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ORIGINAL PAPER

Multi-attribute decision-making based on novel Fermatean fuzzy similarity measure and entropy measure

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Abstract
To deal with situations involving uncertainty, Fermatean fuzzy sets are more effective than Pythagorean fuzzy sets, intuitionistic fuzzy sets, and fuzzy sets. Applications for fuzzy similarity measures can be found in a wide range of fields, including clustering analysis, classification issues, medical diagnosis, etc. The computation of the weights of the criteria in a multi-criteria decision-making problem heavily relies on fuzzy entropy measurements. In this paper, we employ t-conorms to suggest various Fermatean fuzzy similarity measures. We have also discussed all of their interesting characteristics. Using the suggested similarity measurements, we have created some new entropy measures for Fermatean fuzzy sets. By using numerical comparison and linguistic hedging, we have established the superiority of the suggested similarity metrics and entropy measures over the existing measures in the Fermatean fuzzy environment. The usefulness of the proposed Fermatean fuzzy similarity measurements is shown by pattern analysis. Last but not least, a novel multi-attribute decision-making approach is described that tackles a significant flaw in the order preference by similarity to the ideal solution, a conventional approach to decision-making, in a Fermatean fuzzy environment.

Keywords Fuzzy set · Fermatean fuzzy set · t-Conorm · Similarity measure · Entropy measure · Multi-attribute decision-making

1 Introduction

To solve problems involving uncertainty more precisely, a new notion known as the Pythagorean fuzzy set (PFS) was put forward by Yager (2013). PFS is a generalized version of fuzzy (Zadeh 1965) and intuitionistic fuzzy sets (Atanassov 1986) (IFSs). A membership (σ) and a non-membership (ς) degree whose maximum square sum is one ($\sigma^2 + \varsigma^2 \leq 1$) is assigned to each element in a PFS. The Pythagorean fuzzy number and the technique for order of preference by similarity to the ideal solution (TOPSIS) in a Pythagorean fuzzy (PF) environment were proposed by Zhang and Xu (2014). Yager (2014) provided several PF aggregating functions and their usefulness in decision-making. Wei and Lu (2018) developed some PF power aggregation functions. Garg (2016) suggested some new aggregating functions for the PF environment utilizing Einstein operations. Wei (2017) proposed many PF interaction aggregation functions and discussed their applicability to multi-attribute decision-making (MADM). The

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literature has numerous studies (Garg 2017; Lu et al. 2017; Wei et al. 2017; Wei and Lu 2017) on the PF aggregating functions and their applicability. The TODIM (multi-criteria decision-making in Portuguese) in the PF setting was developed by Ren et al. (2016). Many PF measures of information were proposed by Peng et al. (2017). A new PF distance metric was proposed by Peng and Dai (2017). Singh and Ganie (2020) developed a few PF metrics of correlation with their utility. PFSs have been examined and implemented by numerous researchers (Garg 2019a, b; Khan et al. 2019a, b; Akram and Ali 2020; Ejegwa 2020a, b; Rahman et al. 2020; Zhang et al. 2021; Olgun et al. 2021; Mishra et al. 2021; Zeb et al. 2022; Wang et al. 2022; Ganie 2022, 2023; Ganie et al. 2022; Akram and Bibi 2023; Ejegwa et al. 2023; Aldring and Ajay 2023; Kirişci 2023; Akram et al. 2023b; Akram et al. 2023a, c) in various uncertain situations. Chen and Chiou (2015) and Zeng et al. (2020) discussed the applicability of interval-valued IFSs to MADM problems. Although PFSs have many uses in a variety of domains, they are unable to handle circumstances where $\sigma^2 + \varsigma^2 > 1$. For instance, if $\sigma = 0.8$ and $\varsigma = 0.7$, then $\sigma^2 + \varsigma^2 = 1.13 > 1$. Senapati and Yager's (2020) concept of Fermatean fuzzy sets was thus proposed (FFSs). We have $\sigma^3 + \varsigma^3 \leq 1$ in a Fermatean fuzzy set (FFS). FFSs are more robust and effective than FSs, IFSs, and PFSs because these all fall in the space of FFSs. Senapati and Yager (2019) provided a list of FFS aggregation operators and their potential applications in decision-making. Mishra and Rani (2021) proposed the weighted aggregated sum product assessment (WASPAS) method in the Fermatean fuzzy (FF) environment. Keshavarz-Ghorabaee et al. (2020) presented an innovative FF decision-making approach. Garg et al. (2020) demonstrated the use of FF aggregating functions in the COVID-19 testing facility. The continuities and derivatives of FF functions were researched by Yang et al. (2021). Sergi and Sari (2021) suggested a few FF capital budgeting strategies. Sahoo (2021a, b) suggested a few score functions for FFSs and discussed their use in solving transportation-related problems and making decisions. Aydemir and Yilmaz (2020) introduced the FF TOPSIS technique. Some FF aggregating functions based on Einstein's norm were proposed by Akram et al. (2020). There are some investigations of FFSs and their real-world applications in the literature (Salsabeela and John 2021; Aydın 2021; Gul et al. 2021; Hadi et al. 2021; Shit and Ghorai 2021; Rani and Mishra 2021; Akram et al. 2022, 2023d; Mishra et al. 2022a, b, 2023; Ali and Ansari 2022; Zhou et al. 2022; Luqman and Shahzadi 2023). The creation of several FF similarity and entropy measurements is discussed in this study.

Based on the content of equality, two things can be compared very effectively using similarity measurements. Zhang (2016) introduced a PF metric of similarity and applied it to a decision-making problem. Many novel PF measurements of distance and similarity were given by Peng (2019). Based on the cosine function, some PF measurements of similarity were proposed by Wei and Wei (2018). Mohd and Abdullah (2018) created several innovative PF similarity metrics by fusing the Euclidean distance measure with cosine similarity measures. By considering all three membership grades Ejegwa (2020a) presented many PF metrics of similarity and distance. The applicability of some PF metrics of similarity and distance in MADM was shown by Zeng et al. (2018). A Hausdorff PF similarity metric was suggested by Hussain and Yang (2019). Zhang et al. (2019) developed some exponential PF similarity metrics and showed how they may be used for pattern identification, MADM, and medical diagnostics. Li and Lu (2019) provided a few complement-based, matching function-based, and set theoretic-based PF similarity measurements. Wang et al. (2019) provided some PF Dice similarity measures with applications in decision-making. Some trigonometric function-based PF metrics of similarity were created by Verma and Merigo (2019). Peng and Garg (2019) highlighted how several multi-parametric PF measures of similarity can be used in classification challenges.

A PFS's ambiguous content determines its entropy. In a MADM problem involving PF information, the attribute weights are computed using entropy measures. Xue et al. (2018) defined the entropy function for PFSs and its usage in decision-making. Yang and Hussain (2018) provided some probabilistic and nonprobabilistic PF entropy measurements. Thao and Smarandache (2019) introduced the CORPAS MADM approach in the PF environment with the use of a new PF entropy measure. Five FF entropy measurements were introduced by Mishra and Rani (2021).

The following are the main causes that motivated us to carry out this study.

- Several domains, including clustering, decision-making, pattern detection, etc., use fuzzy sets' similarity metrics and their various extensions. The FFS similarity measures, however, have not yet been properly investigated.
- The majority of the proposed formula-level fuzzy similarity metrics, both standard and non-standard, do not adhere to the axiomatic requirements. Thus, there is no technique that can be applied repeatedly to assess similarity.
- The five FF entropy measurements (Olgun et al. 2021) are worthless from the standpoint of linguistic hedging,

and there is no standard method for creating the FF entropy measurements.

- The conventional MADM technique i.e., TOPSIS provides a compromise solution that is most similar to the PIS (positive ideal solution), but not the least similar to the NIS (negative ideal solution). So, a new decision-making method is desirable.

The contribution of this paper is given below:

- We suggest four metrics of similarity for FFSs along with their weighted equivalents.
- We suggest four metrics of entropy for FFSs using the suggested FF metrics of similarity.
- We contrast the performance of the proposed FF information measures with the available measures.
- We apply the FF similarity metrics in classification problems.
- We introduce an innovative decision-making method in the FF setting.

Section 2 of the paper is preliminary. Section 3 lists many unique FF similarity measurements along with desirable characteristics. A few similarity-based FF entropy measurements are introduced in Sect. 4. Section 5 displays a comparison of the proposed FF entropy and similarity metrics with the current FF/PF compatibility metrics. The use of the suggested similarity measures in pattern identification is illustrated in Sect. 6. In Sect. 7, a new MADM technique for the FF environment is proposed. The benefits and implications of the suggested FF similarity metrics, entropy measures, and the new MADM method are covered in Sect. 8. Section 9 provides the conclusion and recommendations for further research.

2 Preliminaries

Let $FFS(B)$ denote the collection of all FFSs of the universal set $B = \{b_1, b_2, \ldots, b_p\}$.

Definition 1 (Yager 2013) A PFS C_1 in B is defined as

$C_1 = \{(b_t, \sigma_{C_1}(b_t), \varsigma_{C_1}(b_t)); b_t \in B\}$.

Here, $\sigma_{C_1}(b_t)$ is the grade of membership and $\varsigma_{C_1}(b_t)$ is the grade of non-membership of the element b_t in C_1 with the conditions that $0 \le \sigma_{C_1}(b_t), \varsigma_{C_1}(b_t) \le 1$ and $0 \le \sigma^2_{C_1}(b_t) + \varsigma^2_{C_1}(b_t) \le 1$. Further, $\tau_{C_1}(b_t) = \sqrt{1 - \sigma^2_{C_1}(b_t) - \varsigma^2_{C_1}(b_t)}$ is the grade of the hesitancy of the element b_t in C_1.

An example of a PFS is $C_1 = \{(b_1, 0.5, 0.6), (b_2, 0.1, 0.3), (b_3, 0.2, 0.6)\}$. Here, we see that $\sigma_{C_1}(b_1) + \varsigma_{C_1}(b_1) = 0.5 + 0.6 = 1.1 > 1$, but $\sigma^2_{C_1}(b_1) + \varsigma^2_{C_1}(b_1) = 0.25 + 0.36 = 0.61 < 1$.

Definition 2 (Senapati and Yager 2020) An FFS C_1 in B is defined as

$C_1 = \{(b_t, \sigma_{C_1}(b_t), \varsigma_{C_1}(b_t)); b_t \in B\}$.

Here, $\sigma_{C_1}(b_t)$ is the grade of membership and $\varsigma_{C_1}(b_t)$ is the grade of non-membership of the element b_t in C_1 with the conditions that $0 \le \sigma_{C_1}(b_t), \varsigma_{C_1}(b_t) \le 1$ and $0 \le \sigma^3_{C_1}(b_t) + \varsigma^3_{C_1}(b_t) \le 1$. Further, $\tau_{C_1}(b_t) = \sqrt[3]{1 - \sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)}$ is the grade of the hesitancy of the element b_t in C_1.

An example of an FFS is $C_1 = \{(b_1, 0.2, 0.4), (b_2, 0.7, 0.8), (b_3, 0.6, 0.5)\}$. Here, we see that $\sigma^2_{C_1}(b_2) + \varsigma^2_{C_1}(b_2) = 0.49 + 0.64 = 1.13 > 1$, but $\sigma^3_{C_1}(b_1) + \varsigma^3_{C_1}(b_1) = 0.343 + 0.512 = 0.855 < 1$.

Definition 3 (Senapati and Yager 2020) Let $C_1, C_2 \in FFS(B)$, then some operations are listed below.

1. $C_1 \cup C_2 = \{(b_t, \max(\sigma_{C_1}(b_t), \sigma_{C_2}(b_t)), \min(\varsigma_{C_1}(b_t), \varsigma_{C_2}(b_t))); b_t \in B\}$.
2. $C_1 \cap C_2 = \{(b_t, \min(\sigma_{C_1}(b_t), \sigma_{C_2}(b_t)), \max(\varsigma_{C_1}(b_t), \varsigma_{C_2}(b_t))); b_t \in B\}$.
3. $C_1 \subseteq C_2$. iff $\sigma_{C_1}(b_t) \le \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \ge \varsigma_{C_2}(b_t) \forall\, b_t \in B$.
4. $(C_1)^c = \{(b_t, \varsigma_{C_1}(b_t), \sigma_{C_1}(b_t)); b_t \in B\}$.

For example, consider the two FFSs $C_1, C_2 \in FFS(B)$ given as

$C_1 = \{(b_1, 0.2, 0.4), (b_2, 0.6, 0.8), (b_3, 0.6, 0.5)\}$,

$C_2 = \{(b_1, 0.8, 0.4), (b_2, 0.7, 0), (b_3, 0.7, 0.2)\}$.

Then,

1.

$$C_1 \cup C_2 = \left\{ \begin{array}{c} (b_1, \max(0.2, 0.8), \min(0.4, 0.4)), (b_2, \max(0.6, 0.7), \min(0.8, 0)), \\ (b_3, \max(0.6, 0.7), \min(0.5, 0.2)) \end{array} \right\} = \{(b_1, 0.8, 0.4), (b_2, 0.7, 0), (b_3, 0.7, 0.2)\}.$$

2.

$$C_1 \cap C_2 = \left\{ \begin{array}{c} (b_1, \min(0.2, 0.8), \max(0.4, 0.4)), (b_2, \min(0.6, 0.7), \max(0.8, 0)), \\ (b_3, \min(0.6, 0.7), \max(0.5, 0.2)) \end{array} \right\} = \{(b_1, 0.2, 0.4), (b_2, 0.6, 0.8), (b_3, 0.6, 0.5)\}.$$

3. $C_1 \subseteq C_2$ because $\sigma_{C_1}(b_t) \leq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \geq \varsigma_{C_2}(b_t) \forall\, t = 1, 2, 3$.
4. $(C_1)^c = \{(b_1, 0.4, 0.2), (b_2, 0.8, 0.6), (b_3, 0.5, 0.6)\}$.

Definition 4 (Peng et al. 2017) A PF similarity measure S_{PF} is a function $S_{\text{PF}} : \text{PFS}(B) \times \text{PFS}(B) \rightarrow [0, 1]$ such that $\forall\, C_1, C_2$ and $C_3 \in \text{PFS}(B)$,

(SM1) $0 \leq S_{\text{PF}}(C_1, C_2) \leq 1$.
(SM2) $S_{\text{PF}}(C_1, C_2) = S_{\text{PF}}(C_2, C_1)$.
(SM3) $S_{\text{PF}}(C_1, C_2) = 1$ iff $C_1 = C_2$.
(SM4) $S_{\text{PF}}(C_1, (C_1)^c) = 0$ iff C_1 is a crisp set, where c denotes the complement.
(SM5) If $C_1 \subseteq C_2 \subseteq C_3$, then $S_{\text{PF}}(C_1, C_2) \geq S_{\text{PF}}(C_1, C_3)$ and $S_{\text{PF}}(C_2, C_3) \geq S_{\text{PF}}(C_1, C_3)$.

An example of a PF similarity measure is given below:

$$S_{\text{PF}}(C_1, C_2) = 1 - \frac{1}{2p} \sum_{t=1}^{p} \left(\left|\sigma^2_{C_1}(b_t) - \sigma^2_{C_2}(b_t)\right| + \left|\varsigma^2_{C_1}(b_t) - \varsigma^2_{C_2}(b_t)\right| + \left|\tau^2_{C_1}(b_t) - \tau^2_{C_2}(b_t)\right| \right).$$

Definition 5 (Peng et al. 2017) A PF distance measure D_{PF} is a function $D_{PF} : \text{PFS}(B) \times \text{PFS}(B) \rightarrow [0, 1]$ such that $\forall\, C_1, C_2$ and $C_3 \in PFS(B)$,

(DM1) $0 \leq D_{PF}(C_1, C_2) \leq 1$.
(DM2) $D_{\text{PF}}(C_1, C_2) = D_{\text{PF}}(C_2, C_1)$.
(DM3) $D_{\text{PF}}(C_1, C_2) = 0$ iff $C_1 = C_2$.
(DM4) $D_{\text{PF}}(C_1, (C_1)^c) = 1$ iff C_1 is a crisp set, where c denotes the complement.
(DM5) If $C_1 \subseteq C_2 \subseteq C_3$, then $D_{PF}(C_1, C_2) \leq D_{PF}(C_1, C_3)$ and $D_{PF}(C_2, C_3) \leq D_{\text{PF}}(C_1, C_3)$.

An example of a PF similarity measure is given below:

$$D_{PF}(C_1, C_2) = \frac{1}{2p} \sum_{t=1}^{p} \left(\left|\sigma^2_{C_1}(b_t) - \sigma^2_{C_2}(b_t)\right| + \left|\varsigma^2_{C_1}(b_t) - \varsigma^2_{C_2}(b_t)\right| + \left|\tau^2_{C_1}(b_t) - \tau^2_{C_2}(b_t)\right| \right).$$

Definition 6 (Mishra and Rani 2021) An FF entropy measure E_{FF} is a function $E_{\text{FF}} : \text{FFS}(B) \rightarrow [0, 1]$ such that $\forall\, C_1, C_2$ and $C_3 \in \text{FFS}(B)$,

(EM1) $0 \leq E_{\text{FF}}(C_1) \leq 1$
(EM2) $E_{\text{FF}}(C_1) = 0$iff C_1 is a crisp set.
(EM3) $E_{\text{FF}}(C_1) = 1$ iff $\sigma_{C_1}(b_t) = \varsigma_{C_1}(b_t) \forall\, b_t \in B$.
(EM4) $E_{\text{FF}}(C_1) = E_{\text{FF}}((C_1)^c)$ where c denotes the complement.
(EM5) $E_{\text{FF}}(C_1) \leq E_{\text{FF}}(C_2)$, if $\sigma_{C_1}(b_t) \leq \sigma_{C_2}(b_t) \leq \varsigma_{C_2}(b_t) \leq \varsigma_{C_1}(b_t)$ or $\sigma_{C_1}(b_t) \geq \sigma_{C_2}(b_t) \geq \varsigma_{C_2}(b_t) \geq \varsigma_{C_1}(b_t) \forall\, b_t \in B$.

An example of FF entropy is given below:

$$E_{\text{FF}}(C_1) = \frac{1}{(\sqrt{2} - 1)p} \sum_{t=1}^{p} \left(\cos\left(\frac{\pi\left(1 + \sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right)}{4} \right) + \cos\left(\frac{\pi\left(1 - \sigma^3_{C_1}(b_t) + \varsigma^3_{C_1}(b_t)\right)}{4} \right) - 1 \right).$$

Definition 7 (Weber 1983) A t-conorm is a function $h : [0, 1] \times [0, 1] \rightarrow [0, 1]$ if $\forall\, i, j, k, l \in [0, 1]$

1. $h(i, j) = h(j, i)$.
2. $h(i, j) \leq h(k, l)$, whenever $i \leq k$ and $j \leq l$.
3. $h(i, 0) = i$.
4. $h(i, h(j, k)) = h(h(i, j), k)$.

An example of a t-conorm is $h(b_1, b_2) = \frac{b_1 + b_2 - 2b_1 b_2}{1 - b_1 b_2}$.

We offer some innovative similarity measures for FFSs along with their characteristics in the following section.

3 New Fermatean fuzzy similarity measures

Here, we suggest a few FF similarity measurements. A similarity metric is first defined in the FF environment.

Definition 8 An FF similarity measure S_{FF} is a function $S_{FF} : \mathrm{FFS}(B) \times \mathrm{FFS}(B) \to \mathbb{R}$ such that $\forall\, C_1, C_2$ and $C_3 \in \mathrm{FFS}(B)$:

(SM1) $0 \le S_{FF}(C_1, C_2) \le 1$.
(SM2) $S_{FF}(C_1, C_2) = S_{FF}(C_2, C_1)$.
(SM3) $S_{FF}(C_1, C_2) = 1$ iff $C_1 = C_2$.
(SM4) $S_{FF}(C_1, (C_1)^c) = 0$ iff C_1 is a crisp set, where c denotes the complement.
(SM5) If $C_1 \subseteq C_2 \subseteq C_3$, then $S_{FF}(C_1, C_2) \ge S_{FF}(C_1, C_3)$ and $S_{FF}(C_2, C_3) \ge S_{FF}(C_1, C_3)$.

We now present a novel technique for deriving FF similarity metrics from t-conorms.

Definition 9 For $C_1, C_2 \in \mathrm{FFS}(B)$, let $S_G : \mathrm{FFS}(B) \times \mathrm{FFS}(B) \to \mathbb{R}$ be a function defined as

$$S_G(C_1, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|\Big). \tag{1}$$

Here, h is a t-conorm.

Theorem 1 *S_G in Eq. (1) is a valid measure of similarity for FFSs.*

Proof We will establish that S_G satisfies the characteristic of a metric of similarity for FFSs listed in Definition 8.

(SM1) It is obvious.

(SM2) $S_G(C_1, C_2) = S_G(C_2, C_1)$ follows from the definition of S_G.

(SM3) $S_G(C_1, C_2) = 1 \iff h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|\Big) = 0\ \forall\, t, \iff \big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big| = 0$ and $\big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big| = 0\ \forall\, t, \iff \sigma^3_{C_1}(b_t) = \sigma^3_{C_2}(b_t)$ and $\varsigma^3_{C_1}(b_t) = \varsigma^3_{C_2}(b_t)\ \forall\, t, \iff C_1 = C_2$.

(SM4) $S_G(C_1, (C_1)^c) = 0 \iff h\Big(\big|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\big|\Big) = 1\ \forall\, t, \iff \big|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\big| = 1$ and $\big|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\big| = 1\ \forall\, t, \iff \big|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\big| = 1\ \forall\, t, \iff \sigma^3_{C_1}(b_t) = 1$ and $\varsigma^3_{C_1}(b_t) = 0$ or $\sigma^3_{C_1}(b_t) = 0$ and $\varsigma^3_{C_1}(b_t) = 1\ \forall\, t, \iff \sigma_{C_1}(b_t) = 1$ and $\varsigma_{C_1}(b_t) = 0$ or $\sigma_{C_1}(b_t) = 0$ and $\varsigma_{C_1}(b_t) = 1\ \forall\, t, \iff C_1$ is a crisp set.

(SM5) Let $C_1 \subseteq C_2 \subseteq C_3$, then $\sigma^3_{C_1}(b_t) \le \sigma^3_{C_2}(b_t) \le \sigma^3_{C_3}(b_t)$ and $\varsigma^3_{C_1}(b_t) \ge \varsigma^3_{C_2}(b_t) \ge \varsigma^3_{C_3}(b_t)\ \forall\, t$. Then, we have

$$\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big| \le \big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_3}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big| \le \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_3}(b_t)\big|,$$

and

$$\big|\sigma^3_{C_2}(b_t) - \sigma^3_{C_3}(b_t)\big| \le \big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_3}(b_t)\big|, \big|\varsigma^3_{C_2}(b_t) - \varsigma^3_{C_3}(b_t)\big| \le \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_3}(b_t)\big|.$$

So, $h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|\Big) \le h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_3}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_3}(b_t)\big|\Big)$ and

$$h\Big(\big|\sigma^3_{C_2}(b_t) - \sigma^3_{C_3}(b_t)\big|, \big|\varsigma^3_{C_2}(b_t) - \varsigma^3_{C_3}(b_t)\big|\Big) \le h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_3}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_3}(b_t)\big|\Big).$$

Thus $S_G(C_1, C_2) \ge S_G(C_1, C_3)$ and $S_G(C_2, C_3) \ge S_G(C_1, C_3)$. Hence, S_G is a similarity measure for FFSs.

Theorem 2 *The measure of similarity S_G in Eq. (1) possesses the following characteristics.*

1. $S_G((C_1)^c, (C_2)^c) = S_G(C_1, C_2)\,\forall\, C_1, C_2 \in \mathrm{FFS}(B)$.
2. $S_G(C_1, (C_2)^c) = S_G((C_1)^c, C_2)\,\forall\, C_1, C_2 \in \mathrm{FFS}(B)$.
3. $S_G(C_1, (C_1)^c) = 1$ *if and only if* $\sigma_{C_1}(b_t) = \varsigma_{C_1}(b_t)\,\forall\, t$.
4. $S_G(C_1 \cap C_2, C_2) \ge S_G(C_1, C_2)$, *for every* $C_1, C_2 \in \mathrm{FFS}(B)$.
5. $S_G(C_1 \cup C_2, C_2) \ge S_G(C_1, C_2)$, *for every* $C_1, C_2 \in \mathrm{FFS}(B)$.

Proof 1. $S_G((C_1)^c, (C_2)^c) = 1 - \frac{1}{p}\sum_{t=1}^{p} h\Big(\big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|, \big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|\Big)$

$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h\Big(\big|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|\Big)$$

$$= S_G(C_1, C_2).$$

2.

$S_G(C_1, (C_2)^c) = 1 - \frac{1}{p}\sum_{t=1}^{p} h\Big(\big|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|, \big|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|\Big)$

$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h\Big(\big|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\big|, \big|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\big|\Big)$$

$= S_G((C_1)^c, C_2)$

3.

$$S_G(C_1,(C_1)^c) = 1 \iff 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\right|\right) = 1$$

$$\iff h\left(\left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\right|\right) = 0 \ \forall \ t$$

$\iff \left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right| = 0$ and $\left|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\right| = 0 \ \forall \ t$

$$\iff \left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right| = 0 \ \forall \ t$$

$$\iff \sigma^3_{C_1}(b_t) = \varsigma^3_{C_1}(b_t) \ \forall \ t$$

$$\iff \sigma_{C_1}(b_t) = \varsigma_{C_1}(b_t) \ \forall \ t.$$

4. $S_G(C_1 \cap C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \begin{pmatrix} \left|min\left(\sigma^3_{C_1}(b_t), \sigma^3_{C_2}(b_t)\right) - \sigma^3_{C_2}(b_t)\right|, \\ \left|max\left(\varsigma^3_{C_1}(b_t), \varsigma^3_{C_2}(b_t)\right) - \varsigma^3_{C_2}(b_t)\right| \end{pmatrix}.$

Now, there are the following cases:

(a) When $\sigma_{C_1}(b_t) \geq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \geq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cap C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_2}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(0, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$\geq 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= S_G(C_1, C_2).$$

(b) When $\sigma_{C_1}(b_t) \geq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \leq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cap C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_2}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_2}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h(0,0) = 1 \geq S_G(C_1, C_2).$$

(c) When $\sigma_{C_1}(b_t) \leq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \geq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cap C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= S_G(C_1, C_2).$$

(d) When $\sigma_{C_1}(b_t) \leq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \leq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cap C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_2}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, 0\right)$$
$$\geq 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= S_G(C_1, C_2).$$

5. $S_G(C_1 \cup C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \begin{pmatrix} \left|\max\left(\sigma^3_{C_1}(b_t), \sigma^3_{C_2}(b_t)\right) - \sigma^3_{C_2}(b_t)\right|, \\ \left|\min\left(\varsigma^3_{C_1}(b_t), \varsigma^3_{C_2}(b_t)\right) - \varsigma^3_{C_2}(b_t)\right| \end{pmatrix}.$

Now, there are the following cases:

(a) When $\sigma_{C_1}(b_t) \geq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \geq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cup C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{2}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, 0\right)$$
$$\geq 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= S_G(C_1, C_2).$$

(b) When $\sigma_{C_1}(b_t) \geq \sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t) \leq \varsigma_{C_2}(b_t) \ \forall \ t$, then

$$S_G(C_1 \cup C_2, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} h \left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right)$$
$$= S_G(C_1, C_2).$$

Table 1 FF similarity measurements

t-Conorms	FF similarity measures
$h(b_1,b_2)=\frac{b_1+b_2-2b_1b_2}{1-b_1b_2}$	$S_{G1}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}\left[\frac{\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right| -2\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}{1-\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}\right]$
$h(b_1,b_2)=b_1+b_2-b_1b_2$	$S_{G2}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}\left[\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right| -\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right]$
$h(b_1,b_2)=\min(1,b_1+b_2)$	$S_{G3}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}\min\left(1,\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right| +\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$
$h(b_1,b_2)=\frac{b_1+b_2}{1+b_1b_2}$	$S_{G4}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}\left[\frac{\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}{1+\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}\right].$

(c) When $\sigma_{C_1}(b_t)\le\sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t)\ge\varsigma_{C_2}(b_t)\ \forall\ t$, then

$$S_G(C_1\cup C_2,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}h\left(\left|\sigma^3_{C_2}(b_t)-\sigma^3_{C_2}(b_t)\right|,\left|\varsigma^3_{C_2}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$$
$$=1-\frac{1}{p}\sum_{t=1}^{p}h(0,0)=1\ge S_G(C_1,C_2).$$

(d) When $\sigma_{C_1}(b_t)\le\sigma_{C_2}(b_t)$ and $\varsigma_{C_1}(b_t)\le\varsigma_{C_2}(b_t)\ \forall\ t$, then

$$S_G(C_1\cup C_2,C_2)$$
$$=1-\frac{1}{p}\sum_{t=1}^{p}h\left(\left|\sigma^3_{C_2}(b_t)-\sigma^3_{C_2}(b_t)\right|,\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$$
$$=1-\frac{1}{p}\sum_{t=1}^{p}h\left(0,\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$$
$$\ge 1-\frac{1}{p}\sum_{t=1}^{p}h\left(\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|,\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$$
$$=S_G(C_1,C_2).$$

Example 1 Table 1 provides some examples of FF similarity measurements.

Next, we construct the weighted metrics of similarity for FFSs.

Definition 10 For $C_1,C_2\in\mathrm{FFS}(B)$, let the function $S^W_G:\mathrm{FFS}(B)\times\mathrm{FFS}(B)\to\mathbb{R}$ be defined as

Table 2 Some examples of weighted FF similarity measurements

t-Conorms	FF-weighted similarity measures
$h(b_1,b_2)=\frac{b_1+b_2-2b_1b_2}{1-b_1b_2}$	$S^W_{G1}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}w_t\left[\frac{\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right| -2\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}{1-\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}\right]$
$h(b_1,b_2)=b_1+b_2-b_1b_2$	$S^W_{G2}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}w_t\left[\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right| -\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right]$
$h(b_1,b_2)=min(1,b_1+b_2)$	$S^W_{G3}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}w_t min\left(1,\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right| +\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|\right)$
$h(b_1,b_2)=\frac{b_1+b_2}{1+b_1b_2}$	$S^W_{G4}(C_1,C_2)=1-\frac{1}{p}\sum_{t=1}^{p}w_t\left[\frac{\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|+\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}{1+\left|\sigma^3_{C_1}(b_t)-\sigma^3_{C_2}(b_t)\right|\left|\varsigma^3_{C_1}(b_t)-\varsigma^3_{C_2}(b_t)\right|}\right].$

$$S_G^W(C_1, C_2) = 1 - \frac{1}{p}\sum_{t=1}^{p} w_t h \left(\left|\sigma^3_{C_1}(b_t) - \sigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \varsigma^3_{C_2}(b_t)\right|\right), \tag{2}$$

where h is a t-conorm.

Theorem 2 *S_G^W given in Eq. (1) is a valid weighted measure of similarity for FFSs.*

Proof Same as Theorem 1.

Example 2 Table 2 provides some examples of weighted FF similarity measurements.

Some similarity-based metrics of entropy for FFSs are given in the next section.

4 Entropy measures based on FF similarity measures

The degree of ambiguity in an FFS is determined using the entropy measures. In this section, we present a technique for creating FF entropy measurements using FF similarity measures.

Definition 11 For $C_1 \in \mathrm{FFS}(B)$, let the function $E_G : FFS(B) \to [0, 1]$ be defined as

$$E_G(C_1) = S_G(C_1, (C_1)^c), \tag{3}$$

with S_G a similarity measure of FFSs.

Theorem 3 *E_G in Eq. (3) is a valid measure of entropy for FFSs.*

Proof We demonstrate that the function E_G has the characteristics of an FF measure of entropy listed in Definition 6.

(EM1) It is obvious as $0 \le S_G(C_1, (C_1)^c) \le 1$.

(EM2) $E_G(C_1) = 0 \Longleftrightarrow S_G(C_1, (C_1)^c) \Longleftrightarrow C_1$ is a crisp set.

(EM3) $E_G(C_1) = 1 \Longleftrightarrow S_G(C_1, (C_1)^c) = 1 \Longleftrightarrow \sigma_{C_1}(b_t) = \varsigma_{C_2}(b_t) \forall\, t$.

(EM4) $E_G(C_1)^c = E_G(C_1)$ follows by the definition of E_G.

(EM5) Consider C_1 to be less fuzzy than C_2, i.e., $\sigma_{C_1}(b_t) \le \sigma_{C_2}(b_t) \le \varsigma_{C_2}(b_t) \le \varsigma_{C_1}(b_t)$ or $\sigma_{C_1}(b_t) \ge \sigma_{C_2}(b_t) \ge \varsigma_{C_2}(b_t) \ge \varsigma_{C_1}(b_t)$.

When $\sigma_{C_1}(b_t) \le \sigma_{C_2}(b_t) \le \varsigma_{C_2}(b_t) \le \varsigma_{C_1}(b_t)$, we have $\left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right| \ge \left|\sigma^3_{C_2}(b_t) - \varsigma^3_{C_2}(b_t)\right|$. So,

$$S_G(C_1, (C_1)^c) = 1 - \tfrac{1}{p}\textstyle\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_1}(b_t) - \varsigma^3_{C_1}(b_t)\right|, \left|\varsigma^3_{C_1}(b_t) - \sigma^3_{C_1}(b_t)\right|\right)$$

$$\le 1 - \frac{1}{p}\sum_{t=1}^{p} h\left(\left|\sigma^3_{C_2}(b_t) - \varsigma^3_{C_2}(b_t)\right|, \left|\varsigma^3_{C_2}(b_t) - \sigma^3_{C_2}(b_t)\right|\right) = S_G(C_2, (C_2)^c).$$

Thus, $E_G(C_1) \le E_G(C_2)$.

Similarly, for $\sigma_{C_1}(b_t) \ge \sigma_{C_2}(b_t) \ge \varsigma_{C_2}(b_t) \ge \varsigma_{C_1}(b_t)$, we have $E_G(C_1) \le E_G(C_2)$. Hence, E_G in Eq. (3) is a valid measure of entropy for FFSs.

Some FF measurements of entropy are provided in Table 3 below using Eq. (3) and the recommended FF measures of similarity.

Now, we contrast several existing PF/FF measures of information with the suggested FF measures of similarity and entropy.

5 Comparative analysis

In this part, we demonstrate that the proposed FF measures of similarity and entropy outperform the majority of existing PF/FF measures of information in terms of accuracy.

5.1 Comparability of the proposed metrics of similarity for FFSs with the several available metrics

For comparability, we first recall the available metrics of distance and similarity. These are shown in Tables 4 and 5 respectively.

Now, we examine three distinct FFS situations, each of which consists of two distinct FFSs. Table 6 displays the values of comparability.

From Table 6, we observe

1. The similarity metrics $S_1, S_4, S_5, S_6, S_9, S_{10}$ and distance metrics $D_1, D_4, D_5, D_6, D_9, D_{10}$ consider the FFSs C_1, C_2 (Case I) and C_1, C_2 (Case II) to be the same, which is unreasonable.
2. The similarity metric S_2 gives 1 as the similarity level between the FFSs (Case III), which is unreasonable as $C_1 \ne C_2$.
3. The level of similarity between the FFSs (Case I and Case II) comes out to be negative by the similarity metrics S_7 and S_8 and thus violates the non-negativity property of a similarity metric.
4. The similarity metrics S_7, S_8, S_9, and S_{10} gives 0 as the similarity level between the FFSs (Case III), which is unreasonable as C_2 is not a complement of C_1.
5. The distance metrics D_7, D_8, D_9, and D_{10} gives 1 as the similarity level between the FFSs (Case III), which is unreasonable as C_2 is not a complement of C_1.

Table 3 Some examples of FF entropy measures

Recommended FF similarity measures	FF entropy measures
S_{G1}	$E_{G1}(C_1) = 1 - \frac{1}{p}\sum_{t=1}^{p} \frac{2\left(\left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right| - \left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|^2\right)}{1 - \left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|^2}$
S_{G2}	$E_{G2}(C_1) = 1 - \frac{1}{p}\sum_{t=1}^{p} 2\left(\left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right| - \left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|^2\right)$
S_{G3}	$E_{G3}(C_1) = 1 - \frac{1}{p}\sum_{t=1}^{p} min\left(1, 2\left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|\right)$
S_{G4}	$E_{G4}(C_1) = 1 - \frac{1}{p}\sum_{t=1}^{p} \frac{2\left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|}{1 + \left|\sigma_{C_1}^3(b_t) - \varsigma_{C_1}^3(b_t)\right|^2}$

6. The suggested metrics of similarity produce satisfactory results in all three cases.

Thus, we conclude that most of the available compatibility measures produce unreasonable results as shown by bold values, whereas the newly suggested measures of similarity $S_{Gt}, t = 1, 2, 3, 4$ give satisfactory results in all three cases. This establishes the effectiveness of the suggested similarity measures.

Next, we contrast the proposed FF entropy measures with the existing PF/FF entropy metrics.

5.2 Comparison of the proposed FF entropy measurements with the existing PF/FF entropy measures

For comparability, we first enumerate the available metrics of entropy for FFSs/PFSs as shown in Table 7.

Now, we demonstrate the usefulness of the recommended FF measurements of entropy using linguistic hedges.

Definition 12 (Senapati and Yager 2020) The modifier of an FFS $C_1 \in FFS(B)$ is given by

$$(C_1)^{\delta} = \left\{\left(b_t, \left(\sigma_{C_1}^3(b_t)\right)^{\delta}, \left(1 - \left(1 - \varsigma_{C_1}^3(b_t)\right)^{\delta}\right)^{\frac{1}{3}}\right), b_t \in B\right\}. \tag{4}$$

Now, we consider the following FFSs:

LARGE: C_1, very LARGE: $(C_1)^2$, quite very LARGE: $(C_1)^3$, very very LARGE: $(C_1)^4$, more or less LARGE: $(C_1)^{\frac{1}{2}}$.

E, the entropy must meet the following criteria since it computes the ambiguity content in an FFS.

$$E\left((C_1)^{\frac{1}{2}}\right) > E(C_1) > E\left((C_1)^2\right) > E\left((C_1)^3\right) > E\left((C_1)^4\right) \tag{5}$$

We now take a look at an illustration of how the aforementioned FFSs compute ambiguity.

Example 3 Consider $C_1 \in FFS(B)$ as

$C_1 = \{(b_1, 0.48, 0.56), (b_2, 0.80, 0.35),$
$(b_3, 0.21, 0.60), (b_4, 0.45, 0.72), (b_5, 0.33, 0.47)\}.$

We form the FFSs $(C_1)^{\frac{1}{2}}, (C_1)^2, (C_1)^3$, and $(C_1)^4$ with the aid of Definition 12. Table 8 displays the amount of ambiguity present in these FFSs.

From Table 8, we observe that

1. $E_1(C_1)^{\frac{1}{2}} \ngtr E_1(C_1)$
2. $E_2(C_1)^{\frac{1}{2}} \ngtr E_2(C_1)$.
3. $E_3(C_1)^{\frac{1}{2}} \ngtr E_3(C_1)$
4. $E_4(C_1)^{\frac{1}{2}} \ngtr E_4(C_1)$
5. $E_5(C_1)^{\frac{1}{2}} \ngtr E_5(C_1)$
6. $E_6(C_1)^{\frac{1}{2}} \ngtr E_6(C_1)$
7. $E_6(C_1)^{\frac{1}{2}} \ngtr E_6(C_1) \ngtr E_6(C_1)^2 \ngtr E_6(C_1)^3 \ngtr E_6(C_1)^4$
8. $E_8(C_1)^{\frac{1}{2}} \ngtr E_8(C_1)$
9. $E_9(C_1)^{\frac{1}{2}} \ngtr E_9(C_1)$
10. $E_{10}(C_1)^{\frac{1}{2}} \ngtr E_{10}(C_1)$
11. $E_{11}(C_1)^{\frac{1}{2}} \ngtr E_{11}(C_1)$
12. $E_{12}(C_1)^2 \ngtr E_{12}(C_1)^3$
13. $E_{13}(C_1)^{\frac{1}{2}} \ngtr E_{13}(C_1)$
14. $E_{14}(C_1)^{\frac{1}{2}} \ngtr E_{14}(C_1)$
15. $E_{15}(C_1)^{\frac{1}{2}} \ngtr E_{15}(C_1)$
16. $E_{Gt}(C_1)^{\frac{1}{2}} > E_{Gt}(C_1) > E_{Gt}(C_1)^2 > E_{Gt}(C_1)^3 > E_{Gt}(C_1)^4, t = 1, 2, 3, 4.$

Therefore, it follows that none of the PF/FF measurements of entropy $E_t, 1 \leq t \leq 15$ that are currently available satisfy the condition stated in Eq. (5). All of our FF entropy

Table 4 Some existing PF measures of similarity (Peng et al. 2017)

Similarity measure	Expression
$S_1(C_1, C_2)$	$1-\frac{1}{2p}\sum_{t=1}^{p}\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|+\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|+\left\|\tau_{C_1}^2(b_t)-\tau_{C_2}^2(b_t)\right\|\right)$
$S_2(C_1, C_2)$	$1-\frac{1}{2p}\sum_{t=1}^{p}\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)-\left(\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right)\right\|\right)$
$S_3(C_1, C_2)$	$1-\frac{1}{4p}\left[\begin{array}{c}\sum_{t=1}^{p}\left(\begin{array}{c}\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|+\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\\+\left\|\tau_{C_1}^2(b_t)-\tau_{C_2}^2(b_t)\right\|\end{array}\right)\\+\sum_{t=1}^{p}\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)-\left(\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right)\right\|\right)\end{array}\right]$
$S_4(C_1, C_2)$	$1-\frac{1}{p}\sum_{t=1}^{p}\max\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|,\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\right)$
$S_5(C_1, C_2)$	$\frac{1}{p}\sum_{t=1}^{p}\frac{1-\max\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|,\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\right)}{1+\max\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|,\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\right)}$
$S_6(C_1, C_2)$	$\frac{\sum_{t=1}^{p}1-\max\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|,\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\right)}{\sum_{t=1}^{p}1+\max\left(\left\|\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\|,\left\|\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\|\right)}$
$S_7(C_1, C_2)$	$x\frac{\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}+y\frac{\sum_{t=1}^{p}\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}, x+y=, x, y\in[0,1]$
$S_8(C_1, C_2)$	$\frac{x}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}+\frac{y}{p}\sum_{t=1}^{p}\frac{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}{\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}, x+y=, x, y\in[0,1]$
$S_9(C_1, C_2)$	$\frac{1}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}$
$S_{10}(C_1, C_2)$	$\frac{\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}$
$S_{11}(C_1, C_2)$	$\frac{1}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}$
$S_{12}(C_1, C_2)$	$\frac{\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}$

metrics $E_{Gt}, 1 \le t \le 4$, however, adhere to the specification outlined in Eq. (5). This demonstrates that the recommended measurements of entropy are more reliable than the ones that are already available from the perspective of a linguistic hedge.

6 The applicability of the proposed similarity metrics in pattern identification

Here, we demonstrate how the suggested FF measures of similarity can be applied to address pattern classification-related issues. In pattern analysis, an unidentified pattern is categorized into one of the recognized patterns by applying several compatibility criteria, such as “similarity”, “distance”, “correlation”, etc. We also compare our findings to the various similarity metrics.

Now, using the examples below, we will solve several pattern analysis-related issues.

Example 4 (Jiang et al. 2019) Consider C_1, C_2, C_3, and C representing patterns in terms of FFSs.

$$C_1 = \{(b_1, 0.01, 0.13), (b_2, 0.21, 0.46), (b_3, 0.33, 0.35)\},$$

$$C_2 = \{(b_1, 0, 0.14), (b_2, 0.20, 0.47), (b_3, 0.35, 0.33)\},$$

$$C_3 = \{(b_1, 0.02, 0.12), (b_2, 0.19, 0.48), (b_3, 0.34, 0.34)\},$$

$$C = \{(b_1, 0.04, 0.10), (b_2, 0.23, 00.44), (b_3, 0.37, 0.31)\}.$$

The challenge is to determine which pattern $C_t, t = 1, 2, 3$ shares the most similarities with C. We use the existing metrics of similarity with the recommended FF similarity measurements for this aim. Table 9 displays the computed results. Table 9 makes it obvious that C should be assigned to C_2 based on the majority of similarity metrics, including the specified FF metrics.

Example 5 Consider C_1, C_2, C_3, and C representing patterns in terms of FFSs.

$$C_1 = \{(b_1, 0.5, 0.3), (b_2, 0.2, 0.5), (b_3, 0.3, 0.4), (b_4, 0.4, 0.3), (b_5, 0.1, 0.3)\},$$

$$C_2 = \{(b_1, 0.3, 0.3), (b_2, 0.1, 0.4), (b_3, 0.2, 0.1), (b_4, 0.2, 0.3), (b_5, 0.7, 0.1)\},$$

$$C_3 = \{(b_1, 0.6, 0.1), (b_2, 0.7, 0), (b_3, 0.4, 0.3), (b_4, 0.5, 0.3), (b_5, 0.4, 0.3)\},$$

$$C = \{(b_1, 0.4, 0.2), (b_2, 0.7, 0.1), (b_3, 0.4, 0.3), (b_4, 0.3, 0.4), (b_5, 0.6, 0.2)\}.$$

The challenge is to determine which pattern $C_t, t = 1, 2, 3$ shares the most similarities with C. We use the existing metrics of similarity with the recommended FF similarity measurements for this aim. Table 10 and Fig. 1 both display the computed results. As can be seen from Table 10, the majority of the measures indicate that C should be assigned to C_3.

From examples 4 and 5, we conclude that the recommended similarity metrics of FFSs are compatible with the current similarity metrics in terms of pattern recognition.

We now demonstrate the usefulness of the FF entropy and similarity measurements in decision-making.

7 An innovative Fermatean fuzzy MADM approach

In this section, we first go over the shortcomings of the conventional Fermatean fuzzy TOPSIS method. Then, under the FF circumstances, we present a novel MADM approach that is similar to TOPSIS.

7.1 Flaws of Fermatean fuzzy TOPSIS method

One of the most popular and efficient approaches for solving MADM problems is the methodology for order preference by similarity to ideal solution (TOPSIS), which was first put out by Hwang and Yoon (1981) and then extended to the fuzzy environment by Chen (2000). The TOPSIS method is predicated on the notion that the best choice should be the one that is farthest from the NIS and closest to the PIS. The selected alternative should have the lowest similarity to NIS and the highest similarity to PIS if we use the similarity metric in TOPSIS rather than the distance measure. The TOPSIS-selected alternative, however, does not have a minimum similarity to NIS, as can be shown in the examples below.

Example 6 Take into consideration a FF decision matrix A_1 with three options $B_t, t = 1, 2, 3$, and two characteristics $C_k, k = 1, 2$.

$$A_1 = \begin{pmatrix} (0.7, 0.6) & (0.5, 0.4) \\ (0.5, 0.6) & (0.7, 0.8) \\ (0, 0.2) & (0.8, 0.4) \end{pmatrix}.$$

Then, PIS B^+ and NIS B^- are $B^+ = \{(0.7, 0.2), (0.8, 0.4)\}$ and $B^- = \{(0, 0.6), (0.5, 0.8)\}$. Table 11 and Fig. 2 display how similar each alternative is to B^+, i.e., $S_{G1}(B_t, B^+)$ and B^- i.e., $S_{G1}(B_t, B^-)$ and their closeness coefficient

$$\mu_t = \frac{S_{G1}(B_t, B^+)}{S_{G1}(B_t, B^+) + S_{G1}(B_t, B^-)}, t = 1, 2, 3.$$

The same Table 11 also displays the alternatives' final rankings. Table 11 makes it evident that there is no minimum similarity between the best alternative B_3 and the NIS B^- as $S_{G1}(B_3, B^-) > S_{G1}(B_1, B^-)$.

Example 7 Take into consideration an FF decision matrix A_2 with three options $B_t, t = 1, 2, 3$, and two characteristics $C_k, k = 1, 2$.

$$A_2 = \begin{pmatrix} (0.9, 0.2) & (0.1, 0.9) \\ (0.7, 0.8) & (0.6, 0.5) \\ (0.1, 0.2) & (0.4, 0.5) \end{pmatrix}.$$

Then PIS B^+ and NIS B^- are $B^+ = \{(0.9, 0.2), (0.6, 0.5)\}$ and $B^- = \{(0.1, 0.8), (0.1, 0.9)\}$. Table 12 and Fig. 3 display how similar each alternative is to B^+, i.e., $S_{G1}(B_t, B^+)$ and B^-, i.e., $S_{G1}(B_t, B^-)$ and their closeness coefficient

$$\mu_t = \frac{S_{G1}(B_t, B^+)}{S_{G1}(B_t, B^+) + S_{G1}(B_t, B^-)}, t = 1, 2, 3.$$

The same Table 12 also displays the alternatives' final rankings. Table 12 makes it evident that there is no minimum similarity between the best alternative B_2 and the NIS B^- as $S_{G1}(B_2, B^-) > S_{G1}(B_3, B^-)$.

The ideal TOPSIS alternative does not possess low similarity with NIS, as seen in Examples 6 and 7. To overcome this severe issue, we suggest the inferior ratio method in the FF environment known as the Fermatean fuzzy inferior ratio (FFIR) method. This is based on the same principle as that of TOPSIS.

Table 5 Some existing PF measures of distance (Peng et al. 2017)

Distance measures	Expression
$D_1(C_1, C_2)$	$\frac{1}{2p}\sum_{t=1}^{p}\left(\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\rvert+\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\rvert+\lvert\tau_{C_1}^2(b_t)-\tau_{C_2}^2(b_t)\rvert\right)$
$D_2(C_1, C_2)$	$\frac{1}{2p}\sum_{t=1}^{p}\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)-\left(\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right)\right\rvert\right)$
$D_3(C_1, C_2)$	$\frac{1}{4p}\left[\begin{array}{c}\sum_{t=1}^{p}\left(\begin{array}{c}\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\rvert+\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\rvert\\+\lvert\tau_{C_1}^2(b_t)-\tau_{C_2}^2(b_t)\rvert\end{array}\right)\\+\sum_{t=1}^{p}\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)-\left(\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right)\right\rvert\right)\end{array}\right]$
$D_4(C_1, C_2)$	$\frac{1}{p}\sum_{t=1}^{p}\max\left(\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\rvert,\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\rvert\right)$
$D_5(C_1, C_2)$	$\frac{2}{p}\sum_{t=1}^{p}\frac{\max\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\rvert,\left\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\rvert\right)}{1+\max\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\rvert,\left\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\rvert\right)}$
$D_6(C_1, C_2)$	$\frac{2\sum_{t=1}^{p}\max\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\rvert,\left\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\rvert\right)}{\sum_{t=1}^{p}1+\max\left(\left\lvert\sigma_{C_1}^2(b_t)-\sigma_{C_2}^2(b_t)\right\rvert,\left\lvert\varsigma_{C_1}^2(b_t)-\varsigma_{C_2}^2(b_t)\right\rvert\right)}$
$D_7(C_1, C_2)$	$1-x\frac{\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}-y\frac{\sum_{t=1}^{p}\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}, x+y=, x, y\in[0,1]$
$D_8(C_1, C_2)$	$1-\frac{x\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}{p\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)}-\frac{y\sum_{t=1}^{p}\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{p\sum_{t=1}^{p}\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}, x+y=, x, y\in[0,1]$
$D_9(C_1, C_2)$	$1-\frac{1}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}$
$D_{10}(C_1, C_2)$	$1-\frac{\sum_{t=1}^{p}min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(\varsigma_{C_1}^2(b_t),\varsigma_{C_2}^2(b_t)\right)}$
$D_{11}(C_1, C_2)$	$1-\frac{1}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}{\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}$
$D_{12}(C_1, C_2)$	$1-\frac{\sum_{t=1}^{p}\min\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\min\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}{\sum_{t=1}^{p}\max\left(\sigma_{C_1}^2(b_t),\sigma_{C_2}^2(b_t)\right)+\max\left(1-\varsigma_{C_1}^2(b_t),1-\varsigma_{C_2}^2(b_t)\right)}$

7.2 Fermatean fuzzy inferior ratio (FFIR) method

Our suggested method produces an alternative that is the least identical to NIS and the most similar to PIS. The algorithm for solving an MCDM problem with p choices $B_t, t=1,2,\ldots,p$ and q criteria $C_k, k=1,2,..,q$ and $w_k, k=1,2,\ldots,q$ as criteria weights, where $0\le w_k\le 1$ and $\sum_{k=1}^{q} w_k = 1$ is given below.

Algorithm Step 1: Create the decision matrix $A=[(\sigma_{tk},\varsigma_{tk})]_{p\times q}$ to convey information about the options concerning the criteria.

Step 2: Create the normalized decision matrix $D=\left[\left(\sigma_{tk}^{'},\varsigma_{tk}^{'}\right)\right]_{p\times q}$ with

$$\left(\sigma_{tk}^{'},\varsigma_{tk}^{'}\right)=\begin{cases}(\sigma_{tk},\varsigma_{tk}),\text{for benefit criteria } C_k\\(\varsigma_{tk},\sigma_{tk}),\text{for cost criteria } C_k.\end{cases}$$

Step 3: Find the criteria weights $w_k, k=1,2,\ldots,q$ using the FF entropy measure as $w_k=\frac{E(C_k)}{\sum_{k=1}^{q}E(C_k)}, k=1,2,\ldots,q,$ with E being a FF measure of entropy.

Step 4: Find the PIS

$$B^{+}=\left\{\left(\sigma_1^{+},\varsigma_1^{+}\right),\left(\sigma_2^{+},\varsigma_2^{+}\right),\ldots,\left(\sigma_q^{+},\varsigma_q^{+}\right)\right\}$$

and the NIS

Table 6 Values of comparability

Compatibility measure	Case I $C_1 = (0.5, 0.3)$ $C_2 = (0.4, 0.3)$	Case I $C_1 = (0.5, 0.2)$ $C_2 = (0.4, 0.3)$	Case I $C_1 = (0, 0)$ $C_2 = (0.5, 0.5)$
S_1	**0.9100**	**0.9100**	0.5000
S_2	0.9550	0.9300	**1**
S_3	0.8650	0.8150	0.7500
S_4	**0.9100**	**0.9100**	0.7500
S_5	**0.8349**	**0.8349**	0.6000
S_6	**0.8349**	**0.8349**	0.6000
S_7	**− 0.5080**	**− 0.1191**	**0**
S_8	**− 0.5080**	**− 0.1191**	**0**
S_9	**0.6400**	**0.6400**	**0**
S_{10}	**0.6400**	**0.6400**	**0**
S_{11}	0.9224	0.8843	0.6000
S_{12}	0.9224	0.8843	0.6000
D_1	**0.0900**	**0.0900**	0.5000
D_2	0.0450	0.0700	0
D_3	0.1350	0.1850	0.2500
D_4	**0.0900**	**0.0900**	0.2500
D_5	**0.1651**	**0.1651**	0.4000
D_6	**0.1651**	**0.1651**	0.4000
D_7	0.1080	0.4969	**1**
D_8	0.1080	0.4969	**1**
D_9	**0.3600**	**0.3600**	**1**
D_{10}	**0.3600**	**0.3600**	**1**
D_{11}	0.0776	0.1157	0.4000
D_{12}	0.0776	0.1157	0.4000
S_{G1}	0.9390	0.9222	0.7778
S_{G2}	0.9390	0.9212	0.7656
S_{G3}	0.9390	0.9200	0.7500
S_{G4}	0.9390	0.9201	0.7538

Unreasonable results are indicated by bold values

$$B^- = \left\{ \left(\sigma_1^-, \varsigma_1^-\right), \left(\sigma_2^-, \varsigma_2^-\right), \ldots, \left(\sigma_q^-, \varsigma_q^-\right) \right\},$$

where $\sigma_k^+ = \max_t(\sigma_{tk})$, $\varsigma_k^+ = \min_t(\sigma_{tk})$ and $\sigma_k^- = \min_t(\sigma_{tk})$, $\varsigma_k^- = \max_t(\sigma_{tk})$, $k = 1, 2, \ldots, q$.

Step 5: Determine the similarity of each alternative $B_t, t = 1, 2, \ldots, p$ with the PIS B^+ and NIS B^- with the help of the newly introduced FF-weighted measures of similarity, i.e., find $S_{Gj}^W(B_t, B^+)$ and $S_{Gj}^W(B_t, B^-), t = 1, 2, \ldots, p$, and $j = 1, 2, 3, 4$.

Step 6: Determine $S_{Gj}^W(B^+)$, where $S_{Gj}^W(B^+) = \max_{1 \le t \le p} S_{Gj}^W(B_t, B^+)$ and so if $S_{Gj}^W(B^+) = S_{Gj}^W(B_t, B^+)$, the alternative B_t has maximum similarity with PIS.

Step 7: Determine $S_{Gj}^W(B^-)$, where $S_{Gj}^W(B^-) = \min_{1 \le t \le p} S_{Gj}^W(B_t, B^-)$ and so if $S_{Gj}^W(B^-) = S_{Gj}^W(B_t, B^-)$, the alternative B_t has minimum similarity with NIS.

Step 8: Determine $v_t = \frac{S_{Gj}^W(B_t, B^+)}{S_{Gj}^W(B^+)} - \frac{S_{Gj}^W(B_t, B^-)}{S_{Gj}^W(B^-)}, 1 \le t \le p$. It is obvious that v_t represents the amount to which an option $B_t, t = 1, 2, \ldots, p$ has the least and greatest similarity with NIS and PIS, respectively, at the same time. The option for which $v_t = 0$ is the best choice.

Step 9: Compute the FFIR $\lambda_t = \frac{v_t}{\min_{1 \le t \le p} v_t}$.

Step 10: In the increasing order of the values of λ_t, we rank the alternatives.

In the example that follows, we apply the suggested FFIR approach to resolve a MADM issue with FF data.

Table 7 Some existing PF/FF measures of entropy (Xue et al. 2018; Yang and Hussain 2018; Thao and Smarandache 2019; Senapati and Yager 2020; Mishra and Rani 2021)

Entropy measure	Expression
E_1	$\frac{-1}{p\times\ln 2}\sum_{t=1}^{p}\begin{pmatrix}\sigma^3_{C_1}(b_t)ln\left(\sigma^3_{C_1}(b_t)\right)+\varsigma^3_{C_1}(b_t)ln\left(\varsigma^3_{C_1}(b_t)\right)\\-\left(1-\tau^3_{C_1}(b_t)\right)ln\left(1-\tau^3_{C_1}(b_t)\right)-\tau^3_{C_1}(b_t)ln2\end{pmatrix}$
E_2	$\frac{1}{2p}\sum_{t=1}^{p}\left(\sin\left(\frac{\sigma^3_{C_1}(b_t)+1-\varsigma^3_{C_1}(b_t)}{2}\right)\pi+\sin\left(\frac{\varsigma^3_{C_1}(b_t)+1-\sigma^3_{C_1}(b_t)}{2}\right)\pi\right)$
E_3	$\frac{1}{(\sqrt{2}-1)p}\sum_{t=1}^{p}\left(\cos\left(\frac{\pi\left(1+\sigma^3_{C_1}(b_t)-\varsigma^3_{C_1}(b_t)\right)}{4}\right)+\cos\left(\frac{\pi\left(1-\sigma^3_{C_1}(b_t)+\varsigma^3_{C_1}(b_t)\right)}{4}\right)-1\right)$
E_4	$\frac{1}{(\sqrt{2}-1)p}\sum_{t=1}^{p}\left(\sin\left(\frac{\pi\left(1+\sigma^3_{C_1}(b_t)-\varsigma^3_{C_1}(b_t)\right)}{4}\right)+\sin\left(\frac{\pi\left(1-\sigma^3_{C_1}(b_t)+\varsigma^3_{C_1}(b_t)\right)}{4}\right)-1\right)$
E_5	$1-\sqrt{\frac{1}{p}\sum_{t=1}^{p}\left(\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right)^2}$
E_6	$\frac{1}{p}\sum_{t=1}^{p}\left(1-\left\lvert\sigma^2_{C_1}(b_t)-\frac{1}{3}\right\rvert-\left\lvert\varsigma^2_{C_1}(b_t)-\frac{1}{3}\right\rvert\right)$
E_7	$\frac{1}{p}\sum_{t=1}^{p}\left(1-\left(\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right)\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert\right)$
E_8	$\frac{1}{p}\sum_{t=1}^{p}\tan\left(\frac{\pi}{4}-\frac{\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert}{4\left(1+\tau^2_{C_1}(b_t)\right)}\pi\right)$
E_9	$\frac{1}{p}\sum_{t=1}^{p}\cot\left(\frac{\pi}{4}+\frac{\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert}{4\left(1+\tau^2_{C_1}(b_t)\right)}\pi\right)$
E_{10}	$\frac{1}{(\sqrt{2}-1)p}\sum_{t=1}^{p}\left(\cos\left(\frac{1+\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)}{4}\pi\right)+\cos\left(\frac{1-\sigma^2_{C_1}(b_t)+\varsigma^2_{C_1}(b_t)}{4}\pi\right)-1\right)$
E_{11}	$\frac{1}{(\sqrt{2}-1)p}\sum_{t=1}^{p}\left(\sin\left(\frac{1+\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)}{4}\pi\right)+\sin\left(\frac{1-\sigma^2_{C_1}(b_t)+\varsigma^2_{C_1}(b_t)}{4}\pi\right)-1\right)$
E_{12}	$\frac{1}{p}\sum_{t=1}^{p}\frac{\min\left(\sigma^2_{C_1}(b_t),\varsigma^2_{C_1}(b_t)\right)}{\max\left(\sigma^2_{C_1}(b_t),\varsigma^2_{C_1}(b_t)\right)}$
E_{13}	$1-\frac{1}{p}\sum_{t=1}^{p}\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert$
E_{14}	$\frac{\sum_{t=1}^{p}\left(1-\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert\right)}{\sum_{t=1}^{p}\left(1+\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert\right)}$
E_{15}	$\frac{1}{p}\sum_{t=1}^{p}\frac{\tau^2_{C_1}(b_t)+1-\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert}{\tau^2_{C_1}(b_t)+1+\left\lvert\sigma^2_{C_1}(b_t)-\varsigma^2_{C_1}(b_t)\right\rvert}$

Example 8 (Singh and Ganie 2022) Consider the issue of choosing a home among the five homes $B_t, t = 1, 2, 3, 4, 5$. Take into account the following factors: C_1: Ventilation, C_2: Purchase price, C_3: Location, C_4: Design, C_5: Ceiling height. The decision matrix A below illustrates how the information regarding the five houses in relation to the five aforementioned criteria is expressed as FFSs.

$$A=\begin{pmatrix}(0.6,0.5) & (0.8,0.3) & (0.4,0.7) & (0.6,0.8) & (0.7,0.5)\\(0.1,0.7) & (0.4,0.6) & (0.2,0.7) & (0.7,0.3) & (0.6,0.6)\\(0.1,0.3) & (0.71,0.3) & (0.6,0.8) & (0.21,0.9) & (0.29,0.8)\\(0.4,0.7) & (0.5,0.6) & (0.1,0.6) & (0.2,0.8) & (0.2,0.9)\\(0.3,0.4) & (0.6,0.6) & (0.4,0.8) & (0.32,0.9) & (0.3,0.9)\end{pmatrix}.$$

Given that the criteria C_2 is a cost attribute, the normalized decision matrix D is provided below using Step 2:

Table 8 Computed values of several measures of entropy concerning Example 3

	$(C_1)^4$	$(C_1)^3$	$(C_1)^2$	C_1	$(C_1)^{\frac{1}{2}}$
E_1	0.5229	0.6056	0.7101	0.8650	0.8455
E_2	0.1422	0.1598	0.1782	0.1919	0.1829
E_3	0.6867	0.7717	0.8602	0.9266	0.8831
E_4	0.6867	0.7717	0.8602	0.9266	0.8831
E_5	0.2731	0.3612	0.4943	0.6880	0.6248
E_6	0.3155	0.4224	0.5296	0.6782	0.6670
E_7	1.5283	1.4081	1.2429	0.9902	0.8592
E_8	0.3640	0.4785	0.5750	0.7412	0.7209
E_9	0.3640	0.4785	0.5750	0.7412	0.7209
E_{10}	12.0414	12.0963	12.1659	12.2385	12.2186
E_{11}	1.2769	1.2909	1.3087	1.3275	1.3222
E_{12}	0.0836	0.1677	0.1497	0.3864	0.4849
E_{13}	0.3155	0.4260	0.5296	0.7311	0.7176
E_{14}	0.1873	0.2706	0.3602	0.5762	0.5596
E_{15}	0.3050	0.4183	0.5062	0.6900	0.6761
E_{G1}	0.3849	0.4778	0.5188	0.6659	0.6692
E_{G2}	0.3241	0.4210	0.4680	0.6339	0.6404
E_{G3}	0.2212	0.3082	0.3731	0.5636	0.6046
E_{G4}	0.2939	0.3890	0.4325	0.6094	0.6238

Table 9 Computed values of several similarity metrics about Example 4

Similarity measure	(C_1, C)	(C_2, C)	(C_3, C)	Result
S_1	0.9824	0.9829	0.9792	C_2
S_2	0.9851	0.9869	0.9833	C_2
S_3	0.9837	0.9849	0.9813	C_2
S_4	0.9824	0.9829	0.9792	C_2
S_5	0.9655	0.9665	0.9595	C_2
S_6	0.0173	0.0168	0.0204	C_3
S_7	0.8431	0.8560	0.8242	C_2
S_8	0.7237	0.7152	0.7494	C_3
S_9	0.5639	0.5503	0.5923	C_3
S_{10}	0.8348	0.8547	0.8188	C_2
S_{11}	0.9694	0.9719	0.9643	C_2
S_{12}	0.9690	0.9728	0.9654	C_2
S_{G1}	0.9854	0.9872	0.9827	C_2
S_{G2}	0.9854	0.9872	0.9826	C_2
S_{G3}	0.9853	0.9872	0.9825	C_2
S_{G4}	0.9853	0.9872	0.9825	C_2

Table 10 Computed values of several similarity metrics concerning Example 5

Similarity measure	(C_1, C)	(C_2, C)	(C_3, C)	Result
S_1	0.7840	0.8000	0.8860	C_3
S_2	0.8590	0.9030	0.9280	C_3
S_3	0.8215	0.8515	0.9070	C_3
S_4	0.7940	0.8260	0.8860	C_3
S_5	0.6868	0.7291	0.8075	C_3
S_6	0.1708	0.1482	0.1023	C_1
S_7	0.3429	0.3147	0.6027	C_3
S_8	0.4036	0.3094	0.4911	C_3
S_9	0.3749	0.4024	0.6498	C_3
S_{10}	0.3041	0.3627	0.6418	C_3
S_{11}	0.7707	0.8130	0.8870	C_3
S_{12}	0.7545	0.8026	0.8873	C_3
S_{G1}	0.8321	0.8619	0.9096	C_3
S_{G2}	0.8257	0.8585	0.9083	C_3
S_{G3}	0.8158	0.8534	0.9068	C_3
S_{G4}	0.8197	0.8552	0.9070	C_3

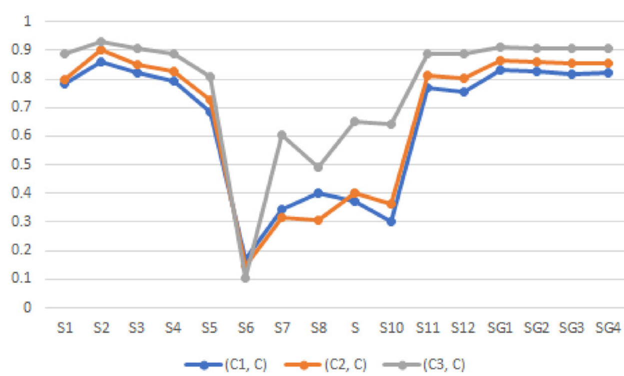

Fig. 1 Similarity values regarding Example 5

Table 11 Computed values related to Example 6

Alternative	$S_{G1}(B_t, B^-)$	$S_{G1}(B_t, B^+)$	μ_t	Ranking
B_1	0.8085	0.8620	0.5160	2
B_2	0.9198	0.7954	0.4637	3
B_3	0.8232	0.8037	0.5205	1

$$D = \begin{pmatrix} (0.6, 0.5) & (0.3, 0.8) & (0.4, 0.7) & (0.6, 0.8) & (0.7, 0.5) \\ (0.1, 0.7) & (0.6, 0.4) & (0.2, 0.7) & (0.7, 0.3) & (0.6, 0.6) \\ (0.1, 0.3) & (0.3, 0.71) & (0.6, 0.8) & (0.21, 0.9) & (0.29, 0.8) \\ (0.4, 0.7) & (0.6, 0.5) & (0.1, 0.6) & (0.2, 0.8) & (0.2, 0.9) \\ (0.3, 0.4) & (0.6, 0.6) & (0.4, 0.8) & (0.32, 0.9) & (0.3, 0.9) \end{pmatrix}.$$

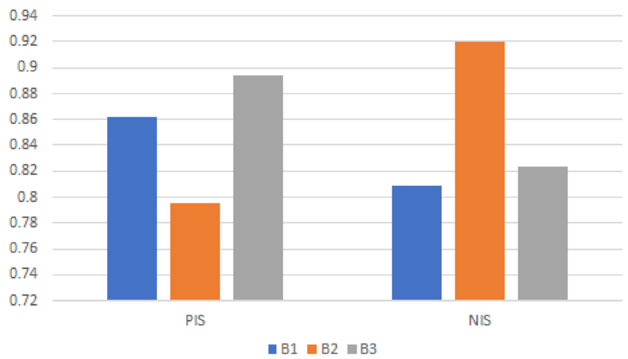

Fig. 2 Alternatives' similarity with PIS and NIS concerning Example 6

Table 12 Computed values related to Example 7

Alternative	$S_{G1}(B_t, B^-)$	$S_{G1}(B_t, B^+)$	μ_t	Ranking
B_1	0.8051	0.8378	0.5099	3
B_2	0.7531	0.8460	0.5291	1
B_3	0.7202	0.7814	0.5204	2

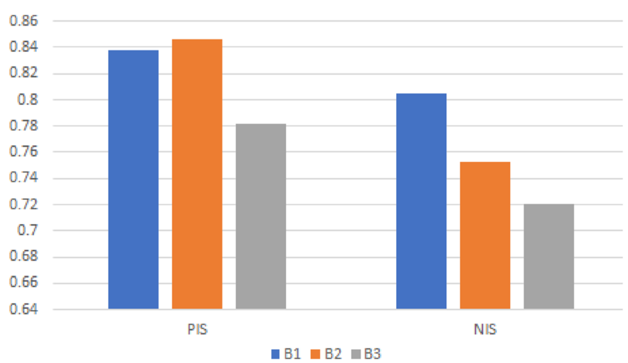

Fig. 3 Alternatives' similarity with PIS and NIS concerning Example 7

We acquire the following criteria weights using Step 3 and the suggested entropy measure E_{G1} presented in Table 3:$w_1 = 0.2714$, $w_2 = 0.2464, w_3 = 0.1897, w_4 = 0.1250$, and $w_5 = 0.1675$.

The FF PIS B^+ and FF NIS B^- are then provided using Step 4.

$$B^+ = \{(0.6, 0.3), (0.6, 0.4), (0.6, 0.6), (0.6, 0.4), (0.6, 0.3)\}$$

$$B^- = \{(0.1, 0.7), (0.3, 0.8), (0.1, 0.8), (0.2, 0.9), (0.2, 0.9)\}.$$

Then using the suggested similarity measure S^W_{G1} we calculate the similarity values of each alternative $B_t, t = 1, 2, 3, 4, 5$ with the FF PIS B^+ and FF NIS B^-. Then we get $S^W_{G1}(B^+) = \max_{1 \le t \le p} S^W_{G1}(B_t, B^+) = 0.9594 = S^W_{G1}(B_2, B^+)$ and $S^W_{G1}(B^-) = \min_{1 \le t \le p} S^W_{G1}(B_t, B^-) = 0.9307 = S^W_{G1}(B_2, B^-)$

Then for each alternative, we determine ν_t and $\lambda_t, t = 1, 2, 3, 4, 5$. Finally, we rank the alternatives in increasing order of the values of λ_t. All these computations are listed in Table 13 and shown in Fig. 4. Table 13 also provides the ranking results for the other there suggested weighted similarity measures $S^W_{Gj}, j = 2, 3, 4$.

We conclude from Table 13 and Figs. 5, 6, 7 and 8 that B_2 is the most practical choice because all recommended FF similarity measures show the same results. We can see from Table 13 and Figs. 5, 6, 7 and 8 that the optimal alternative B_2 is most similar to PIS B^+ while being least similar to NIS B^-.

8 Discussion and comparative analysis

In the vast quantity of studies on the topic, applications for fuzzy and non-standard fuzzy information measures can be found in MADM, pattern recognition, clustering analysis, picture segmentation, etc. In a certain situation, both activities appear to have the same outcome. Yet, it could provide a variety of results. For instance, in a MADM situation, the ranks of the alternatives may vary depending on the fuzzy entropy or fuzzy knowledge metrics used. When evaluating the compatibility of two fuzzy sets, we may obtain different results with alternate fuzzy similarity/distance/accuracy metrics. This seems to be due to the fuzzy/non-standard fuzzy information measure's failure to accurately represent the ambiguity or precision present in the fuzzy/non-standard fuzzy sets under consideration. As a result, when modeling a specific fuzzy system, we must carefully assess the fuzzy/non-standard fuzzy information measurements. There are many reasons given in the literature for picking a fuzzy information/compatibility measure in a certain situation. The noteworthy ones involve computations for weight, similarity/distance, and linguistic hedging, among other things. The significance of our recommended similarity and entropy metrics are then justified.

We have devised a method for creating the FF similarity measures from t-conorms in Sect. 3 . In Theorem 2, we went through a few of their fresh features. By computing the similarity between several FFSs, we have demonstrated in Sect. 5 that the suggested similarity metrics are preferable. Table 6 has three different FFS situations, each of which consists of two different FFSs. In these three circumstances, while calculating the degree of similarity between the FFSs, we found that the majority of the existing distance and similarity metrics did not produce the desired results, and some of them even failed to meet the

Table 13 Computed values for Example 8 based on the suggested FF similarity measure $S^W_{Gj}, j = 1, 2, 3, 4$

Similarity measure	Alternative	$S^W_{Gj}(B_t, B^-)$	$S^W_{Gj}(B_t, B^+)$	μ_t	Ranking
S^W_{G1}(proposed)	B_1	0.9414	0.9472	− 0.0243	2
	B_2	0.9307	0.9594	0	1
	B_3	0.9595	0.9216	− 0.0703	5
	B_4	0.9571	0.9304	− 0.0586	3
	B_5	0.9610	0.9260	− 0.0674	4
S^W_{G2}(proposed)	B_1	0.9365	0.9438	− 0.0248	2
	B_2	0.9255	0.9562	0	1
	B_3	0.9594	0.9165	− 0.0781	5
	B_4	0.9551	0.9249	− 0.0648	3
	B_5	0.9590	0.9210	− 0.0730	4
S^W_{G3}(proposed)	B_1	0.9256	0.9374	− 0.0285	2
	B_2	0.9128	0.9512	0	1
	B_3	0.9592	0.9047	− 0.0998	3
	B_4	0.9515	0.9115	− 0.0842	4
	B_5	0.9559	0.9047	− 0.0932	5
S^W_{G4}(proposed)	B_1	0.9325	0.9409	− 0.0520	2
	B_2	0.9216	0.9534	0	1
	B_3	0.9593	0.9125	− 0.0838	5
	B_4	0.9535	0.9208	− 0.0688	3
	B_5	0.9573	0.9174	− 0.0765	4

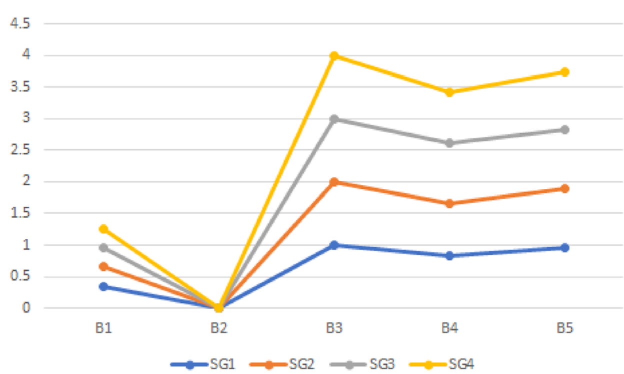

Fig. 4 FFIR values corresponding to the proposed four FF similarity measures

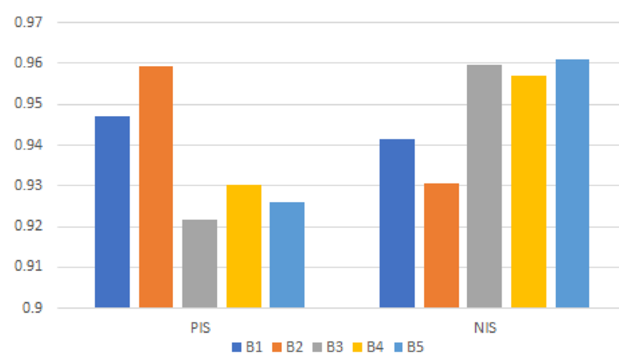

Fig. 5 Alternatives' similarity using S^W_{G1}

necessary axiomatic requirements. Nonetheless, the suggested similarity measures handled all three scenarios correctly and without creating any illogical circumstances. We have demonstrated in Sect. 6 how the recommended similarity measures can be applied to the classification problems and we have also noted that the suggested similarity metrics produce satisfactory results.

With the suggested FF similarity measures, we have demonstrated how to create several innovative FF entropy measures in Sect. 4. Additionally, we have proven that these entropy measurements meet all axiomatic constraints for FF entropy measures. We used linguistic hedges in Sect. 5.2 to demonstrate how the suggested entropy metrics outperform the current entropy measures. Only the suggested entropy measures are found to perform in accordance with the desired condition stated in Eq. (5) in the numerical example studied in this Sect. 5.2. Also, we've shown how to use them to calculate attribute weights in a MADM problem in Sect. 7.2 (Step 3 of Algorithm).

Finding the best option out of all those that are offered is the ultimate goal of a MADM technique. The alternative that is most similar to the positive ideal solution (PIS) and least similar to the negative ideal solution is the ideal

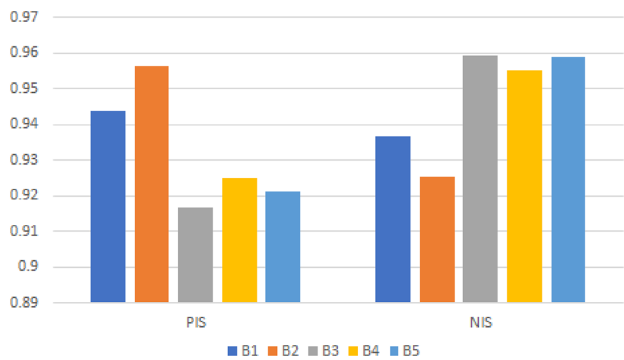

Fig. 6 Alternatives' similarity using S_{G2}^{W}

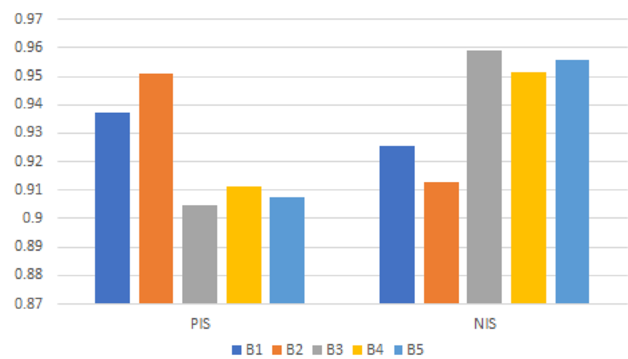

Fig. 7 Alternatives' similarity using S_{G3}^{W}

alternative (NIS). In contrast, we have demonstrated in

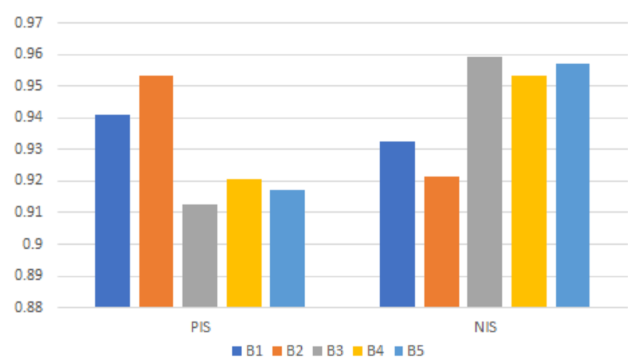

Fig. 8 Alternatives' similarity using S_{G4}^{W}

Examples 6 and 7 that the optimal option resulting from TOPSIS has maximum similarity to PIS but not least resemblance to NIS. However, the suggested FFIR method gives us the optimal option that is close to PIS and at the same time is far from the NIS as can be seen in Table 13.

The benefits of the novel MADM technique, the suggested entropy metrics, and similarity measures are outlined below.

- The suggested method of obtaining the new similarity measures from t-conorms can be used in many recent generalizations of fuzzy sets.
- Since the present entropy methods yield inaccurate results, the ambiguity content of FFSs can be calculated using similarity-based entropy measurements.
- The suggested similarity metrics can be used for image processing, identifying construction materials, and bidirectional approximate reasoning, among other things.
- The TOPSIS method has a significant flaw that causes it to provide irrational results, hence the newly developed MADM approach, the FFIR method, can be used in its substitute.

In addition to the benefits already described, one drawback of the suggested methods is that they are challenging to implement in real-world situations with the crisp data found in repositories and other websites that are similar to them. The suggested actions can be put into practice by either developing a linguistic database or applying certain conversion procedures.

9 Conclusion

This article presents a novel method for the creation of various similarity metrics and entropy measures for FFSs. First, four new similarity measures were created, and then utilizing the suggested similarity measurements, four new entropy measures were established. The proposed measures of similarity outperform the majority of the PF distance/similarity metrics reported in the literature in terms of the distance or degree of similarity between different PFSs/FFSs. The suggested entropy metrics for FFSs are also more dependable than the existing PF/FF entropy measures from the linguistic hedge standpoint. The suggested FF similarity measurements have achieved satisfactory results in pattern analysis. A compromise solution that has the most similarity to PIS and the least similarity to NIS was created using the recently proposed MADM methodology, commonly known as the FFIR method.

We will further present examples of clustering and medical diagnostics using the proposed FFS similarity metrics. Additionally, we will add some more recent generalizations of FSs, such as picture fuzzy sets (Cuong and Kreinovich 2013), spherical fuzzy sets (Mahmood et al. 2019), complex fuzzy sets (Ramot et al. 2002), etc., into the suggested method for calculating similarity and entropy measurements. We will also extend the newly introduced MADM method to the recent extensions of fuzzy sets.

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Author contributions All authors contributed equally.

Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

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