n$, which picks out the nearest integer smaller

[1] A very compact expression for $D=4$ maximal helicity violating N–gluon amplitudes has been derived in [13].

than or equal to its argument. The α'–dependence of (3.2) is encoded in the kinematic invariants $s_{ij} = \alpha'(k_i + k_j)^2$ with the external gluon momenta k_i satisfying the on-shell condition $k_i^2 = 0$. For further details we refer the reader to Refs. [22,23].

The result (3.1) is valid in any space–time dimension D, for any compactification and any amount of supersymmetry. Furthermore, the expression (3.1) does not make any reference to any kinematical or helicity choices. Hence, the same is true for our results throughout this article. The integrals (3.2) share a very interesting mathematical structure [14,23]. For a given N the functions (3.2) represent integrals on the moduli space of Riemann spheres with N marked points $\mathcal{M}_{0,N}$ [24,25]. These spaces have an N–fold symmetry following from N–fold cyclic transformations on the disk, *cf.* [23] for more details. The lowest terms of the α'–expansion of the functions F^σ assume the form [23]

$$\begin{aligned} F^\sigma &= 1 + \alpha'^2\ p_2^\sigma\ \zeta(2) + \alpha'^3\ p_3^\sigma\ \zeta(3) + \ldots \quad , \quad \sigma = (23\ldots N-2)\ , \\ F^\sigma &= \alpha'^2\ p_2^\sigma\ \zeta(2) + \alpha'^3\ p_3^\sigma\ \zeta(3) + \ldots \quad , \quad \sigma \neq (23\ldots N-2)\ , \end{aligned} \tag{3.3}$$

with some polynomials p_n^σ of degree n in the dimensionful kinematic invariants $\hat s_{ij} = (k_i + k_j)^2 = s_{ij}/\alpha'$ and $\hat s_{i\ldots l} = (k_i + \ldots + k_l)^2 = s_{i\ldots l}/\alpha'$. Note that starting at $N \geq 7$ subsets of F^σ start at even higher order in α', *i.e.* $p_2^\sigma, \ldots, p_\nu^\sigma = 0$ for some $\nu \geq 2$. In Refs. [24,25] it is proven, that at lowest order in α' these integrals always lead to linear $\mathbf{Q}$ combinations of MZVs of weight $w \leq N-3$.

In the following let us discuss the cases $N = 4$ and $N = 5$ in more detail before moving to the general case afterwards.

3.1. $N = 4$

For $N = 4$ *Eq.* (3.1) becomes:

$$\mathcal{A}(1,2,3,4) = A_{YM}(1,2,3,4)\ \frac{\Gamma(1-s)\ \Gamma(1-u)}{\Gamma(1-s-u)}\ , \tag{3.4}$$

with the two kinematic invariants $s = \alpha'(k_1+k_2)^2$ and $u = \alpha'(k_1+k_4)^2$. With the identities

$$\begin{aligned} \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1}\ \zeta_{2n+1} &= \frac{1}{2}\ \ln\left\{ e^{-2\gamma_E x}\ \frac{\Gamma(1-x)}{\Gamma(1+x)} \right\}\ , \\ \pi\ \frac{s\ u}{s+u}\ \frac{\sin[\pi(s+u)]}{\sin(\pi s)\ \sin(\pi u)} &= \exp\left\{ 2\sum_{n=1}^\infty \frac{\zeta_{2n}}{2n}\ [\ s^{2n} + u^{2n} - (s+u)^{2n}\ ] \right\}\ , \end{aligned}$$

we may bring (3.4) into the following form

$$\mathcal{A}(1,2,3,4) = P\ \exp\left\{ \sum_{n\geq 1} \zeta_{2n+1}\ M_{2n+1} \right\}\ A_{YM}(1,2,3,4)\ , \tag{3.5}$$

with:

$$P=\exp\left\{\sum_{n=1}^{\infty}\frac{\zeta_{2n}}{2n}\ [\ s^{2n}+u^{2n}-(s+u)^{2n}\ ]\right\}\ ,$$
$$M_{2n+1}=\frac{1}{2n+1}\ \left[\ s^{2n+1}+u^{2n+1}-(s+u)^{2n+1}\ \right]. \tag{3.6}$$

In (3.5) we observe a disentanglement of Riemann zeta functions of even and odd arguments. Furthermore, no MZVs of depth greater than one $r>1$ appear.

3.2. $N=5$

For $N=5$ we have a basis of two color ordered superstring amplitudes $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$. According to (3.1) they take the form:

$$\begin{aligned}\mathcal{A}(1,2,3,4,5)&=A_{YM}(1,2,3,4,5)\ F_1+A_{YM}(1,3,2,4,5)\ F_2\ ,\\ \mathcal{A}(1,3,2,4,5)&=A_{YM}(1,3,2,4,5)\ \widetilde{F}_1+A_{YM}(1,2,3,4,5)\ \widetilde{F}_2\ ,\end{aligned} \tag{3.7}$$

with the functions (3.2)

$$\begin{aligned}F_1:=F^{(23)}_{(12345)}&=s_{12}\ s_{34}\ \int_0^1 dx\int_0^1 dy\ x^{s_{45}}\ y^{s_{12}-1}\ (1-x)^{s_{34}-1}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}}\ ,\\ F_2:=F^{(32)}_{(12345)}&=s_{13}\ s_{24}\ \int_0^1 dx\int_0^1 dy\ x^{s_{45}}\ y^{s_{12}}\ (1-x)^{s_{34}}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}-1},\end{aligned} \tag{3.8}$$

where $s_i\equiv\alpha'(k_i+k_{i+1})^2$ subject to cyclic identification $k_{i+N}\equiv k_i$. Furthermore, we have

$$\widetilde{F}_1=\left.F_1\right|_{2\leftrightarrow 3}\quad ,\quad \widetilde{F}_2=\left.F_2\right|_{2\leftrightarrow 3}\ . \tag{3.9}$$

When investigating the α'–expansions of (3.7) one makes the following intriguing observation[2]:

$$\mathcal{A}=P\ Q\ \ :\exp\left\{\sum_{n\geq 1}\zeta_{2n+1}\ M_{2n+1}\right\}:\ \ A\ , \tag{3.10}$$

with the vectors

$$A=\begin{pmatrix}A_{YM}(1,2,3,4,5)\\ A_{YM}(1,3,2,4,5)\end{pmatrix}\quad ,\quad \mathcal{A}=\begin{pmatrix}\mathcal{A}(1,2,3,4,5)\\ \mathcal{A}(1,3,2,4,5)\end{pmatrix}\ , \tag{3.11}$$

and the matrices

$$M_{2n+1}=\left.\begin{pmatrix}F_1 & F_2\\ \widetilde{F}_2 & \widetilde{F}_1\end{pmatrix}\right|_{\zeta_{2n+1}}\ , \tag{3.12}$$

$$P=\begin{pmatrix}\sum\limits_{m\geq 0}p_{2m}\ \zeta_2^m & \sum\limits_{m\geq 0}q_{2m}\ \zeta_2^m\\ \sum\limits_{m\geq 0}\widetilde{q}_{2m}\ \zeta_2^m & \sum\limits_{m\geq 0}\widetilde{p}_{2m}\ \zeta_2^m\end{pmatrix}=1+\sum_{n\geq 1}\zeta_2^n\ P_{2n}\ ,$$

[2] We have tested this formula up to weight 16. Work beyond this order is in progress [26].

where $\widetilde{p}_{2m} = p_{2m}|_{2\leftrightarrow 3}$, $\widetilde{q}_{2m} = q_{2m}|_{2\leftrightarrow 3}$. Furthermore, we have the matrix:

$$Q = 1 + \sum_{n\geq 8} Q_n \ , \tag{3.13}$$

with:

$$Q_8 = \frac{1}{5}\ \zeta_{3,5}\ [M_5, M_3] \quad , \quad Q_9 = 0 \ ,$$

$$Q_{10} = \left\{\ \frac{3}{14}\ \zeta_5^2 + \frac{1}{14}\ \zeta_{3,7}\ \right\}\ [M_7, M_3] \ ,$$

$$Q_{11} = \left\{\ 9\ \zeta_2\ \zeta_9 + \frac{6}{25}\ \zeta_2^2\ \zeta_7 - \frac{4}{35}\ \zeta_2^3\ \zeta_5 + \frac{1}{5}\ \zeta_{3,3,5}\ \right\}\ [M_3, [M_5, M_3]] \ ,$$

$$\begin{aligned} Q_{12} = &\left\{\ \frac{2}{9}\ \zeta_5\ \zeta_7 + \frac{1}{27}\ \zeta_{3,9}\ \right\}\ [M_9, M_3] \\ &+ \frac{48}{691}\ \left\{\ \frac{18}{35}\ \zeta_2^3\ \zeta_3^2 + \frac{1}{5}\ \zeta_2^2\ \zeta_3\ \zeta_5 - 10\ \zeta_2\ \zeta_3\ \zeta_7 - \frac{7}{2}\ \zeta_2\ \zeta_5^2 - \frac{3}{5}\ \zeta_2^2\ \zeta_{3,5} - 3\ \zeta_2\ \zeta_{3,7} \right. \\ &\left. - \frac{1}{12}\ \zeta_3^4 - \frac{467}{108}\ \zeta_5\ \zeta_7 + \frac{799}{72}\ \zeta_3\ \zeta_9 + \frac{2665}{648}\ \zeta_{3,9} + \zeta_{1,1,4,6}\ \right\}\ \{\ [M_9, M_3] - 3\ [M_7, M_5]\ \}\ , \end{aligned}$$

$$\begin{aligned} Q_{13} = &\left\{\ \frac{11}{4}\ \zeta_2\ \zeta_{11} - \frac{2}{35}\ \zeta_2^2\ \zeta_9 - \frac{16}{245}\ \zeta_2^3\ \zeta_7 - \frac{3}{35}\ \zeta_{3,5,5} + \frac{1}{14}\ \zeta_{3,3,7}\ \right\}\ [M_3, [M_7, M_3]] \\ &+ \left\{\ \frac{11}{2}\ \zeta_2\ \zeta_{11} + \frac{2}{5}\ \zeta_2^2\ \zeta_9 + \frac{1}{5}\ \zeta_5\ \zeta_{3,5} + \frac{1}{25}\ \zeta_{3,5,5}\ \right\}\ [M_5, [M_5, M_3]] \ , \end{aligned}$$

$$\begin{aligned} Q_{14} = &\left\{\ 4\ \zeta_2\ \zeta_5\ \zeta_7 + \frac{4}{175}\ \zeta_2^3\ \zeta_{3,5} - \frac{647287}{11880}\ \zeta_7^2 - \frac{12775}{198}\ \zeta_5\ \zeta_9 + \frac{232}{81}\ \zeta_{5,9} \right. \\ &\left. + \frac{2}{3}\ \zeta_2\ \zeta_{3,9} - \frac{12841}{1188}\ \zeta_{3,11} + \frac{1}{5}\ \zeta_{3,3,3,5}\ \right\}\ [M_3, [M_3, [M_5, M_3]]] \\ &+ \left\{\ -\frac{235}{396}\ \zeta_7^2 - \frac{23}{33}\ \zeta_5\ \zeta_9 + \frac{1}{27}\ \zeta_{5,9} - \frac{23}{198}\ \zeta_{3,11}\ \right\}\ [M_{11}, M_3] \\ &+ \left\{\ \frac{55}{36}\ \zeta_7^2 + \frac{5}{3}\ \zeta_5\ \zeta_9 + \frac{5}{18}\ \zeta_{3,11} - \frac{2}{27}\ \zeta_{5,9}\ \right\}\ [M_9, M_5] \ , \end{aligned}$$

$$\begin{aligned} Q_{15} = &\left\{\ \frac{1339}{30}\ \zeta_2\ \zeta_{13} + \frac{128}{45}\ \zeta_2^2\ \zeta_{11} - \frac{236}{4725}\ \zeta_2^3\ \zeta_9 - \frac{184}{2625}\ \zeta_4^2\ \zeta_7 - \frac{64}{5775}\ \zeta_2^5\ \zeta_5 \right. \\ &\left. - \frac{2}{45}\ \zeta_5^3 - \frac{1}{15}\ \zeta_7\ \zeta_{3,5} - \frac{2}{45}\ \zeta_5\ \zeta_{3,7} + \frac{1}{27}\ \zeta_{3,3,9}\ \right\}\ [M_3, [M_9, M_3]] \\ &+ \left\{\ -\frac{143}{20}\ \zeta_2\ \zeta_{13} - \frac{11}{35}\ \zeta_2^2\ \zeta_{11} + \frac{68}{1225}\ \zeta_2^3\ \zeta_9 + \frac{11}{70}\ \zeta_5^3 + \frac{24}{875}\ \zeta_2^4\ \zeta_7 + \frac{48}{13475}\ \zeta_2^5\ \zeta_5 \right. \end{aligned}$$

$$+\frac{1}{5}\ \zeta_7\ \zeta_{3,5} + \frac{3}{35}\ \zeta_5\ \zeta_{3,7} - \frac{1}{70}\ \zeta_{5,3,7}\ \Big\}\ [M_5,[M_7,M_3]] + \frac{2}{15}\ \zeta_{5,3,7}\ [M_3,[M_7,M_5]]$$

$$+\ \frac{48}{7601}\ \Big\{ -8\ \zeta_2\ \zeta_3\ \zeta_5^2 + \frac{21}{2}\ \zeta_2\ \zeta_5\ \zeta_{3,5} - \frac{14}{5}\ \zeta_2\ \zeta_{3,5,5} + 2\ \zeta_2\ \zeta_{3,3,7} - 26\ \zeta_2\ \zeta_3^2\ \zeta_7$$

$$-\ \frac{6417649}{2880}\ \zeta_2\ \zeta_{13} - 6\ \zeta_2\ \zeta_3\ \zeta_{3,7} - \frac{8495287}{15120}\ \zeta_2^2\ \zeta_{11} - \frac{23}{10}\ \zeta_2^2\ \zeta_3^2\ \zeta_5 - \frac{8}{5}\ \zeta_2^2\ \zeta_3\ \zeta_{3,5}$$

$$+\ 4\ \zeta_2^2\ \zeta_{3,3,5} + \frac{12}{35}\ \zeta_2^3\ \zeta_3^3 + \frac{54263011}{396900}\ \zeta_2^3\ \zeta_9 + \frac{57847}{15750}\ \zeta_2^4\ \zeta_7 - \frac{1714624}{121275}\ \zeta_2^5\ \zeta_5$$

$$+\ \frac{1451972}{716625}\ \zeta_2^6\ \zeta_3 + \frac{1185701}{30240}\ \zeta_5^3 - \frac{74}{3}\ \zeta_3\ \zeta_5\ \zeta_7 - \frac{1}{15}\ \zeta_3^5 + \frac{6775}{144}\ \zeta_3^2\ \zeta_9 + \frac{2188}{945}\ \zeta_5\ \zeta_{3,7}$$

$$-\frac{12199}{720}\ \zeta_7\ \zeta_{3,5} + \frac{29}{9}\ \zeta_3\ \zeta_{3,9} + \zeta_3\ \zeta_{1,1,4,6} + \frac{17203}{3360}\ \zeta_{5,3,7} - \frac{853}{648}\ \zeta_{3,3,9} + \zeta_{1,1,3,4,6}\ \Big\}$$

$$\times \Big\{\ [M_3,[M_9,M_3]] - 3\ [M_3,[M_7,M_5]]\ \Big\}\ ,$$

$$Q_{16} = \frac{1}{50}\ \zeta_{3,5}^2\ ([M_5,M_3])^2 + \Big\{\ \frac{210}{121}\ \zeta_9\ \zeta_7 + \frac{9}{11}\ \zeta_{11}\ \zeta_5 - \frac{5}{242}\ \zeta_{5,11} + \frac{3}{22}\ \zeta_{3,13}\ \Big\}\ [M_{11},M_5]$$

$$+\Big\{\ -\frac{1275}{1573}\ \zeta_9\ \zeta_7 - \frac{57}{143}\ \zeta_{11}\ \zeta_5 + \frac{3}{242}\ \zeta_{5,11} - \frac{19}{286}\ \zeta_{3,13}\Big\}\ [M_{13},M_3]$$

$$+\Big\{\ \frac{24}{35}\ \zeta_7\ \zeta_5\ \zeta_2^2 + \frac{6}{245}\ \zeta_5^2\ \zeta_2^3 + \frac{2}{245}\ \zeta_{3,7}\ \zeta_2^3 + \frac{4}{35}\ \zeta_{3,9}\ \zeta_2^2 + \frac{967}{56}\ \zeta_7^2\ \zeta_2$$

$$+\ \frac{363}{14}\ \zeta_9\ \zeta_5\ \zeta_2 - \frac{47}{42}\ \zeta_{5,9}\ \zeta_2 + \frac{121}{28}\ \zeta_{3,11}\ \zeta_2 - \frac{2272973}{330330}\ \zeta_9\ \zeta_7 - \frac{601677}{40040}\ \zeta_{11}\ \zeta_5$$

$$+\frac{23181}{67760}\ \zeta_{5,11} - \frac{200559}{80080}\ \zeta_{3,13} - \frac{3}{35}\ \zeta_{3,3,5,5} + \frac{1}{14}\ \zeta_{3,3,3,7}\ \Big\}\ [M_3,[M_3,[M_7,M_3]]]$$

$$+\Big\{\ -\frac{8}{25}\ \zeta_7\ \zeta_5\ \zeta_2^2 - \frac{2}{35}\ \zeta_5^2\ \zeta_2^3 - \frac{4}{75}\ \zeta_{3,9}\ \zeta_2^2 - \frac{333}{20}\ \zeta_7^2\ \zeta_2$$

$$-\ 21\ \zeta_9\ \zeta_5\ \zeta_2 + \zeta_{5,9}\ \zeta_2 - \frac{7}{2}\ \zeta_{3,11}\ \zeta_2 - \frac{299373}{7150}\ \zeta_9\ \zeta_7 - \frac{21033}{1300}\ \zeta_{11}\ \zeta_5$$

$$+\frac{909}{2200}\ \zeta_{5,11} - \frac{7011}{2600}\ \zeta_{3,13} + \frac{1}{5}\ \zeta_5\ \zeta_{3,3,5} + \frac{1}{25}\ \zeta_{3,3,5,5}\ \Big\}\ [M_3,[M_5,[M_5,M_3]]]$$

$$+\ \frac{720}{3617}\ \Big\{-\frac{21331}{525}\ \zeta_5\ \zeta_7\ \zeta_2^2 - \frac{284}{245}\ \zeta_5^2\ \zeta_2^3 + \frac{108}{875}\ \zeta_{3,5}\ \zeta_2^4 - \frac{62}{245}\ \zeta_{3,7}\ \zeta_2^3$$

$$-\ \frac{8954}{1575}\ \zeta_{3,9}\ \zeta_2^2 - \frac{78201}{140}\ \zeta_7^2\ \zeta_2 - \frac{12443}{14}\ \zeta_5\ \zeta_9\ \zeta_2 + \frac{697}{21}\ \zeta_{5,9}\ \zeta_2 - \frac{1991}{14}\ \zeta_{3,11}\ \zeta_2$$

$$
-137\ \zeta_{11}\ \zeta_3\ \zeta_2 - \frac{11}{7}\ \zeta_9\ \zeta_3\ \zeta_2^2 + \frac{848}{245}\ \zeta_7\ \zeta_3\ \zeta_2^3 + \frac{48}{35}\ \zeta_5\ \zeta_3\ \zeta_2^4 + \frac{408}{2695}\ \zeta_3^2\ \zeta_2^5
$$

$$
-\frac{4}{7}\ \zeta_3^2\ \zeta_5^2 - \frac{1}{3}\ \zeta_3^3\ \zeta_7 + \frac{4850713}{6600}\ \zeta_7\ \zeta_9 + \frac{455534}{525}\ \zeta_5\ \zeta_{11} + \frac{8497}{42}\ \zeta_3\ \zeta_{13} + \frac{1}{7}\ \zeta_3^2\ \zeta_{3,7}
$$

$$
-\frac{114307}{7392}\ \zeta_{5,11} + \frac{2217053}{16800}\ \zeta_{3,13} - \frac{2}{5}\ \zeta_5\ \zeta_{3,3,5} - \frac{6}{7}\ \zeta_3\ \zeta_{3,5,5} + \frac{5}{7}\ \zeta_{3,3,7}\ \zeta_3
$$

$$
\left.+\frac{542}{175}\ \zeta_{3,3,5,5} - \frac{19}{7}\ \zeta_{3,3,3,7} + \zeta_{1,1,6,8}\right\}\ \left\{\ \frac{7}{11}\ [M_{11}, M_5] - \frac{2}{11}\ [M_{13}, M_3] - [M_9, M_7]\right.
$$

$$
\left.+\frac{6493}{9240}\ [M_3, [M_3, [M_7, M_3]]] - \frac{751}{100}\ [M_3, [M_5, [M_5, M_3]]]\ \right\}\ . \tag{3.14}
$$

Finally, in (3.10) the ordering colons $:\ \ldots\ :$ are defined such that matrices with larger subscript multiply matrices with smaller subscript from the left,

$$
:\ M_i\ M_j\ := \begin{cases} M_i\ M_j\ , & i \geq j\ , \\ M_j\ M_i\ , & i < j\ . \end{cases} \tag{3.15}
$$

The generalization to iterated matrix products $:M_{i_1} M_{i_2} \ldots M_{i_p}:$ is straightforward. The expression (3.10) allows to conveniently extract any order in α' of the superstring amplitude by simple matrix manipulations. *E.g.* at weight eight from (3.10) we obtain the expressions

$$
\begin{aligned}
\mathcal{A}\ |_{\zeta_3 \zeta_5} &= M_5\ M_3\ A\ , \\
\mathcal{A}\ |_{\zeta_{3,5}} &= \frac{1}{5}\ [M_5, M_3]\ A\ , \\
\mathcal{A}\ |_{\zeta_2 \zeta_3^2} &= \frac{1}{2}\ P_2\ M_3\ M_3\ A\ , \\
\mathcal{A}\ |_{\zeta_2^4} &= P_8\ A\ ,
\end{aligned} \tag{3.16}
$$

while for weight ten we get:

$$
\begin{aligned}
\mathcal{A}\ |_{\zeta_3 \zeta_7} &= M_7\ M_3\ A\ , \\
\mathcal{A}\ |_{\zeta_{3,7}} &= \frac{1}{14}\ [M_7, M_3]\ A\ , \\
\mathcal{A}\ |_{\zeta_5^2} &= \left(\frac{1}{2}\ M_5\ M_5 + \frac{3}{14}\ [M_7, M_3]\right)\ A\ , \\
\mathcal{A}\ |_{\zeta_2 \zeta_3 \zeta_5} &= P_2\ M_5\ M_3\ A\ , \\
\mathcal{A}\ |_{\zeta_2 \zeta_{3,5}} &= \frac{1}{5}\ P_2\ [M_5, M_3]\ A\ , \\
\mathcal{A}\ |_{\zeta_2^2 \zeta_3^2} &= \frac{1}{2}\ P_4\ M_3\ M_3\ A\ , \\
\mathcal{A}\ |_{\zeta_2^5} &= P_{10}\ A\ .
\end{aligned} \tag{3.17}
$$

Above P_{2n} means taking the coefficient $\zeta(2)^n$ of the matrix P, *i.e.*

$$P_{2n} = P|_{\zeta_2^n} \ . \tag{3.18}$$

The terms M_5M_3A in (3.16) and $M_7M_3A,\ P_2M_5M_3$ in (3.17) use the ordering prescription (3.15) introduced in (3.10) for the matrices M_i stemming from the exponential.

3.3. General N

For generic N in (3.1) we have a basis of $(N-3)!$ color ordered superstring amplitudes $\mathcal{A}(1, 2_\sigma, \ldots, (N-2)_\sigma, N-1, N)$. Putting these $(N-3)!$ amplitudes into an $(N-3)!$–dimensional vector $\mathcal{A}$ according to (3.1) the latter can be expressed by an $(N-3)!\times(N-3)!$–matrix F acting on the vector A encoding an $(N-3)!$–dimensional YM–basis as:

$$\mathcal{A} = F\ A\ . \tag{3.19}$$

The matrix F encodes the full α'–dependence of the superstring amplitude (3.19). We conjecture, that the α'–dependence of the latter assumes the same form (3.10) as for the case $N = 5$

$$\mathcal{A} = P\ Q\ \ : \exp\left\{\sum_{n\geq 1} \zeta_{2n+1}\ M_{2n+1}\right\} :\ A\ , \tag{3.20}$$

with the matrices P, M and Q now being $(N-3)! \times (N-3)!$ matrices, following from

$$M_{2n+1} = F\ |_{\zeta_{2n+1}}\ , \tag{3.21}$$
$$P = 1 + \sum_{n\geq 1} \zeta_2^n\ P_{2n} := 1 + \sum_{n\geq 1} \zeta_2^n\ \ F\ |_{\zeta_2^n}\ ,$$

with $P_{2n} = P|_{\zeta_2^n}$ and Q given in (3.13). The polynomial structure of the matrices M, P and Q is further exhibited in [26].

What makes the form (3.20) appealing is the disentanglement of the full α'–expansion into several contributions accounting for different classes of MZVs: P comprising powers of ζ_2, M accounting for ζ_{2n+1} and powers thereof and Q encapsulating the MZVs $\zeta_{n_1,\ldots,n_r}$ of depth $r > 1$ greater than one. As we shall see in section 4 the specific form (3.20) is bolstered by the decomposition of motivic MZVs. It is interesting to note, that in (3.13) MZVs of depth greater than one $r > 1$ appear with commutators as:

$$\zeta_{n_1,\ldots,n_r}\ \ [M_{n_2}, [M_{n_3}, \ldots, [M_{n_r}, M_{n_1}]] \ldots]\ . \tag{3.22}$$

This property turns out to have a crucial impact on the closed string amplitude, *cf.* section 6.

At weight 16 in (3.13) the term $\frac{1}{50}\ \zeta_{3,5}^2\ ([M_5, M_3])^2$ gives rise to speculate, that all terms in Q follow from expanding an exponential:

$$Q = \exp\left\{\frac{1}{5}\ \zeta_{3,5}\ [M_5, M_3] + \left(\frac{3}{14}\ \zeta_5^2 + \frac{1}{14}\ \zeta_{3,7}\right)\ [M_7, M_3] + \ldots\right\}\ . \tag{3.23}$$

In fact, at weight 18 we find the following terms[3]

$$\begin{aligned} \mathcal{A}|_{\zeta_{3,5}\zeta_{3,7}} &= \frac{1}{5}\ \frac{1}{14}\ [M_7, M_3]\ [M_5, M_3] + \frac{208926}{894845}\ [M_3, [M_3, [M_7, M_5]]] \\ &- \frac{69642}{894845}\ [M_3, [M_3, [M_9, M_3]]]\ , \\ \mathcal{A}|_{\zeta_{3,5}\zeta_5^2} &= \frac{1}{2}\ \frac{1}{5}\ [M_5, M_3] M_5^2 + \frac{1}{5}\ \frac{3}{14}\ [M_7, M_3]\ [M_5, M_3] + \frac{1}{5}\ [M_5, [M_5, M_3]] M_5 \\ &+ \frac{1800}{43867}\ [M_{11}, M_7] - \frac{22500}{570271}\ [M_{13}, M_5] + \frac{7200}{570271}\ [M_{15}, M_3] \\ &- \frac{7044111243797}{6415252209080}\ [M_3, [M_3, [M_7, M_5]]] + \frac{2792059}{5702710}\ [M_5, [M_5, [M_5, M_3]]] \\ &- \frac{2432943}{7983794}\ [M_5, [M_3, [M_7, M_3]]] - \frac{2818807834641}{6415252209080}\ [M_3, [M_3, [M_9, M_3]]]\ , \end{aligned} \tag{3.24}$$

in agreement with the Ansatz (3.23).

Obviously, for $N = 4$ in (3.20) we have $Q = 1$ as all commutators vanish for the scalars M_{2n+1} given in (3.6). With this information (3.20) boils down to (3.5). So far, for $N = 6$ we have verified (3.20) up to α'^8. Further tests are in progress [26].

3.4. Minimal depth representation with Euler sums

The choice of basis elements may follow some minimal intrinsic representation guided by the minimal depth representation and the choice of positive odd indices only. For MZVs this is achieved by also allowing for Euler sums as basis elements:

$$\zeta(\epsilon_1 n_1, \ldots, \epsilon_r n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} \epsilon_l^{k_l}\ k_l^{-n_l} \quad , \quad n_l \in \mathbf{N}^+\ ,\ n_r \geq 2\ . \tag{3.25}$$

with signs $\epsilon_l = \pm 1$. For $M_{w,r}$ being the number of basis elements for MZVs when expressed in terms of Euler sums in a minimal depth representation at weight $w > 2$ and depth r we have $M_{12,2} = 2, M_{12,4} = 0,\ M_{15,3} = 3,\ M_{15,5} = 0, M_{16,2} = 3$ and $M_{16,4} = 2$ [20]. At

[3] Note the commutator relations: $[M_7, M_3][M_5, M_3] = [M_5, M_3][M_7, M_3]$ and $[M_3, [M_5, [M_7, M_3] = [M_5, [M_3, [M_7, M_3]$.

weight 12 one may get rid of the basis element $\zeta_{1,1,4,6}$ with even entries at the cost of the introducing the Euler sum $\zeta_{\overline{5},\overline{7}} := \zeta(-5,-7)$ [19]:

$$\begin{aligned}\zeta_{1,1,4,6} = &-\frac{5045}{648}\ \zeta_{3,9} + 3\ \zeta_2\ \zeta_{3,7} + \frac{3}{5}\ \zeta_2^2\ \zeta_{3,5} - \frac{799}{72}\ \zeta_3\ \zeta_9 - \frac{5747}{432}\ \zeta_5\ \zeta_7 + 10\ \zeta_2\ \zeta_3\ \zeta_7 \\ &+ \frac{7}{2}\ \zeta_2\ \zeta_5^2 - \frac{1}{5}\ \zeta_2^2\ \zeta_3\ \zeta_5 + \frac{1}{12}\ \zeta_3^4 - \frac{18}{35}\ \zeta_2^3\ \zeta_3^2 + \frac{694891}{2837835}\ \zeta_2^6 - \frac{64}{27}\ \zeta_{\overline{5},\overline{7}}\ . \end{aligned} \tag{3.26}$$

Similarly, we may use the Euler sum $\zeta_{\overline{3},\overline{9}} := \zeta(-3,-9)$ to arrive at [27]:

$$\begin{aligned}\zeta_{1,1,4,6} = &\frac{371}{144}\ \zeta_{3,9} + 3\ \zeta_2\ \zeta_{3,7} + \frac{3}{5}\ \zeta_2^2\ \zeta_{3,5} - \frac{3131}{144}\ \zeta_3\ \zeta_9 + \frac{107}{24}\ \zeta_5\ \zeta_7 + 10\ \zeta_2\ \zeta_3\ \zeta_7 \\ &+ \frac{7}{2}\ \zeta_2\ \zeta_5^2 - \frac{1}{5}\ \zeta_2^2\ \zeta_3\ \zeta_5 + \frac{1}{12}\ \zeta_3^4 - \frac{18}{35}\ \zeta_2^3\ \zeta_3^2 - \frac{117713}{2627625}\ \zeta_2^6 + \frac{64}{9}\ \zeta_{\overline{3},\overline{9}}\ . \end{aligned} \tag{3.27}$$

In [19] the object $A_{5,7}$

$$A_{5,7} = \zeta_{\overline{5},\overline{7}} + \zeta_{5,7} \tag{3.28}$$

has been argued to play a special status within the Euler sums, since it is quite similar to the MZVs. With this (3.26) can be written:

$$\begin{aligned}\zeta_{1,1,4,6} = &-\frac{7967}{1944}\ \zeta_{3,9} + 3\ \zeta_2\ \zeta_{3,7} + \frac{3}{5}\ \zeta_2^2\ \zeta_{3,5} - \frac{799}{72}\ \zeta_3\ \zeta_9 + \frac{11431}{1296}\ \zeta_5\ \zeta_7 + 10\ \zeta_2\ \zeta_3\ \zeta_7 \\ &+ \frac{7}{2}\ \zeta_2\ \zeta_5^2 - \frac{1}{5}\ \zeta_2^2\ \zeta_3\ \zeta_5 + \frac{1}{12}\ \zeta_3^4 - \frac{18}{35}\ \zeta_2^3\ \zeta_3^2 - \frac{5607853}{6081075}\ \zeta_2^6 - \frac{64}{27}\ A_{5,7}\ . \end{aligned} \tag{3.29}$$

Clearly, the above three equations (3.26), (3.27) and (3.29) are related by the identities:

$$\begin{aligned}\zeta_{5,7} &= \frac{14}{9}\ \zeta_{3,9} + \frac{28}{3}\ \zeta_5\ \zeta_7 - \frac{776224}{1576575}\ \zeta_2^6\ , \\ \zeta_{\overline{3},\overline{9}} &= -\frac{1}{3}\ \zeta_{\overline{5},\overline{7}} - \frac{13429}{9216}\ \zeta_{3,9} + \frac{1533}{1024}\ \zeta_3\ \zeta_9 - \frac{7673}{3072}\ \zeta_5\ \zeta_7 + \frac{10275263}{252252000}\ \zeta_2^6\ . \end{aligned} \tag{3.30}$$

We can write the weight 12 part Q_{12} of (3.13) in terms of Euler sums in a minimal depth representation and positive odd indices only in the following three ways corresponding to (3.26), (3.27) and (3.29), respectively;

$$\begin{aligned}Q_{12} = &\left\{\frac{2}{9}\ \zeta_5\ \zeta_7 + \frac{1}{27}\ \zeta_{3,9}\right\}\ [M_9, M_3] + \frac{48}{691}\ \{\ [M_9, M_3] - 3\ [M_7, M_5]\ \} \\ &\times \left\{\ \frac{694891}{2837835}\ \zeta_2^6 - \frac{7615}{432}\ \zeta_5\ \zeta_7 - \frac{595}{162}\ \zeta_{3,9} - \frac{64}{27}\ \zeta_{\overline{5},\overline{7}}\ \right\} \\ = &\left\{\ \frac{2}{9}\ \zeta_5\ \zeta_7 + \frac{1}{27}\ \zeta_{3,9}\right\}\ [M_9, M_3] + \frac{48}{691}\ \{\ [M_9, M_3] - 3\ [M_7, M_5]\ \} \\ &\times \left\{\ -\frac{117713}{2627625}\ \zeta_2^6 + \frac{29}{216}\ \zeta_5\ \zeta_7 - \frac{511}{48}\ \zeta_3\ \zeta_9 + \frac{8669}{1296}\ \zeta_{3,9} + \frac{64}{9}\ \zeta_{\overline{3},\overline{9}}\ \right\}\end{aligned}$$

$$= \left\{ \frac{2}{9}\ \zeta_5\ \zeta_7 + \frac{1}{27}\ \zeta_{3,9} \right\}\ [M_9, M_3] + \frac{48}{691}\ \{\ [M_9, M_3] - 3\ [M_7, M_5]\ \}$$

$$\times \left\{ -\frac{5607853}{6081075}\ \zeta_2^6 + \frac{5827}{1296}\ \zeta_5\ \zeta_7 + \frac{7}{486}\ \zeta_{3,9} - \frac{64}{27}\ A_{5,7} \right\}\ . \tag{3.31}$$

At weight 15 in (3.13) one may get rid of the basis element $\zeta_{1,1,3,4,6}$ with even entries at the cost of the introducing the Euler sum $\zeta_{\overline{3},\overline{5},\overline{7}} := \zeta(-3,-5,-7)$ [19]:

$$\begin{aligned}
\zeta_{1,1,3,4,6} = &\frac{16663}{11664}\ \zeta_{3,3,9} + \frac{150481}{68040}\ \zeta_{5,3,7} - \frac{20651486329}{4082400}\ \zeta_{15} + \frac{1903}{120}\ \zeta_7\ \zeta_{3,5} - \frac{101437}{38880}\ \zeta_5\ \zeta_{3,7}\\
&- \frac{1520827}{38880}\ \zeta_5^3 + 10\ \zeta_3\ \zeta_{1,1,4,6} + \frac{162823}{3888}\ \zeta_3\ \zeta_{3,9} - \frac{93619}{1296}\ \zeta_3\ \zeta_5\ \zeta_7 + \frac{3601}{48}\ \zeta_3^2\ \zeta_9\\
&- \frac{17}{20}\ \zeta_3^5 + \frac{14}{5}\ \zeta_2\ \zeta_{3,5,5} - 2\ \zeta_2\ \zeta_{3,3,7} + \frac{31753363}{12960}\ \zeta_2\zeta_{13} - \frac{21}{2}\ \zeta_2\ \zeta_5\ \zeta_{3,5}\\
&- 27\ \zeta_2\ \zeta_3\ \zeta_{3,7} - \frac{61}{2}\ \zeta_2\ \zeta_3\ \zeta_5^2 - 84\ \zeta_2\ \zeta_3^2\ \zeta_7 - 4\ \zeta_2^2\ \zeta_{3,3,5} + \frac{979621}{1701}\ \zeta_2^2\ \zeta_{11}\\
&- 5\ \zeta_2^2\ \zeta_3\ \zeta_{3,5} + \frac{9}{2}\ \zeta_2^2\ \zeta_3^2\ \zeta_5 - \frac{490670609}{3572100}\ \zeta_2^3\ \zeta_9 + \frac{186}{35}\ \zeta_2^3\ \zeta_3^3 - \frac{1455253}{283500}\ \zeta_2^4\ \zeta_7\\
&+ \frac{4049341}{311850}\ \zeta_2^5\ \zeta_5 + \frac{12073102}{1488375}\ \zeta_2^6\ \zeta_3 + \frac{1408}{81}\ A_{3,5,7}\ .
\end{aligned} \tag{3.32}$$

More precisely, with the relations (3.32) and (3.29) the combination $\zeta_3\zeta_{1,1,4,6} + \zeta_{1,1,3,4,6}$ can be eliminated to cast the weight 15 part Q_{15} in terms of Euler sums in a minimal depth representation and positive odd indices only:

$$\begin{aligned}
Q_{15} = &\left\{ \frac{1339}{30}\ \zeta_2\ \zeta_{13} + \frac{128}{45}\ \zeta_2^2\ \zeta_{11} - \frac{236}{4725}\ \zeta_2^3\ \zeta_9 - \frac{184}{2625}\ \zeta_4^2\ \zeta_7 - \frac{64}{5775}\ \zeta_2^5\ \zeta_5 \right.\\
&\left. -\frac{2}{45}\ \zeta_5^3 - \frac{1}{15}\ \zeta_7\ \zeta_{3,5} - \frac{2}{45}\ \zeta_5\ \zeta_{3,7} + \frac{1}{27}\ \zeta_{3,3,9} \right\}\ [M_3, [M_9, M_3]]\\
&+ \left\{ -\frac{143}{20}\ \zeta_2\ \zeta_{13} - \frac{11}{35}\ \zeta_2^2\ \zeta_{11} + \frac{68}{1225}\ \zeta_2^3\ \zeta_9 + \frac{11}{70}\ \zeta_5^3 + \frac{24}{875}\ \zeta_2^4\ \zeta_7 + \frac{48}{13475}\ \zeta_2^5\ \zeta_5 \right.\\
&\left. +\frac{1}{5}\ \zeta_7\ \zeta_{3,5} + \frac{3}{35}\ \zeta_5\ \zeta_{3,7} - \frac{1}{70}\ \zeta_{5,3,7} \right\}\ [M_5, [M_7, M_3]] + \frac{2}{15}\ \zeta_{5,3,7}\ [M_3, [M_7, M_5]]\\
&+ \frac{48}{7601}\ \left\{ \frac{1408}{81}\ A_{3,5,7} - \frac{704}{27}\ A_{5,7}\ \zeta_3 - \frac{20651486329}{4082400}\ \zeta_{15} + \frac{1149577}{5184}\ \zeta_2\ \zeta_{13} \right.\\
&+ \frac{1912097}{136080}\ \zeta_2^2\ \zeta_{11} - \frac{230351}{357210}\ \zeta_2^3\ \zeta_9 - \frac{414007}{283500}\ \zeta_2^4\ \zeta_7 - \frac{45779}{39690}\ \zeta_2^5\ \zeta_5 - \frac{24257}{3869775}\ \zeta_2^6\ \zeta_3\\
&+ \frac{77}{648}\ \zeta_3\ \zeta_5\ \zeta_7 + \frac{77}{3888}\ \zeta_{3,9}\ \zeta_3 + \frac{319}{3402}\ \zeta_5^3 - \frac{15983}{54432}\ \zeta_{3,7}\ \zeta_5 - \frac{781}{720}\ \zeta_{3,5}\ \zeta_7\\
&\left. +\frac{1995367}{272160}\ \zeta_{5,3,7} + \frac{1309}{11664}\ \zeta_{3,3,9} \right\}\ \Big\{\ [M_3, [M_9, M_3]] - 3\ [M_3, [M_7, M_5]]\ \Big\}\ .
\end{aligned} \tag{3.33}$$

4. Motivic multiple zeta values

In this section we want to compare our findings (3.20) with the beautiful work of F. Brown on the decomposition of motivic multiple zeta values [5]. For this purpose after reviewing some aspects of motivic MZVs we determine the decomposition of motivic MZVs for the weights 11 until 16.

4.1. Motivic aspects of multiple zeta values

An important question is to explicitly describe the structure of the algebra $\mathcal{Z}$, which eventually allows to get a grip on all algebraic MZV identities over $\mathbf{Q}$. For this purpose the actual MZVs (2.1) are replaced by symbols (or motivic MZVs), which are elements of a certain algebra.

In this section we review some aspects of motivic MZVs [5]. The task is to lift the ordinary iterated integrals I_γ given in (2.5) to motivic versions I^m such that the standard relations are fulfilled. With an embedding $\sigma : F \hookrightarrow \mathbf{C}$ the iterated integrals I_γ can be upgraded to a framed mixed Tate motive over F (motivic iterated integral)

$$I^m(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{H}(F) \quad , \quad a_0, \ldots, a_{n+1} \in F \ , \tag{4.1}$$

with $p_\sigma(I^m(a_0; a_1, \ldots, a_n; a_{n+1})) = I(\sigma(a_0); \sigma(a_1), \ldots, \sigma(a_n); \sigma(a_{n+1}))$ [4] and some number field F. The latter is a finite degree field extension of the field of rational numbers $\mathbf{Q}$. The symbols (4.1) are elements of a commutative graded Hopf algebra $\mathcal{H}(F)$:

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \ . \tag{4.2}$$

The Hopf algebra[4] $\mathcal{H}$ implies a product given by the shuffle product

$$I^m(x; a_1, \ldots, a_r; y) \cdot I^m(x; a_{r+1}, \ldots, a_{r+s}; y) = \sum_{\sigma \in \Sigma(r,s)} I^m(x; a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; y) \ , \tag{4.3}$$

with $\Sigma(r, s) = \{\sigma \in \Sigma(r + s) \mid \sigma^{-1}(1) < \ldots < \sigma^{-1}(r) \cap \sigma^{-1}(r + 1) < \ldots < \sigma^{-1}(r + s)\}$ and $a_i, x, y \in \{0, 1\}$ and the coproduct Δ acting on the elements I^m as [4]:

$$\begin{aligned} \Delta\ I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = & \sum_{0=i_0<i_1<\ldots<i_k<i_{k+1}=n+1} I^m(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \\ & \otimes \prod_{p=0}^{k} I^m(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \ , \end{aligned} \tag{4.4}$$

[4] A Hopf algebra is an algebra $\mathcal{A}$ with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, *i.e.* $\mu(x_1 \otimes x_2) = x_1 \cdot x_2$ and associativity. At the same time it is also a coalgebra with coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and coassociativity such that the product and coproduct are compatible: $\Delta(x_1 \cdot x_2) = \Delta(x_1) \otimes \Delta(x_2)$, with $x_1, x_2 \in \mathcal{A}$.

with $0 \leq k \leq n$ and $a_i \in F$. As in (2.7) by (4.1) with $a_i \in \{0,1\}$ we may define the motivic versions $\zeta^m_{n_1,\ldots,n_r}$ of the MZVs $\zeta_{n_1,\ldots,n_r}$, *i.e.* by (4.1) the motivic MZVs are defined as

$$\zeta^m_{n_1,\ldots,n_r} = (-1)^r \ I^m(0;\rho(n_1,\ldots,n_r);1) \in \mathcal{H}_w(\mathbf{Z}) \ , \tag{4.5}$$

with the weight $w = \sum_{l=1}^r n_l$ and ρ given in (2.6). Any symbol $I^m(a_0;a_1,\ldots,a_n;a_{n+1})$, with $a_i \in \{0,1\}$, can be reduced to a linear combination of elements of the form (4.5), with $n_i \geq 1$, $n_r \geq 2$ and $w = N$. The dimension of the space of motivic MZVs of weight k is equal to d_k, *i.e.* $\dim_{\mathbf{Q}}(\mathcal{H}_k) = d_k$. The map $\mathcal{H}_k \to \mathcal{Z}_k$ is surjective, *i.e.* $\dim_{\mathbf{Q}}(\mathcal{Z}_k) \leq \dim_{\mathbf{Q}}(\mathcal{H}_k) = d_k$ [28,21]. By this certain identities between MZVs can be lifted to their motivic versions [5].

There is a non–canonical isomorphism

$$\mathcal{H} \simeq \mathcal{A} \ \otimes_{\mathbf{Q}} \ \mathbf{Q}[\zeta^m_2] \quad , \quad \mathcal{A} = \mathcal{H}/\zeta^m_2\mathcal{H} \ , \tag{4.6}$$

with the first factor graded by the weight, *i.e.* $\mathcal{A} = \bigoplus\limits_{n\geq 0} \mathcal{A}_n$.

To explicitly describe the structure of $\mathcal{H}$ one introduces the (trivial) algebra–comodule:

$$\mathcal{U} = \mathbf{Q}\langle f_3, f_5, \ldots\rangle \ \otimes_{\mathbf{Q}} \ \mathbf{Q}[f_2] \ . \tag{4.7}$$

The first factor $\mathcal{U}' = \mathcal{U}/f_2\mathcal{U}$ is a cofree Hopf–algebra on the cogenerators f_{2r+1} in degree $2r+1 \geq 3$, whose basis consists of all non–commutative words in the f_{2i+1}. The multiplication on $\mathcal{U}'$ is given by the shuffle product $\sqcup\!\sqcup$

$$f_{i_1}\ldots f_{i_r} \sqcup\!\sqcup f_{i_{r+1}}\ldots f_{i_{r+s}} = \sum_{\sigma\in\Sigma(r,s)} f_{i_{\sigma(1)}}\ldots f_{i_{\sigma(r+s)}} \ , \tag{4.8}$$

with $\Sigma(r,s)$ given after *Eq.* (4.3). The Hopf–algebra $\mathcal{U}'$ is isomorphic to the space of non–commutative polynomials in f_{2i+1}. The element f_2 commutes with all f_{2r+1}. Again, there is a grading $\mathcal{U}_k$ on $\mathcal{U}$, with $\dim(\mathcal{U}_k) = d_k$. Then, there exists a morphism ϕ of graded algebra–comodules

$$\phi: \ \mathcal{H} \longrightarrow \mathcal{U} \ , \tag{4.9}$$

normalized by:

$$\phi(\zeta^m_n) = f_n \quad , \quad n \geq 2 \ . \tag{4.10}$$

The map (4.9) sends every motivic MZV to a non–commutative polynomial in the f_i. Furthermore, (4.9) respects the shuffle multiplication rule (4.8):

$$\phi(x_1x_2) = \phi(x_1) \sqcup\!\sqcup \phi(x_2) \quad , \quad x_1, x_2 \in \mathcal{H} \ . \tag{4.11}$$

It is believed, that the isomorphism $\mathcal{Z}_k \simeq \mathcal{U}_k$ of graded algebras over $\mathbf{Q}$ holds.

The motivic MZVs have a hidden structure, which is revealed by the action of motivic derivations. The latter are derived from the coaction $\Delta : \mathcal{H} \to \mathcal{A} \otimes_{\mathbf{Q}} \mathcal{H}$ [21,5]

$$\Delta\ I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{\substack{0=i_0<i_1<\ldots< \\ <i_k<i_{k+1}=n+1}} \Pi \left(\prod_{p=0}^{k} I^m(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I^m(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})\ , \tag{4.12}$$

which represents a modification of the coproduct (4.4). Here, Π is the projector $\Pi : \mathcal{H} \to \mathcal{A}$ acting on ζ_2^m as $\zeta_2^m \overset{\Pi}{\longrightarrow} 0$. The derivations $D_r : \mathcal{H}_n \to \mathcal{A}_r \otimes_{\mathbf{Q}} \mathcal{H}_{n-r} \overset{\pi\otimes id}{\to} \mathcal{L}_r \otimes_{\mathbf{Q}} \mathcal{H}_{n-r}$ on $\mathcal{H}$ are defined as the infinitesimal version of the coaction (4.12) [5]

$$D_r\ I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{p=0}^{n-r} \pi\left(I^a(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1})\right) \otimes I^m(a_0; a_1, \ldots, a_p, a_{p+r+1}, \ldots, a_n; a_{n+1})\ , \tag{4.13}$$

with the projection $\pi : \mathcal{A} \to \mathcal{L}$ onto the Lie coalgebra $\mathcal{L} = \frac{\mathcal{A}_{>0}}{\mathcal{A}_{>0}\mathcal{A}_{>0}}$ describing all indecomposable (irreducible) elements of $\mathcal{A}$. By this we have $D_{2r} I^m \equiv 0$.

4.2. On the decomposition of motivic multi zeta values

The coalgebra structure (4.7) underlying the motivic MZVs can be used to decompose any MZV into a basis. Let us now describe the decomposition of motivic MZVs up to some weight $M \geq 2$ [5].

We are looking for decompositions in the $\mathbf{Q}$–vector space $\mathcal{H}_N,\ 2 \leq N \leq M$ spanned by the symbols (4.5), with $w = N$ and $n_i \geq 1,\ n_r \geq 2$. To check, that a (conjectural) polynomial basis B of motivic MZVs $\bigoplus_{2\leq n\leq M} \mathcal{H}_n$ up to weight M indeed represents a polynomial basis of motivic MZVs up to weight M for $n \leq N$ for each set B_n of elements of B of weight n one constructs the map (4.9):

$$\phi :\ B_n \longrightarrow\ \mathcal{U}_n \quad , \quad n \leq N\ . \tag{4.14}$$

This map assigns to every element of our basis B (of weight at most N) a $\mathbf{Q}$–linear combination of monomials

$$f_{2i_1+1} \ldots f_{2i_r+1}\ f_2^k \quad , \quad r, k \geq 0, i_1, \ldots, i_r \geq 1\ ,\ 2\ (i_1 + \ldots + i_r) + r + 2k = n\ , \tag{4.15}$$

which are basis elements of the $\mathbf{Q}$–vector space $\mathcal{U}_n$ supplemented by the multiplication rule $ш : \mathcal{U}_m \times \mathcal{U}_n \to \mathcal{U}_{n+m}$ given in (4.8). Actually, ϕ can be extended to the vector space $\mathcal{H}_n$:

$$\phi :\ \mathcal{H}_n \longrightarrow\ \mathcal{U}_n \quad , \quad n \leq N\ . \tag{4.16}$$

For the basis B we must have: $\dim_{\mathbf{Q}}(\langle B\rangle_N) = d_N,\ 2 \leq N \leq M$, with $\langle B\rangle_N$ the $\mathbf{Q}$–vector space spanned by monomials in the elements of B of total additive weight N. Furthermore, we have

$$B \supset B^0 = \{\zeta_2^m\} \cup \{\zeta_3^m, \ldots, \zeta_{2r+1}^m\}\ , \tag{4.17}$$

with $r = \lfloor (M-1)/2 \rfloor$. For the elements of B^0 the map ϕ is given by (4.10). For the remaining elements of B the explicit construction of ϕ is performed inductively, *i.e.* from (4.14) the case $n = N+1$ is determined. To find $\phi(\xi)$ for a general $\xi \in B_{N+1}$, with $\xi = I^m(a_0; a_1, \ldots, a_{N+1}; a_{N+2})$ according to (4.5), we need to compute the coefficients

$$\begin{aligned}\xi_{2r+1} = &\sum_{p=0}^{N-2r} c^{\phi}_{2r+1}\left(I^m(a_p; a_{p+1}, \ldots, a_{p+2r+1}; a_{p+2r+2})\right) \\ &\times \phi\left(I^m(a_0; a_1, \ldots, a_p, a_{p+2r+2}, \ldots, a_{N+1}; a_{N+2})\right) \in \mathcal{U}_{N-2r} \quad , \quad 3 \leq 2r+1 \leq N\end{aligned} \tag{4.18}$$

in the expansion:

$$\phi(\xi) = \sum_{3 \leq 2r+1 \leq N} f_{2r+1}\ \xi_{2r+1} \in \mathcal{U}_{N+1}\ . \tag{4.19}$$

Above the operator $c^{\phi}_{2r+1}(\xi)$, with $\xi \in \mathcal{H}_{2r+1}$ determines the rational coefficient of f_{2r+1} in the monomial $\phi(\xi) \in \mathcal{U}_{2r+1}$. Note, that the right hand side of (4.18) only involves elements I^m from $\mathcal{H}_{\leq N}$ for which ϕ has already been determined.

The above construction allows to assign a $\mathbf{Q}$–linear combination of monomials to every element $\zeta^m_{n_1,\ldots,n_r}$. The map[5] ϕ sends every motivic MZV of weight less or equal to N to a non–commutative polynomial in the f_i's. Inverting this map gives the decomposition of $\zeta^m_{n_1,\ldots,n_r}$ w.r.t. the basis B_n, with $n = \sum_{l=1}^r n_l$. In other words, the derivations (4.20) are used to detect elements in $\mathcal{U}$ and to decompose any motivic MZV ξ into a candidate basis B.

In [5] the map (4.14) and the decomposition are explicitly worked out up to weight 10. *E.g.* one finds

$$\phi(\zeta^m_{3,5}) = -5\ f_5 f_3 \quad , \quad \phi(\zeta^m_{3,7}) = -14\ f_7 f_3 - 6\ f_5 f_5\ , \tag{4.21}$$

[5] The choice of ϕ describes for each weight $2r+1$ the motivic derivation operators ∂^{ϕ}_{2r+1} acting on the space of motivic MZVs $\partial^{\phi}_{2r+1} : \mathcal{H} \to \mathcal{H}$ [5]

$$\partial^{\phi}_{2r+1} = (c^{\phi}_{2r+1} \otimes id) \circ D_{2r+1}\ , \tag{4.20}$$

with D_{2n+1} given in (4.13) and the coefficient function c^{ϕ}_{2r+1}, introduced above.

and at weight 10 one has for $\xi_{10} \in \mathcal{H}_{10}$ the following decomposition

$$\begin{aligned}\xi_{10} = a_0\ (\zeta_2^m)^5 + a_1\ (\zeta_2^m)^2\ (\zeta_3^m)^2 + a_2\ \zeta_2^m\ \zeta_3^m\ \zeta_5^m \\ + a_3\ (\zeta_5^m)^2 + a_4\ \zeta_2^m\ \zeta_{3,5}^m + a_5\ \zeta_3^m\ \zeta_7^m + a_6\ \zeta_{3,7}^m\ ,\end{aligned} \tag{4.22}$$

with the operators:

$$\begin{aligned}a_1 &= \frac{1}{2}\ c_2^2\ \partial_3^2,\ a_2 = c_2\ \partial_5\partial_3,\ a_3 = \frac{1}{2}\ \partial_5^2 + \frac{3}{14}\ [\partial_7, \partial_3]\ , \\ a_4 &= \frac{1}{5}\ c_2\ [\partial_5, \partial_3],\ a_5 = \partial_7\partial_3,\ a_6 = \frac{1}{14}\ [\partial_7, \partial_3]\ .\end{aligned} \tag{4.23}$$

acting on $\phi(\xi_{10})$. The derivation operators $\partial_{2n+1} : \mathcal{U} \to \mathcal{U}$ are defined as [5]:

$$\partial_{2n+1}(f_{i_1} \ldots f_{i_r}) = \begin{cases} f_{i_2} \ldots f_{i_r}\ , & i_1 = 2n+1\ , \\ 0\ , & \text{otherwise}\ , \end{cases} \tag{4.24}$$

with $\partial_{2n+1} f_2 = 0$. Furthermore, we have the product rule for the shuffle product:

$$\partial_{2n+1}(a \,ш\, b) = \partial_{2n+1} a \,ш\, b + a \,ш\, \partial_{2n+1} b \quad , \quad a, b \in \mathcal{U}'\ . \tag{4.25}$$

Finally, c_2^n takes the coefficient of f_2^n.

It seems very amusing, that the coefficients (4.23) and the commutator structure agree exactly with (3.17). Therefore, motivic multi zeta values encapsulate the α'–expansion of the open superstring amplitude.

4.3. Decomposition of motivic multi zeta values for weights 11 *through* 16

In order to bolster this connection, in the following subsections we determine the decompositions ξ_w of any motivic MZV for the weights $11 \leq w \leq 16$.

For a given weight w we proceed as described in [5]: in lines of the Tables 1–3 at weight w we first detect the new elements B_w to be added to constitute the conjectural basis B up to weight w. For these new elements B_w we then compute their coefficients (4.18) or motivic derivations ∂_{2r+1}^{ϕ} by applying the relations $(R0) - (R4)$ given in section 5.1 of [5]. Equipped with these results we then determine the map (4.19) by using the findings from the lower weights. After having derived the map (4.19) for all d_w basis elements of $\langle B \rangle_w$ we can construct the basis for $\mathcal{U}_w$ and eventually the operator ξ_w.

For the depth two case ζ_{n_1,n_2}^m there exists a closed formula, which computes the map $\phi(\zeta_{n_1,n_2}^m)$, directly [29]. Our results for $\phi(\zeta_{3,9}^m), \phi(\zeta_{3,11}^m), \phi(\zeta_{5,9}^m), \phi(\zeta_{3,13}^m)$ and $\phi(\zeta_{5,11}^m)$ agree with what this formula gives. However, as it will become clear in the following, beyond depth two the computations involve new aspects and become rather involved.

4.3.1. Decomposition at weight 11

At weight 11 we take the following set of motivic MZVs

$$B = \{\ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m\ \} \tag{4.26}$$

as independent algebra generators up to weight 11. In [5] up to weight $n \leq 10$ to each element of B an element of $\mathcal{U}$ is associated by the map ϕ given in (4.14). Hence, we only need to compute $\phi(\zeta_{3,3,5}^m)$, which according to (4.18) requires the following derivatives:

$$\begin{aligned} &\partial_3^\phi \zeta_{3,3,5}^m = 0\ , && \partial_5^\phi \zeta_{3,3,5}^m = -5\ \zeta_{3,3}^m = -\frac{5}{2}\ (\zeta_3^m)^2 + \frac{4}{7}\ (\zeta_2^m)^3\ , \\ &\partial_7^\phi \zeta_{3,3,5}^m = -\frac{6}{5}\ (\zeta_2^m)^2\ , && \partial_9^\phi \zeta_{3,3,5}^m = -45\ \zeta_2^m\ . \end{aligned} \tag{4.27}$$

From these results the expression (4.19) gives rise to:

$$\phi(\zeta_{3,3,5}^m) = -\frac{5}{2}\ f_5(f_3 ш f_3) + \frac{4}{7}\ f_5 f_2^3 - \frac{6}{5}\ f_7 f_2^2 - 45\ f_9 f_2\ . \tag{4.28}$$

Gathering the information about the lower weight basis $\mathcal{U}_{k\leq 10}$ with (4.28) we can construct the following basis for $\mathcal{U}_{11}$:

$$\begin{aligned} &-\frac{5}{2}\ f_5(f_3 ш f_3) + \frac{4}{7}\ f_5 f_2^3 - \frac{6}{5}\ f_7 f_2^2 - 45\ f_9 f_2\ , \\ &-5\ (f_5 f_3) ш f_3,\ f_{11},\ f_3 ш f_3 ш f_5,\ f_3 ш f_3 ш f_3 f_2\ , \\ &f_9 f_2,\ f_7 f_2^2,\ f_5 f_2^3,\ f_3 f_2^4\ . \end{aligned} \tag{4.29}$$

This basis gives rise to the following decomposition of any motivic MZV ξ_{11} of weight 11

$$\begin{aligned} \xi_{11} = a_1\ \zeta_{3,3,5}^m + a_2\ \zeta_{3,5}^m\ \zeta_3^m + a_3\ \zeta_{11}^m + a_4\ (\zeta_3^m)^2\ \zeta_5^m + a_5\ \zeta_2^m\ (\zeta_3^m)^3 \\ + a_6\ \zeta_2^m\ \zeta_9^m + a_7\ (\zeta_2^m)^2\ \zeta_7^m + a_8\ (\zeta_2^m)^3\ \zeta_5^m + a_9\ (\zeta_2^m)^4\ \zeta_3^m \end{aligned} \tag{4.30}$$

with[6] the following operators

$$\begin{aligned} &a_1 = \frac{1}{5}\ [\partial_3,[\partial_5,\partial_3]],\ a_2 = \frac{1}{5}\ [\partial_5,\partial_3]\partial_3\ , \\ &a_3 = \partial_{11},\ a_4 = \frac{1}{2}\ \partial_5\partial_3^2,\ a_5 = \frac{1}{6}\ c_2\ \partial_3^3\ , \\ &a_6 = c_2\ \partial_9 + 9\ [\partial_3,[\partial_5,\partial_3]],\ a_7 = c_2^2\ \partial_7 + \frac{6}{25}\ [\partial_3,[\partial_5,\partial_3]]\ , \\ &a_8 = c_2^3\ \partial_5 - \frac{4}{35}\ [\partial_3,[\partial_5,\partial_3]],\ a_9 = c_2^4\ \partial_3 \end{aligned} \tag{4.31}$$

acting on $\phi(\xi_{11})$.

[6] The following relations $[\partial_3,[\partial_5,\partial_3]] f_3 ш f_3 ш f_5 = 0$ and $[\partial_3,[\partial_5,\partial_3]] f_5 f_3 ш f_3 = 0$ are useful. More generally, we have: $[\partial_a,[\partial_b,\partial_c]] f_a ш f_b ш f_c = 0$ and $[\partial_a,[\partial_b,\partial_a]] f_b f_a ш f_a = 0$.

4.3.2. Decomposition at weight 12

Next, at weight 12 we take the set of motivic MZVs

$$B = \{\ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m\ \} \tag{4.32}$$

as independent algebra generators up to weight 12. We need to compute $\phi(\zeta_{3,9}^m)$ and $\phi(\zeta_{1,1,4,6}^m)$, which require the following derivatives

$$\begin{aligned} &\partial_3^\phi \zeta_{3,9}^m = 0\ , && \partial_7^\phi \zeta_{3,9}^m = -15\ \zeta_5^m\ ,\\ &\partial_5^\phi \zeta_{3,9}^m = -6\ \zeta_7^m\ , && \partial_9^\phi \zeta_{3,9}^m = -27\ \zeta_3^m\ , \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \partial_3^\phi \zeta_{1,1,4,6}^m &= \frac{1}{3}\ (\zeta_3^m)^3 - \frac{799}{72}\ \zeta_9^m + 10\ \zeta_7^m\zeta_2^m - \frac{1}{5}\ \zeta_5^m(\zeta_2^m)^2 - \frac{36}{35}\ \zeta_3^m(\zeta_2^m)^2\ ,\\ \partial_5^\phi \zeta_{1,1,4,6}^m &= 29\ \zeta_7^m - 11\ \zeta_5^m\zeta_2^m - \frac{16}{5}\ \zeta_3^m(\zeta_2^m)^2\ ,\\ \partial_7^\phi \zeta_{1,1,4,6}^m &= \frac{1133}{16}\ \zeta_5^m - 32\ \zeta_3^m\zeta_2^m\ ,\\ \partial_9^\phi \zeta_{1,1,4,6}^m &= \frac{1799}{18}\ \zeta_3^m\ , \end{aligned} \tag{4.34}$$

respectively. With the derivatives (4.33) and (4.34) we determine the following maps:

$$\begin{aligned} \phi(\zeta_{3,9}^m) &= -6\ f_5f_7 - 15\ f_7f_5 - 27\ f_9f_3\ ,\\ \phi(\zeta_{1,1,4,6}^m) &= \frac{1799}{18}\ f_9f_3 - 32\ f_7f_3f_2 + \frac{1133}{16}\ f_7f_5 + 29\ f_5f_7 - 11\ f_5^2f_2 - \frac{16}{5}\ f_5f_3f_2^2\\ &\quad + \frac{1}{3}\ f_3(f_3 ш f_3 ш f_3) - \frac{799}{72}\ f_3f_9 + 10\ f_3f_7f_2 - \frac{1}{5}\ f_3f_5f_2^2 - \frac{36}{35}\ f_3^2f_2^3\ . \end{aligned} \tag{4.35}$$

Inspecting the lower weight basis $\mathcal{U}_{k\leq 12}$ with (4.35) we have the following basis for $\mathcal{U}_{12}$:

$$\begin{aligned} &\frac{1799}{18}\ f_9f_3 - 32\ f_7f_3f_2 + \frac{1133}{16}\ f_7f_5 + 29\ f_5f_7 - 11\ f_5^2f_2 - \frac{16}{5}\ f_5f_3f_2^2\\ &+ \frac{1}{3}\ f_3(f_3 ш f_3 ш f_3) - \frac{799}{72}\ f_3f_9 + 10\ f_3f_7f_2 - \frac{1}{5}\ f_3f_5f_2^2 - \frac{36}{35}\ f_3^2f_2^3,\\ &-6\ f_5f_7 - 15\ f_7f_5 - 27\ f_9f_3,\ f_3 ш f_9,\ f_5 ш f_7,\ f_3 ш f_3 ш f_3 ш f_3,\\ &(-14f_7f_3 - 6f_5^2)f_2,\ -5\ f_5f_3f_2^2,\ f_5 ш f_5f_2,\ f_3 ш f_7f_2,\\ &f_3 ш f_5f_2^2,\ f_3 ш f_3f_2^3,\ f_2^6\ . \end{aligned} \tag{4.36}$$

Therefore, the decomposition of any motivic MZV ξ_{12} of weight 12 assumes the form

$$\begin{aligned}\xi_{12} &= a_1\ \zeta^m_{1,1,4,6} + a_2\ \zeta^m_{3,9} + a_3\ \zeta^m_9\ \zeta^m_3 + a_4\ \zeta^m_7\ \zeta^m_5 + a_5\ (\zeta^m_3)^4 + a_6\ \zeta^m_{3,7}\ \zeta^m_2 \\ &+ a_7\ \zeta^m_{3,5}\ (\zeta^m_2)^2 + a_8\ (\zeta^m_5)^2\ \zeta^m_2 + a_9\ \zeta^m_7\ \zeta^m_3\ \zeta^m_2 + a_{10}\ \zeta^m_5\ \zeta^m_3\ (\zeta^m_2)^2 \\ &+ a_{11}\ (\zeta^m_3)^2\ (\zeta^m_2)^3 + a_{12}\ (\zeta^m_2)^6\ ,\end{aligned} \tag{4.37}$$

with the following operators

$$\begin{aligned} &a_1 = \frac{48}{691}\ ([\partial_9,\partial_3] - 3\ [\partial_7,\partial_5])\ ,\ a_2 = \frac{1}{27}\ [\partial_9,\partial_3] + \frac{2665}{648}\ a_1, \\ &a_3 = \partial_9\partial_3 + \frac{799}{72}\ a_1,\ a_4 = \partial_7\partial_5 + \frac{2}{9}\ [\partial_9,\partial_3] - \frac{467}{108}\ a_1,\ a_5 = \frac{1}{24}\ \partial_3^4 - \frac{1}{12}\ a_1, \\ &a_6 = \frac{1}{14}\ c_2\ [\partial_7,\partial_3] - 3\ a_1,\ a_7 = \frac{1}{5}\ c_2^2\ [\partial_5,\partial_3] - \frac{3}{5}\ a_1, \\ &a_8 = c_2\ \left(\frac{1}{2}\ \partial_5^2 + \frac{3}{14}\ [\partial_7,\partial_3]\right) - \frac{7}{2}\ a_1,\ a_9 = c_2\ \partial_7\partial_3 - 10\ a_1, \\ &a_{10} = c_2^2\ \partial_5\partial_3 + \frac{1}{5}\ a_1,\ a_{11} = \frac{1}{2}\ c_2^3\ \partial_3^2 + \frac{18}{35}\ a_1,\ a_{12} = c_2^6 \end{aligned} \tag{4.38}$$

acting on $\phi(\xi_{12})$.

4.3.3. Decomposition at weight 13

At weight 13 the following set of motivic MZVs

$$B = \{\ \zeta^m_2,\ \zeta^m_3,\ \zeta^m_5,\ \zeta^m_7,\ \zeta^m_{3,5},\ \zeta^m_9,\ \zeta^m_{3,7},\ \zeta^m_{11},\ \zeta^m_{3,3,5},\ \zeta^m_{3,9},\ \zeta^m_{1,1,4,6},\ \zeta^m_{3,3,7},\ \zeta^m_{3,5,5}\ \} \tag{4.39}$$

represents independent algebra generators up to weight 13. We need to compute $\phi(\zeta^m_{3,3,7})$ and $\phi(\zeta^m_{3,5,5})$, which require the following derivatives

$$\begin{aligned} &\partial^\phi_3 \zeta^m_{3,3,7} = 0\ , && \partial^\phi_9 \zeta^m_{3,3,7} = -\frac{56}{5}\ (\zeta^m_2)^2\ , \\ &\partial^\phi_5 \zeta^m_{3,3,7} = -6\ \zeta^m_{3,5}\ , && \\ &\partial^\phi_7 \zeta^m_{3,3,7} = -7\ (\zeta^m_3)^2 + \frac{32}{35}\ (\zeta^m_2)^3\ , && \partial^\phi_{11} \zeta^m_{3,3,7} = -\frac{407}{2}\ \zeta^m_2\ , \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} &\partial^\phi_3 \zeta^m_{3,5,5} = 0\ , && \partial^\phi_9 \zeta^m_{3,5,5} = -10\ (\zeta^m_2)^2\ , \\ &\partial^\phi_5 \zeta^m_{3,5,5} = -5\ \zeta^m_{3,5}\ , && \partial^\phi_{11} \zeta^m_{3,5,5} = -\frac{275}{2}\ \zeta^m_2\ , \\ &\partial^\phi_7 \zeta^m_{3,5,5} = 0\ , && \end{aligned} \tag{4.41}$$

respectively. The derivatives (4.40) and (4.41) give rise to the maps:

$$\begin{aligned}\phi(\zeta^m_{3,3,7}) &= 30\ f_5^2 f_3 - 7\ f_7(f_3 \sqcup\!\sqcup f_3) + \frac{32}{35}\ f_7 f_2^3 - \frac{56}{5}\ f_9 f_2^2 - \frac{407}{2}\ f_{11} f_2\ ,\\ \phi(\zeta^m_{3,5,5}) &= 25\ f_5^2 f_3 - 10\ f_9 f_2^2 - \frac{275}{2}\ f_{11} f_2\ .\end{aligned} \tag{4.42}$$

Collecting the information about the lower weight basis $\mathcal{U}_{k\leq 13}$ with (4.42) we have the following basis for $\mathcal{U}_{13}$:

$$\begin{aligned}&30\ f_5^2 f_3 - 7\ f_7(f_3 \sqcup\!\sqcup f_3) + \frac{32}{35}\ f_7 f_2^3 - \frac{56}{5}\ f_9 f_2^2 - \frac{407}{2}\ f_{11} f_2\ ,\\ &25\ f_5^2 f_3 - 10\ f_9 f_2^2 - \frac{275}{2}\ f_{11} f_2,\ f_{13},\ (-14 f_7 f_3 - 6 f_5^2) \sqcup\!\sqcup f_3\ ,\\ &-5\ (f_5 f_3) \sqcup\!\sqcup f_5,\ f_7 \sqcup\!\sqcup f_3 \sqcup\!\sqcup f_3,\ f_5 \sqcup\!\sqcup f_5 \sqcup\!\sqcup f_5,\\ &-\frac{5}{2}\ f_5(f_3 \sqcup\!\sqcup f_3) f_2 + \frac{4}{7}\ f_5 f_2^4 - \frac{6}{5}\ f_7 f_2^3 - 45\ f_9 f_2^2,\ -5(f_5 f_3) \sqcup\!\sqcup f_3 f_2,\\ &f_{11} f_2,\ f_5 \sqcup\!\sqcup f_3 \sqcup\!\sqcup f_3 f_2,\ f_3 \sqcup\!\sqcup f_3 \sqcup\!\sqcup f_3 f_2^2,\ f_9 f_2^2,\ f_7 f_2^3,\ f_5 f_2^4,\ f_3 f_2^5\ .\end{aligned} \tag{4.43}$$

Therefore, we have the following decomposition of any motivic MZV ξ_{13} of weight 13:

$$\begin{aligned}\xi_{13} = a_1\ \zeta^m_{3,3,7} + a_2\ \zeta^m_{3,5,5} + a_3\ \zeta^m_{13} + a_4\ \zeta^m_{3,7}\ \zeta^m_3 + a_5\ \zeta^m_{3,5}\ \zeta^m_5 + a_6\ \zeta^m_7 (\zeta^m_3)^2 \qquad (4.44)\\ + a_7\ (\zeta^m_5)^2 \zeta^m_3 + a_8\ \zeta^m_{3,3,5}\ \zeta^m_2 + a_9\ \zeta^m_{3,5}\ \zeta^m_3\ \zeta^m_2 + a_{10}\ \zeta^m_{11}\ \zeta^m_2 + a_{11}\ \zeta^m_5\ (\zeta^m_3)^2\ \zeta^m_2\\ + a_{12}\ (\zeta^m_3)^3\ (\zeta^m_2)^2 + a_{13}\ \zeta^m_9\ (\zeta^m_2)^2 + a_{14}\ \zeta^m_7\ (\zeta^m_2)^3 + a_{15}\ \zeta^m_5\ (\zeta^m_2)^4 + a_{16}\ \zeta^m_3\ (\zeta^m_2)^5\ ,\end{aligned}$$

with the following operators

$$\begin{aligned}&a_1 = \frac{1}{14}\ [\partial_3, [\partial_7, \partial_3]],\ a_2 = \frac{1}{25}\ [\partial_5, [\partial_5, \partial_3]] - \frac{3}{35}\ [\partial_3, [\partial_7, \partial_3]],\ a_3 = \partial_{13},\\ &a_4 = \frac{1}{14}\ [\partial_7, \partial_3]\partial_3,\ a_5 = \frac{1}{5}\ \partial_5[\partial_5, \partial_3],\ a_6 = \frac{1}{2}\partial_7\partial_3^2\ , a_7 = \frac{3}{14}\ [\partial_7, \partial_3]\partial_3 + \frac{1}{2}\ \partial_5^2\partial_3,\\ &a_8 = \frac{1}{5}\ c_2\ [\partial_3, [\partial_5, \partial_3]],\ a_9 = \frac{1}{5}\ c_2[\partial_5, \partial_3]\partial_3,\\ &a_{10} = c_2\partial_{11} + \frac{11}{2}\ [\partial_5, [\partial_5, \partial_3]] + \frac{11}{4}\ [\partial_3, [\partial_7, \partial_3]],\ a_{11} = \frac{1}{2}\ c_2\partial_5\partial_3^2,\ a_{12} = \frac{1}{6}\ c_2^2\partial_3^3,\\ &a_{13} = c_2^2\partial_9 + 9\ c_2[\partial_3, [\partial_5, \partial_3]] + \frac{2}{5}\ [\partial_5, [\partial_5, \partial_3]] - \frac{2}{35}\ [\partial_3, [\partial_7, \partial_3]],\\ &a_{14} = c_2^3\partial_7 + \frac{6}{25}\ c_2[\partial_3, [\partial_5, \partial_3]] - \frac{16}{245}\ [\partial_3, [\partial_7, \partial_3]],\\ &a_{15} = c_2^4\partial_5 - \frac{4}{35}\ c_2[\partial_3, [\partial_5, \partial_3]],\ a_{16} = c_2^5\partial_3\end{aligned} \tag{4.45}$$

acting on $\phi(\xi_{13})$.

4.3.4. Decomposition at weight 14

At weight 14 we take the following set of motivic MZVs

$$B = \{\ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m,\ \zeta_{3,3,7}^m,\ \zeta_{3,5,5}^m,\ \zeta_{3,3,3,5}^m,\ \zeta_{3,11}^m,\ \zeta_{5,9}^m\ \} \tag{4.46}$$

as independent algebra generators up to weight 14. Hence, we only need to compute the maps $\phi(\zeta_{3,11}^m)$, $\phi(\zeta_{5,9}^m)$ and $\phi(\zeta_{3,3,3,5}^m)$, which require the following derivatives

$$\begin{aligned} &\partial_3^\phi \zeta_{3,11}^m = 0\ , & &\partial_9^\phi \zeta_{3,11}^m = -28\ \zeta_5^m\ ,\\ &\partial_5^\phi \zeta_{3,11}^m = -6\ \zeta_9^m\ , & &\partial_{11}^\phi \zeta_{3,11}^m = -44\ \zeta_3^m\ ,\\ &\partial_7^\phi \zeta_{3,11}^m = -15\ \zeta_7^m\ , \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} &\partial_3^\phi \zeta_{5,9}^m = 0\ , & &\partial_9^\phi \zeta_{5,9}^m = -69\ \zeta_5^m\ ,\\ &\partial_5^\phi \zeta_{5,9}^m = 0\ , & &\partial_{11}^\phi \zeta_{3,5,5}^m = -165\ \zeta_3^m\ ,\\ &\partial_7^\phi \zeta_{5,9}^m = -15\ \zeta_7^m\ , \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} &\partial_3^\phi \zeta_{3,3,3,5}^m = 0\ , & &\partial_9^\phi \zeta_{3,3,3,5}^m = -\frac{405}{2}\ \zeta_5^m + 90\ \zeta_3^m\ \zeta_2^m\ ,\\ &\partial_5^\phi \zeta_{3,3,3,5}^m = -\frac{5}{6}\ (\zeta_3^m)^3 - \frac{5}{3}\ \zeta_9^m + \frac{4}{7}\ \zeta_3^m\ (\zeta_2^m)^3\ , & &\partial_{11}^\phi \zeta_{3,3,3,5}^m = -15\ \zeta_3^m\ ,\\ &\partial_7^\phi \zeta_{3,3,3,5}^m = -51\ \zeta_7^m + 30\ \zeta_5^m\ \zeta_2^m\ , \end{aligned} \tag{4.49}$$

respectively. These derivatives give rise to :

$$\begin{aligned} \phi(\zeta_{3,11}^m) &= -6\ f_5 f_9 - 15\ f_7^2 - 28\ f_9 f_5 - 44\ f_{11}\ f_3\ ,\\ \phi(\zeta_{5,9}^m) &= -15\ f_7^2 - 69\ f_9 f_5 - 165\ f_{11}\ f_3\ ,\\ \phi(\zeta_{3,3,3,5}^m) &= -\frac{5}{6}\ f_5\ (f_3 ш f_3 ш f_3) - \frac{5}{3}\ f_5 f_9 + \frac{4}{7}\ f_5 f_3 f_2^3 - 51 f_7^2\\ &\quad + 30\ f_7 f_5 f_2 - \frac{405}{2}\ f_9 f_5 + 90\ f_9 f_3 f_2 - 15\ f_{11} f_3\ , \end{aligned} \tag{4.50}$$

respectively. Gathering the information about the lower weight basis $\mathcal{U}_{k\leq 13}$ with (4.50) we can construct the basis for $\mathcal{U}_{14}$ displayed in (A.1). This basis (A.1) gives rise to the following decomposition of any motivic MZV ξ_{14} of weight 14

$$\xi_{14} = a_1\ \zeta_{3,3,3,5}^m + a_2\ \zeta_{3,11}^m + a_3\ \zeta_{5,9}^m + a_4\ \zeta_{3,3,5}^m\ \zeta_3^m + a_5\ \zeta_{3,5}^m\ (\zeta_3^m)^2 + a_6\ \zeta_3^m\ \zeta_{11}^m$$

$$+ a_7\ (\zeta_3^m)^3\ \zeta_5^m + a_8\ \zeta_5^m\ \zeta_9^m + a_9\ (\zeta_7^m)^2 + a_{10}\ \zeta_{1,1,4,6}^m\ \zeta_2^m + a_{11}\ \zeta_{3,9}^m\ \zeta_2^m$$
$$+ a_{12}\ \zeta_3^m\ \zeta_9^m\ \zeta_2^m + a_{13}\ \zeta_5^m\ \zeta_7^m\ \zeta_2^m + a_{14}\ (\zeta_3^m)^4\ \zeta_2^m + a_{15}\ \zeta_{3,7}^m\ (\zeta_2^m)^2$$
$$+ a_{16}\ \zeta_{3,5}^m\ (\zeta_2^m)^3 + a_{17}\ (\zeta_5^m)^2\ (\zeta_2^m)^2 + a_{18}\ \zeta_7^m\ \zeta_3^m\ (\zeta_2^m)^2 + a_{19}\ \zeta_5^m\ \zeta_3^m\ (\zeta_2^m)^3$$
$$+ a_{20}\ (\zeta_3^m)^2\ (\zeta_2^m)^4 + a_{21}\ (\zeta_2^m)^7 \tag{4.51}$$

with the operators a_i acting on $\phi(\xi_{14})$ and given in (A.2).

4.3.5. Decomposition at weight 15

At weight 15 we have the following set of motivic MZVs

$$B = \{\ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m,\ \zeta_{3,3,7}^m,\ \zeta_{3,5,5}^m,\ \zeta_{3,3,3,5}^m,\ \zeta_{3,11}^m,\ \zeta_{5,9}^m,\ \zeta_{5,3,7}^m,\ \zeta_{3,3,9}^m,\ \zeta_{1,1,3,4,6}^m\} \tag{4.52}$$

as independent algebra generators up to weight 15. Hence, we only need to compute the maps $\phi(\zeta_{5,3,7}^m)$, $\phi(\zeta_{3,3,9}^m)$ and $\phi(\zeta_{1,1,3,4,6}^m)$, which require the following derivatives

$$\begin{aligned}
&\partial_3^\phi \zeta_{5,3,7}^m = 0\ , && \partial_9^\phi \zeta_{5,3,7}^m = \frac{136}{35}\ (\zeta_2^m)^3\ ,\\
&\partial_5^\phi \zeta_{5,3,7}^m = -3\ (\zeta_5^m)^2 + \frac{96}{385}\ (\zeta_2^m)^5 + 6\ \zeta_{3,7}^m\ , && \partial_{11}^\phi \zeta_{5,3,7}^m = -22\ (\zeta_2^m)^2\ ,\\
&\partial_7^\phi \zeta_{5,3,7}^m = -14\ \zeta_{5,3}^m = -14\ \zeta_3^m\ \zeta_5^m + 14\ \zeta_{3,5}^m + \frac{48}{25}\ (\zeta_2^m)^4\ , && \partial_{13}^\phi \zeta_{5,3,7}^m = -\frac{1001}{2}\ \zeta_2^m
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
&\partial_3^\phi \zeta_{3,3,9}^m = 0\ , && \partial_9^\phi \zeta_{3,3,9}^m = -\frac{27}{2}\ (\zeta_3^m)^2 - \frac{116}{35}\ (\zeta_2^m)^3\ ,\\
&\partial_5^\phi \zeta_{3,3,9}^m = -6\ \zeta_{3,7}^m\ , && \partial_{11}^\phi \zeta_{3,3,9}^m = -\frac{252}{5}\ (\zeta_2^m)^2\ ,\\
&\partial_7^\phi \zeta_{3,3,9}^m = -\frac{72}{175}\ (\zeta_2^m)^4 - 15\ \zeta_{3,5}^m\ , && \partial_{13}^\phi \zeta_{3,3,9}^m = -\frac{1209}{2}\ \zeta_2^m\ ,
\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
\partial_3^\phi \zeta_{1,1,3,4,6}^m &= \frac{74}{3}\ \zeta_5^m \zeta_7^m - 83\ \zeta_3^m \zeta_9^m - \frac{29}{9}\ \zeta_{3,9}^m - \zeta_{1,1,4,6}^m + 6\ \zeta_{3,7}^m\ \zeta_2^m + \frac{8}{5}\ \zeta_{3,5}^m\ (\zeta_2^m)^2\\
&\quad + 8\ (\zeta_5^m)^2\ \zeta_2^m + 42\ \zeta_3^m \zeta_7^m \zeta_2^m + \frac{24}{5}\ \zeta_3^m \zeta_5^m (\zeta_2^m)^2 - \frac{1451972}{716625}\ (\zeta_2^m)^6\\
\partial_5^\phi \zeta_{1,1,3,4,6}^m &= -\frac{12263}{112}\ (\zeta_5^m)^2 - \frac{245}{2}\ \zeta_3^m \zeta_7^m + \frac{145}{112}\ \zeta_{3,7}^m - \frac{25}{2}\ \zeta_{3,5}^m \zeta_2^m + \frac{87}{2}\ \zeta_3^m \zeta_5^m \zeta_2^m
\end{aligned}$$

$$+\frac{15}{2}\ (\zeta_3^m)^2(\zeta_2^m)^2+\frac{19939}{1617}\ (\zeta_2^m)^5\ ,$$

$$\partial_7^\phi\zeta_{1,1,3,4,6}^m=\frac{31}{4}\ \zeta_3^m\zeta_5^m+\frac{481}{20}\ \zeta_{3,5}^m-12\ (\zeta_3^m)^2\zeta_2^m+\frac{6404}{2625}\ (\zeta_2^m)^4\ ,$$

$$\partial_9^\phi\zeta_{1,1,3,4,6}^m=-\frac{5599}{72}\ (\zeta_3^m)^2+\frac{25687}{630}\ (\zeta_2^m)^3\ ,$$

$$\partial_{11}^\phi\zeta_{1,1,3,4,6}^m=\frac{28519}{60}\ (\zeta_2^m)^2\ ,$$

$$\partial_{13}^\phi\zeta_{1,1,3,4,6}^m=\frac{56717}{120}\ \zeta_2^m\ , \tag{4.55}$$

respectively. These derivatives give rise to :

$$\begin{aligned}\phi(\zeta_{5,3,7}^m)&=-3\ f_5\ (f_5 ш f_5)+\frac{96}{385}\ f_5f_2^5-6\ f_5\ (14f_7f_3+6f_5^2)-14\ f_7(f_3 ш f_5)\\&-70\ f_7f_5f_3+\frac{48}{25}\ f_7f_2^4+\frac{136}{35}\ f_9f_2^3-22\ f_{11}f_2^2-\frac{1001}{2}\ f_{13}f_2\ ,\\ \phi(\zeta_{3,3,9}^m)&=6\ f_5\ (14f_7f_3+6f_5^2)-\frac{72}{175}\ f_7f_2^4+75\ f_7f_5f_3-\frac{27}{2}\ f_9\ (f_3 ш f_3)\\&-\frac{116}{35}\ f_9f_2^3-\frac{252}{5}\ f_{11}f_2^2-\frac{1209}{2}\ f_{13}f_2\ ,\\ \phi(\zeta_{1,1,3,4,6}^m)&=-\frac{29}{9}\ f_3\ \phi(\zeta_{3,9}^m)-f_3\ \phi(\zeta_{1,1,4,6}^m)+\frac{74}{3}\ f_3(f_5 ш f_7)-83\ f_3(f_3 ш f_9)\\&-6\ f_3(14f_7f_3+6f_5^2)f_2-8\ f_3f_5f_3f_2^2+8\ f_3(f_5 ш f_5)f_2\\&+42\ f_3(f_3 ш f_7)f_2+\frac{24}{5}\ f_3(f_3 ш f_5)f_2^2-\frac{1451972}{716625}\ f_3f_2^6\\&-\frac{12263}{112}\ f_5(f_5 ш f_5)-\frac{245}{2}\ f_5(f_3 ш f_7)-\frac{145}{112}\ f_5(14f_7f_3+6f_5^2)\\&+\frac{125}{2}\ f_5^2f_3f_2+\frac{87}{2}\ f_5(f_3 ш f_5)f_2+\frac{15}{2}\ f_5(f_3 ш f_3)f_2^2+\frac{19939}{1617}\ f_5f_2^5\ ,\\&+\frac{31}{4}\ f_7(f_3 ш f_5)-\frac{481}{4}\ f_7f_5f_3-12\ f_7(f_3 ш f_3)f_2+\frac{6404}{2625}\ f_7f_2^4\ ,\\&-\frac{5599}{72}\ f_9(f_3 ш f_3)+\frac{25687}{630}\ f_9f_2^3+\frac{28519}{60}\ f_{11}\ f_2^2+\frac{56717}{120}\ f_{13}f_2\ ,\end{aligned} \tag{4.56}$$

respectively. The maps $\phi(\zeta_{3,9}^m)$ and $\phi(\zeta_{1,1,4,6}^m)$ are given in (4.35). With the information about the lower weight basis $\mathcal{U}_{k\leq 14}$ with (4.56) we can construct the basis for $\mathcal{U}_{15}$ shown in (A.6). This basis (A.6) gives rise to the following decomposition of any motivic MZV ξ_{15} of weight 15

$$\xi_{15}=a_1\ \zeta_{1,1,3,4,6}^m+a_2\ \zeta_{3,3,9}^m+a_3\ \zeta_{5,3,7}^m+a_4\ \zeta_{15}^m+a_5\ \zeta_{1,1,4,6}^m\ \zeta_3^m+a_6\ \zeta_{3,9}^m\ \zeta_3^m$$

$$+ a_7\ \zeta_9^m\ (\zeta_3^m)^2 + a_8\ \zeta_3^m\ \zeta_5^m\ \zeta_7^m + a_9\ (\zeta_3^m)^5 + a_{10}\ \zeta_{3,7}^m\ \zeta_5^m + a_{11}\ (\zeta_5^m)^3 + a_{12}\ \zeta_{3,5}^m\ \zeta_7^m$$
$$+ a_{13}\ \zeta_2^m\ \zeta_{3,3,7}^m + a_{14}\ \zeta_2^m\ \zeta_{3,5,5}^m + a_{15}\ \zeta_2^m\ \zeta_{13}^m + a_{16}\ \zeta_2^m\ \zeta_3^m\ \zeta_{3,7}^m + a_{17}\ \zeta_2^m\ \zeta_5^m\ \zeta_{3,5}^m$$
$$+ a_{18}\ \zeta_2^m\ (\zeta_3^m)^2\ \zeta_7^m + a_{19}\ \zeta_2^m\ \zeta_3^m\ (\zeta_5^m)^2 + a_{20}\ (\zeta_2^m)^2\ \zeta_{3,3,5}^m + a_{21}\ (\zeta_2^m)^2\ \zeta_3^m\ \zeta_{3,5}^m$$
$$+ a_{22}\ (\zeta_2^m)^2\ \zeta_{11}^m + a_{23}\ \zeta_5^m\ (\zeta_2^m)^2\ (\zeta_3^m)^2 + a_{24}\ (\zeta_2^m)^3\ (\zeta_3^m)^3 + a_{25}\ (\zeta_2^m)^3\ \zeta_9^m$$
$$+ a_{26}\ (\zeta_2^m)^4\ \zeta_7^m + a_{27}\ (\zeta_2^m)^5\ \zeta_5^m + a_{28}\ (\zeta_2^m)^6\ \zeta_3^m\ , \tag{4.57}$$

with the operators a_i acting on $\phi(\xi_{15})$ and collected in (A.7).

4.3.6. Decomposition at weight 16

Finally, at weight 16 we have the following set of motivic MZVs

$$B = \{\ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m,\ \zeta_{3,3,7}^m,\ \zeta_{3,5,5}^m,\ \zeta_{3,3,3,5}^m,\ \zeta_{3,11}^m,\ \zeta_{5,9}^m,\ \zeta_{5,3,7}^m,\ \zeta_{3,3,9}^m,\ \zeta_{1,1,3,4,6}^m,\ \zeta_{3,3,3,7}^m,\ \zeta_{3,3,5,5}^m,\ \zeta_{3,13}^m,\ \zeta_{5,11}^m,\ \zeta_{1,1,6,8}^m\} \tag{4.58}$$

as independent algebra generators up to weight 16. Hence, we only need to compute the maps $\phi(\zeta_{3,3,3,7}^m),\ \phi(\zeta_{3,3,5,5}^m),\ \phi(\zeta_{3,13}^m),\ \phi(\zeta_{5,11}^m)$ and $\phi(\zeta_{1,1,6,8}^m)$, which require the following derivatives

$$\begin{aligned}
&\partial_3^\phi \zeta_{3,3,3,7}^m = 0, \quad \partial_5^\phi \zeta_{3,3,3,7}^m = -6\ \zeta_{3,3,5}^m\ ,\\
&\partial_7^\phi \zeta_{3,3,3,7}^m = -\frac{775}{6}\ \zeta_9^m - \frac{7}{3}\ (\zeta_3^m)^3 + 63\ \zeta_7^m \zeta_2^m + \frac{36}{5}\ \zeta_5^m (\zeta_2^m)^2 + \frac{8}{5}\ \zeta_3^m (\zeta_2^m)^3\ ,\\
&\partial_9^\phi \zeta_{3,3,3,7}^m = -476\ \zeta_7^m + 280\ \zeta_5^m \zeta_2^m\ ,\\
&\partial_{11}^\phi \zeta_{3,3,3,7}^m = -\frac{3723}{4}\ \zeta_5^m + 407\ \zeta_3^m \zeta_2^m, \quad \partial_{13}^\phi \zeta_{3,3,3,7}^m = -165\ \zeta_3^m\ ,
\end{aligned} \tag{4.59}$$

$$\begin{aligned}
&\partial_3^\phi \zeta_{3,3,5,5}^m = 0, \quad \partial_5^\phi \zeta_{3,3,5,5}^m = -5\ \zeta_{3,3,5}^m\ ,\\
&\partial_7^\phi \zeta_{3,3,5,5}^m = \frac{381}{2}\ \zeta_9^m - 105\ \zeta_7^m \zeta_2^m - 6\ \zeta_5^m (\zeta_2^m)^2,\\
&\partial_9^\phi \zeta_{3,3,5,5}^m = 70\ \zeta_7^m + 25\ \zeta_5^m \zeta_2^m - 36\ \zeta_3^m (\zeta_2^m)^2\ ,\\
&\partial_{11}^\phi \zeta_{3,3,5,5}^m = -\frac{1881}{4}\ \zeta_5^m + 275\ \zeta_3^m \zeta_2^m, \quad \partial_{13}^\phi \zeta_{3,3,5,5}^m = \frac{99}{2}\ \zeta_3^m\ ,
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&\partial_3^\phi \zeta_{3,13}^m = 0\ , && \partial_9^\phi \zeta_{3,13}^m = -28\ \zeta_7^m\ ,\\
&\partial_5^\phi \zeta_{3,13}^m = -6\ \zeta_{11}^m\ , && \partial_{11}^\phi \zeta_{3,13}^m = -45\ \zeta_5^m\ ,\\
&\partial_7^\phi \zeta_{3,13}^m = -15\ \zeta_9^m\ , && \partial_{13}^\phi \zeta_{3,13}^m = -65\ \zeta_3^m\ ,
\end{aligned} \tag{4.61}$$

$$\begin{aligned} \partial_3^\phi \zeta^m_{5,11} &= 0 \ , & \partial_9^\phi \zeta^m_{5,11} &= -70 \ \zeta^m_7 \ , \\ \partial_5^\phi \zeta^m_{5,11} &= 0 \ , & \partial_{11}^\phi \zeta^m_{5,11} &= -209 \ \zeta^m_5 \ , \\ \partial_7^\phi \zeta^m_{5,11} &= -15 \ \zeta^m_9 \ , & \partial_{13}^\phi \zeta^m_{5,11} &= -429 \ \zeta^m_3 \ , \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} \partial_3^\phi \zeta^m_{1,1,6,8} &= -\frac{5}{7} \ \zeta^m_{3,3,7} + \frac{6}{7} \ \zeta^m_{3,5,5} - \frac{2}{7} \ \zeta^m_3 \zeta^m_{3,7} - \frac{8497}{42} \ \zeta^m_{13} + \frac{8}{7} \ \zeta^m_3 (\zeta^m_5)^2 + (\zeta^m_3)^2 \zeta^m_7 \\ &\quad + 137 \ \zeta^m_{11} \zeta^m_2 + \frac{11}{7} \ \zeta^m_9 (\zeta^m_2)^2 - \frac{848}{245} \ \zeta^m_7 (\zeta^m_2)^3 - \frac{48}{35} \ \zeta^m_5 (\zeta^m_2)^4 - \frac{816}{2695} \ \zeta^m_3 (\zeta^m_2)^5 \ , \\ \partial_5^\phi \zeta^m_{1,1,6,8} &= -\frac{2}{5} \ \zeta^m_{3,3,5} - \frac{18211}{240} \ \zeta^m_{11} + (\zeta^m_3)^2 \zeta^m_5 + \frac{71}{2} \ \zeta^m_9 \zeta^m_2 + \frac{163}{25} \ \zeta^m_7 (\zeta^m_2)^2 \\ &\quad + \frac{36}{35} \ \zeta^m_5 (\zeta^m_2)^3 - \frac{132}{175} \ \zeta^m_3 (\zeta^m_2)^4 \ , \\ \partial_7^\phi \zeta^m_{1,1,6,8} &= 72 \ \zeta^m_9 + (\zeta^m_3)^3 - 22 \ \zeta^m_7 \zeta^m_2 - 7 \ \zeta^m_5 (\zeta^m_2)^2 - \frac{116}{35} \ \zeta^m_3 (\zeta^m_2)^3 \ , \\ \partial_9^\phi \zeta^m_{1,1,6,8} &= \frac{26921}{72} \ \zeta^m_7 - \frac{277}{2} \ \zeta^m_5 \zeta^m_2 - 41 \ \zeta^m_3 (\zeta^m_2)^2 \ , \\ \partial_{11}^\phi \zeta^m_{1,1,6,8} &= \frac{11536}{15} \ \zeta^m_5 - \frac{727}{2} \ \zeta^m_3 \zeta^m_2, \quad \partial_{13}^\phi \zeta^m_{1,1,6,8} = \frac{28513}{25} \ \zeta^m_3 \ , \end{aligned} \tag{4.63}$$

respectively. These derivatives give rise to :

$$\begin{aligned} \phi(\zeta^m_{3,3,3,7}) &= -6 \ f_5 \ \phi(\zeta^m_{3,3,5}) - \frac{775}{6} \ f_7 f_9 - \frac{7}{3} \ f_7 (f_3 ш f_3 ш f_3) + 63 \ f_7^2 f_2 + \frac{36}{5} \ f_7 f_5 f_2^2 \\ &\quad + \frac{8}{5} \ f_7 f_3 f_2^3 - 476 \ f_9 f_7 + 280 \ f_9 f_5 f_2 - \frac{3723}{4} \ f_{11} f_5 + 407 \ f_{11} f_3 f_2 - 165 \ f_{13} f_3, \\ \phi(\zeta^m_{3,3,5,5}) &= -5 \ f_5 \ \phi(\zeta^m_{3,3,5}) + \frac{381}{2} \ f_7 f_9 - 105 \ f_7^2 f_2 - 6 \ f_7 f_5 f_2^2 + 70 \ f_9 f_7 + 25 \ f_9 f_5 f_2 \\ &\quad - 36 \ f_9 f_3 f_2^2 - \frac{1881}{4} \ f_{11} f_5 + 275 \ f_{11} f_3 f_2 + \frac{99}{2} \ f_{13} f_3 \ , \\ \phi(\zeta^m_{3,13}) &= -6 \ f_5 f_{11} - 15 \ f_7 f_9 - 28 \ f_9 f_7 - 45 \ f_{11} f_5 - 65 \ f_{13} f_3 \ , \\ \phi(\zeta^m_{5,11}) &= -15 \ f_7 f_9 - 70 \ f_9 f_7 - 209 \ f_{11} f_5 - 429 \ f_{13} f_3 \ , \end{aligned} \tag{4.64}$$

$$\begin{aligned} \phi(\zeta^m_{1,1,6,8}) &= -\frac{5}{7} \ f_3 \ \phi(\zeta^m_{3,3,7}) + \frac{6}{7} \ f_3 \ \phi(\zeta^m_{3,5,5}) + \frac{2}{7} \ f_3 [f_3 ш (14 f_7 f_3 + 6 f_5^2)] - \frac{8497}{42} \ f_3 f_{13} \\ &\quad + \frac{8}{7} \ f_3 (f_3 ш f_5 ш f_5) + f_3 (f_3 ш f_3 ш f_7) + 137 \ f_3 f_{11} f_2 + \frac{11}{7} \ f_3 f_9 f_2^2 \\ &\quad - \frac{848}{245} \ f_3 f_7 f_2^3 - \frac{48}{35} \ f_3 f_5 f_2^4 - \frac{816}{2695} \ f_3 f_3 f_2^5 - \frac{2}{5} \ f_5 \ \phi(\zeta^m_{3,3,5}) - \frac{18211}{240} \ f_5 f_{11} \end{aligned}$$

$$+ f_5(f_3 \,ш\, f_3 \,ш\, f_5) + \frac{71}{2}\; f_5 f_9 f_2 + \frac{163}{25}\; f_5 f_7 f_2^2 + \frac{36}{35}\; f_5 f_5 f_2^3 - \frac{132}{175}\; f_5 f_3 f_2^4\;,$$

$$+ 72\; f_7 f_9 + f_7(f_3 \,ш\, f_3 \,ш\, f_3) - 22\; f_7 f_7 f_2 - 7\; f_7 f_5 f_2^2 - \frac{116}{35}\; f_7 f_3 f_2^3 + \frac{26921}{72}\; f_9 f_7$$

$$- \frac{277}{2}\; f_9 f_5 f_2 - 41\; f_9 f_3 f_2^2 + \frac{11536}{15}\; f_{11} f_5 - \frac{727}{2}\; f_{11} f_3 f_2 + \frac{28513}{25}\; f_{13} f_3,$$

respectively. The maps $\phi(\zeta^m_{3,3,5}), \phi(\zeta^m_{3,3,7})$ and $\phi(\zeta^m_{3,5,5})$ are given in (4.28) and (4.42), respectively. Gathering the information about the lower weight basis $\mathcal{U}_{k\leq 15}$ with (4.64) we can construct the basis for $\mathcal{U}_{16}$ shown in (A.8). This basis (A.8) gives rise to the following decomposition of any motivic MZV ξ_{16} of weight 16

$$\begin{aligned}
\xi_{16} = {} & a_1\; \zeta^m_{1,1,6,8} + a_2\; \zeta^m_{3,3,3,7} + a_3\; \zeta^m_{3,3,5,5} + a_4\; \zeta^m_{3,13} + a_5\; \zeta^m_{5,11} + a_6\; \zeta^m_3\; \zeta^m_{3,3,7} \\
& + a_7\; \zeta^m_3\; \zeta^m_{3,5,5} + a_8\; \zeta^m_{13}\; \zeta^m_3 + a_9\; \zeta^m_{3,7}\; (\zeta^m_3)^2 + a_{10}\; \zeta^m_{3,5}\; \zeta^m_3\; \zeta^m_5 + a_{11}\; \zeta^m_7 (\zeta^m_3)^3 \\
& + a_{12}\; (\zeta^m_5)^2\; (\zeta^m_3)^2 + a_{13}\; \zeta^m_9\; \zeta^m_7 + a_{14}\; (\zeta^m_{3,5})^2 + a_{15}\; \zeta^m_{11}\; \zeta^m_5 + a_{16}\; \zeta^m_{3,3,5}\; \zeta^m_5 \\
& + a_{17}\; \zeta^m_2\; \zeta^m_3\; \zeta^m_{3,3,5} + a_{18}\; \zeta^m_{3,5}\; (\zeta^m_3)^2\; \zeta^m_2 + a_{19}\; \zeta^m_{11}\; \zeta^m_3\; \zeta^m_2 + a_{20}\; \zeta^m_5\; (\zeta^m_3)^3\; \zeta^m_2 \\
& + a_{21}\; (\zeta^m_3)^4\; (\zeta^m_2)^2 + a_{22}\; \zeta^m_9\; \zeta^m_3\; (\zeta^m_2)^2 + a_{23}\; \zeta^m_7\; \zeta^m_3\; (\zeta^m_2)^3 + a_{24}\; \zeta^m_5\; \zeta^m_3\; (\zeta^m_2)^4 \\
& + a_{25}\; (\zeta^m_3)^2\; (\zeta^m_2)^5 + a_{26}\; \zeta^m_{3,3,3,5}\; \zeta^m_2 + a_{27}\; \zeta^m_{3,11}\; \zeta^m_2 + a_{28}\; \zeta^m_{5,9}\; \zeta^m_2 + a_{29}\; \zeta^m_9\; \zeta^m_5\; \zeta^m_2 \\
& + a_{30}\; (\zeta^m_7)^2\; \zeta^m_2 + a_{31}\; \zeta^m_{1,1,4,6}\; (\zeta^m_2)^2 + a_{32}\; \zeta^m_{3,9}\; (\zeta^m_2)^2 + a_{33}\; \zeta^m_7\; \zeta^m_5\; (\zeta^m_2)^2 \\
& + a_{34}\; \zeta^m_{3,7}\; (\zeta^m_2)^3 + a_{35}\; \zeta^m_{3,5}\; (\zeta^m_2)^4 + a_{36}\; (\zeta^m_5)^2\; (\zeta^m_2)^3 + a_{37}\; (\zeta^m_2)^8\;,
\end{aligned} \tag{4.65}$$

with the operators a_i acting on $\phi(\xi_{16})$ and listed in (A.9).

4.3.7. Comments on regularizing the coproduct and the map ϕ

Some terms in the sum of the coproduct (4.4) may imply divergences [16,4,6]. Divergences of multiple polylogarithms are end–point divergences, *i.e.* the poles in the integrand (2.5) coincide with the endpoints of the path γ. A canonical regularization has been introduced in [16] by shifting the endpoints by a small parameter ϵ:

$$I^m(0; a_1, \ldots, a_n; 1) \rightarrow I^m(\epsilon; a_1, \ldots, a_n; 1-\epsilon)\;. \tag{4.66}$$

Expanding the latter w.r.t. small ϵ gives a polynomial in $\ln\epsilon$. Its constant term defines the regularized value $\hat{I}^m(0; a_1, \ldots, a_n; 1)$. The coproduct in the non–generic case is defined by replacing in the sum of (4.4) every multiple polylogarithm $I^m(0; a_1, \ldots, a_n; 1)$ by its regularized value $\hat{I}^m(0; a_1, \ldots, a_n; 1)$ [16,4].

Also the coaction (4.12) and therefore (4.13) and (4.18) may be plagued by divergences. We have regularized the terms in the sum (4.18) in the same way as described above for the coproduct (4.4). The problem, which only affects the first factor $c^{\phi}_{2r+1}(\ldots)$ of the terms in (4.18), occurs only in the computation of the maps $\phi(\zeta^m_{1,1,4,6}), \phi(\zeta^m_{1,1,3,4,6})$ and $\phi(\zeta^m_{1,1,6,8})$. In addition, in the above three cases $c^{\phi}_{2r+1}(\ldots)$ computes the coefficient of ζ^m_{2r+1}, which does not depend on the regularization, *i.e.* it is independent on ϵ.

Let us demonstrate the regularization at the computation of $\partial^{\phi}_3(\zeta^m_{1,1,6,8})$, whose result is given in (4.63). With $\zeta^m_{1,1,6,8} = I^m(0;1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0;1)$ computing (4.18) for $r=1$ yields:

$$\begin{aligned}\xi_3 = c^{\phi}_3\left[I^m(0;1,1,0;1) + I^m(0;1,0,1;1)\right]\ I^m(0;1,0,0,0,0,1,0,0,0,0,0,0,0;1)\\ -\ c^{\phi}_3\left[I^m(0;0,0,1;1)\right]\ I^m(0;1,1,0,0,0,1,0,0,0,0,0,0,0;1)\ .\end{aligned} \tag{4.67}$$

Above, the integral $I^m(0;1,0,1;1)$ has to be replaced by its regularized value $\hat{I}^m(0;1,0,1;1)$. The latter is computed from expanding

$$\begin{aligned}I^m(\epsilon;1,0,1;1-\epsilon) &\simeq \int_{\epsilon}^{1-\epsilon}\frac{dt_3}{1-t_3}\int_{\epsilon}^{t_3}\frac{dt_2}{t_2}\int_{\epsilon}^{t_2}\frac{dt_1}{1-t_1}\\ &= -\zeta^m_2\ \ln\epsilon - 2\ \zeta^m_3 + \left[2+\zeta^m_2-(\ln\epsilon)^2\right]\ \epsilon + \mathcal{O}(\epsilon^2)\end{aligned} \tag{4.68}$$

w.r.t. small ϵ. Hence, we have[7]:

$$\hat{I}^m(0;1,0,1;1) = -2\ \zeta^m_3\ . \tag{4.69}$$

Note, that this agrees, with what one would obtain by applying the shuffle rule (4.3)

$$I^m(0;1,0;1)\ I^m(0;1;1) = I^m(0;1,0,1;1) + 2\ I^m(0;1,1,0;1)\ , \tag{4.70}$$

from which we obtain:

$$I^m(0;1,0,1;1) = I^m(0;1,0;1)\ I^m(0;1;1) - 2\ I^m(0;1,1,0;1)\ . \tag{4.71}$$

With $I^m(0;1,1,0;1) = \zeta^m_{1,2} = \zeta^m_3$ the two expressions (4.68) and (4.71) give the same finite piece. This is a consequence of the fact, that the shuffle relation also holds for the canonical regularization of multiple polylogarithms [16]. An other way to arrive at the conclusion (4.69) follows from simply identifying $I^m(a_0;a_1;a_2) \simeq 0$ for $a_i \in \{0,1\}$ in the shuffle relation (4.70), *cf.* [5].

[7] With this result *Eq.* (4.67) becomes: $\xi_3 = c^{\phi}_3\left(\zeta^m_{1,2} - 2\zeta^m_3\right)\ \phi(\zeta^m_{5,8}) + c^{\phi}_3\left(-\zeta^m_3\right)\ \phi(\zeta^m_{1,4,8}) = -\phi(\zeta^m_{5,8}) - \phi(\zeta^m_{1,4,8})$.

4.4. Motivic decomposition operators and powers in α'

By comparing the decomposition operators ξ_l given for $l = 10, \ldots, 16$ in (4.23), (4.31), (4.38), (4.45), (A.2), (A.7) and (A.9), respectively with the corresponding order α'^l in the expansion of (3.20) (with the operators (3.21) and (3.14)) we see an exact match in the coefficient and commutator structure by identifying the motivic derivation operators (4.20) and the matrix operators (3.21)

$$\partial_{2n+1} \simeq M_{2n+1} \ , \tag{4.72}$$

and the coefficient operator c_2 with the matrix operators (3.18):

$$c_2^k \simeq P_{2k} \quad , \quad k \geq 1 \ . \tag{4.73}$$

We can further strengthen this connection. Let $\mathcal{F}$ be the free graded Lie algebra (some vector space over some field F) freely generated by the generators e_{2r+1} of degree $-(2r+1)$ with the Lie–bracket $(e_i, e_j) \mapsto [e_i, e_j]$ and the Jacobi relations:

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0 \ . \tag{4.74}$$

E.g. at weight 11 the elements e_{11} and $[e_3, [e_3, e_5]]$ generate $\mathcal{F}_{11}$. For $f_3, f_5, \ldots$ being the functionals on the vector space generated by the vectors $e_3, e_5, \ldots$ such that $\langle f_i, e_j \rangle = \delta_{ij}$ the dual to the universal enveloping algebra $U(\mathcal{F})$ is isomorphic to the space $\mathcal{U}$ of non–commutative polynomials in f_{2n+1} with the shuffle product [16,30].

In fact, the Lie algebra generators e_i can be identified with the matrices M_i introduced in (3.21), *i.e.*

$$e_i \simeq M_i \ , \tag{4.75}$$

and of course the matrices M_i fulfill the Jacobi identity (4.74):

$$[M_i, [M_j, M_k]] + [M_j, [M_k, M_i]] + [M_k, [M_i, M_j]] = 0 \ . \tag{4.76}$$

To conclude, motivic MZVs encapsulate the α'–expansion of the open superstring amplitude.

5. Motivic structure of the open superstring amplitude

The symbol of a transcendental function represents a motivic road map encoding all the relevant information about the function without further specifying the latter explicitly in terms of multiple polylogarithms [3,2,31]. In particular, the various relations among different multiple polylogarithms become simple algebraic identities in the corresponding tensor algebra. In this section we show, that the isomorphism ϕ, which is induced by the coaction (4.12), encapsulates all the relevant information of the α'–expansion of the open superstring amplitude without further specifying the latter explicitly in terms of MZVs. By passing from the MZVs $\zeta \in \mathcal{Z}$ to their motivic versions $\zeta^m \in \mathcal{H}$ and then mapping the latter to elements $\phi(\zeta^m)$ of the Hopf algebra $\mathcal{U}$ the map ϕ endows the superstring amplitude with its motivic structure: it maps the α'–expansion into a very short and intriguing form in terms of the non–commutative Hopf algebra $\mathcal{U}$. In particular, the various relations among different MZVs become simple algebraic identities in the Hopf algebra $\mathcal{U}$. Moreover, in this writing the final result for the superstring expansion does not depend on the choice of a specific set[8] of MZVs as basis elements.

5.1. Motivic structure up to weight 16

In this section we apply the isomorphism ϕ to the motivic version the open superstring amplitude expression (3.20), with the matrices P, M defined in (3.21) and Q given in (3.14). The action (4.9) of ϕ on the motivic MZVs is explained in the previous section. The first hint of a simplification under ϕ occurs in (3.16) at weight $w = 8$, where the commutator term $[M_5, M_3]$ together with the prefactor $\frac{1}{5}\zeta^m_{3,5}$ conspires into:

$$\phi\left(\zeta_3^m\zeta_5^m\ M_5M_3 + \frac{1}{5}\ \zeta^m_{3,5}\ [M_5, M_3] \right)\ A\ =\ (f_3f_5\ M_5M_3\ +\ f_5f_3\ M_3M_5)\ A\ . \tag{5.1}$$

The right hand side obviously treats the objects f_3, M_3 and f_5, M_5 in a democratic way. The effect of the map ϕ becomes even more drastic at weight $w = 11$ at the permutations of $M_3M_3M_5$:

$$\begin{aligned}\phi(\ \mathcal{A}|_{w=11}\ ) = \phi\Bigg(& \frac{1}{5}\ \zeta^m_{3,3,5}\ [M_3,[M_5,M_3]] + \frac{1}{5}\ \zeta^m_{3,5}\zeta^m_3\ [M_5,M_3]M_3 + \zeta^m_{11}\ M_{11} \\ & + \zeta^m_3(\zeta^m_2)^4 P_8M_3 + \frac{1}{2}\ (\zeta^m_3)^2\zeta^m_5\ M_5M_3^2\ +\ \frac{1}{6}\ (\zeta^m_3)^3\zeta^m_2\ P_2M_3^3 \\ & + \zeta^m_9\zeta^m_2\ \ \{P_2M_9 + 9\ [M_3,[M_5,M_3]]\} \\ & + \zeta^m_7(\zeta^m_2)^2\ \left\{P_4M_7 + \frac{6}{25}\ [M_3,[M_5,M_3]]\right\} \\ & + \zeta^m_5(\zeta^m_2)^3\ \left\{P_6M_5 - \frac{4}{35}\ [M_3,[M_5,M_3]]\right\}\Bigg)\ A \end{aligned} \tag{5.2}$$

[8] For instance instead of taking a basis containing the depth one elements ζ^m_{2n+1} one also could choose the set of Lyndon words in the Hoffman elements $\zeta^m_{n_1,\ldots,n_r}$, with $n_i = 2, 3$ [21,19].

$$= \big[f_{11}\ M_{11} + f_3^2 f_5\ M_5 M_3^2 + f_3 f_5 f_3\ M_3 M_5 M_3 + f_5 f_3^2\ M_3^2 M_5$$
$$+ P_2 f_2\ \big(f_9\ M_9\ +\ f_3^3\ M_3^3\big) + P_4 f_2^2 f_7\ M_7 + P_6 f_2^3 f_5\ M_5 + P_8 f_2^4 f_3\ M_3\big]\ A\ .$$

We observe from (5.1) and (5.2) that in the Hopf algebra $\mathcal{U}$, every non–commutative word of odd letters f_{2k+1} multiplies the associated reverse product of matrices M_{2k+1}. Powers f_2^k of the commuting generator f_2 are accompanied by P_{2k}, which multiplies all the operators M_{2k+1} from the left. Most notably, in contrast to the representation in terms of motivic MZVs, the numerical factors become unity, *i.e.* all the rational numbers in (3.13) drop out. Our explicit results confirm, that the beautiful structure with the combination of operators $M_{i_p}\ldots M_{i_2} M_{i_1}$ accompanying the word $f_{i_1} f_{i_2}\ldots f_{i_p}$, continues to hold through weight $w = 16$:

$$\begin{aligned}
\phi\,(\mathcal{A}) = \big(&\ 1 + f_2 P_2 + f_2^2 P_4 + f_2^3 P_6 + f_2^4 P_8 + f_2^5 P_{10} + f_2^6 P_{12} + f_2^7 P_{14} + f_2^8 P_{16} + \ldots\ \big)\\
\times \big\{&\ 1 + f_3\ M_3 + f_5\ M_5 + f_3^2\ M_3^2 + f_7\ M_7 + f_3 f_5\ M_5 M_3 + f_5 f_3\ M_3 M_5\\
&+ f_9\ M_9 + f_3^3\ M_3^3 + f_5^2\ M_5^2 + f_3 f_7\ M_7 M_3 + f_7 f_3\ M_3 M_7 + f_{11}\ M_{11}\\
&+ f_3^2 f_5\ M_5 M_3^2 + f_3 f_5 f_3\ M_3 M_5 M_3 + f_5 f_3^2\ M_3^2 M_5 + f_3^4\ M_3^4 + f_3 f_9\ M_9 M_3\\
&+ f_9 f_3\ M_3 M_9 + f_5 f_7\ M_7 M_5 + f_7 f_5\ M_5 M_7 + f_{13}\ M_{13} + f_3^2 f_7\ M_7 M_3^2\\
&+ f_3 f_7 f_3\ M_3 M_7 M_3 + f_7 f_3^2\ M_3^2 M_7 + f_3 f_5^2\ M_5^2 M_3 + f_5 f_3 f_5\ M_5 M_3 M_5\\
&+ f_5^2 f_3\ M_3 M_5^2 + f_7^2\ M_7^2 + f_3 f_{11}\ M_{11} M_3 + f_{11} f_3\ M_3 M_{11} + f_5 f_9\ M_9 M_5\\
&+ f_9 f_5\ M_5 M_9 + f_3^3 f_5\ M_5 M_3^2 + f_3^2 f_5 f_3\ M_3 M_5 M_3^2 + f_3 f_5 f_3^2\ M_3^2 M_5 M_3\\
&+ f_5 f_3^3\ M_3^3 M_5 + f_{15}\ M_{15} + f_5^3\ M_5^3 + f_3^5\ M_3^5 + f_3^2 f_9\ M_9 M_3^2 + f_3 f_9 f_3\ M_3 M_9 M_3\\
&+ f_9 f_3^2\ M_3^2 M_9 + f_3 f_5 f_7\ M_7 M_5 M_3 + f_3 f_7 f_5\ M_5 M_7 M_3 + f_5 f_3 f_7\ M_7 M_3 M_5\\
&+ f_5 f_7 f_3\ M_3 M_7 M_5 + f_7 f_3 f_5\ M_5 M_3 M_7 + f_7 f_5 f_3\ M_3 M_5 M_7 + f_7 f_9\ M_9 M_7\\
&+ f_9 f_7\ M_7 M_9 + f_{11} f_5\ M_5 M_{11} + f_5 f_{11}\ M_{11} M_5 + f_3 f_{13}\ M_{13} M_3 + f_{13} f_3\ M_3 M_{13}\\
&+ f_3^2 f_5^2\ M_5^2 M_3^2 + f_5^2 f_3^2\ M_3^2 M_5^2 + f_3 f_5^2 f_3\ M_3 M_5^2 M_3 + f_5 f_3^2 f_5\ M_5 M_3^2 M_5\\
&+ f_3 f_5 f_3 f_5\ M_5 M_3 M_5 M_3 + f_5 f_3 f_5 f_3\ M_3 M_5 M_3 M_5 + f_3^3 f_7\ M_7 M_3^3\\
&+ f_3^2 f_7 f_3\ M_3 M_7 M_3^2 + f_3 f_7 f_3^2\ M_3^2 M_7 M_3 + f_7 f_3^3\ M_3^3 M_7 + \ldots\ \big\}\ A.
\end{aligned} \tag{5.3}$$

This writing of the amplitude in terms of elements of the algebra comodule $\mathcal{U}$ encodes all the information contained in (3.14).

5.2. Motivic structure at general weight

Motivated by the observation that every non–commutative word constructed from odd generators f_{2k+1} shows up in (5.3) we conjecture the following all–weight–formula for image $\phi(\mathcal{A})$:

$$\phi(\mathcal{A}) = \left(\sum_{k=1}^{\infty} f_2^k\ P_{2k}\right)\left(\sum_{p=0}^{\infty}\sum_{\substack{i_1,\ldots,i_p\\ \in 2\mathbf{N}^+-1}} f_{i_1}f_{i_2}\ldots f_{i_p}M_{i_p}\ldots M_{i_2}M_{i_1}\right)\ A\ . \tag{5.4}$$

In (5.4) the sum over the combinations $f_{i_1}f_{i_2}\ldots f_{i_p}M_{i_p}\ldots M_{i_2}M_{i_1}$ includes *all* possible non–commutative words $f_{i_1}f_{i_2}\ldots f_{i_p}$ with coefficients $M_{i_p}\ldots M_{i_2}M_{i_1}$ graded by their length p. Matrices P_{2k} associated with the powers f_2^k always act by left multiplication. The commutative nature of f_2 with respect to the odd generators f_{2k+1} ties in with the fact that the P_{2k} matrices have the well defined place left of M_{2k+1} in the matrix ordering. One can easily check that (5.4) is compatible through weights ≤ 16, see (5.3), and we shall give further evidence that the validity extends to higher weights.

We have already pointed out in subsection 4.4 that the decomposition formulae for MZVs ξ_w of weight w exactly match the corresponding α'^w part of the superstring amplitude subject to the replacements (4.72) and (4.73) in the differential operators a_i. If this mapping holds to arbitrary weight, then the simplicity of our final result (5.4) reflects the role of $\phi(\xi_w)$ being the unit operator projected to weight w, *e.g.*

$$\begin{aligned}\phi(\xi_{10}) =\ & f_2^5c_2^5\ +\ f_2^2f_3^2\,c_2^2\partial_3^2\ +\ f_2\left(f_3f_5\,\partial_5\partial_3\ +\ f_5f_3\,\partial_3\partial_5\right)c_2\\ & + f_5^2\,\partial_5^2\ +\ f_7f_3\,\partial_3\partial_7\ +\ f_3f_7\,\partial_7\partial_3\ =\ id\,\Big|_{w=10}\end{aligned}$$

maps any non-commutative weight ten polynomial of f_2, f_3, f_5, f_7, f_9 to itself. More generally, the differential operator $c_2^k\partial_{i_p}\ldots\partial_{i_2}\partial_{i_1}$ annihilates all $\mathcal{U}$ elements except for $f_2^kf_{i_1}f_{i_2}\ldots f_{i_p}$. Hence, the weight w identity operator is given by

$$\begin{aligned}\phi(\xi_w)\ =\ & \sum_{k=1}^{\infty} f_2^kc_2^k\ \sum_{p=0}^{\infty}\sum_{\substack{i_1,\ldots,i_p\\ \in 2\mathbf{N}^+-1}} f_{i_1}f_{i_2}\ldots f_{i_p}\,\partial_{i_p}\ldots\partial_{i_2}\partial_{i_1}\\ & \times\,\delta\big(2(i_1+i_2+\ldots i_p)\ +\ p\ +\ 2k-w\big)\ =\ id\,\Big|_w\end{aligned} \tag{5.6}$$

where the $\delta(\ldots)$ function makes sure that the correct weight is picked out. Clearly, (5.6) maps to the weight w contributions of (5.4) under the replacements (4.72) and (4.73). In this sense, the image under ϕ of the disk amplitude at weight w is closely related to the identity operator in the algebra comodule $\mathcal{U}$, restricted to weight w.

6. Closed superstring amplitude

The string world–sheet describing the tree–level string S–matrix of N gravitons is described by a complex sphere with N (integrated) insertions of graviton vertex operators.

One of the key properties of graviton amplitudes in string theory is that at tree–level they can be expressed as sum over squares of (color ordered) gauge amplitudes in the left– and right–moving sectors. This map, known as Kawai–Lewellen–Tye (KLT) relations [9], gives a relation between a closed string tree–level amplitude on the sphere and a sum of squares of (partial ordered) open string disk amplitudes. We may write these relations in matrix notation as follows

$$\mathcal{M} = \mathcal{A}^t\ S\ \mathcal{A}\ , \tag{6.1}$$

with the vector $\mathcal{A}$ encoding the $(N-3)!$ independent color ordered open string subamplitudes and some $(N-3)! \times (N-3)!$ matrix S. The latter encodes the sin–factors from the KLT relations [9] and the contributions from the monodromy relations [10,11] to express the both left– and right–movers in terms of the same open string basis $\mathcal{A}$. Hence, in superstring theory the tree–level computation of graviton amplitudes boils down to considering squares of tree–level gauge amplitudes $\mathcal{A}$ given in (3.1). For this sector the explicit expression (3.20) and subsequent results from the previous sections can be used. The KLT relations are insensitive to the compactification details or the amount of supersymmetries of the superstring background. Hence, the following discussions and results are completely general.

In the sequel we shall discuss the implication of (3.20) to the closed string amplitude $\mathcal{M}(1,\ldots,N)$ involving N closed strings. Especially, we shall be interested in the structure of its α'–expansion. The latter has been already investigated up to the order α'^8 for the cases $N=4,5$ and $N=6$ with the remarkable observation, that the graviton amplitudes do not seem to allow for Riemann zeta functions involving even entries or MZVs of depth higher than one [32]. With the explicit expression (3.20) for the open superstring amplitude we are able to bolster these findings.

6.1. $N=4$

For $N=4$ the KLT relation (6.1) can be written as:

$$\mathcal{M}(1,2,3,4) = \mathcal{A}^t\ S\ \mathcal{A}\ , \tag{6.2}$$

with the basis $\mathcal{A} = \mathcal{A}(1,2,3,4)$ of open string amplitudes (3.5) and the scalar:

$$S = \sin(\pi s)\ \frac{\sin(\pi u)}{\sin(\pi t)}\ . \tag{6.3}$$

With (3.5) and

$$P = \left\{\pi\ \frac{s\ u}{s+u}\ \frac{\sin[\pi(s+u)]}{\sin(\pi s)\ \sin(\pi u)}\right\}^{1/2}\ , \tag{6.4}$$

Eq. (6.2) yields:

$$\mathcal{M}(1,2,3,4) = \pi \ \frac{su}{s+u} \ \exp\left\{2\sum_{n\geq 1}\zeta_{2n+1}\ M_{2n+1}\right\}\ |A|^2\ , \tag{6.5}$$

with the YM subamplitude $A = A_{YM}(1,2,3,4)$ and M_{2n+1} given in (3.6). Obviously, in the four–graviton amplitude (6.5), not any Riemann zeta function with even entries shows up.

The field–theory contribution from (6.2) arises from $P = 1$ and $\mathcal{A} = A$, *i.e.*

$$\mathcal{M}(1,2,3,4)|_{FT} = A^t\ S_0\ A\ , \tag{6.6}$$

with

$$S_0 \equiv S|_{FT} = -\pi\ s\ \frac{u}{t}\ . \tag{6.7}$$

We observe, that:

$$P^t S P = S_0\ . \tag{6.8}$$

This equation guarantees the absence of powers of ζ_2 in (6.5). Stated differently, the absence of powers of ζ_2 in (6.2) allows to determine P from the equation (6.8) as:

$$P = S_0^{1/2}\ (S^{-1})^{1/2}\ . \tag{6.9}$$

6.2. $N = 5$

For $N = 5$ the closed string amplitude (6.1) can be cast into

$$\mathcal{M}(1,2,3,4,5) = \mathcal{A}^t\ S\ \mathcal{A}\ , \tag{6.10}$$

with the basis $\mathcal{A}$ of open string amplitudes given in (3.11) and the symmetric matrix S encoding the diagonal matrix $\text{diag}\{\sin(\pi s_{12})\sin(\pi s_{34}),\ \sin(\pi s_{13})\sin(\pi s_{24})\}$ from the KLT relation [9] and further sin–factors from the monodromy relations [10,11] expressing the string amplitudes $\mathcal{A}(2,1,4,3,5)$ and $\mathcal{A}(3,1,4,2,5)$ in terms of the basis elements $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$. More precisely, we have

$$S = [\sin(\pi s_{35})\ \sin(\pi s_{25})\ \sin(\pi s_{14})]^{-1}\ \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12} & \Sigma_{22}\end{pmatrix}\ , \tag{6.11}$$

with:

$$\begin{aligned}
\Sigma_{11} &= \frac{1}{4}\ \sin(\pi s_1)\ \sin(\pi s_3)\ [\ \sin\pi(s_1-s_2-s_3) - \sin\pi(s_1+s_2-s_3)\\
&\quad + \sin\pi(s_1+s_2+s_3) + \sin\pi(s_1+s_2-s_3-2s_4)\\
&\quad + \sin\pi(-s_1+s_2+s_3-2s_5) - \sin\pi(s_1+s_2+s_3-2s_4-2s_5)\ ]\ ,\\
\Sigma_{12} &= -\sin(\pi s_1)\ \sin(\pi s_3)\ \sin(\pi s_{13})\ \sin(\pi s_{24})\ \sin\pi(s_4+s_5)\ ,\\
\Sigma_{22} &= \frac{1}{4}\ \sin(\pi s_{13})\ \sin(\pi s_{24})\ [\ \sin\pi(s_1+s_2-s_3-s_4-s_5)\\
&\quad - \sin\pi(s_1+s_2+s_3-s_4-s_5) + \sin\pi(s_1-s_2-s_3-s_4+s_5)\\
&\quad - \sin\pi(s_1-s_2-s_3+s_4+s_5) + \sin\pi(s_1+s_2+s_3+s_4+s_5)\ ]\ .
\end{aligned} \tag{6.12}$$

The field–theory contribution from (6.10) arises from $P=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\mathcal{A}=A$, with the YM basis vector A given in (3.11), *i.e.*

$$\mathcal{M}(1,2,3,4,5)|_{FT}=A^t\ S_0\ A\ , \tag{6.13}$$

with

$$S_0\equiv S|_{FT}=(s_{25}\ s_{35}\ s_{14})^{-1}\ \begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{12}&\sigma_{22}\end{pmatrix}\ , \tag{6.14}$$

and:

$$\begin{aligned}
\sigma_{11}&=s_1s_3\ [\ s_4(s_3-s_5)(-s_2+s_4+s_5)+s_1(-s_3(s_4+s_5)+s_5(-s_2+s_4+s_5)\ ]\ ,\\
\sigma_{12}&=-s_1s_3\ s_{13}\ s_{24}\ (s_4+s_5)\ ,\\
\sigma_{22}&=-s_{13}\ s_{24}\ [\ s_1s_4(s_2+s_3)+s_1s_3s_5+s_2s_5(s_3+s_4)\ ]\ .
\end{aligned}\tag{6.15}$$

By considering the closed superstring amplitude (6.10) and analyzing its α'–expansion [32] we find, that the following matrix equation holds:

$$P^t\ SP=S_0\ . \tag{6.16}$$

We have checked the validity of (6.16) up to the order α'^{18}. In addition, we find the relations:

$$\begin{aligned}
S_0\ Q_{(2)}+Q^t_{(2)}\ S_0&=0\quad ,\quad Q_{(2)}=[M_l,M_m]\ ,\\
S_0\ Q_{(3)}-Q^t_{(3)}\ S_0&=0\quad ,\quad Q_{(3)}=[M_l,[M_m,M_n]]\ ,\\
S_0\ Q_{(4)}+Q^t_{(4)}\ S_0&=0\quad ,\quad Q_{(4)}=[M_k,[M_l,[M_m,M_n]]]\ .
\end{aligned}\tag{6.17}$$

We have verified the above relations up to weight 19 for the following commutators

$$\begin{aligned}
Q_{(2)}&=[M_5,M_3],\ [M_7,M_3],\ [M_9,M_3],\ [M_7,M_5],\ [M_9,M_5],\ [M_{11},M_3],\ [M_9,M_7]\ ,\\
&\quad [M_{11},M_5],\ [M_{13},M_3]\ ,\\
Q_{(3)}&=[M_3,[M_5,M_3]],\ [M_3,[M_7,M_3]],\ [M_5,[M_5,M_3]],\ [M_3,[M_7,M_5]],\ [M_5,[M_7,M_3]]\ ,\\
&\quad [M_3,[M_9,M_3]],\ [M_7,[M_7,M_3]],\ [M_7,[M_7,M_5]]\ ,\\
Q_{(4)}&=[M_3,[M_3,[M_5,M_3]]],\ [M_3,[M_3,[M_7,M_3]]],\ [M_3,[M_5,[M_5,M_3]]]\ ,
\end{aligned}\tag{6.18}$$

respectively. The alternating sign in (6.17) can be explained through the transposition

$$[M_{n_2},[M_{n_3},\ldots,[M_{n_r},M_{n_1}]]\ldots]^t=(-1)^{r-1}[M^t_{n_2},[M^t_{n_3},\ldots,[M^t_{n_r},M^t_{n_1}]]\ldots] \tag{6.19}$$

of $(r-1)$–fold nested commutators.

To conclude: the consequence of the relations (6.16) and (6.17) is, that in the α'–expansion of (6.10) the zeta function ζ_2 or powers thereof and MZVs $\zeta_{n_1,\ldots,n_r}$ of depth greater than one $r>1$ drop out. This result is in agreement with the observation made in [32]. We now have verified this observation through weight 18.

6.3. General N

The general form of the closed string amplitude is given in (6.1),

$$\mathcal{M}(1,\ldots,N)=\mathcal{A}^t\ S\ \mathcal{A}\ , \tag{6.20}$$

with the $(N-3)!\times(N-3)!$ matrix S specified above and the vector $\mathcal{A}$ encoding the $(N-3)!$ open string subamplitudes (3.20). Let us now phrase the cancellation of ζ_2 or powers thereof and MZVs $\zeta_{n_1,\ldots,n_r}$ of depth greater than one $r>1$ in the α'–expansion of (6.20). With the explicit expression for (3.20) the following equation guarantees the absence of any power of ζ_2

$$P^t\ SP=S_0\ , \tag{6.21}$$

with:

$$S_0\equiv S|_{FT}\ . \tag{6.22}$$

On the other hand, the vanishing of any MZV $\zeta_{n_1,\ldots,n_r}$ of depth greater than one $r>1$ demands for a cancellation of all commutator terms in Q. This fact translates into

$$S_0\ Q_{(r)}+(-1)^r\ Q^t_{(r)}S_0=0\ , \tag{6.23}$$

with:

$$Q_{(r)}=[M_{n_2},[M_{n_3},\ldots,[M_{n_r},M_{n_1}]]\ldots]\ .$$

With these informations (6.21) and (6.23) the closed superstring (6.20) amplitude for any number N of external assumes the generic form

$$\mathcal{M}(1,\ldots,N)=A^t\ \left(:\exp\left\{\sum_{r\geq1}\zeta_{2r+1}\ M_{2r+1}\right\}:\right)^t\ S_0\ \ :\exp\left\{\sum_{s\geq1}\zeta_{2s+1}\ M_{2s+1}\right\}:\ A\ , \tag{6.24}$$

with the $(N-3)!$ dimensional vector A specifying a YM basis $A\equiv A_{YM}$, the $(N-3)!\times(N-3)!$ matrix S_0 introduced in (6.1) and the $(N-3)!\times(N-3)!$ matrices M_{2n+1} defined in (3.21). The ordering colons enclosing the exponentials are defined in (3.15).

We would like to mention two final remarks: Similarly as for the $N=4$ case (6.9) it should be possible to directly determine P from the matrix equation (6.21). Furthermore, *Eq.* (6.23) gives rise to a recursive relation allowing to determine higher M_{2l+1} from lower M_{2m+1}. These properties are further investigated and exhibited in more details in [26].

Of course, with the explicit expression for P and M the relations (6.21) and (6.23) and hence (6.24) can be verified to all orders.

6.4. Motivic structure of the closed superstring amplitude

Experiencing the simplicity in the open string sector suggests to also investigate the image under ϕ of the gravity amplitude (6.20). We insert the result (5.4) for $\phi(\mathcal{A})$ into (6.20). The multiplication rule (4.11) of the isomorphism ϕ yields:

$$\phi(\mathcal{M}) = A^t \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbf{N}^+-1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1} \right\}^t \\ ш\ S_0 \left\{ \sum_{q=0}^{\infty} \sum_{\substack{j_1,\ldots,j_q \\ \in 2\mathbf{N}^+-1}} f_{j_1} f_{j_2} \ldots f_{j_q} M_{j_q} \ldots M_{j_2} M_{j_1} \right\} A\ . \tag{6.25}$$

The sum over $f_2^k P_{2k}$ in the open string amplitudes $\mathcal{A}^t, \mathcal{A}$ has already been dropped taking into account the cancellation (6.21) of the matrix P.

The representation (6.25) for $\phi(\mathcal{M})$ has the shortcoming that it hides the cancellation of MZVs of depth higher than one. In the motivic picture in terms of motivic MZVs the cancelation of MZVs of depth ≥ 2 has its origin in the commutation relations (6.23). Indeed, these commutation relations are required to show that all the ordered products in $f_{i_1} f_{i_2} \ldots f_{i_p}$ in (6.25) can effectively be replaced by shuffle products. The difference between the two products is multiplied by commutators of M_{2k+1} which cancel around S_0 through (6.23). Once all the f_{2k+1} products are of shuffle type, they can be identified as the image under ϕ of single zetas:

$$f_{i_1} \,ш\, f_{i_2} \,ш\, \ldots \,ш\, f_{i_p} = \phi\Big(\prod_{j=1}^{p} \zeta^m_{i_j} \Big)\ . \tag{6.26}$$

An alternative approach is to start from (6.24), where the reduction to single zetas is already manifest. One then obtains a representation in $\mathcal{U}$, which appears to differ from (6.25). However, the difference between the two is again composed of M_{2k+1} commutators which ultimately drop out thanks to the relations (6.23).

7. Conclusion

In this work we have investigated the structure of the α'–expansion of the open and closed superstring amplitude at tree–level with particular emphasis on their transcendentality properties. The strict matching of powers α'^w with their associated MZV prefactors of weight w constituting a well–confirmed pattern has been considerably refined.

The main point is to replace the $\mathbf{C}$ valued MZVs ζ by more abstract versions thereof, the so–called motivic MZVs ζ^m, which are endowed by a coalgebra structure. Furthermore, through the isomorphism ϕ the motivic MZVs are mapped into an algebra comodule generated by the non–commutative words in generators $f_3, f_5, f_7, \ldots$ and an additional element f_2. In the same way as the symbol conveniently captures patterns of field theory amplitudes the isomorphism ϕ yields a strikingly simple and compact expression (5.4) for the open superstring disk amplitude: the systematics of the α'–dependence is written in closed and short form to all weights. In contrast to the symbol, the map ϕ does not lose any information and can be inverted to recover the tree amplitude in terms of motivic MZVs.

In the closed superstring sector the properties of the matrix P encoded in (6.21) and the commutation relations (6.23) between different matrices M_{2r+1} result in the compact form (6.24), with any power of ζ_2 and all MZVs of depth greater than one cancelled. On the other hand, after applying the map ϕ this result turns into (6.25), in which all matrices M_{2r+1} and Hopf–algebra generators f_{2s+1} are treated democratically without the necessity for the ordering prescription (3.15) in (6.24)

The polynomial structure of the matrices M and P and various other aspects of α'–expansions are further elaborated in [26].

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Appendix A. Decomposition of motivic multi zeta values

A.1. Decomposition at weight 14

Gathering the information about the lower weight basis $\mathcal{U}_{k\leq 13}$ with (4.50) we can construct the following basis for $\mathcal{U}_{14}$:

$$\begin{aligned}
&-\frac{5}{6}\ f_5\ (f_3 ш f_3 ш f_3) - \frac{5}{3}\ f_5 f_9 + \frac{4}{7}\ f_5 f_3 f_2^3 - 51 f_7^2 + 30\ f_7 f_5 f_2 - \frac{405}{2}\ f_9 f_5 \\
&+ 90\ f_9 f_3 f_2 - 15\ f_{11} f_3\ , \\
&-6\ f_5 f_9 - 15\ f_7^2 - 28\ f_9 f_5 - 44\ f_{11}\ f_3,\ -15\ f_7^2 - 69\ f_9 f_5 - 165\ f_{11}\ f_3, \\
&\left(-\frac{5}{2}\ f_5(f_3 ш f_3) + \frac{4}{7}\ f_5 f_2^3 - \frac{6}{5}\ f_7 f_2^2 - 45\ f_9 f_2\right) ш f_3,\ -5\ (f_5 f_3) ш f_3 ш f_3, \\
&f_{11} ш f_3,\ f_3 ш f_3\ \ ш f_5\ \ ш f_3,\ f_9 ш f_5,\ f_7 ш f_7, \qquad\qquad (A.1) \\
&\left(\frac{1799}{18}\ f_9 f_3 - 32\ f_7 f_3 f_2 + \frac{1133}{16}\ f_7 f_5 + 29\ f_5 f_7 - 11\ f_5^2 f_2 - \frac{16}{5}\ f_5 f_3 f_2^2\right. \\
&\left.+\frac{1}{3}\ f_3(f_3 ш f_3 ш f_3) - \frac{799}{72}\ f_3 f_9 + 10\ f_3 f_7 f_2 - \frac{1}{5}\ f_3 f_5 f_2^2 - \frac{36}{35}\ f_3^2 f_2^3\right) f_2, \\
&(-6\ f_5 f_7 - 15\ f_7 f_5 - 27\ f_9 f_3)\, f_2,\ f_9 ш f_3 f_2,\ f_7 ш f_5 f_2,\ f_3 ш f_3 ш f_3 ш f_3 f_2, \\
&(-14 f_7 f_3 - 6 f_5^2) f_2^2,\ -5\ f_5 f_3 f_2^3,\ f_5 ш f_5 f_2^2,\ f_3 ш f_7 f_2^2,\ f_3 ш f_5 f_2^3,\ f_3 ш f_3 f_2^4,\ f_2^7\ .
\end{aligned}$$

The operators a_i of the decomposition (4.51) are:

$$\begin{aligned}
&a_1 = \frac{1}{5}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]],\ a_2 = -\frac{23}{198}\ [\partial_{11}, \partial_3] + \frac{5}{18}\ [\partial_9, \partial_5] - \frac{12841}{1188}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]], \\
&a_3 = -\frac{2}{27}\ [\partial_9, \partial_5] + \frac{1}{27}\ [\partial_{11}, \partial_3] + \frac{232}{81}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]], \\
&a_4 = \frac{1}{5}\ [\partial_3, [\partial_5, \partial_3]]\partial_3,\ a_5 = \frac{1}{10}\ [\partial_5, \partial_3]\partial_3^2,\ a_6 = \partial_{11}\partial_3,\ a_7 = \frac{1}{6}\ \partial_5 \partial_3^3, \\
&a_8 = \partial_9 \partial_5 - \frac{23}{33}\ [\partial_{11}, \partial_3] + \frac{5}{3}\ [\partial_9, \partial_5] - \frac{12775}{198}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]]\ , \\
&a_9 = \frac{1}{2}\ \partial_7^2 - \frac{235}{396}\ [\partial_{11}, \partial_3] + \frac{55}{36}\ [\partial_9, \partial_5] - \frac{647287}{11880}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]] \\
&a_{10} = c_2 a_0,\ a_{11} = c_2 \left(\frac{1}{27}\ [\partial_9, \partial_3] + \frac{2665}{648}\ a_0\right) + \frac{2}{3}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]], \\
&a_{12} = c_2 \left(\partial_9 \partial_3 + \frac{799}{72}\ a_0\right) + 9\ [\partial_3, [\partial_5, \partial_3]]\partial_3,
\end{aligned}$$

$$a_{13} = c_2 \left(\partial_7\partial_5 + \frac{2}{9}\ [\partial_9, \partial_3] - \frac{467}{108}\ a_0\right) + 4\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]],$$

$$a_{14} = c_2\ \left(\frac{1}{24}\partial_3^4 - \frac{1}{12}\ a_0\right),\ a_{15} = \frac{1}{14}\ c_2^2\ [\partial_7, \partial_3] - 3\ c_2 a_0,$$

$$a_{16} = \frac{1}{5} c_2^3\ [\partial_5, \partial_3] - \frac{3}{5}\ c_2\ a_0 + \frac{4}{175}\ [\partial_3, [\partial_3, [\partial_5, \partial_3]]],$$

$$a_{17} = \frac{1}{2}\ c_2^2 \left(\partial_5^2 + \frac{3}{7}\ [\partial_7, \partial_3]\right) - \frac{7}{2}\ c_2 a_0,$$

$$a_{18} = c_2^2\ \partial_7\partial_3 - 10\ c_2 a_0 + \frac{6}{25}\ [\partial_3, [\partial_5, \partial_3]]\partial_3,$$

$$a_{19} = c_2^3\ \partial_5\partial_3 + \frac{1}{5}\ c_2 a_0 - \frac{4}{35}\ [\partial_3, [\partial_5, \partial_3]]\partial_3,$$

$$a_{20} = \frac{1}{2}\ c_2^4 \partial_3^2 + \frac{18}{35}\ c_2 a_0,\ a_{21} = c_2^7 \tag{A.2}$$

acting on $\phi(\xi_{14})$. Above we have introduced the operator:

$$a_0 = \frac{48}{691}\ \left(\ [\partial_9, \partial_3] - 3\ [\partial_7, \partial_5]\ \right)\ . \tag{A.3}$$

Furthermore, we have used some useful formulae exhibited in the following. Nested commutators involving the derivatives ∂_3 and ∂_5 acting on various products of f_3 and f_5 have a "diagonal" structure:

$$\begin{aligned}
[\partial_3, [\partial_3, [\partial_3, \partial_5]]]\, f_5 f_3 f_3 f_3 &= 1\ , \\
[\partial_3, [\partial_3, \partial_5]]\, \partial_3\, (f_5 f_3 f_3) \,ш\, f_3 &= 1\ , \\
[\partial_3, \partial_5]\, \partial_3^2\, (f_5 f_3) \,ш\, f_3 \,ш\, f_3 &= 2\ , \\
\partial_5\, \partial_3^3\, f_5 \,ш\, f_3 \,ш\, f_3 \,ш\, f_3 &= 6\ .
\end{aligned} \tag{A.4}$$

On the other hand, all the other combinations of differential operators

$$[\partial_3, [\partial_3, [\partial_3, \partial_5]]],\ [\partial_3, [\partial_3, \partial_5]]\partial_3,\ [\partial_3, \partial_5]\partial_3^2,\ \partial_5\, \partial_3^3$$

acting on the products $\{f_5 f_3^3,\ (f_5 f_3 f_3) \,ш\, f_3,\ (f_5 f_3) \,ш\, f_3 \,ш\, f_3,\ f_5 \,ш\, f_3 \,ш\, f_3 \,ш\, f_3\}$ vanish. *E.g.* $[\partial_3, [\partial_3, [\partial_3, \partial_5]]]$ annihilates all of $(f_5 f_3 f_3) \,ш\, f_3,\ (f_5 f_3) \,ш\, f_3 \,ш\, f_3,\ f_5 \,ш\, f_3 \,ш\, f_3 \,ш\, f_3$. More generally, we have:

$$\underbrace{[\partial_3, [\partial_3, [\ldots, [\partial_3, \partial_5]\ldots]]]}_{(k-p)-\text{fold commutator}}\ \partial_3^p\, (f_5 f_3^{k-q})\, (\,ш\, f_3)^q\ =\ p!\ \delta_{p,q}\ . \tag{A.5}$$

A.2. Decomposition at weight 15

At weight 15 we collect the information about the lower weight basis $\mathcal{U}_{k\leq 14}$ and with (4.56) we can construct the following basis for $\mathcal{U}_{15}$:

$$\begin{aligned}
&\phi(\zeta^m_{1,1,3,4,6}),\ \phi(\zeta^m_{3,3,9}),\ \phi(\zeta^m_{5,3,7}),\ f_{15},\ f_3 ш \phi(\zeta^m_{1,1,4,6}),\ f_3 ш \phi(\zeta^m_{3,9}),\\
&f_9 ш f_3 ш f_3,\ f_3 ш f_5 ш f_7,\ f_3 ш f_3 ш f_3 ш f_3 ш f_3,\\
&(-14 f_7 f_3 - 6 f_5^2) ш f_5,\ f_5 ш f_5 ш f_5,\ (-5 f_5 f_3) ш f_7,\\
&\phi(\zeta^m_{3,3,7}) f_2,\ \phi(\zeta^m_{3,5,5}) f_2,\ f_{13} f_2,\ (-14 f_7 f_3 - 6 f_5^2) ш f_3 f_2,\ (-5 f_5 f_3) ш f_5 f_2,\\
&f_7 ш f_3 ш f_3 f_2,\ f_5 ш f_5 ш f_3 f_2,\\
&\phi(\zeta^m_{3,3,5}) f_2^2,\ (-5 f_5 f_3) ш f_3 f_2^2,\ f_{11} f_2^2,\ f_5 ш f_3 ш f_3 f_2^2,\ f_3 ш f_3 ш f_3 f_2^3,\\
&f_9 f_2^3,\ f_7 f_2^4,\ f_5 f_2^5,\ f_3 f_2^6\ ,
\end{aligned} \tag{A.6}$$

with $\phi(\zeta^m_{3,3,5}), \phi(\zeta^m_{3,9}), \phi(\zeta^m_{1,1,4,6}), \phi(\zeta^m_{3,3,7}), \phi(\zeta^m_{3,5,5}), \phi(\zeta^m_{1,1,3,4,6}), \phi(\zeta^m_{3,3,9})$ and $\phi(\zeta^m_{5,3,7})$ given in (4.28), (4.35), (4.42) and (4.56), respectively. The operators a_i of the decomposition (4.57) are:

$$\begin{aligned}
a_1 &= \frac{48}{7601}\ \left([\partial_3,[\partial_9,\partial_3]] - 3\ [\partial_3,[\partial_7,\partial_5]]\right),\ a_2 = \frac{1}{27}\ [\partial_3,[\partial_9,\partial_3]] - \frac{853}{648}\ a_1,\\
a_3 &= \frac{2}{15}\ [\partial_3,[\partial_7,\partial_5]] - \frac{1}{70}\ [\partial_5,[\partial_7,\partial_3]] + \frac{17203}{3360}\ a_1,\ a_4 = \partial_{15},\\
a_5 &= a_1 + a_0\ \partial_3,\ a_6 = \frac{1}{27}\ [\partial_9,\partial_3]\partial_3 + \frac{2665}{648}\ a_0\partial_3 + \frac{29}{9}\ a_1\ ,\\
a_7 &= \frac{1}{2}\ \partial_9\partial_3^2 + \frac{799}{72}\ a_0\partial_3 + \frac{6775}{144}\ a_1,\\
a_8 &= \partial_7\partial_5\partial_3 + \frac{2}{9}\ [\partial_9,\partial_3]\partial_3 - \frac{467}{108}\ a_0\partial_3 - \frac{74}{3}\ a_1,\\
a_9 &= \frac{1}{5!}\ \partial_3^5 - \frac{1}{12}\ a_0\partial_3 - \frac{1}{15}\ a_1,\\
a_{10} &= \frac{1}{14}\ [\partial_7,\partial_3]\partial_5 + \frac{2188}{945}\ a_1 + \frac{3}{35}\ [\partial_5,[\partial_7,\partial_3]] - \frac{2}{45}\ [\partial_3,[\partial_9,\partial_3]],\\
a_{11} &= \frac{1}{6}\ \partial_5^3 + \frac{3}{14}\ [\partial_7,\partial_3]\partial_5 + \frac{1185701}{30240}\ a_1 + \frac{11}{70}\ [\partial_5,[\partial_7,\partial_3]] - \frac{2}{45}\ [\partial_3,[\partial_9,\partial_3]]\\
a_{12} &= \frac{1}{5}\ [\partial_5,\partial_3]\partial_7 - \frac{12199}{720}\ a_1 + \frac{1}{5}\ [\partial_5,[\partial_7,\partial_3]] - \frac{1}{15}\ [\partial_3,[\partial_9,\partial_3]]\\
a_{13} &= \frac{1}{14}\ c_2\ [\partial_3,[\partial_7,\partial_3]] + 2\ a_1,\\
a_{14} &= -\frac{3}{35}\ c_2\ [\partial_3,[\partial_7,\partial_3]] + \frac{1}{25}\ c_2\ [\partial_5,[\partial_5,\partial_3]] - \frac{14}{5}\ a_1,
\end{aligned}$$

$$a_{15} = c_2\ \partial_{13} - \frac{6417649}{2880}\ a_1 - \frac{143}{20}\ [\partial_5,[\partial_7,\partial_3]] + \frac{1339}{30}\ [\partial_3,[\partial_9,\partial_3]],$$

$$a_{16} = \frac{1}{14}\ c_2\ [\partial_7,\partial_3]\partial_3 - 3\ a_0\partial_3 - 6\ a_1,$$

$$a_{17} = \frac{1}{5}\ c_2\ [\partial_5,\partial_3]\partial_5 + \frac{1}{5}\ c_2\ [\partial_5,[\partial_5,\partial_3]] + \frac{21}{2}\ a_1,$$

$$a_{18} = \frac{1}{2}\ c_2\ \partial_7\partial_3^2 - 10\ a_0\partial_3 - 26\ a_1,$$

$$a_{19} = \frac{1}{2}\ c_2\ \ \partial_5^2\partial_3 + \frac{3}{14}\ c_2\ [\partial_7,\partial_3]\partial_3 - \frac{7}{2}\ a_0\partial_3 - 8\ a_1,$$

$$a_{20} = \frac{1}{5}\ c_2^2\ [\partial_3,[\partial_5,\partial_3]] + 4\ a_1,\ \ a_{21} = \frac{1}{5}\ c_2^2\ [\partial_5,\partial_3]\partial_3 - \frac{3}{5}\ a_0\partial_3 - \frac{8}{5}\ a_1,$$

$$a_{22} = c_2^2\ \partial_{11} + \frac{11}{4}\ c_2\ [\partial_3,[\partial_7,\partial_3]] + \frac{11}{2}\ c_2\ [\partial_5,[\partial_5,\partial_3]] - \frac{8495287}{15120}\ a_1$$
$$- \frac{11}{35}\ [\partial_5,[\partial_7,\partial_3]] + \frac{128}{45}\ [\partial_3,[\partial_9,\partial_3]],\ \ a_{23} = \frac{1}{2}\ c_2^2\ \partial_5\partial_3^2 + \frac{1}{5}\ a_0\partial_3 - \frac{23}{10}\ a_1,$$

$$a_{24} = \frac{1}{6}\ c_2^3\ \partial_3^3 + \frac{18}{35}\ a_0\partial_3 + \frac{12}{35}\ a_1,$$

$$a_{25} = c_2^3\ \partial_9 + 9\ c_2^2\ [\partial_3,[\partial_5,\partial_3]] - \frac{2}{35}\ c_2\ [\partial_3,[\partial_7,\partial_3]] + \frac{2}{5}\ c_2\ [\partial_5,[\partial_5,\partial_3]]$$
$$+ \frac{54263011}{396900}\ a_1 + \frac{68}{1225}\ [\partial_5,[\partial_7,\partial_3]] - \frac{236}{4725}\ [\partial_3,[\partial_9,\partial_3]],$$

$$a_{26} = c_2^4\ \partial_7 + \frac{6}{25}\ c_2^2\ [\partial_3,[\partial_5,\partial_3]] - \frac{16}{245}\ c_2\ [\partial_3,[\partial_7,\partial_3]] + \frac{57847}{15750}\ a_1$$
$$+ \frac{24}{875}\ [\partial_5,[\partial_7,\partial_3]] - \frac{184}{2625}\ [\partial_3,[\partial_9,\partial_3]],$$

$$a_{27} = c_2^5\ \partial_5 - \frac{4}{35}\ c_2^2\ [\partial_3,[\partial_5,\partial_3]] - \frac{1714624}{121275}\ a_1 + \frac{48}{13475}\ [\partial_5,[\partial_7,\partial_3]],$$
$$- \frac{64}{5775}\ [\partial_3,[\partial_9,\partial_3]],\ \ a_{28} = c_2^6\ \partial_3 + \frac{1451972}{716625}\ a_1 \tag{A.7}$$

acting on $\phi(\xi_{15})$. Above we have used the operator a_0 defined in (A.3).

A.3. Decomposition at weight 16

Gathering the information about the lower weight basis $\mathcal{U}_{k\leq 15}$ with (4.56) we can construct the following basis for $\mathcal{U}_{16}$:

$$\phi(\zeta^m_{1,1,6,8}),\ \phi(\zeta^m_{3,3,3,7}),\ \phi(\zeta^m_{3,3,5,5}),\ \phi(\zeta^m_{3,13}),\ \phi(\zeta^m_{5,11})$$

$$f_3 \,ш\, \phi(\zeta^m_{3,3,7}),\ f_3 \,ш\, \phi(\zeta^m_{3,5,5}),\ f_3 \,ш\, f_{13},\ (-14f_7f_3 - 6f_5^2) \,ш\, f_3 \,ш\, f_3,$$
$$(-5f_5f_3) \,ш\, f_3 \,ш\, f_5,\ f_3 \,ш\, f_3 \,ш\, f_3 \,ш\, f_7,\ f_3 \,ш\, f_3 \,ш\, f_5 \,ш\, f_5,\ f_7 \,ш\, f_9,$$
$$25\ (f_5f_3) \,ш\, (f_5f_3),\ f_5 \,ш\, f_{11},\ f_5 \,ш\, \phi(\zeta^m_{3,3,5}),$$
$$f_3 \,ш\, \phi(\zeta^m_{3,3,5})f_2,\ f_3 \,ш\, f_3 \,ш\, (-5f_5f_3)f_2,\ f_3 \,ш\, f_{11}f_2,\ f_3 \,ш\, f_3 \,ш\, f_3 \,ш\, f_5f_2$$
$$f_3 \,ш\, f_3 \,ш\, f_3 \,ш\, f_3f_2^2,\ f_3 \,ш\, f_9f_2^2,\ f_3 \,ш\, f_7f_2^3,\ f_3 \,ш\, f_5f_2^4,\ f_3 \,ш\, f_3f_2^5,$$
$$\phi(\zeta^m_{3,3,3,5})f_2,\ \phi(\zeta^m_{3,11})f_2,\ \phi(\zeta^m_{5,9})f_2,\ f_5 \,ш\, f_9f_2,\ f_7 \,ш\, f_7f_2\ ,$$
$$\phi(\zeta^m_{1,1,4,6})f_2^2,\ \phi(\zeta^m_{3,9})f_2^2,\ f_5 \,ш\, f_7f_2^2,\ (-14f_7f_3 - 6f_5^2)f_2^3, -5f_5f_3f_2^4,$$
$$f_5 \,ш\, f_5f_2^3,\ f_2^8\ , \tag{A.8}$$

with the maps $\phi(\zeta^m_{3,3,5}),\ \phi(\zeta^m_{3,9}), \phi(\zeta^m_{1,1,4,6}),\ \phi(\zeta^m_{3,3,7}), \phi(\zeta^m_{3,5,5}),\ \phi(\zeta^m_{3,3,3,5}), \phi(\zeta^m_{3,11}), \phi(\zeta^m_{5,9})$ $\phi(\zeta^m_{3,3,3,7}), \phi(\zeta^m_{3,3,5,5}), \phi(\zeta^m_{3,13}), \phi(\zeta^m_{5,11})$ and $\phi(\zeta^m_{1,1,6,8})$ given in (4.28), (4.35), (4.42), (4.50) and (4.64), respectively. The operators a_i of the decomposition (4.65) are:

$$a_1 = \frac{720}{3617}\left\{ \frac{7}{11}\ [\partial_{11}, \partial_5] - \frac{2}{11}\ [\partial_{13}, \partial_3] - [\partial_9, \partial_7] + \frac{6493}{9240}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]] \right.$$
$$\left. -\frac{751}{100}\ [\partial_3, [\partial_5, [\partial_5, \partial_3]]] \right\},\ a_2 = -\frac{19}{7}\ a_1 + \frac{1}{14}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]],$$
$$a_3 = \frac{542}{175}\ a_1 - \frac{3}{35}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]] + \frac{1}{25}\ [\partial_3, [\partial_5, [\partial_5, \partial_3]]],$$
$$a_4 = -\frac{19}{286}\ [\partial_{13}, \partial_3] + \frac{3}{22}\ [\partial_{11}, \partial_5] + \frac{2217053}{16800}\ a_1 - \frac{200559}{80080}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]]$$
$$- \frac{7011}{2600}\ [\partial_3, [\partial_5, [\partial_5, \partial_3]]]$$
$$a_5 = \frac{3}{242}\ [\partial_{13}, \partial_3] - \frac{5}{242}\ [\partial_{11}, \partial_5] - \frac{114307}{7392}\ a_1 + \frac{23181}{67760}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]]$$
$$+ \frac{909}{2200}\ [\partial_3, [\partial_5, [\partial_5, \partial_3]]],\ a_6 = -\frac{1}{14}\ [\partial_3, [\partial_3, \partial_7]]\partial_3 + \frac{5}{7}\ a_1,$$
$$a_7 = -\frac{1}{25}\ [\partial_5, [\partial_3, \partial_5]]\partial_3 + \frac{3}{35}\ [\partial_3, [\partial_3, \partial_7]]\partial_3 - \frac{6}{7}\ a_1,\ a_8 = \partial_{13}\partial_3 + \frac{8497}{42}\ a_1,$$
$$a_9 = \frac{1}{28}\ [\partial_7, \partial_3]\partial_3^2 + \frac{1}{7}\ a_1,\ a_{10} = \frac{1}{5}\ [\partial_5, \partial_3]\partial_5\partial_3 + \frac{1}{5}\ [\partial_5, [\partial_5, \partial_3]]\partial_3,$$
$$a_{11} = \frac{1}{3!}\ \partial_7\partial_3^3 - \frac{1}{3}\ a_1,\ a_{12} = \frac{1}{2}\ \left(\frac{1}{2}\ \partial_5^2 + \frac{3}{14}\ [\partial_7, \partial_3]\right)\partial_3^2 - \frac{4}{7}\ a_1,$$
$$a_{13} = \partial_9\partial_7 + \frac{4850713}{6600}\ a_1 - \frac{2272973}{330330}\ [\partial_3, [\partial_3, [\partial_7, \partial_3]]] - \frac{299373}{7150}\ [\partial_3, [\partial_5, [\partial_5, \partial_3]]]$$
$$- \frac{1275}{1573}\ [\partial_{13}, \partial_3] + \frac{210}{121}\ [\partial_{11}, \partial_5],\ a_{14} = \frac{1}{50}\ [\partial_5, \partial_3]^2,$$

$$a_{15} = \partial_{11}\partial_5 + \frac{455534}{525}\ a_1 - \frac{601677}{40040}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{21033}{1300}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]]$$
$$- \frac{57}{143}\ [\partial_{13},\partial_3] + \frac{9}{11}\ [\partial_{11},\partial_5],$$
$$a_{16} = \frac{1}{5}\ [\partial_3,[\partial_5,\partial_3]]\partial_5 + \frac{1}{5}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]] - \frac{2}{5}\ a_1,$$
$$a_{17} = \frac{1}{5}\ c_2\ [\partial_3,[\partial_5,\partial_3]]\partial_3,\ a_{18} = \frac{1}{10}\ c_2\ [\partial_5,\partial_3]\partial_3^2,$$
$$a_{19} = c_2\ \partial_{11}\partial_3 - \frac{11}{4}\ [\partial_3,[\partial_3,\partial_7]]\partial_3 - \frac{11}{2}\ [\partial_5,[\partial_3,\partial_5]]\partial_3 - 137\ a_1,$$
$$a_{20} = \frac{1}{3!}\ c_2\ \partial_5\partial_3^3,\ a_{21} = \frac{1}{4!}\ c_2^2\ \partial_3^4 - \frac{1}{12}\ c_2^2\ a_0,$$
$$a_{22} = c_2^2\ \partial_9\partial_3 + 9\ c_2\ [\partial_3,[\partial_5,\partial_3]]\partial_3 + \frac{799}{72}\ c_2^2\ a_0 - \frac{2}{35}\ [\partial_3,[\partial_7,\partial_3]]\partial_3$$
$$+ \frac{2}{5}\ [\partial_5,[\partial_5,\partial_3]]\partial_3 - \frac{11}{7}\ a_1,$$
$$a_{23} = c_2^3\ \partial_7\partial_3 + \frac{6}{25}\ c_2\ [\partial_3,[\partial_5,\partial_3]]\partial_3 - 10\ c_2^2\ a_0 - \frac{16}{245}\ [\partial_3,[\partial_7,\partial_3]]\partial_3 + \frac{848}{245}\ a_1,$$
$$a_{24} = c_2^4\ \partial_5\partial_3 - \frac{4}{35}\ c_2\ [\partial_3,[\partial_5,\partial_3]]\partial_3 + \frac{1}{5}\ c_2^2\ a_0 + \frac{48}{35}\ a_1,$$
$$a_{25} = \frac{1}{2}\ c_2^5\ \partial_3^2 + \frac{18}{35}\ c_2^2\ a_0 + \frac{408}{2695}\ a_1,\ a_{26} = \frac{1}{5}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_{27} = -\frac{23}{198}\ c_2\ [\partial_{11},\partial_3] + \frac{5}{18}\ c_2\ [\partial_9,\partial_5] - \frac{12841}{1188}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] - \frac{1991}{14}\ a_1$$
$$+ \frac{121}{28}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{7}{2}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{28} = \frac{1}{27}\ c_2\ [\partial_{11},\partial_3] - \frac{2}{27}\ c_2\ [\partial_9,\partial_5] + \frac{232}{81}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] + \frac{697}{21}\ a_1$$
$$- \frac{47}{42}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] +\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{29} = c_2\ \partial_9\partial_5 + 9\ [\partial_3,[\partial_5,\partial_3]]\partial_5 - \frac{23}{33}\ c_2\ [\partial_{11},\partial_3] + \frac{5}{3}\ c_2\ [\partial_9,\partial_5] - \frac{12443}{14}\ a_1$$
$$- \frac{12775}{198}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] + \frac{363}{14}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - 21\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{30} = \frac{1}{2}\ c_2\ \partial_7^2 - \frac{235}{396}\ c_2\ [\partial_{11},\partial_3] + \frac{55}{36}\ c_2\ [\partial_9,\partial_5] - \frac{647287}{11880}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]]$$

$$
\begin{aligned}
&- \frac{78201}{140}\ a_1 + \frac{967}{56}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{333}{20}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]],\ a_{31} = c_2^2\ a_0, \\
a_{32} &= \frac{1}{27}\ c_2^2\ [\partial_9,\partial_3] + \frac{2665}{648}\ c_2^2\ a_0 + \frac{2}{3}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] - \frac{8954}{1575}\ a_1 \\
&+ \frac{4}{35}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{4}{75}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]], \\
a_{33} &= c_2^2\ \partial_7\partial_5 + \frac{2}{9}\ c_2^2\ [\partial_9,\partial_3] + \frac{6}{25}\ [\partial_3,[\partial_5,\partial_3]]\partial_5 - \frac{467}{108}\ c_2^2\ a_0 + 4\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] \\
&- \frac{21331}{525}\ a_1 + \frac{24}{35}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{8}{25}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]], \\
a_{34} &= \frac{1}{14}\ c_2^3\ [\partial_7,\partial_3] - 3\ c_2^2\ a_0 - \frac{62}{245}\ a_1 + \frac{2}{245}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]], \\
a_{35} &= \frac{1}{5}\ c_2^4\ [\partial_5,\partial_3] - \frac{3}{5}\ c_2^2\ a_0 + \frac{4}{175}\ c_2\ [\partial_3,[\partial_3,[\partial_5,\partial_3]]] + \frac{108}{875}\ a_1, \\
a_{36} &= c_2^3\ \left(\frac{1}{2}\ \partial_5^2 + \frac{3}{14}\ [\partial_7,\partial_3]\right) - \frac{4}{35}\ [\partial_3,[\partial_5,\partial_3]]\partial_5 - \frac{7}{2}\ c_2^2\ a_0 \\
&- \frac{284}{245}\ a_1 + \frac{6}{245}\ [\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \frac{2}{35}\ [\partial_3,[\partial_5,[\partial_5,\partial_3]]],\ a_{37} = c_2^8\ ,
\end{aligned} \tag{A.9}
$$

acting on $\phi(\xi_{16})$. Again, we have used the operator a_0 defined in (A.3).

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