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MOVING HOMOLOGY CLASSES TO INFINITY

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Abstract. Let $q : \tilde{X} \to X$ be a regular covering over a finite polyhedron with free abelian group of covering translations. Each nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$ with $q^*\xi = 0$ determines a notion of *"infinity"* of the noncompact space $\tilde{X}$. In this paper we characterize homology classes z in $\tilde{X}$ which can be realized in arbitrary small neighborhoods of infinity in $\tilde{X}$. This problem was motivated by applications in the theory of critical points of closed 1-forms initiated in [2], [3].

1. Introduction

Consider a regular covering $q : \tilde{X} \to X$ over a finite polyhedron X with free abelian group of covering translations $H \simeq \mathbf{Z}^r$. In the case of infinite cyclic coverings $r = 1$ the space $\tilde{X}$ has two ends [5] and one may specify an end of $\tilde{X}$ by choosing a nonzero homomorphism $H \to \mathbf{R}$ viewed up to a positive multiplicative constant. According to the formalism of [5], in the case $r > 1$ the space $\tilde{X}$ has only one end. Nevertheless, as we show in this paper, for any $r \geq 1$ one has the notion of a neighborhood of infinity in $\tilde{X}$ with respect to a nonzero linear map $H \to \mathbf{R}$ viewed up to a positive multiplicative constant. In other words, each nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$ with $q^*\xi = 0$ determines a notion of *"infinity"* of the noncompact space $\tilde{X}$.

In this paper we characterize homology classes $z \in H_i(\tilde{X}; \mathbf{k})$ which can be realized by cycles lying in arbitrary small neighborhoods of infinity of $\tilde{X}$ with respect to a given nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$. This problem was motivated by applications in the theory of critical points of closed 1-forms initiated in [2], [3] where it was used (in the case of rank one classes only) to obtain cohomological lower bounds for $\mathrm{cat}(X, \xi)$. The main result of this paper states that a homology class z with coefficients in a field $\mathbf{k}$ is movable to infinity of $\tilde{X}$ with respect to a cohomology class ξ of full rank $r = \mathrm{rk}(H)$ if and only if z is $\mathbf{k}[H]$-torsion. Surprisingly, this statement shows that the property of a homology class to be movable to infinity is in some sense independent of the direction ξ.

We also study the problem of describing the set of all "directions" $\mathcal{M}_z \subset H^1(X; \mathbf{R})$ with respect to which a given homology class is movable to infinity.

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We show that this set is either empty or it is very large, see Theorem 2 for the exact statement.

The results of this paper generalize earlier work [2] which treats the simplest case of infinite cyclic covers $H \simeq \mathbf{Z}$. The main technical tool of our proof is an exact sequence (see (8)) relating the Novikov homology [7] with the $\lim^0$ and $\lim^1$ terms of an inverse system associated to neighborhoods of infinity. We later show that the $\lim^1$-term vanishes.

It would be interesting to generalize the results of [3] where the problem of movability to infinity of homology classes with integral coefficients was studied for cyclic covers.

Techniques similar to those used here were employed in [6], [7], [8].

2. The Abel - Jacobi map.

Let X be a connected finite cell complex and $q : \tilde{X} \to X$ a regular covering having a free abelian group of covering transformations $H \simeq \mathbf{Z}^r$. Because of the exact sequence

$$1 \to \pi_1(\tilde{X}) \to \pi_1(X) \to H \to 0,$$

the dual vector space $H^* = \mathrm{Hom}(H, \mathbf{R})$ can be identified with the subgroup of $H^1(X; \mathbf{R}) = \mathrm{Hom}(\pi_1(X); \mathbf{R})$ consisting of all cohomology classes $\xi \in H^1(X; \mathbf{R})$ such that $q^*(\xi) = 0 \in H^1(\tilde{X}; \mathbf{R})$. Denote $H_{\mathbf{R}} = H \otimes \mathbf{R}$; it is a vector space of dimension r containing H as a lattice.

Proposition 1. *There exists a canonical Abel - Jacobi map $A : \tilde{X} \to H_{\mathbf{R}}$ having the following properties:*

- *(a) A is H-equivariant; here H acts on $\tilde{X}$ by covering transformations and it acts on $H_{\mathbf{R}}$ by translations.*[1]
- *(b) A is proper (i.e. the preimage of a compact subset of $H_{\mathbf{R}}$ is compact).*
- *(c) A is determined uniquely up to replacing it by a map $A' : \tilde{X} \to H_{\mathbf{R}}$ of the form $A' = A + F \circ q$ where $F : X \to H_{\mathbf{R}}$ is a continuous map.*

Proof. The Abel - Jacobi map A can be constructed as follows. Let $e_1, \dots, e_r$ be a free basis of H and let $\xi_1, \dots, \xi_r$ be the dual basis of H^*. Choose a continuous closed 1-form[2] ω_i on X lying in class ξ_i. Then the induced form $q^*(\omega_i) = df_i$ on $\tilde{X}$ is exact; the function $f_i : \tilde{X} \to \mathbf{R}$ is determined uniquely up to adding a constant. One defines the Abel - Jacobi map A as follows

$$A(x) = \sum_{i=1}^{r} e_i \otimes f_i(x) \in H_{\mathbf{R}}, \quad x \in \tilde{X}. \tag{1}$$

[1] We use additive notation for the group operation in H when H is viewed as a subgroup of $H_{\mathbf{R}}$; in all other circumstances we use multiplicative notation for the group operation of H. With these conventions statement (a) can be expressed by $A(gx) = A(x) + g$.

[2] For the notion of a continuous closed 1-form on a topological space we refer to [2, 4]. Briefly, a continuous closed 1-form on X is given by an open cover $\mathcal{U} = \{U\}$ of X and a continuous function $f_U : U \to \mathbf{R}$ for each open set $U \in \mathcal{U}$ such that for $U, V \in \mathcal{U}$ the difference $f_U - f_V$ is locally constant on $U \cap V$.

Let ω_i' be a different closed 1-form on X lying in cohomology class ξ_i. Then $\omega_i' = \omega_i + d\phi_i$ where $\phi_i : X \to \mathbf{R}$ and $q^*(\omega_i') = df_i'$ with $f_i' = f_i + \phi_i \circ q$. Hence we see that the new map A' has the form $A' = A + F \circ q$ where

$$F(x) = \sum_{i=1}^{r} e_i \otimes \phi_i(x), \quad x \in X.$$

To show that A (given by formula (1)) is H-equivariant note that for $g \in H$ and $x \in \tilde{X}$ one has $f_i(gx) = f_i(x) + \xi_i(g)$. Substituting into (1) one finds $A(gx) = A(x) + g$ which proves (a).

Any equivariant map $A' : \tilde{X} \to H_{\mathbf{R}}$ is of the the form (1) and hence the argument given above proves part (c).

To show that A is proper it is enough to show that any sequence $x_n \in \tilde{X}$ such that $A(x_n)$ converges to a vector $h \in H_{\mathbf{R}}$ has a convergent subsequence. If $h = \sum_{i=1}^{r} h_i e_i$ then our assumption is that for any $i = 1, \dots, r$ the sequence $f_i(x_n)$ converges to h_i. Since X is compact we may assume that the sequence $q(x_n) \in X$ converges to a point $x_0 \in X$. Let $U \subset X$ be a small connected neighborhood of x_0 such that $q^{-1}(U)$ is a disjoint union $\bigcup_j U_j$ and each U_j is mapped homeomorphically onto U by q. We will also assume that U is so small that for any $x, x' \in U_j$ one has

$$\big|f_i(x) - f_i(x')\big| < 1/2, \quad i = 1, 2, \dots, r. \tag{2}$$

Let us show that for large n all points x_n lie in the same set U_j. This would clearly imply that the sequence x_n has a limit point. Suppose that $x_n \in U_j$ and $x_m \in gU_j$ where $g \in H$, $g \neq 0$. Find $i = 1, \dots, r$ such that $|\xi_i(g)| \geq 1$. Then we have $|f_i(gx_n) - f_i(x_m)| < 1/2$ and $|f_i(gx_n) - f_i(x_n)| = |\xi_i(g)| \geq 1$ which together imply $|f_i(x_n) - f_i(x_m)| \geq 1/2$. The last inequality cannot be satisfied for large n and m as $f_i(x_n)$ converges to $h_i \in \mathbf{R}$, see above.

This completes the proof. □

3. Neighborhoods of infinity in $\tilde{X}$.

Consider again a connected finite cell complex X and a regular covering $q : \tilde{X} \to X$ with group of covering translations $H \simeq \mathbf{Z}^r$. Let $\xi \in H^1(X; \mathbf{R})$ be a nonzero cohomology class with the property $q^*(\xi) = 0$. As we observed above, ξ determines a linear functional $\xi_{\mathbf{R}} : H_{\mathbf{R}} \to \mathbf{R}$.

Definition 2. *A subset $N \subset \tilde{X}$ is called a neighborhood of infinity in $\tilde{X}$ with respect to the cohomology class ξ if N contains the set*

$$\{x \in \tilde{X};\ \xi_{\mathbf{R}}(A(x)) > c\} \subset N, \tag{3}$$

for some real $c \in \mathbf{R}$. Here $A : \tilde{X} \to H_{\mathbf{R}}$ is an Abel - Jacobi map for the covering $q : \tilde{X} \to X$.

This notion is independent of the choice of the Abel - Jacobi map $A : \tilde{X} \to H_{\mathbf{R}}$. Indeed, any other Abel - Jacobi map $A' : \tilde{X} \to H_{\mathbf{R}}$ has the form $A' = A + F \circ q$ where $F : X \to H_{\mathbf{R}}$ is continuous. Since X is compact, there exists a constant $C > 0$ such that $|\xi_{\mathbf{R}}(F(y))| < C$ for all $y \in X$. Hence

$\xi_{\mathbf{R}}(A(x)) > c$ implies $\xi_{\mathbf{R}}(A'(x)) > c - C$ and similarly $\xi_{\mathbf{R}}(A'(x)) > c$ implies $\xi_{\mathbf{R}}(A(x)) > c - C$ where $x \in \tilde{X}$.

Let $N \subset \tilde{X}$ be a neighborhood of infinity with respect to ξ. For a covering transformation $g \in H$ the set gN is also a neighborhood of infinity with respect to ξ.

Lemma 3. *Fix a cell structure on X and consider the induced cell structure on $\tilde{X}$. Then for any nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$ with $q^*(\xi) = 0$ there exists a neighborhood of infinity $N \subset \tilde{X}$ with respect to ξ having the following properties:*

(A) N is a cell subcomplex of $\tilde{X}$;
(B) The complement $\tilde{X} - N$ is a neighborhood of infinity with respect to the cohomology class $-\xi$;
(C) For every cell e of X there is a lift $\tilde{e}$ lying in N such that for $g \in H$ the cell $g\tilde{e} \subset N$ if and only if $\xi(g) \geq 0$;
(D) If $N' \subset \tilde{X}$ is an arbitrary neighborhood of infinity with respect to ξ and if $g \in H$ is an arbitrary element with $\xi(g) > 0$ then there exists $n > 0$ such that $g^n N \subset N'$.

Proof. For each cell e_i of X choose a cell $\tilde{e}_i$ of $\tilde{X}$ covering e_i, where $i = 1, 2, \ldots, k$. We claim that one may choose the lifts $\tilde{e}_i$ such that the following property holds: the boundary of any cell $\tilde{e}_i$ lies in the union

$$\partial \tilde{e}_i \subset \bigcup_j g_j \tilde{e}_j, \quad \dim \tilde{e}_j = \dim \tilde{e}_i - 1, \tag{4}$$

where

$$\xi(g_j) \geq 0. \tag{5}$$

Note that the same cell $\tilde{e}_j$ may appear in the union (4) several times with different covering translations g_j. The union (4) has finitely many terms. Lifts $\tilde{e}_i$ with the above property can be constructed inductively with respect to $\dim e_i$. For an arbitrary choice of lifts $\tilde{e}_i$ condition (4) is satisfied and to achieve (5) one may have to replace $\tilde{e}_i$ by $g^n \tilde{e}_i$ where $g \in H$, $\xi(g) > 0$ and n is large enough.

Now we set

$$N = \bigcup_{i=1}^{k} \left(\bigcup_{\xi(g) \geq 0} g\tilde{e}_i \right).$$

Here the union is taken with respect to all $g \in H$ with $\xi(g) > 0$ and with respect to $i = 1, \ldots, k$. Let us show that N satisfies conditions (A)-(D) of Lemma 3. Condition (A) is satisfies because of (4), (5). Condition (C) is obvious by construction. Now we want to show that N is a neighborhood of infinity with respect to ξ and that the complement $\tilde{X} - N$ is a neighborhood of infinity with respect to $-\xi$. Let $A : \tilde{X} \to H_{\mathbf{R}}$ be an Abel - Jacobi map. Since the closures of the cells $\tilde{e}_i$ are compact there exist constants

$L < K$ such that for any $i = 1, \dots, k$ and for any $x \in \tilde{e}_i$ one has $L < \xi_{\mathbf{R}}(A(x)) < K$. Then we see that $\{x \in \tilde{X}; \xi_{\mathbf{R}}(A(x)) > K\}$ is contained in N, and $\{x \in \tilde{X}; -\xi(A(x)) > -L\}$ is contained in $\tilde{X} - N$. We are left to prove statement (D). We know that $N \subset \{x \in \tilde{X}; \xi_{\mathbf{R}}(A(x)) > L\}$. If $c \in \mathbf{R}$ is such that $N' \supset \{x \in \tilde{X}; \xi(A(x)) > c\}$ then $g^n N \subset N'$ assuming that $n\xi(g) + L > c$. This completes the proof. □

4. Homology classes movable to infinity.

Let X be a connected finite cell complex and $q : \tilde{X} \to X$ be a regular covering with free abelian group of covering translations $H \simeq \mathbf{Z}^r$. Let $\mathbf{k}$ be a field. The following definition was introduced in [2] in the case $r = 1$.

Definition 4. *A homology class $z \in H_i(\tilde{X}; \mathbf{k})$ is said to be movable to infinity of $\tilde{X}$ with respect to a nonzero cohomology class $\xi \in H^1(X; \mathbf{R})$, $q^*(\xi) = 0$, if in any neighborhood N of infinity with respect to ξ there exists a (singular) cycle representing z.*

Equivalently, a homology class $z \in H_i(\tilde{X}; \mathbf{k})$ is movable to infinity with respect to $\xi \in H^1(X; \mathbf{R})$ if z lies in the intersection

$$\bigcap_N \operatorname{Im} \left[H_i(N; \mathbf{k}) \to H_i(\tilde{X}; \mathbf{k}) \right] \tag{6}$$

where N runs over all neighborhoods of infinity in $\tilde{X}$ with respect to ξ. This can also be expressed by saying that z lies in the kernel of the natural homomorphism

$$H_i(\tilde{X}; \mathbf{k}) \to \varprojlim H_i(\tilde{X}, N; \mathbf{k}) \tag{7}$$

where in the inverse limit N runs over all neighborhoods of infinity in $\tilde{X}$ with respect to ξ.

Example 5. Let X be a closed smooth manifold admitting a closed 1-form ω with no zeros. Then any homology class $z \in H_i(\tilde{X}; \mathbf{k})$ is movable to infinity with respect to $\xi = [\omega] \in H^1(X; \mathbf{R})$ assuming that $q^*(\xi) = 0$.

5. Statement of the main result.

The following theorem gives an explicit description of all movable homology classes. It generalizes the results of [2], §5 which treats the case of infinite cyclic covers $q : \tilde{X} \to X$.

Theorem 1. *Let X be a finite cell complex and $q : \tilde{X} \to X$ be a regular covering having a free abelian group of covering transformations $H \simeq \mathbf{Z}^r$. Let $\xi \in H^1(X; \mathbf{R})$ be a nonzero cohomology class of rank r satisfying $q^*(\xi) = 0$. The following properties (A), (B), (C) of a nonzero homology class $z \in H_i(\tilde{X}; \mathbf{k})$ (where $\mathbf{k}$ is a field) are equivalent:*

(A) z is movable to infinity with respect to ξ.

(B) Any singular cycle c in $\tilde{X}$ realizing the class z bounds an infinite singular chain c' in $\tilde{X}$ containing only finitely many simplices lying outside every neighborhood of infinity $N \subset \tilde{X}$ with respect to ξ.

(C) There exists a nonzero element $x \in \mathbf{k}[H]$ such that $x \cdot z = 0$.

Note that the implications (C) $\Rightarrow$ (B) $\Rightarrow$ (A) are straightforward (see below); the only nontrivial statement is the implication (A) $\Rightarrow$ (C). Let us explain why (C) $\Rightarrow$ (B). Suppose that the class $z \in H_i(\tilde{X}; \mathbf{k})$ is torsion, i.e. $x \cdot z = 0$ where $x \in \mathbf{k}[H]$, $x \neq 0$. Without loss of generality we may assume that $x = 1 - y$ where $y \in \mathbf{k}[H]$ has the form $y = \sum a_j g_j$ where $a_j \in \mathbf{k}$, $g_j \in H$ and $\xi(g_j) > 0$. Let c be a chain representing the class z. Then the cycle $x \cdot c$ bounds, i.e. $(1 - y) \cdot c = \partial c_1$ where c_1 is a finite chain in $\tilde{X}$. Set $c' = c_1 + yc_1 + y^2 c_1 + \dots$. Then $\partial c' = c$ and c' has finitely many simplices lying outside every neighborhood of infinity $N \subset \tilde{X}$ with respect to ξ.

The proof of Theorem 1 will be completed in §9. Note that property (C) does not involve the class ξ; we derive some implications of this fact in §10.

6. The $\lim^1$ Exact sequence.

In this section we describe an exact sequence which will be used in the proof of Theorem 1. As above, let $q : \tilde{X} \to X$ be a regular covering of a finite cell complex X. We assume that the group of covering translations is free abelian $H \simeq Z^r$. Let $\xi \in H^1(X; \mathbf{R})$ be a nonzero cohomology class with $q^*(\xi) = 0$. Such ξ can be viewed as a group homomorphism $\xi : H \to \mathbf{R}$, where $\xi(gg') = \xi(g) + \xi(g')$ for $g, g' \in H$. Note that now we use multiplicative notation for the group operation in H. Consider the group ring $\mathbf{k}[H]$ and its Novikov completion $\widehat{\mathbf{k}[H]}_\xi$. Elements of $\mathbf{k}[H]$ are finite sums of the form $\sum a_i g_i$ where $a_i \in \mathbf{k}$ and $g_i \in H$. Elements of $\widehat{\mathbf{k}[H]}_\xi$ are countable sums $\sum a_i g_i$ having the property $\lim_{i \to +\infty} \xi(g_i) = +\infty$.

Note that the Novikov completion $\widehat{\mathbf{k}[H]}_\xi$ *is a field* assuming that $\xi : H \to \mathbf{R}$ is injective, i.e. $\mathrm{rk}(\xi) = r$. Indeed, if $x \in \widehat{\mathbf{k}[H]}_\xi$ is nonzero then x can be written in form $\sum a_i g_i$ with $a_i \neq 0$ and $\xi(g_1) < \xi(g_2) < \xi(g_3) < \dots$ tends to infinity. Hence $a_1^{-1} g_1^{-1} x = 1 - y$ where $y = \sum b_j g'_j \in \widehat{\mathbf{k}[H]}_\xi$ is such that $\xi(g'_j) \geq \epsilon > 0$. We see that $x^{-1} = (1 + y + y^2 + \dots) \cdot g_1^{-1} a_1^{-1}$ and the series converges in the completed ring $\widehat{\mathbf{k}[H]}_\xi$.

Proposition 6. *Assume that $\xi \in H^1(X; \mathbf{R})$ is nonzero and $q^*(\xi) = 0$. Then there is an exact sequence*

$$(8) \quad 0 \to \varprojlim{}^1 H_{q+1}(\tilde{X}, N; \mathbf{k}) \to H_q(\tilde{X}; \widehat{\mathbf{k}[H]}_\xi) \to \varprojlim H_q(\tilde{X}, N; \mathbf{k}) \to 0,$$

where N runs over all neighborhoods of infinity in $\tilde{X}$ with respect to ξ.

A similar exact sequence appears (in a slightly different context) in the thesis of J.-C. Sikorav. In Proposition 13 below we show that the $\lim^1$ term in (8) vanishes assuming that ξ has maximal rank.

Proof. For any real number $a \in \mathbf{R}$ denote $H_a = \{g \in H; \xi(g) \geq a\}$. If $a \geq 0$ then $H_a \subset H$ is a sub-semigroup. Consider the semigroup ring $B = \mathbf{k}[H_0]$; here H_0 is H_a with $a = 0$. We view the whole group ring $\mathbf{k}[H]$ as a $(\mathbf{k}[H] - B)$-bimodule which we denote M. Similarly, for $a \in \mathbf{R}$, $a > 0$ denote $M_a = \mathbf{k}[H_a] \subset M$; it is an B-submodule of M. We have an isomorphism

$$\widehat{\mathbf{k}[H]}_\xi = \varprojlim M/M_a \tag{9}$$

where in the inverse limit a tends to $+\infty$. Note that in the last equation both sides are $(\mathbf{k}[H] - B)$-bimodules and (9) is an isomorphism of bimodules.[3]

Fix a cellular neighborhood of infinity N in $\tilde{X}$ with respect to ξ as constructed in Lemma 3. The semigroup H_0 acts on N preserving the cell structure. Let $C_*(N)$ denote the cellular chain complex of N; it is a chain complex of B-modules. Note that $C_i(N)$ is a finitely generated free B-module by Lemma 3, condition (C). We have

$$M \otimes_B C_*(N) = C_*(\tilde{X}); \tag{10}$$

this is an isomorphism of $\mathbf{k}[H]$-modules.

Fix a group element $g \in H$ with $\xi(g) > 0$. For $n \to \infty$ the sets $g^n N$ form a basis of neighborhoods of infinity with respect to ξ (by statement (4) of Lemma 3). Let C_n denote the chain complex $C_*(\tilde{X})/C_*(g^n N)$ where $n \in \mathbf{Z}$, $n \to +\infty$. The map $C_{n+1} \to C_n$ is an epimorphism and hence the inverse system $\{C_n\}$ satisfies the Mittag - Leffler condition, see [9]. We obtain using (10) that the inverse limit chain complex can be identified with

$$\varprojlim C_n = \left(\varprojlim M/M_a\right) \otimes_B C_*(N)$$

$$= \widehat{\mathbf{k}[H]}_\xi \otimes_B C_*(N) = \widehat{\mathbf{k}[H]}_\xi \otimes_{\mathbf{k}H} C_*(\tilde{X})$$

By Theorem 3.5.2 from [9] we have an exact sequence

$$0 \to \varprojlim{}^1 H_{q+1}(C_n) \to H_q(\varprojlim C_n) \to \varprojlim H_q(C_n) \to 0.$$

As the discussion above shows, the middle term can be identified with the module $H_q(\tilde{X}; \widehat{\mathbf{k}[H]}_\xi)$, the term on the right is $\varprojlim H_q(\tilde{X}, N; \mathbf{k})$ and the term on the left is $\varprojlim{}^1 H_{q+1}(\tilde{X}, N; \mathbf{k})$. This completes the proof. □

Proposition 6 is true for any commutative ring $\mathbf{k}$.

7. Some commutative algebra.

In this section we discuss curious properties of a localization of the ring $B = \mathbf{k}[H_0]$ which appeared in the previous section. Recall our notations: $\mathbf{k}$ is a field; $H = \mathbf{Z}^r$ is a free abelian group of rank $r \geq 1$; $\xi : H \to \mathbf{R}$ is a group homomorphism and finally $H_0 = \{h \in H; \xi(h) \geq 0\}$. In this section

[3]Note that M/M_a is not a left $\mathbf{k}[H]$-module but the limit is.

we will require ξ to be injective. Clearly H_0 is a sub-semigroup of H. For $a \in \mathbf{R}$, $a \geq 0$, we denote

$$H_{\geq a} = \{g \in H; \xi(g) \geq a\}, \quad H_{>a} = \{g \in H; \xi(g) > a\}.$$

These are subsemigroups of $H_0 = H_{\geq 0}$. We also denote

$$J_{\geq a} = \mathbf{k}[H_{\geq a}], \quad J_{>a} = \mathbf{k}[H_{>a}];$$

these are ideals of B. One has $J_{\geq a} \supset J_{>a}$ and the ideals $J_{\geq a}$ and $J_{>a}$ coincide assuming that $a \notin \xi(H)$.

Let $S = 1 + J_{>0}$ be the set of all elements of the form $1 + x$ where $x \in J_{>0}$. Obviously $S \subset B$ is a multiplicative subset. The ring B' is obtained by inverting elements of S, i.e. $B' = S^{-1}B$. Any ideal $J \subset B$ determines ideal $J' = S^{-1}J \subset B'$. In particular, the ideals $J_{\geq a}, J_{>a} \subset B$ determine the ideals $J'_{\geq a}$, $J'_{>a} \subset B'$.

For $r = 1$ the ring B is simply the polynomial ring $\mathbf{k}[T]$ in one variable. It follows that for $r = 1$ the rings B and B' are principal ideal domains. For $r > 1$ the rings B and B' are not Noetherian, see below. However even for $r > 1$ the ring B' retains some of the properties of principal ideal domains. For instance, we show that every finitely generated ideal of B' is principal. Also, we give a full classification of finitely presented B'-modules.

Lemma 7.

(A) For $a \in \xi(H)$, $a \geq 0$, the ideal $J'_{\geq a} \subset B'$ is principal.
(B) If $r > 1$ the ideal $J'_{>a} \subset B'$ is not finitely generated.
(C) Any finitely generated ideal of B' coincides with $J'_{\geq a}$ for some $a \geq 0$, $a \in \xi(H)$. Hence any finitely generated ideal of B' is principal.
(D) Let $r > 1$. Then any ideal of B' which is not finitely generated coincides with one of the ideals $J'_{>a}$ for some $a > 0$.

Proof. If $g \in H$ is such that $\xi(g) = a$ then g generates $\mathbf{k}[H_{\geq a}] = J_{\geq a}$; this proves (A). If $r > 1$ then the image $\xi(H) \subset \mathbf{R}$ is dense. One has

$$J'_{>a} = \bigcup_{b \in \xi(H),\, b > a} J'_{\geq b}.$$

This implies (B) as any finite set of elements of $J'_{>a}$ lies in some $J'_{\geq b}$, where $b > a$, and hence it cannot generate $J'_{>a}$ since $J'_{\geq b} \subsetneq J'_{>a}$.

Any nonzero element $x \in B'$ can be written in the form $x = gu$ where $g \in H_0$ and u is an invertible element of B'. Moreover, this representation is unique as the only group element $g \in H_0$ which is invertible in B' is $g = 1$. Now assume that $x_1, \dots, x_k \in B$ generate an ideal $J \subset B'$. Write $x_i = g_i u_i$ where $g_i \in H$ and u_i is a unit. We obtain that J is also generated by the group elements $g_1, \dots, g_k \in H_0$. Therefore, J coincides with $J'_{\geq a}$ where $a = \min\{\xi(g_i); i = 1, \dots, k\}$. This proves (C).

Let $J \subset B'$ be an ideal which has no finite generating set. Let $\{x_i\}_{i \in I}$ be an infinite generating set where $x_i \in B'$. As above, we may write $x_i = g_i u_i$

where $g_i \in H_0$ and u_i is a unit. Let a be the infimum of the numbers $\xi(g_i)$. If $a = \xi(g_i)$ for some i then x_i generates J and $J = J'_{\geq a}$ is principal. If $a < \xi(g_i)$ for all $i \in I$ then clearly $J = J'_{>a}$. This proves (D). □

Lemma 8. *Let $f : F_1 \to F_0$ be a homomorphism between finitely generated free B'-modules. Denote $m = \mathrm{rk}(F_1)$, $n = \mathrm{rk}(F_0)$. Then there exist free basis $\alpha_1, \dots, \alpha_m \in F_1$ and $\beta_1, \dots, \beta_n \in F_0$ and an integer $k \leq \min\{m, n\}$ such that*

$$f(\alpha_i) = g_i\beta_i, \quad i = 1, \dots, k, \tag{11}$$

and

$$f(\alpha_i) = 0, \quad i = k+1, \dots, m. \tag{12}$$

Here $g_i \in H$ are group elements satisfying $0 \leq \xi(g_1) \leq \xi(g_2) \leq \cdots \leq \xi(g_k)$.

Proof. Start with arbitrary free basis $\alpha_1, \dots, \alpha_m \in F_1$ and $\beta_1, \dots, \beta_n \in F_0$ and write

$$f(\alpha_i) = \sum_{j=1}^{n} a_{ij}\beta_j, \quad \text{where} \quad i = 1, \dots, m, \quad a_{ij} \in B'.$$

Find the largest number $a \geq 0$ such that one has $a_{ij} \in J'_a$ for all i, j. Without loss of generality we may assume that the element $a_{11} \in B'$ is such that $a_{11} = gu$ where $g \in H$, $\xi(g) = a$ and $u \in B'$ is a unit. Now we may replace the initially chosen basis by $\alpha'_1, \dots, \alpha'_m \in F_1$ and $\beta'_1, \dots, \beta'_n \in F_0$ where

$$\begin{aligned}
\alpha'_1 &= \alpha_1, \\
\alpha'_i &= \alpha_i - a_{i1}g^{-1}u^{-1}\alpha_1, \quad i = 2, \dots, n, \\
\beta'_1 &= u\beta_1 + \sum_{j=2}^{n} a_{1j}g^{-1}\beta_j, \\
\beta'_i &= \beta_i, \quad i = 2, \dots, m.
\end{aligned}$$

Note that $a_{ij}g^{-1}u^{-1}$ and $a_{1j}g^{-1}$ are well defined as elements of B'. In the new basis we have

$$f(\alpha'_1) = g\beta'_1 \quad \text{and} \quad f(\alpha'_i) = \sum_{i=2}^{n} a'_{ij}\beta'_j.$$

Our statement now follows by induction. □

Corollary 9. *Let $f : F_1 \to F_0$ be a homomorphism between free B'-modules of finite rank. Then the kernel* $\mathrm{Ker}(f)$ *is free and its rank is at most* $\mathrm{rk}(F_1)$*. Similarly, the image* $\mathrm{Im}(f)$ *is free and its rank is at most* $\mathrm{rk}(F_0)$*.*

Corollary 10. *Any finitely presented B'-module M is isomorphic to a finite direct sum of the form $B' \oplus B' \oplus \cdots \oplus B' \oplus C_1 \oplus \cdots \oplus C_k$ where each module C_i has the form $C_i = B'/(g_iB')$ with $g_i \in H$, $\xi(g_i) > 0$.*

Note that the isomorphism type of each "cyclic" module $C_i = B'/(g_i B')$ is fully determined by the real number $a_i = \xi(g_i) > 0$.

Corollary 11. *Let M be a finitely presented B'-module. Then there exists a group element $g \in H$ with $\xi(g) \geq 0$ such that for any torsion element $m \in M$ one has $gm = 0$.*

Proof. Take $g \in H$ such that $\xi(g) \geq \max\{\xi(g_i), i = 1, \dots, k\}$ where $g_i \in H$ appear in the previous Corollary. □

Corollary 12. *Let C_* be a free chain complex of B'-modules such that each module C_i is finitely generated. Then each homology module $H_i(C)$ is finitely presented.*

Proof. Let $d : C_i \to C_{i-1}$ be the boundary homomorphism. Using Corollary 9 we find that the module of cycles $Z_i = \mathrm{Ker}(d)$ as well as the module of boundaries $B_{i-1} = \mathrm{Im}(d)$ are finitely generated free modules. Therefore $H_i(C) = Z_i/B_i$ is finitely presented. □

8. Vanishing of the $\lim^1$ term.

We now show that under certain conditions the $\lim^1$ term of the exact sequence (8) vanishes.

Proposition 13. *Let X be a finite CW-complex and $q : \tilde{X} \to X$ a normal covering with free abelian group of covering translations $H \simeq \mathbf{Z}^r$. Let $\xi \in H^1(X; \mathbf{R})$ be a cohomology class of rank $r = \mathrm{rk}(H)$ such that $q^*(\xi) = 0$. Then for any field $\mathbf{k}$ one has*

$$\varprojlim{}^1 H_q(\tilde{X}, N; \mathbf{k}) = 0 \tag{13}$$

where N runs over all neighborhoods of infinity in $\tilde{X}$ with respect to ξ.

To prove Proposition 13 we show that the inverse system $H_q(\tilde{X}, N; \mathbf{k})$ satisfies *the Mittag-Leffler condition* (see [9], Proposition 3.5.7), i.e. that for any neighborhood of infinity $N \subset \tilde{X}$ there exists a neighborhood of infinity $N' \subset N$ such that for any neighborhood of infinity $N'' \subset N'$ one has

$$\mathrm{Im}\left[H_q(\tilde{X}, N'') \to H_q(\tilde{X}, N)\right] = \mathrm{Im}\left[H_q(\tilde{X}, N') \to H_q(\tilde{X}, N)\right]. \tag{14}$$

In (14) all homology groups are with coefficients in the field $\mathbf{k}$ and all neighborhoods of infinity are with respect to a fixed cohomology class ξ, see above. Equality (14) follows from the following slightly stronger statement:

Proposition 14. *Let $N \subset \tilde{X}$ be a neighborhood of infinity with respect to ξ. Then there exists a neighborhood of infinity $N' \subset N$ such that*

$$\mathrm{Im}\left[H_q(\tilde{X}) \to H_q(\tilde{X}, N)\right] = \mathrm{Im}\left[H_q(\tilde{X}, N') \to H_q(\tilde{X}, N)\right], \tag{15}$$

where all homology groups have $\mathbf{k}$ (i.e. a field) as the ring of coefficients.

This can be expressed by saying that for any N there exists $N' \subset N$ such that any cycle of $\tilde{X}$ relative to N which can be refined to a cycle relative to N' can be refined to an absolute cycle in $\tilde{X}$. Proposition 14 clearly implies the Mittag-Leffler condition and hence Proposition 13.

Proof of Proposition 14. We will first prove Proposition 14 assuming that N is a neighborhood of infinity as in Lemma 3.

Let $C_*(N)$ and $C_*(\tilde{X})$ denote the cellular chain complexes of N and $\tilde{X}$ with coefficients in $\mathbf{k}$. They are naturally B-modules where $B = \mathbf{k}[H_0]$. As above, let $S \subset B$ denote the multiplicative subset of all elements of the form $1 + x$ where $x \in J_{>0}$. Consider the localized complexes $S^{-1}C_*(N)$ and $S^{-1}C_*(\tilde{X})$ which we will denote by $C'_*(N)$ and $C'_*(\tilde{X})$ correspondingly. The canonical inclusions $C_*(N) \to C'_*(N)$ and $C_*(\tilde{X}) \to C'_*(\tilde{X})$ determine a chain homomorphism $F : C_*(\tilde{X})/C_*(N) \to C'_*(\tilde{X})/C'_*(N)$. We claim that F is an isomorphism. Injectivity of F is equivalent to $C_*(\tilde{X}) \cap C'_*(N) = C_*(N)$ (which is obvious) and surjectivity of F is equivalent to $C_*(\tilde{X}) + C'_*(N) = C'_*(\tilde{X})$. The latter follows from the following observation: if $c \in C_*(\tilde{X})$ and $x \in J_{>0}$ then the "fractional" chain $\frac{c}{1-x}$ can be written as

$$\frac{c}{1-x} = [c + xc + x^2c + \cdots + x^{k-1}c] + \frac{x^k c}{1-x} \tag{16}$$

where the term in the square brackets lies in $C_*(\tilde{X})$ and the fraction in the RHS lies in $C'_*(N)$ for k large enough.

The short exact sequence of chain complexes over the ring $B' = S^{-1}B$

$$0 \to C'_*(N) \to C'_*(\tilde{X}) \to C_*(\tilde{X})/C_*(N) \to 0$$

gives the exact sequence

$$\cdots \to H'_q(N) \xrightarrow{i_*} H'_q(\tilde{X}) \to H_q(\tilde{X}, N) \xrightarrow{\partial} H'_{q-1}(N) \to \dots$$

where $H'_*(N)$ denotes homology of the complex $C'_*(N)$ and similarly for $H'_*(\tilde{X})$; the symbol $H_q(\tilde{X}, N)$ denotes $H_q(\tilde{X}, N; \mathbf{k})$. Note that each homology module $H'_*(N)$ is finitely presented over B' (by Corollary 12) and the kernel of the homomorphism $i_* : H'_q(N) \to H'_q(\tilde{X})$ coincides with the torsion submodule[4] of $H'_q(N)$. Applying Corollary 11 we obtain that there exists a group element $g \in H$ with $\xi(g) \geq 0$ such that multiplication by g annihilates the torsion submodule of $H'_q(N)$. Hence we obtain that the composition

$$H_{q+1}(\tilde{X}, N) \xrightarrow{\partial} H'_q(N) \xrightarrow{g} H'_q(N) \tag{17}$$

is trivial. This implies triviality of the composition

$$H_{q+1}(\tilde{X}, N) \xrightarrow{g} H_{q+1}(\tilde{X}, N) \xrightarrow{\partial} H'_q(N) \tag{18}$$

and thus the image $\mathrm{Im}[g : H_{q+1}(\tilde{X}, N) \to H_{q+1}(\tilde{X}, N)]$ coincides with $\mathrm{Im}[H'_{q+1}(\tilde{X}) \to H_{q+1}(\tilde{X}, N)]$.

[4]Since $H'_q(\tilde{X}) = Q(\mathbf{k}[H]) \otimes_{S^{-1}B} H'_q(N)$ where $Q(\mathbf{k}[H])$ is the field of fractions of $\mathbf{k}[H]$.

Notice that $\mathrm{Im}[H'_{q+1}(\tilde{X}) \to H_{q+1}(\tilde{X}, N)] = \mathrm{Im}[H_{q+1}(\tilde{X}) \to H_{q+1}(\tilde{X}, N)]$ as easily follows from (16).

This proves the claim of Proposition 14 for the specially chosen neighborhood N if one sets $N' = gN \subset N$.

To prove Proposition 14 for an arbitrary neighborhood of infinity $N_1 \subset \tilde{X}$ we note that for some $g_1 \in H$ with $\xi(g_1) > 0$ one has $g_1 N \subset N_1$, see Lemma 3. Then $g_1 N' \subset N_1$ and we have

$$\mathrm{Im}[H_q(\tilde{X}, g_1 N') \to H_q(\tilde{X}, N_1)] =$$

$$\mathrm{Im}[H_q(\tilde{X}, g_1 N') \to H_q(\tilde{X}, g_1 N) \to H_q(\tilde{X}, N_1)] =$$

$$\mathrm{Im}[H_q(\tilde{X}) \to H_q(\tilde{X}, g_1 N) \to H_q(\tilde{X}, N_1)] =$$

$$\mathrm{Im}[H_q(\tilde{X}) \to H_q(\tilde{X}, N_1)]. \quad \square$$

9. Proof of Theorem 1.

Let $C_* = C_*(\tilde{X})$ be the cellular chain complex of $\tilde{X}$ with coefficients in $\mathbf{k}$. It is free and finitely generated over $\mathbf{k}[H]$. Consider two larger rings

$$\mathbf{k}[H] \subset Q(\mathbf{k}[H]) \subset \widehat{\mathbf{k}[H]}_\xi$$

where $Q(\mathbf{k}[H])$ is the field of quotients of $\mathbf{k}[H]$. Accordingly, we have three chain complexes $C_* \subset C'_* \subset C''_*$ where $C_* = C_*(\tilde{X})$ and $C'_* = Q(\mathbf{k}[H]) \otimes_{\mathbf{k}[H]} C_*$ is obtained by localization while $C''_* = \widehat{\mathbf{k}[H]}_\xi \otimes_{\mathbf{k}[H]} C_*$ is obtained by completion. The induced homomorphisms on homology are

$$H_i(\tilde{X}; \mathbf{k}) \xrightarrow{\alpha} H_i(C') \xrightarrow{\beta} H_i(C''). \tag{19}$$

We claim (1) that β is injective and (2) that the kernel of the composition $\beta \circ \alpha$ coincides with the set of homology classes $z \in H_i(\tilde{X}; \mathbf{k})$ which are movable to infinity with respect to ξ. Claim (1) follows since $\widehat{\mathbf{k}[H]}_\xi$ contains $Q(\mathbf{k}[H])$ as a subfield. Claim (2) follows since $H_i(C'') \simeq \varprojlim H_i(\tilde{X}, N; \mathbf{k})$ by Propositions 6 and 13, and (7).

From these two claims it follows that the set of homology classes $z \in H_i(\tilde{X}; \mathbf{k})$ which are movable to infinity with respect to ξ coincides with the kernel of α. Since α is a localization homomorphism with respect to the set of all nonzero elements of the ring $\mathbf{k}[H]$ we obtain that a homology class $z \in H_i(\tilde{X}; \mathbf{k})$ lies in its kernel if and only if it is torsion.

10. The set of directions with respect to which a given homology class is movable to infinity.

Let X be a finite cell complex and $q : \tilde{X} \to X$ be a regular covering having a free abelian group of covering transformations $H \simeq \mathbf{Z}^r$. Fix a homology class $z \in H_i(\tilde{X}; \mathbf{k})$ and consider the set

$$\mathcal{M}_z \subset H^* \subset H^1(X; \mathbf{R})$$

of all cohomology classes $\xi \in H^1(X; \mathbf{R})$ with $q^*(\xi) = 0$ such that z is movable to infinity with respect to ξ. In this section we discuss the structure of the set $\mathcal{M}_z$.

Theorem 2. *If $\mathcal{M}_z$ contains a cohomology class $\xi \in H^1(X; \mathbf{R})$ with $\operatorname{rk}(\xi) = r$ then $\mathcal{M}_z$ contains a set of the form $H^* - \bigcup_{j=1}^k Q_j$ where each $Q_j \subset H^*$ is an integral hyperplane*

$$Q_j = \{\xi \in H^*; \, \xi(\gamma_j) = 0\} = \gamma_j^\perp, \quad j = 1, \dots, k, \tag{20}$$

where $\gamma_j \in H$, $\gamma_j \neq 1 \in H$. In particular, if $\mathcal{M}_z$ contains a single rank r cohomology class then it contains all classes $\xi' \in H^*$ with $\operatorname{rk}(\xi') = r$.

Proof. Let z be movable to infinity with respect to ξ, where $\operatorname{rk}(\xi) = r$. By Theorem 1 there exists a nonzero element $x \in \mathbf{k}[H]$ such that $x \cdot z = 0$. We can write $x = \sum_{h \in H} a_h h, \quad a_h \in \mathbf{k}$ with only finitely many $a_h \neq 0$. Let $\operatorname{supp} x = \{h \in H : a_h \neq 0\}$. Consider the finite set $\{h_1 h_2^{-1} \in H; h_1, h_2 \in \operatorname{supp} x, \; h_1 \neq h_2\}$; we can write this set as $\{\gamma_1, \dots, \gamma_k\}$ for some integer k; here $\gamma_j \in H$. Each γ_j determines the hyperplane Q_j given by (20).

Let us show that any class $\xi' \in H^*$ which does not lie in the union of hyperplanes Q_j belongs to $\mathcal{M}_z$. Indeed, if $\xi' \in H^* - \bigcup_{j=1}^k Q_j$ then $\xi'(h_1) \neq \xi'(h_2)$ for any pair of distinct $h_1, h_2 \in \operatorname{supp} x$. Now we can use the relation $x \cdot z = 0$ and repeat the argument of the easy part (C) ⇒ (B) ⇒ (A) of the proof of Theorem 1 (see §5) with the class ξ' replacing ξ. □

Theorem 2 implies that if $\mathcal{M}_z$ contains a cohomology class of rank r then it also contains a cohomology class of rank 1. The next statement shows that the converse also holds.

Proposition 15. *If $\mathcal{M}_z$ contains a cohomology class of rank 1 then it contains a cohomology class of rank r and hence the conclusion of Theorem 2 holds.*

Proof. The proof is similar to the proof of [2, Lm.5.3]. Let ξ be a rank 1 cohomology class such that z is movable to infinity with respect to ξ. We show that there exists a nonzero $x \in \mathbf{k}[H]$ such that $x \cdot z = 0$; the result then follows from Theorem 1. Let $\Gamma \subset H$ denote the kernel of the surjective homomorphism $\xi : H \to \mathbf{Z}$. Then $\mathbf{k}[H] = \mathbf{k}[\Gamma][t, t^{-1}]$ where t is mapped to 1 under ξ. Choose a finite subcomplex $K \subset \tilde{X}$ with $H \cdot K = \tilde{X}$ such that $z \in \operatorname{Im}(i_* : H_q(K; \mathbf{k}) \to H_q(\tilde{X}; \mathbf{k}))$.

Note that $H_q(\Gamma \cdot K; \mathbf{k})$ is a finitely generated $\mathbf{k}[\Gamma]$-module. Here $\Gamma \cdot K$ denotes the union $\bigcup_{g \in \Gamma} gK$. This claim follows since $\mathbf{k}[\Gamma]$ is Noetherian and the complex $\Gamma \cdot K$ has finitely many Γ-orbits of cells.

By the choice of K we know that z can be represented in $H_q(\Gamma \cdot K; \mathbf{k})$. Since z is movable to infinity with respect to ξ we get that z can be represented in $H_q(\bigcup_{k \geq N} t^k \Gamma \cdot K; \mathbf{k})$ for every $N \in \mathbf{Z}$.

Let N be a positive integer such that $\Gamma \cdot K \cap t^N \Gamma \cdot K = \emptyset$. Let us show that for every $k \geq N$ the class z lies in $\mathrm{Im}(i_* : H_q(t^k \Gamma \cdot K; \mathbf{k}) \to H_q(\tilde{X}; \mathbf{k}))$.

Let $B = \bigcup_{r \geq k} t^r \Gamma \cdot K$ and $C = \bigcup_{r \leq k} t^r \Gamma \cdot K$. Then $\tilde{X} = B \cup C$ and by a suitable choice of K we can achieve that $B \cap C = t^k \Gamma \cdot K$. Our claim now follows from the Mayer-Vietoris sequence

$$\ldots \longrightarrow H_q(B \cap C; \mathbf{k}) \longrightarrow H_q(B; \mathbf{k}) \oplus H_q(C; \mathbf{k}) \longrightarrow H_q(\tilde{X}; \mathbf{k}) \longrightarrow \ldots$$

since z can be represented in $H_q(B; \mathbf{k})$ and $H_q(C; \mathbf{k})$.

Denote

$$V \quad = \quad \bigcap_{k \geq N} \mathrm{Im} \left(i_* : H_q(t^k \Gamma \cdot K; \mathbf{k}) \to H_q(\tilde{X}; \mathbf{k}) \right).$$

Since $\mathbf{k}[\Gamma]$ is Noetherian we get that V is a finitely generated $\mathbf{k}[\Gamma]$-module containing z. Furthermore t^{-1} induces a $\mathbf{k}[\Gamma]$-endomorphism $t^{-1} : V \to V$. By the Cayley-Hamilton Theorem, see Eisenbud [1, Thm.4.3], there is a polynomial $p(x) \in \mathbf{k}[\Gamma][x]$ such that $p(t^{-1}) = 0 : V \to V$. In particular $p(t^{-1})z = 0$. But $p(t^{-1})$ can be interpreted as an element of $\mathbf{k}[H]$ with the corresponding action. Thus we can choose $x = p(t^{-1}) \in \mathbf{k}[H]$. □

11. Examples

1. Consider first the case $X = \Sigma_g$ – the orientable surface of genus $g > 1$. Let $q : \tilde{\Sigma}_g \to \Sigma_g$ be the universal abelian cover. In this case the covering translation group $H \simeq \mathbf{Z}^{2g}$. Clearly, $H_2(\tilde{\Sigma}_g; \mathbf{k}) = 0$ and as it is well-known $H_1(\tilde{\Sigma}_g; \mathbf{k})$ has no $\mathbf{k}[H]$-torsion. Hence by Theorem 1 there are no nonzero homology classes in $\tilde{\Sigma}_g$ which are movable to infinity. Note that the rank of the $\mathbf{k}[H]$-module $H_1(\tilde{\Sigma}_g; \mathbf{k})$ equals $2g - 2 > 0$.

2. Consider now the universal abelian cover $q : \tilde{X} \to X$ where $X = \Sigma_g \times S^1$. Now $H = \mathbf{Z}^{2g+1}$ and $\tilde{X} = \tilde{\Sigma}_g \times \mathbf{R}$. We find that $H_1(\tilde{X}; \mathbf{k}) = H_1(\tilde{\Sigma}_g; \mathbf{k}) \neq 0$ (see the previous example). If $t : \tilde{X} \to \tilde{X}$ denotes the translation corresponding to the shift by 1 on $\mathbf{R}$ then for any homology class $z \in H_1(\tilde{X}; \mathbf{k})$ one has $tz = z$, i.e. $(t - 1) \cdot z = 0$. We see that any homology class z is torsion with annihilating polynomial $x = t - 1$. Repeating the proof of Theorem 2 one finds that any homology class $z \in H_1(\tilde{X}; \mathbf{k})$ is movable to infinity with respect to any cohomology class $\xi \in H^1(X; \mathbf{R})$ such that $\xi|_{S^1} \neq 0 \in H^1(S^1; \mathbf{R})$. On the contrary, if $\xi|_{S^1} = 0$ then no nonzero homology class is movable to infinity. We see that in this case the set $\mathcal{M}_z \subset H^1(X; \mathbf{R}) = \mathbf{R}^{2g+1}$ of all directions in which a nonzero class z is movable coincides with the complement to one hyperplane $\xi|_{S^1} = 0$ in H^*.

3. Next we briefly show how one may construct examples where $H = \mathbf{Z}^2$ and the set $\mathcal{M}_z \subset H^*$ coincides with the complement to any prescribed finite set of lines $Q_1, \ldots, Q_k$ on the plane H^* having rational slopes. Here we will use the technique developed in [4], pages 25 - 28.

Assume that the line Q_j passes through the point (n_j, m_j) where n_j and m_j are integers. Consider the polynomial

$$P = \prod_{j=1}^{k} \left(1 - t_1^{n_j} t_2^{m_j}\right) \; = \; \sum_{i,j} L_{ij}\, t_1^i t_2^j. \tag{21}$$

The numbers L_{ij} are determined by this relation. Denote by $\mathcal{L} \subset \mathbf{R}^2$ the set of points with at least one coordinate integral. One constructs a closed curve w lying on $\mathcal{L} \subset \mathbf{R}^2$ with the property that its winding number with the point $(i + \frac{1}{2}, j + \frac{1}{2})$ equals L_{ij} for any i, j. This closed curve can be interpreted as a word w in two letters which is a product of commutators. One then uses this word to build a two dimensional cell complex having one zero-dimensional cell, two one-dimensional cells, and one two-dimensional cell glued according to the word w. Computations explained in [4], pages 25 - 28 show that $H_1(\tilde{X}; \mathbf{k})$ as a $\mathbf{k}[H]$-module is isomorphic to the factor of $\mathbf{k}[H]$ with respect to the ideal generated by P. Let $z \in H_1(\tilde{X}; \mathbf{k})$ be the generator. As in the proof of Theorem 2 above one finds that the class z is movable to infinity with respect to any cohomology class $\xi \in H^* - \cup_{j=1}^k Q_j$. This argument shows that $\mathcal{M}_z$ contains $H^* - \cup_{j=1}^k Q_j$. With more effort one can show that $\mathcal{M}_z$ coincides with $H^* - \cup_{j=1}^k Q_j$. We omit the details.

References

[1] D. Eisenbud, *Commutative algebra. With a view towards algebraic geometry*, Springer - Verlag, New York, 1995.

[2] M. Farber, *Zeros of closed 1-forms, homoclinic orbits and Lusternik-Schnirelman theory*, Topol. Methods Nonlinear Anal. 19 (2002), 123-152.

[3] M. Farber, *Lusternik – Schnirelman theory and dynamics,* "Lusternik - Schnirelmann Category and Related Topics", Contemporary Mathematics, 316(2002), 95 - 111.

[4] M. Farber, *Topology of closed one-forms.* Mathematical Surveys and Monographs, 108. American Mathematical Society, Providence, RI, 2004.

[5] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds, Boston, 1968, 115-133.

[6] W. Neumann, *Signature related invariants of manifolds. I. Monodromy and γ-invariants*, Topology, **18**(1979), 147-172.

[7] S. Novikov, *Multi-valued functions and functionals. An analogue of Morse theory.* Soviet. Math. Doklady, 24(1981), 222-226.

[8] J.Cl. Sikorav, *Problèmes d'intersections et de points fixes en géométrie hamiltonienne*, Comment. Math. Helv., **62**(1987), 62-73.

[9] C. A. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, 1994

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