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Article

Moving Singular Points and the Van der Pol Equation, as Well as the Uniqueness of Its Solution

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Abstract: The article considers the Van der Pol equation nonlinearity aspect related to a moving singular point. The fact of the existence of moving singular points and the uniqueness of their solution for complex domains have been proved. An answer to the question about the existence of moving singular points in the real domain was obtained. The proof of existence and uniqueness is based on an author's modification of the technology of the classical Cauchy theorem. A priori estimates of the analytical approximate solution in the vicinity of a moving singular point are obtained. Calculations of a numerical experiment are presented.

Keywords: Van der Pol equation; moving singular points; analytical approximate solution; a priori estimate

MSC: 34G20; 35A05

1. Introduction

Many problems of the theory of nonlinear oscillations lead to the Van der Pol Equation [1–6] and relaxation oscillations [7]. In recent publications, the Van der Pol equation is used in modeling human movement in synchronization when processing and transmitting information in neural networks [8]. To study this equation, methods of qualitative and asymptotic theory of differential equations are used [9–11]. If the applied methods of qualitative [9,10] and asymptotic [11] as well as analytical [12] theories to nonlinear differential equations allow for establishing the fact of the existence of moving singular points, then the following questions remain open: 1. On the structure of an analytical approximate solution in the neighborhood of a moving singular point; 2. The question of the error of this approximate solution; 3. The question of the size of the very neighborhood of a moving singular point; and 4. The question of the technology for obtaining these moving singular points. The Van der Pol equation is a non-linear differential equation that is generally not solvable in quadratures. It should be noted that publications on the Van der Pol equation do not include terminology of a moving singular point, although some studies have considered the existence of such points. This category of differential equations actually includes: (1) solvability options in quadratures for specific cases and is presented in publications [13–19]; (2) a version of the analytical approximate solution method, effectively implemented for the study of a number of classes of nonlinear differential Equations [20–24] developed by the author of this paper. The author's version of the modification of the classical Cauchy majorant method acts as the basis in the proofs of the theorems of existence and uniqueness of the solution, which allows for, in the end, solving all the problems listed above. Considering the Van der Pol equation in the complex domain, on the basis of the author's modification of the majorant method, we prove the existence theorem for the uniqueness of the solution in the neighborhood of a moving singular point. As a result of the theorem, the existence of the moving singular point is also proved. Based on the analysis of the parameter of the equation, as in a particular case, we obtain an answer

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to the question of the existence of a moving singular point in the real region. It follows from the proved theorem that, in the real region, there are no moving singular points for positive values of the equation parameter, but there are singular points for negative values of the equation parameter. It is possible to construct an analytical approximate solution, and an a priori error estimate is obtained. It also presents a technology for optimizing a priori error estimates based on a posteriori one. The theoretical results are accompanied by a numerical experiment.

2. Research Method and Result

For the Van der Pol equation, consider the Cauchy problem:

$$w''(z) = -a(w^2 - 1)w' - w, \tag{1}$$

$$w(z_0) = w_0, \quad w'(z) = w_1, \tag{2}$$

where $a = constant$ – parameter.

Let us create a proof of the existence and uniqueness theorem for the solution of the Cauchy problem (1)–(2) in the vicinity of a moving singular point z^*. As a consequence, we will have the fact of the existence of the moving singular point.

Theorem 1. *Let z^*—moving singular point of solving the Cauchy problem (1)–(2). Then, there is a unique solution to the problem (1)–(2) in the vicinity of a moving singular point z^* as*

$$w(z) = (z^* - z)^{-\frac{1}{2}} \sum_{n=0}^{\infty} C_n (z^* - z)^{\frac{n}{2}} \tag{3}$$

for $|z^* - z| < 1/|a+1|$.

Proof. Consider the representation of the solution for Equation (1) in the vicinity of a moving singular point in the form

$$w(z) = (z^* - z)^{\rho} \sum_{n=0}^{\infty} C_n (z^* - z)^{\frac{n}{2}}. \tag{4}$$

From (4), we obtain expressions of the first and second derivatives:

$$w'(z) = -\sum_{n=0}^{\infty} C_n \left(\frac{n}{2} + \rho\right)(z^* - z)^{\frac{n}{2}+\rho-1}, \tag{5}$$

$$w''(z) = \sum_{n=0}^{\infty} C_n \left(\frac{n}{2} + \rho\right)\left(\frac{n}{2} + \rho - 1\right)(z^* - z)^{\frac{n}{2}+\rho-2}. \tag{6}$$

Substitute (4)–(6) in (1), we obtain

$$\sum_{n=0}^{\infty} C_n \left(\frac{n}{2} + \rho\right)\left(\frac{n}{2} + \rho - 1\right)(z^* - z)^{\frac{n}{2}+\rho-2}$$

$$= a\left(\left(\sum_{n=0}^{\infty} C_n (z^* - z)^{\frac{n}{2}+\rho}\right)^2 - 1\right) \sum_{n=0}^{\infty} C_n \left(\frac{n}{2} + \rho\right)(z^* - z)^{\frac{n}{2}+\rho-1} - \sum_{n=0}^{\infty} C_n (z^* - z)^{\frac{n}{2}+\rho}$$

or

$$\sum_{n=0}^{\infty} C_n \left(\frac{n}{2} + \rho\right)\left(\frac{n}{2} + \rho - 1\right)(z^* - z)^{\frac{n}{2}+\rho-2}$$

$$= a\sum_{n=0}^{\infty} C_n^{***} (z^* - z)^{\frac{n}{2}+3\rho-1} - \sum_{n=0}^{\infty} C_n (z^* - z)^{\frac{n}{2}+\rho}, \tag{7}$$

where

$$C_n^{***} = \sum_{i=0}^{n} C_i^{**} C_{n-i} \left(\frac{n-1}{2} \right), \quad C_n^{**} = C_n^{*}, \quad \forall n \neq 2, \quad C_2^{**} = C_2^{*} - 1,$$

$$C_n^{*} = \sum_{i=0}^{n} C_i C_{n-i}.$$

From (7), by virtue of the identity requirement, the degrees of the left and right sides must be equal, which leads to the condition

$$\frac{n}{2} + \rho - 2 = \frac{n}{2} + 3\rho - 1. \tag{8}$$

Then, from (8), we obtain $\rho = -1/2$.

At the next stage, due to the identity requirement in (7), we obtain recurrent relations for the coefficients C_n:

$$C_n \left(\frac{n}{2} - \frac{1}{2} \right) \left(\frac{n}{2} - \frac{3}{2} \right) = aC_n^{***}, \quad \forall n = 0,\ 1,\ 2,\ 3 \tag{9}$$

and

$$C_n \left(\frac{n}{2} - \frac{1}{2} \right) \left(\frac{n}{2} - \frac{3}{2} \right) = aC_n^{***} - C_{n-4}, \quad \forall n = 4,\ 5,\ \ldots\ . \tag{10}$$

The relations (9) and (10) allow us to unambiguously obtain the expressions of coefficients C_n for a series in (3). Then, from (9), we obtain

$$C_0 \cdot \frac{3}{4} = -aC_0^3 \cdot \frac{1}{2}, \tag{11}$$

whence follows

$$C_0 = \pm i \sqrt{\frac{3}{2a}}.$$

For C_1 from (9) follows the value $C_1 = 0$. Similarly, from (9) we obtain expressions for C_2 and C_3:

$$C_2 = -a \cdot \frac{1}{4} C_0, \quad C_3 = 0.$$

Subsequent expressions, due to the recurrent ratio (10), determine the conformity—odd coefficients $C_{2n+1} = 0$. The formulas (8) and (9) enable uniquely determining all the coefficients in solution (3). Thus, we obtain a proof of the uniqueness of solution (3). We pass to the second part of the proof of the theorem, the convergence of the correct part of the series in (3). We prove the validity of the estimate for coefficients with an even index for $n \geq 2$

$$|C_{2n}| \leq \frac{\sqrt{\frac{3}{2a}} |a+1|^n}{\left(\frac{2n-1}{2} \right) \left(\frac{2n-3}{2} \right)}. \tag{12}$$

From (10), we have

$$C_{2n+2} \left(n + \frac{1}{2} \right) \left(n - \frac{1}{2} \right) = aC_{2n+2}^{***} - C_{2n-4}$$

or, taking into consideration the formula,

$$C_n^{***} = \sum_{i=0}^{n} C_i^{**} C_{n-i} \left(\frac{n-1}{2} \right),$$

we obtain

$$C_{2n+2}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)-aC_0^2C_{2n+2}\left(n+\frac{1}{2}\right)=a\sum_{i=1}^{2n+1}C_i^{**}C_{2n+2-i}\left(n+\frac{1}{2}\right)-C_{2n-4}.$$

After transformation from the last one, we have

$$C_{2n+2}\left(n+\frac{1}{2}\right)(n+1)=a\sum_{i=1}^{2n+1}C_i^{**}C_{2n+2-i}\left(n+\frac{1}{2}\right)-C_{2n-4}. \tag{13}$$

Hence, from (13), we have

$$|C_{2n+2}|=\left|\frac{1}{\left(n+\frac{1}{2}\right)(n+1)}\left(a\sum_{i=1}^{2n+1}C_i^{**}C_{2n+2-i}\left(n+\frac{1}{2}\right)-C_{2n-4}\right)\right|$$

$$=\left|\frac{1}{\left(n+\frac{1}{2}\right)(n+1)}\left(a\sum_{i=1}^{2n+1}\left(\sum_{j=1}^{i}C_jC_{i-j}\right)\left(n+\frac{1}{2}\right)C_{2n+2-i}-C_{2n-4}\right)\right|.$$

From the latter, taking into account (12), we obtain

$$|C_{2n+2}|\leq\left|\frac{1}{\left(n+\frac{1}{2}\right)(n+1)}\left(a\sum_{i=1}^{2n+1}\left(\sum_{j=1}^{i}\frac{\sqrt{\frac{3}{2a}}(a+1)^{\frac{j}{2}}}{\left(\frac{j-1}{2}\right)^*\left(\frac{j-3}{2}\right)^*}\frac{\sqrt{\frac{3}{2a}}(a+1)^{\frac{i-j}{2}}}{\left(\frac{i-j-1}{2}\right)^*\left(\frac{i-j-3}{2}\right)^*}\right)\right.\right.$$

$$\left.\left.\times\frac{\sqrt{\frac{3}{2a}}(a+1)^{\frac{2n+2-i}{2}}}{\left(\frac{2n+2-i-1}{2}\right)^*\left(\frac{2n+2-i-3}{2}\right)^*}-\frac{\sqrt{\frac{3}{2a}}(a+1)^{\frac{2n-4}{2}}}{\left(\frac{2n-4-1}{2}\right)\left(\frac{2n-4-3}{2}\right)}\right)\right|$$

for $n\geq 4$, or, after a series of transformations in the latter,

$$|C_{2n+2}|\leq\frac{\sqrt{\frac{3}{2a}}|a+1|^{n+1}}{\left(n+\frac{1}{2}\right)(n+1)}\leq\frac{\sqrt{\frac{3}{2a}}|a+1|^{n+1}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}=A_{2n+2},$$

in this connection

$$(j-1)^*=\begin{cases}1, & j=1;\\ j-1, & j\neq 1,\end{cases}\qquad (j-3)^*=\begin{cases}1, & j=3;\\ j-3, & j\neq 3,\end{cases}$$

$$(i-j-1)^*=\begin{cases}1, & i-j=1;\\ i-j-1, & i-j\neq 1,\end{cases}\qquad (i-j-3)^*=\begin{cases}1, & i-j=3;\\ i-j-3, & i-j\neq 3,\end{cases}$$

$$(2n+2-i-1)^*=\begin{cases}1, & 2n+2-i=1;\\ 2n+2-i-1, & 2n+2-i\neq 1,\end{cases}$$

$$(2n+2-i-3)^*=\begin{cases}1, & 2n+2-i=3;\\ 2n+2-i-3, & 2n+2-i\neq 3.\end{cases}$$

Consider a series of

$$\sum_{n=1}^{\infty}A_n(z^*-z)^{\frac{n-1}{2}}, \tag{14}$$

is a majorant for the correct part of the series in (3). Given that the series (14) is convergent in the domain, on the basis of the sufficient feature of fraction series convergence

$$|z^* - z| < \frac{1}{|a+1|} = \rho, \tag{15}$$

then, therefore, the correct part of the series (3) will also converge in the region (15). □

Comment 1. Theorem 1 is a theorem for the existence of a moving singular point z^*.

Comment 2. The Van der Pol equation has no moving singular points of algebraic type in the real domain for the case $\alpha > 0$. This follows from (11) that all coefficients $C_n \equiv 0$.

Comment 3. For $\alpha < 0$, the Van der Pol equation has moving singular points of algebraic type in the real domain.

Theorem 1 allows us to construct an analytical approximate solution in the vicinity of a moving singular point z^*:

$$w_N(z) = (z^* - z)^{-\frac{1}{2}} \sum_{n=0}^{N} C_n (z^* - z)^{\frac{n}{2}}. \tag{16}$$

The following Theorem 2 allows us to obtain an a priori estimate for the analytical approximate solution (16).

Theorem 2. *For an analytical approximate solution (16) in the domain (15) the error estimate is valid*

$$\Delta w_N \leq \frac{\sqrt{\frac{3}{2\alpha}}|\alpha+1|^{\frac{N+1}{2}}|z^*-z|^{\frac{N}{2}}}{1-|a+1||z^*-z|}.$$

Proof. Taking into account (12), we have in the case $N+1 = 2k$

$$\Delta w_N(z) = |w(z) - w_N(z)| = \left| \sum_{n=0}^{\infty} C_n (z^*-z)^{\frac{n-1}{2}} - \sum_{n=0}^{N} C_n (z^*-z)^{\frac{n-1}{2}} \right|$$

$$= \left| \sum_{n=N+1}^{\infty} C_n (z^*-z)^{\frac{n-1}{2}} \right| \leq \left| \sum_{n=N+1}^{\infty} \frac{\sqrt{\frac{3}{2\alpha}}|\alpha+1|^{\frac{n}{2}}}{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)} (z^*-z)^{\frac{n-1}{2}} \right|$$

$$\leq \frac{\sqrt{\frac{3}{2a}}|a+1|^{\frac{N+1}{2}}|z^*-z|^{\frac{N}{2}}}{1-|a+1||z^*-z|}$$

given that $|z^* - z| < 1/|a+1|$. Given the area (15) for the structure of the analytical approximate solution (16), we obtain the area (15) for the a priori error estimation. □

3. Discussion

Consider the Cauchy problem (1)–(2): $a = 2$, $w(0) = i$, $w'(0) = -i$, $z^* = 0.8571773685$, $\rho = 0.33333333$, $z_1 = 0.6571773685$, $z_2 = 0.83$. Consider the option

$$C_0 = \sqrt{\frac{3}{2a}}i.$$

The calculations are presented in Table 1, in which $w_4(z_i)$—approximate solution (16); Δ_1—a priori estimation of error, theorem 2; and Δ_2—a posteriori estimation of error.

Table 1. Characteristics of the calculations.

z_i	$w_4(z_i)$	Δ_1	Δ_2
$z_1 = 0.6571773685$	$1.5075188i$	0.6185	0.0104
$z_2 = 0.83$	$4.8204059i$	0.000584	$7 \cdot 10^{-7}$

The values z_1 and z_2 fall into the domain determined by Theorem 2 condition.

In the case z_1 of a posteriori evaluation $\Delta_2 = 0.0104$ in the structure of the approximate solution (16), the value required is $N = 20$. The summands from 6 to 20 in the sum do not exceed the required accuracy. Therefore, $w_4(z_1)$ has a precision $\varepsilon = 0.0104$. In the case z_2 for a posteriori evaluation $\Delta_2 = 7 \cdot 10^{-7}$ similar to Theorem 2, $N = 10$ is required. The summands from 6 to 10 in the sum do not exceed the required accuracy. Therefore, $w_4(z_2)$ has an accuracy of $\varepsilon = 7 \cdot 10^{-7}$. In the above example, two argument values are considered. The above calculations show how the structure of the analytical approximate solution is related to the accuracy of the calculation result. The closer the argument is to the value of the moving singular point, the more accurate the result. The structure of the analytical approximate solution (3) requires fewer terms. The a posteriori estimate substantially improves the a priori estimate.

4. Conclusions

In this paper, for the Van der Pol equation in the complex domain, the theorem of the existence and uniqueness of the solution in the vicinity of a moving singular point of algebraic type is proved. Along with the fact of the existence of a moving singular point, a condition is set for the parameter of the equation "a", with which the equation under consideration will not have the indicated type of moving singular points in the real domain. In this case, it is possible to apply classical numerical methods to the Van der Pol equation solution. An analytical approximate solution is obtained in the vicinity of a moving singular point. A numerical experiment was performed to confirm the theoretical results. The technology of optimization of a priori estimates of an analytical approximate solution using a posteriori estimates is presented.

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