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ORIGINAL PAPER

Multi-attribute decision-making for electronic waste recycling using interval-valued Fermatean fuzzy Hamacher aggregation operators

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Abstract

The utilization of electrical and electronics equipments in waste recycling has become a paramount for various countries. The waste electrical and electronics equipment (WEEE) recyclers own a crucial position in the environmental growth of a country as they help to minimize the carbon emissions during the recycling of WEEE in the most eco-friendly way. Therefore, the selection and assessment of an appropriate WEEE recycling partner has become a most important part of DM (decision-making) applications. The collusion of numerous quantitative and qualitative factors makes the recycling partner selection problem, a multifaced and significant decision for the managerial experts. The main objective of this work is to propose MADM (multi-attribute decision-making) techniques to evaluate the WEEE recycling partners under interval-valued Fermatean fuzzy (IVFF) information. In this regard, certain Hamacher AOs (aggregation operators) are proposed to develop the required DM method. These AOs include Hamacher weighted averaging, ordered weighted averaging, weighted geometric, ordered weighted geometric, generalized Einstein weighted averaging, generalized Einstein ordered weighted averaging, generalized Einstein weighted geometric, etc. Then, these averaging operators are utilized to come up with a MADM techniques under IVFF environment. Furthermore, the constructed technique is applied to a case study in China to incorporate with the e-waste recycling partner selection problem. Moreover, a brief comparison of the proposed with is presented with various existing techniques to manifest the productivity and coherence of the proposed model. Finally, the accuracy and consistency of results shows that the proposed technique is fully compatible and applicable to handle any MADM problem.

Keywords IVFFSs · Hamacher AOs · Weighted averaging operators · Weighted geometric operators · MADM · Electronic waste recycling in China

1 Introduction

The rapid progress in equipment capabilities and features results to short the consumer life of electronic devices. This causes a large stream of waste of superannuated electronic apparatus, i.e., electronic waste (e-waste). Although, there exist various typical disposal techniques for e-waste, but from environmental as well as from economic point of view, these existing methods own various disadvantages nd drawbacks. Therefore, some other updated waste management options should be considered such as, recycling. Due to short history of e-waste recycling, there is a lack of solid infrastructure in locale. The most common and primary method for the disposal of e-waste in China is contemplated as incineration and disposal in landfills. In increase of requisite for landfills may cause a serious freight towards our environs. Due to the scarcity of landfills proportions and growing apprehension about condition of environment produce the desire of new waste management methods. The updated techniques of e-waste disposal should divert the end-of-life (EOL) electronic devices from carbonize and landfills. Nevertheless, distinct elements involve in the evolution of new strategies. The developed strategy must contain the following characteristics: technically feasible, economically sustainable, and must contain a sensible degree of social support for the mission. The most important feature of a developed technique must be recycling and reusing of EOL electronic devices.

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At present, it can be noted that e-waste recycling possesses a very short history in China, that is the reason of absence of fixed and broad infrastructure in the area. According to the estimation of "International Association of Electronics Recyclers" (IAER), there were just over 700 employees in US e-waste recycling industry. Various modern technologies are combined to produce the electronic devices and consist of different components and materials. This implies that various parties and technologies should be involved in the recycling of EOL electronics. Crucial elements in the recycling of e-waste are assembling, categorizing and reclamation, recycling and disposal, as shown in Fig. 1.

MADM (Multi-attribute decision-making) is considered as the bulk accelerated and developing area of research that provides the most appropriate possible choice from a collection of finite choices corresponding to certain attributes. The indeterminacy in decision experts' opinions leads to certain issues while evaluating the information of candidates. Such kind of hassles are handled through the coined notion of fuzzy sets (FSs) introduced by Zadeh (1965), that is utilized to deal with uncertainty. Then, the notion of FSs was generalized by intuitionistic fuzzy sets (IFSs) exposing by the belonging and non-belonging degrees by Atanassov (1986). The main characteristic of IFSs is that the summation of these values should be less than one. Although, there arises many situations in which the sum of both degrees must greater than 1, Pythagorean fuzzy sets (PFSs) were defined to handle such kind of phenomena by Yager (2014). The constraint of PFSs relaxes the space of membership grades as the square sum of these values should be less than one. Therefore, PFSs are contemplated as an extra flexible and appropriate framework to handle the complicated MADM situations. Furthermore, a more inclusive class of PFSs named as, q-rung orthopair fuzzy sets (q-ROFSs), was proposed. In these sets the sum of the qth power of the belonging and non-belonging grades should be less than or equal to one (Yager 2017). After that, Fermatean fuzzy sets (FFSs) were presented as a special case of q-ROFSs where $q = 3$ (Senapati and Yager 2020). In FFSs, the cube addition of both grades (i.e., belonging and non-belonging) is less than or equal to unity. The constraining relationship between the belonging and non-belonging grades is considered as a zestful variance between IFSs, PFSs and FFSs. Thus, the FFSs has been proven more powerful and effective framework as differentiated to IFSs and PFSs to cope with uncertain MADM problems. While handling various decision-making applications beneath FFSs, it becomes very demanding for the decision makers to particularize their opinions more precisely using classical evaluations due to dearth in accessible data and knowledge. Similar to these situations, it is considered advantageous for decision makers to furnish their recommendations using an interval number within [0, 1]. Thus, it is very useful to consider the notion of interval-valued Fermatean fuzzy sets (IVFFSs), that warranty the belongingness grades to be assumed as interval values (Rani et al. 2022a).

1.1 Problem statement

Even though there are conventional disposal techniques for e-waste, those techniques have risks from both the economic and environmental viewpoints. As a result, new e-waste control options need to be considered, for instance, recycling. But electronic recycling has quick records, so there isn't yet a strong infrastructure in the area. Keeping in view the E-waste recycling partner selection problem, this study first introduces the certain Hamacher AOs under IVFF environment. The basic motive of the developed study is the initiation of a MADM framework to select the most appropriate e-waste recycling partner. At present times, China is not simplest a massive consumption nation of electrical merchandise, but additionally a largest importer of e-waste. Keeping in view these reasons, we consider a case study of e-waste recycling partner selection in China. In this work, we've got targeted at the usage of an revolutionary approach so that it would oblige the company's shareholders to appraise and choose for the maximum appropriate alternative. All the necessary details are presented in Sect. 10 that how we choose the criteria, the decision experts preferences, and alternatives to evaluate. The same section presents the problem, the method that is applied, and the results which are obtained.

1.2 Motivation

Being motivated by the key features of IVFFSs, we introduce the certain aggregation operators for IVFFSs. The crucial ultimatums at the back of the present work are provided as follows:

1. The Einstein and Hamacher aggregation operators (AOs) possess much flexibility and effectiveness as compared to the elementary operations and results. In spite of this, there is no literature about the Hamacher AOs under IVFFSs to accumulate the IVFF data.
2. In the previous studies, no research endeavors have been perform to discuss the various characteristics of AOs for IVFFSs.
3. The traditional decision-making methods based on AOs are extended through various vague and uncertain environments but bulk of the AOs propose obstruction and limitations to manipulate with IVFF information.
4. The existing techniques introduce various methods to evaluate the e-waste recycling partner selection problem but these frameworks have limitations and drawbacks while dealing with uncertain e-waste recycler partner selection problem from IVFFSs.

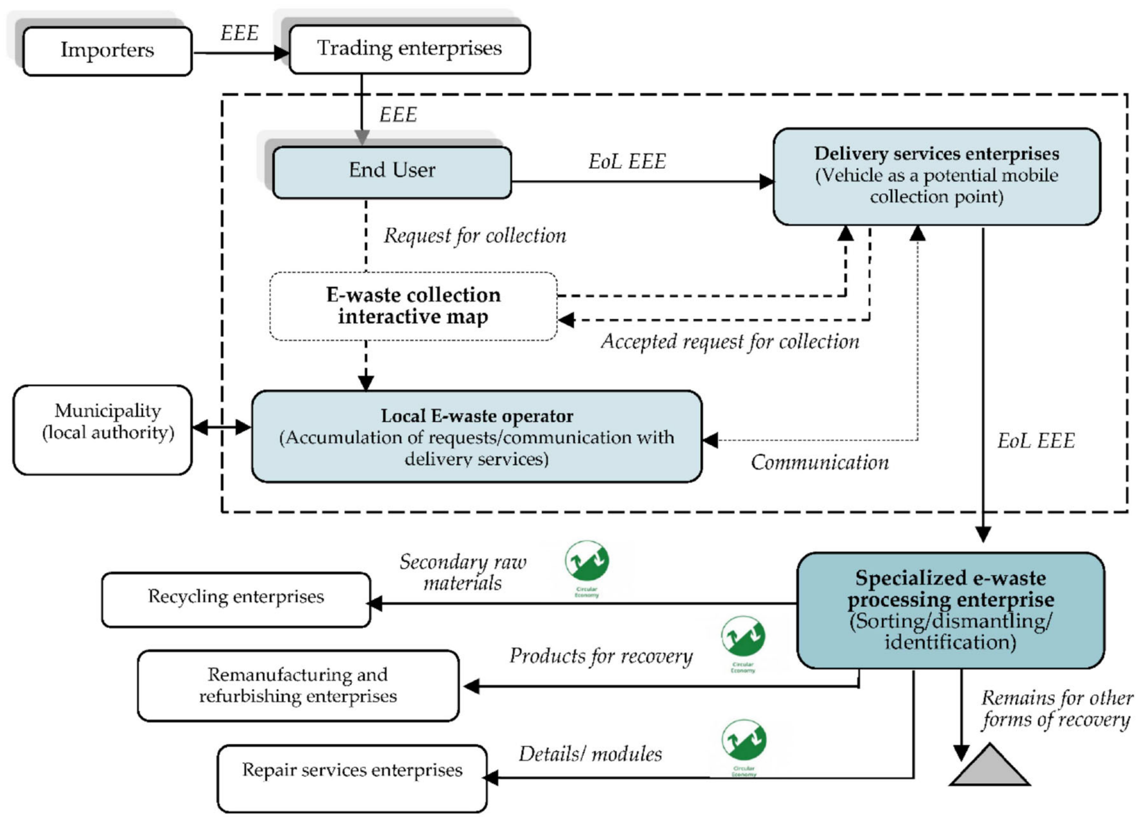


Fig. 1 Key factors in the recycling of e-waste

1.3 Novelty and contribution

The novelty and vital endowments of the proposed work are narrated as:

1. A novel function to calculate the score values for IVFFSs is utilized to compare any number of IVFF numbers (IVFFNs).
2. Certain AOs, including Hamacher weighted averaging operator, Hamacher weighted geometric operator, Hamacher ordered weighted operator, etc are introduced to aggregate the IVFFSs.
3. Some useful and productive properties of proposed operators are discussed briefly.
4. A composite IVFF-based framework is proposed developing the amalgamation of the score function, Einstein aggregation operators and Hamacher aggregation operators to solve the MADM problems with unspecified decision makers and unknown weights of criteria.
5. To showcase the steadiness and permanence of proposed model, a case study of e-waste recycling partner choice in China is taken underneath IVFF surroundings.
6. The robustness and authenticity of our developed work is demonstrated through comparing the developed method with the previous knowledge.

1.4 Structure of the study

The remaining of this study is prepared as: a comprehensive review about the proposed study is presented in Sect. 2. Some fundamental concepts of FFSs and IVFFSs are discussed in Sect. 3. We talk about certain operations and distance measures for IVFFSs. In Sect. 4, the FF Hamacher AOs has been introduced. Hamacher weighted and ordered weighted averaging operators under IVFFSs are introduced in Sect. 5. Section 6 discusses the IVFF Hamacher weighted and ordered weighted geometric operators. Furthermore, generalized IVFF Einstein weighted, ordered weighted, weighted geometric, and ordered weighted geometric AOs are introduced and discussed in Sects. 7 and 8, respectively. In Sect. 9, MADM technique has been performed through IVFF AOs as proposed in previous sections. Then, a case study regarding the e-waste recycler partner selection problem in China has been performed in Sect. 10 through the technique described in the previous section. In Sect. 11, we present a comprehensive comparison of the proposed approach with some existing techniques. Section 12 deals with conclusions, limitations of the proposed work and future research directions.

2 Literature review

A comprehensive review about the proposed study is presented in this section.

2.1 Interval-valued Fermatean fuzzy sets

Human's cognitions, pressure of time, and the restricted data or knowledge are considered as the key factors that make difficult the usage of crisp numbers in modeling the work conditions through the data mining and processing methodologies. To handle these problems, the concept of FSs was introduced by Zadeh (1965), in this set every element is expressed utilizing the membership function to minimize the enigma of data and knowledge. After that, various extended forms of FSs have been presented, included IFSs (Atanassov 1986), PFSs (Yager 2014), q-ROFSs (Yager 2014), FFSs (Senapati and Yager 2020), etc. Having the capability to depict the useful information, FFSs have been proven more generalized as contrasted to IFSs and PFSs and its applications are more wider. For instance, Aydin (2021) utilized the entropy theory and developed a novel FF entropy formula that is based on Euclidean distance among FFNs and its compliment. Then, they developed the four distinct FF cosine similarity measures. The author discussed the numerical example for third party logistic (3PL) firm assessment application in cold chain management to prove the applicability of the proposed method. Deng and Wang (2022) initiated two novel distance measure techniques for FFSs to address the medical diagnosis and pattern recognition problems. One of which is based on Hellinger distance and the other one is based on the triangular divergence of FFSs. Numerical examples are illustrated to show the effectiveness and accuracy of the proposed techniques. The other similarity measures and their properties for FFSs were discussed by Xu and Shen (2021). Furthermore, the authors provided the interpretative MADM application and two examples related to medical diagnosis to reveal the effectiveness and viability of the proposed technique. To analyze the MADM problems under FFNs, Gül (2021) proposed three novel techniques, named as, SAW (Simple Additive Weighting), ARAS (Additive Ratio Assessment), and VIKOR (VIse KriterijumsaOptimiz acija I Kompromisno Resenje). He also presented an example of proposed techniques for the selection of the best testing laboratory to diagnose Covid-19 infected patients. An improved generalized score function (IGSF) for FFNs was proposed by Mishra et al. (2021). They proposed a FF- CRITIC (CRiteria Importance Through Intercriteria Correlation)-EDAS (Evaluation based on Distance from Average Solution) framework to handle MADM problems. Akram et al. (2021b, a) proposed two modified and novel techniques, named as, Pythagorean fuzzy hybrid Order of Preference by Similarity to an Ideal Solution (PFH-TOPSIS) method and Pythagorean fuzzy hybrid ELimination and Choice Translating REality I (PFH-ELECTRE I) method, to rank risk evaluations in failure modes and effects analysis (FMEA). An integrated ELECTRE-I approach to rank risks in FMEA under hesitant PF information was proposed by Akram et al. (2022a, b, c, d, e, f, g). A novel MCGDM technique for the selection of an antivirus mask under FF and FF soft information was presented by Shahzadi and Akram (2020 and 2021). Recently, Akram et al. (2022a, b, c, d, e, f, g) proposed an integrated MULTIMOORA method combined with 2-tuple linguistic FFSs and illustrated the application of proposed method in selection of urban quality of life. Shahzadi et al. (2022) applied MOORA method under FF information in the selection of intelligent manufacturing system (IMS). In the context of IVFFSs, generalization of capital budgeting techniques under IVFFSs was proposed by Sergi et al. (2022). Further, Jeevaraj (2021) proposed certain new concepts, including ordering principle, distance measure and similarity measure on the class of IVFFNs. Rani and Mishra (2022a, b) proposed the score and accuracy functions for IVFFNs. They also proposed two aggregation operators (AOs) to aggregate the IVFFSs information and discussed some axioms.

2.2 Einstein AOs

Many researchers have been studied the concept of AOs from the MADM perspectives. In recent years, the fuzzy AOs are fundamentally becoming fascinating in various eras of research and are designated penetrating attention amidst the academic circles. Einstein sum and product are contained in Einstein operators that are proved to be notable choices towards the algebraic sum and product, respectively. For instance, Rahman et al. (2020) initiated the idea of Einstein AOs, including the interval-valued PF (IVPF) Einstein weighted averaging AO and the IVPF Einstein ordered weighted averaging AO and discussed certain properties of these operators. Certain complex Einstein weighted geometric AOs and their applications in supplier chain management under IVPFSs were proposed by Ali et al. (2021). Rani and Mishra (2021) studied FF Einstein weighted operators and a new MULTIMOORA method was proposed to solve MADM problems. Some other AOs for FFSs and IVFFSs are studied in Rani and Mishra (2022a, b) and Rani et al. (2022a). Based on Einstein operations, Kamaci et al. (2021) described the notion of interval-valued picture hesitant fuzzy set (IVPHFS) and its operational laws. Moreover, the authors derived some dynamic IVPHF AOs (based on Einstein operators) to aggregate these sets collected at different time periods. The authors proposed the new concepts, including product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy and discussed various of its desirable characteristics in the background of Lq-ROFGs which are based on the Einstein operator. Garg et al. (2020) merged the valuable properties of the FFS with the Yager operator and developed certain AOs, including FF Yager weighted average (FFYWA), FF Yager ordered weighted average (FFYOWA), FF Yager hybrid weighted average (FFYHWA), FF Yager weighted geometric (FFYWG), FF Yager ordered weighted geometric (FFYOWG), and FF Yager hybrid weighted geometric (FFYHWG) operators. Furthermore, new classes of linguistic FF Hamy mean operators, namely, the linguistic FF Hamy mean operator, the linguistic FF dual Hamy mean operator, the linguistic FF weighted Hamy mean operator, and the linguistic FF weighted dual Hamy mean operator were introduced by Akram et al. (2022a, b, c, d, e, f, g). Akram et al. (2022a, b, c, d, e, f, g) defined a triangular interval-valued Fermatean fuzzy number (TIVFFN) and discussed its arithmetic operations.

2.3 Hamacher AOs

Wei (2019) utilized Hamacher operations and power AOs to develop some PF Hamacher power AOs, including PF Hamacher power average (PFHPA) operator, PF Hamacher power geometric (PFHPG) operator, PF Hamacher power weighted average (PFHPWA) operator, PF Hamacher power weighted geometric (PFHPWG) operator, PF Hamacher power ordered weighted average (PFHPOWA) operator, PF Hamacher power ordered weighted geometric (PFHPOWG) operator, PF Hamacher power hybrid average (PFHPHA) operator and PF Hamacher power hybrid geometric (PFHPHG) operator. Wu and Wei (2017) proposed PF hamacher AOs and discussed their application in MADM. PF interaction AOs and their applications in MADM were proposed and discussed by Wei (2017). Certain new concepts of IF Einstein hybrid AOs and their application to MADM were proposed by Zhao and Wei (2013). Wang and Liu (2012) aggregated the IF data and information using Einstein AOs. PF interactive Hamacher power AOs to assess the express service quality using entropy weights were utilized in Wang et al. (2021). Hadi et al. (2021) deviced novel operations on FFSs by applying Hamacher T-conorm and T-norm and discussed their basic operations. Induced by the Hamacher operations and FFS, the authors proposed FF Hamacher arithmetic and geometric AOs. Shahzadi et al. (2021a, b) proposed various AOs, for instance, Dombi, Einstein, and Hamacher under FFSs and utilized these AOs in the MADM process to evaluate the given alternatives. Senapati and Chen (2021) introduced various novel IVPF AOs based on Hamacher triangular norms and discussed their application in MADM issues. Shahzadi et al. (2021a, b) defined Hamacher interactive hybrid weighted AOs under FFNs. Certain Hamacher AOs based on the IVIFNs and their application to MADM were proposed by Liu (2013). Other valuable contributions on Hamacher AOs can be seen from Donyatalab et al. (2020), Akram et al. (2022a, b, c, d, e, f, g) and Jan et al. (2021).

3 Preliminaries

This section recalls some definitions including IVPFS, FFS, IVFFS and score functions related to IVFFS.

Definition 1 (Senapati and Yager 2020) A FFS $\mathcal{T}$ on non-empty set $\mathcal{V}$ is given by

$$\mathcal{T} = \{\langle x, \rho_{\mathcal{T}}(x), \varrho_{\mathcal{T}}(x)\rangle\},$$

where $\rho_{\mathcal{T}} : \mathcal{V} \to [0,1]$, $\varrho_{\mathcal{T}} : \mathcal{V} \to [0,1]$ and $\varpi_{\mathcal{T}}(x) = \sqrt[3]{1-(\rho_{\mathcal{T}}(x))^3-(\varrho_{\mathcal{T}}(x))^3}$ specify MD, NMD and InD, respectively. FFNs are components of the FFS.

Definition 2 (Peng and Yang 2016) An IVPFS $\mathcal{P}$ on non-empty set $\mathcal{V}$ is given by

$$\mathcal{P} = \{\langle x, [\rho^{lb}_{\mathcal{P}}(x), \rho^{ub}_{\mathcal{P}}(x)], [\varrho^{lb}_{\mathcal{P}}(x), \varrho^{ub}_{\mathcal{P}}(x)]\rangle\},$$

where $0 \le \rho^{lb}_{\mathcal{P}}(x) \le \rho^{ub}_{\mathcal{P}}(x) \le 1$, $0 \le \varrho^{lb}_{\mathcal{P}}(x) \le \varrho^{ub}_{\mathcal{P}}(x) \le 1$ and $(\rho^{ub}_{\mathcal{P}}(x))^3 + (\varrho^{ub}_{\mathcal{P}}(x))^3 \le 1$. $\rho_{\mathcal{P}}(x) = [\rho^{lb}_{\mathcal{P}}(x), \rho^{ub}_{\mathcal{P}}(x)]$ and $\varrho_{\mathcal{P}}(x) = [\varrho^{lb}_{\mathcal{P}}(x), \varrho^{ub}_{\mathcal{P}}(x)]$ symbolize the interval valued membership and non-membership degree respectively. $\varpi_{\mathcal{P}}(x) = [\varpi^{lb}_{\mathcal{P}}(x), \varpi^{ub}_{\mathcal{P}}(x)]$ is the interval valued hesitancy degree, where $\varpi^{lb}_{\mathcal{P}}(x) = \sqrt[3]{1-(\rho^{lb}_{\mathcal{P}}(x))^3-(\varrho^{lb}_{\mathcal{P}}(x))^3}$ and $\varpi^{ub}_{\mathcal{P}}(x) = \sqrt[3]{1-(\rho^{ub}_{\mathcal{P}}(x))^3-(\varrho^{ub}_{\mathcal{P}}(x))^3}$.

Definition 3 (Rani and Mishra 2022a, b) An IVFFS $\mathcal{G}$ on non-empty set $\mathcal{V}$ is given by

$$\mathcal{G} = \{\langle x, [\rho^{lb}_{\mathcal{G}}(x), \rho^{ub}_{\mathcal{G}}(x)], [\varrho^{lb}_{\mathcal{G}}(x), \varrho^{ub}_{\mathcal{G}}(x)]\rangle\},$$

where $0 \le \rho^{lb}_{\mathcal{T}}(x) \le \rho^{ub}_{\mathcal{T}}(x) \le 1$, $0 \le \varrho^{lb}_{\mathcal{T}}(x) \le \varrho^{ub}_{\mathcal{T}}(x) \le 1$ and $(\rho^{ub}_{\mathcal{T}}(x))^3 + (\varrho^{ub}_{\mathcal{T}}(x))^3 \le 1$. $\rho_{\mathcal{T}}(x) = [\rho^{lb}_{\mathcal{T}}(x), \rho^{ub}_{\mathcal{T}}(x)]$ and $\varrho_{\mathcal{T}}(x) = [\varrho^{lb}_{\mathcal{T}}(x), \varrho^{ub}_{\mathcal{T}}(x)]$ symbolize the interval valued membership and non-membership degree respectively. $\varpi_{\mathcal{G}}(x) = [\varpi^{lb}_{\mathcal{G}}(x), \varpi^{ub}_{\mathcal{G}}(x)]$ is the interval valued hesitancy degree, where $\varpi^{lb}_{\mathcal{G}}(x) = \sqrt[3]{1-(\rho^{lb}_{\mathcal{G}}(x))^3-(\varrho^{lb}_{\mathcal{G}}(x))^3}$ and $\varpi^{ub}_{\mathcal{G}}(x) = \sqrt[3]{1-(\rho^{ub}_{\mathcal{G}}(x))^3-(\varrho^{ub}_{\mathcal{G}}(x))^3}$.

Definition 4 (Rani and Mishra 2022a, b) The score function and accuracy function for IVFFN $\mathcal{G} = \langle[\rho^{lb}_{\mathcal{G}}, \rho^{ub}_{\mathcal{G}}], [\varrho^{lb}_{\mathcal{G}}, \varrho^{lb}_{\mathcal{G}}]\rangle$ are represented by

$$\mathcal{S}(\mathcal{G}) = \frac{1}{2}[(\rho^{lb}_{\mathcal{G}})^3 + (\rho^{ub}_{\mathcal{G}})^3 - (\varrho^{lb}_{\mathcal{G}})^3 - (\varrho^{ub}_{\mathcal{G}})^3],$$
$$\mathcal{A}(\mathcal{G}) = \frac{1}{2}[(\rho^{lb}_{\mathcal{G}})^3 + (\rho^{ub}_{\mathcal{G}})^3 + (\varrho^{lb}_{\mathcal{G}})^3 + (\varrho^{ub}_{\mathcal{G}})^3],$$

where $S(\mathcal{G}) \in [-1,1]$ and $\mathcal{A}(\mathcal{G}) \in [0,1]$.

Definition 5 (Rani and Mishra 2022a, b) Consider two IVFFNs $\mathcal{G}_1 = \langle[\rho_{\mathcal{G}_1}^{lb}, \rho_{\mathcal{G}_1}^{ub}], [\varrho_{\mathcal{G}_1}^{lb}, \varrho_{\mathcal{G}_1}^{ub}]\rangle$ and $\mathcal{G}_2 = \langle[\rho_{\mathcal{G}_2}^{lb}, \rho_{\mathcal{G}_2}^{ub}], [\varrho_{\mathcal{G}_2}^{lb}, \varrho_{\mathcal{G}_2}^{ub}]\rangle$. Then

1. If $S(\mathcal{G}_1) < S(\mathcal{G}_2)$, then $\mathcal{G}_1 \prec \mathcal{G}_2$;
2. If $S(\mathcal{G}_1) > S(\mathcal{G}_2)$, then $\mathcal{G}_1 \succ \mathcal{G}_2$;
3. If $S(\mathcal{G}_1) = S(\mathcal{G}_2)$, then

(a) If $\mathcal{A}(\mathcal{G}_1) < \mathcal{A}(\mathcal{G}_2)$, then $\mathcal{G}_1 \prec \mathcal{G}_2$;

(b) If $\mathcal{A}(\mathcal{G}_1) > \mathcal{A}(\mathcal{G}_2)$, then $\mathcal{G}_1 \succ \mathcal{G}_2$;

(c) If $\mathcal{A}(\mathcal{G}_1) = \mathcal{A}(\mathcal{G}_2)$, then $\mathcal{G}_1 \sim \mathcal{G}_2$.

4 Hamacher operations for IVFFNs

The Hamacher operations for IVFFNs are given by:

Definition 6 Let $\mathcal{G} = \langle[\rho^{lb}, \rho^{ub}], [\varrho^{lb}, \varrho^{ub}]\rangle$, $\mathcal{G}_1 = \langle[\rho_1^{lb}, \rho_1^{ub}], [\varrho_1^{lb}, \varrho_1^{ub}]\rangle$ and $\mathcal{G}_2 = \langle[\rho_2^{lb}, \rho_2^{ub}], [\varrho_2^{lb}, \varrho_2^{ub}]\rangle$ be IVFFNs and $\gamma > 0$, then

(i)

$$\overline{\mathcal{G}} = \langle[\varrho_{\mathcal{G}}^{lb}, \varrho_{\mathcal{G}}^{ub}],\ [\rho_{\mathcal{G}}^{lb}, \rho_{\mathcal{G}}^{ub}]\rangle$$

(ii)

$$\mathcal{G}_1 \wedge_\epsilon \mathcal{G}_2 = \langle[\min\{\rho_1^{lb}, \rho_2^{lb}\}, \min\{\rho_1^{ub}, \rho_2^{ub}\}], [\max\{\varrho_1^{lb}, \varrho_2^{lb}\}, \max\{\varrho_1^{ub}, \varrho_2^{ub}\}]\rangle$$

(iii)

$$\mathcal{G}_1 \vee_\epsilon \mathcal{G}_2 = \langle[\max\{\rho_1^{lb}, \rho_2^{lb}\}, \max\{\rho_1^{ub}, \rho_2^{ub}\}], [\min\{\varrho_1^{lb}, \varrho_2^{lb}\}, \min\{\varrho_1^{ub}, \varrho_2^{ub}\}]\rangle$$

(iv)

$$\mathcal{G}_1 \oplus \mathcal{G}_2 = \Bigg\langle\Bigg[\sqrt[3]{\frac{(\rho_1^{lb})^3 + (\rho_2^{lb})^3 + (\rho_1^{lb})^3(\rho_2^{lb})^3 - (1-\gamma)(\rho_1^{lb})^3(\rho_2^{lb})^3}{1-(1-\gamma)(\rho_1^{lb})^3(\rho_2^{lb})^3}}, \sqrt[3]{\frac{(\rho_1^{ub})^3 + (\rho_2^{ub})^3 - (\rho_1^{ub})^3(\rho_2^{ub})^3 - (1-\gamma)(\rho_1^{ub})^3(\rho_2^{ub})^3}{1-(1-\gamma)(\rho_1^{ub})^3(\rho_2^{ub})^3}}\Bigg], \Bigg[\frac{\varrho_1^{lb}\varrho_2^{lb}}{\sqrt[3]{\gamma + (1-\gamma)((\varrho_1^{lb})^3 + (\varrho_2^{lb})^3 - (\varrho_1^{lb})^3(\varrho_2^{lb})^3)}}, \frac{\varrho_1^{ub}\varrho_2^{ub}}{\sqrt[3]{\gamma + (1-\gamma)((\varrho_1^{ub})^3 + (\varrho_2^{ub})^3 - (\varrho_1^{ub})^3(\varrho_2^{ub})^3)}}\Bigg]\Bigg\rangle$$

(v)

$$\mathcal{G}_1 \otimes \mathcal{G}_2 = \Bigg\langle\Bigg[\frac{\rho_1^{lb}\rho_2^{lb}}{\sqrt[3]{\gamma + (1-\gamma)((\rho_1^{lb})^3 + (\rho_2^{lb})^3 - (\rho_1^{lb})^3(\rho_2^{lb})^3)}}, \frac{\rho_1^{ub}\rho_2^{ub}}{\sqrt[3]{\gamma + (1-\gamma)((\rho_1^{ub})^3 + (\rho_2^{ub})^3 - (\rho_1^{ub})^3(\rho_2^{ub})^3)}}\Bigg], \Bigg[\sqrt[3]{\frac{(\varrho_1^{lb})^3 + (\varrho_2^{lb})^3 + (\varrho_1^{lb})^3(\varrho_2^{lb})^3 - (1-\gamma)(\varrho_1^{lb})^3(\varrho_2^{lb})^3}{1-(1-\gamma)(\varrho_1^{lb})^3(\varrho_2^{lb})^3}}, \sqrt[3]{\frac{(\varrho_1^{ub})^3 + (\varrho_2^{ub})^3 - (\varrho_1^{ub})^3(\varrho_2^{ub})^3 - (1-\gamma)(\varrho_1^{ub})^3(\varrho_2^{ub})^3}{1-(1-\gamma)(\varrho_1^{ub})^3(\varrho_2^{ub})^3}}\Bigg]\Bigg\rangle$$

(vi)

$$\lambda\mathcal{G} = \Bigg\langle\Bigg[\sqrt[3]{\frac{(1+(\gamma-1)(\rho^{lb})^3)^\lambda - (1-(\rho^{lb})^3)^\lambda}{(1+(\gamma-1)(\rho^{lb})^3)^\lambda + (\gamma-1)(1-(\rho^{lb})^3)^\lambda}}, \sqrt[3]{\frac{(1+(\gamma-1)(\rho^{ub})^3)^\lambda - (1-(\rho^{ub})^3)^\lambda}{(1+(\gamma-1)(\rho^{ub})^3)^\lambda + (\gamma-1)(1-(\rho^{ub})^3)^\lambda}}\Bigg], \Bigg[\frac{\sqrt[3]{\gamma}(\varrho^{lb})^\lambda}{\sqrt[3]{(1+(\gamma-1)(1-(\varrho^{lb})^3))^\lambda + (\gamma-1)((\varrho^{lb})^3)^\lambda}}, \frac{\sqrt[3]{\gamma}(\varrho^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(1-(\varrho^{ub})^3))^\lambda + (\gamma-1)((\varrho^{ub})^3)^\lambda}}\Bigg]\Bigg\rangle$$

(vii)

$$\mathcal{G}^\lambda = \Bigg\langle\Bigg[\frac{\sqrt[3]{\gamma}(\rho^{lb})^\lambda}{\sqrt[3]{(1+(\gamma-1)(1-(\rho^{lb})^3))^\lambda + (\gamma-1)((\rho^{lb})^3)^\lambda}}, \frac{\sqrt[3]{\gamma}(\rho^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(1-(\rho^{ub})^3))^\lambda + (\gamma-1)((\rho^{ub})^3)^\lambda}}\Bigg], \Bigg[\sqrt[3]{\frac{(1+(\gamma-1)(\varrho^{lb})^3)^\lambda - (1-(\varrho^{lb})^3)^\lambda}{(1+(\gamma-1)(\varrho^{lb})^3)^\lambda + (\gamma-1)(1-(\varrho^{lb})^3)^\lambda}}, \sqrt[3]{\frac{(1+(\gamma-1)(\varrho^{ub})^3)^\lambda - (1-(\varrho^{ub})^3)^\lambda}{(1+(\gamma-1)(\varrho^{ub})^3)^\lambda + (\gamma-1)(1-(\varrho^{ub})^3)^\lambda}}\Bigg]\Bigg\rangle$$

Theorem 1 *Suppose $\mathcal{G} = \langle[\rho_{\mathcal{G}}^{lb}, \rho_{\mathcal{G}}^{ub}], [\varrho_{\mathcal{G}}^{lb}, \varrho_{\mathcal{G}}^{ub}]\rangle$, $\mathcal{G}_1 = \langle[\rho_1^{lb}, \rho_1^{ub}], [\varrho_1^{lb}, \varrho_1^{ub}]\rangle$ and $\mathcal{G}_2 = \langle[\rho_2^{lb}, \rho_2^{ub}], [\varrho_2^{lb}, \varrho_2^{ub}]\rangle$ are three IVFFNs; then, $\mathcal{G}_3 = \mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_4 = \lambda\mathcal{G}$ are also IVFFNs.*

Proof As $\mathcal{G}$ is an IVFFN and $\lambda > 0$, therefore, $0 \leq \rho_{\mathcal{G}}^{lb}, \rho_{\mathcal{G}}^{ub}, \varrho_{\mathcal{G}}^{lb}, \varrho_{\mathcal{G}}^{ub} \leq 1$ and $0 \leq (\rho_{\mathcal{G}}^{ub})^3 + (\varrho_{\mathcal{G}}^{ub})^3 \leq 1$, therefore $1 - (\rho_{\mathcal{G}}^{ub})^3 \geq (\varrho_{\mathcal{G}}^{ub})^3 \geq 0$, $1 - (\varrho_{\mathcal{G}}^{ub})^3 \geq (\rho_{\mathcal{G}}^{ub})^3 \geq 0$, and $(1 - (\rho_{\mathcal{G}}^{ub})^3)^\lambda \geq (\varrho_{\mathcal{G}}^{ub})^3$, then

$$\sqrt[3]{\frac{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda - (1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda + (\gamma-1)(1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}} \leq \sqrt[3]{\frac{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda - (\gamma-1)(\varrho_{\mathcal{G}}^{ub})^{3\lambda}}{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda + (\varrho_{\mathcal{G}}^{ub})^3)^\lambda}}$$

and

$$\frac{\sqrt[3]{\gamma}(\varrho_{\mathcal{G}}^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(\varrho_{\mathcal{G}}^{ub})^3)^\lambda + (\gamma-1)((\varrho_{\mathcal{G}}^{ub})^3)^\lambda}} \leq \frac{\sqrt[3]{\gamma}(\varrho_{\mathcal{G}}^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda + ((\varrho_{\mathcal{G}}^{ub})^3)^\lambda}}.$$

Thus,

$$\left(\sqrt[3]{\frac{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda-(1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda+(\gamma-1)(1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}}\right)^3 +\left(\frac{\sqrt[3]{\gamma}(\varrho_{\mathcal{G}}^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(\varrho_{\mathcal{G}}^{ub})^3)^\lambda+(\gamma-1)((\varrho_{\mathcal{G}}^{ub})^3)^\lambda}}\right)^3 \leq 1.$$

Furthermore,

$$\left(\sqrt[3]{\frac{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda-(1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}{(1+(\gamma-1)(\rho_{\mathcal{G}}^{ub})^3)^\lambda+(\gamma-1)(1-(\rho_{\mathcal{G}}^{ub})^3)^\lambda}}\right)^3 +\left(\frac{\sqrt[3]{\gamma}(\varrho_{\mathcal{G}}^{ub})^\lambda}{\sqrt[3]{(1+(\gamma-1)(\varrho_{\mathcal{G}}^{ub})^3)^\lambda+(\gamma-1)((\varrho_{\mathcal{G}}^{ub})^3)^\lambda}}\right)^3 = 0$$

iff $\rho_{\mathcal{G}}^{ub}=\varrho_{\mathcal{G}}^{ub}=0$. Thus, result is proved. □

Theorem 2 *Suppose* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then*

(i) $\mathcal{G}_1 \oplus \mathcal{G}_2 = \mathcal{G}_2 \oplus \mathcal{G}_1$
(ii) $\mathcal{G}_1 \otimes \mathcal{G}_2 = \mathcal{G}_1 \otimes \mathcal{G}_2$
(iii) $\lambda(\mathcal{G}_1 \oplus \mathcal{G}_2) = \lambda\mathcal{G}_1 \oplus \lambda\mathcal{G}_2$
(iv) $(\mathcal{G}_1 \otimes \mathcal{G}_2)^\lambda = \mathcal{G}_1^\lambda \otimes \mathcal{G}_2^\lambda$
(v) $\lambda_1\mathcal{G} \oplus \lambda_2\mathcal{G} = (\lambda_1+\lambda_2)\mathcal{G}$
(vi) $\mathcal{G}^{\lambda_1} \otimes \mathcal{G}^{\lambda_2} = \mathcal{G}^{(\lambda_1+\lambda_2)}$

Proof □

Theorem 3 *Suppose* $\mathcal{G}_1 = \langle\rho_1, \varrho_1\rangle$ *and* $\mathcal{G}_2 = \langle\rho_2, \varrho_2\rangle$ *are IVFFNs, then*

(i) $\mathcal{G}_1^c \wedge \mathcal{G}_2^c = (\mathcal{G}_1 \vee \mathcal{G}_2)^c$
(ii) $\mathcal{G}_1^c \vee \mathcal{G}_2^c = (\mathcal{G}_1 \wedge \mathcal{G}_2)^c$
(iii) $\mathcal{G}_1^c \oplus \mathcal{G}_2^c = (\mathcal{G}_1 \otimes \mathcal{G}_2)^c$
(iv) $\mathcal{G}_1^c \otimes \mathcal{G}_2^c = (\mathcal{G}_1 \oplus \mathcal{G}_2)^c$
(v) $(\mathcal{G}_1 \vee \mathcal{G}_2) \oplus (\mathcal{G}_1 \wedge \mathcal{G}_2) = \mathcal{G}_1 \oplus \mathcal{G}_2$
(vi) $(\mathcal{G}_1 \vee \mathcal{G}_2) \otimes (\mathcal{G}_1 \wedge \mathcal{G}_2) = \mathcal{G}_1 \otimes \mathcal{G}_2$

Proof It is obvious. □

Theorem 4 *Suppose* $\mathcal{G}_1 = \langle\rho_1, \varrho_1\rangle$, $\mathcal{G}_2 = \langle\rho_2, \varrho_2\rangle$ *and* $\mathcal{G}_3 = \langle\rho_3, \varrho_3\rangle$ *are three IVFFNs, then*

(i) $(\mathcal{G}_1 \vee \mathcal{G}_2) \wedge \mathcal{G}_3 = (\mathcal{G}_1 \wedge \mathcal{G}_3) \vee (\mathcal{G}_2 \wedge \mathcal{G}_3)$
(ii) $(\mathcal{G}_1 \wedge \mathcal{G}_2) \vee \mathcal{G}_3 = (\mathcal{G}_1 \vee \mathcal{G}_3) \wedge (\mathcal{G}_2 \vee \mathcal{G}_3)$
(iii) $(\mathcal{G}_1 \vee \mathcal{G}_2) \oplus \mathcal{G}_3 = (\mathcal{G}_1 \oplus \mathcal{G}_3) \vee (\mathcal{G}_2 \oplus \mathcal{G}_3)$
(iv) $(\mathcal{G}_1 \wedge \mathcal{G}_2) \oplus \mathcal{G}_3 = (\mathcal{G}_1 \oplus \mathcal{G}_3) \wedge (\mathcal{G}_2 \oplus \mathcal{G}_3)$
(v) $(\mathcal{G}_1 \vee \mathcal{G}_2) \otimes \mathcal{G}_3 = (\mathcal{G}_1 \otimes \mathcal{G}_3) \vee (\mathcal{G}_2 \otimes \mathcal{G}_3)$
(vi) $(\mathcal{G}_1 \wedge_\epsilon \mathcal{G}_2) \otimes_\epsilon \mathcal{G}_3 = (\mathcal{G}_1 \otimes_\epsilon \mathcal{G}_3) \wedge (\mathcal{G}_2 \otimes \mathcal{G}_3)$

Proof The proof is trivial, so we omit it. □

5 Some Hamacher weighted averaging operators based on IVFFSs

The interval valued Fermatean fuzzy Hamacher weighted averaging operators are given as:

Definition 7 Suppose $\mathcal{G}_{\mathrm{j}} = \langle[\rho_{\mathrm{j}}^{lb}, \rho_{\mathrm{j}}^{ub}], [\varrho_{\mathrm{j}}^{lb}, \varrho_{\mathrm{j}}^{ub}]\rangle (\mathrm{j} = 1, 2, \ldots, \mathfrak{s})$ is a collection of IVFFNs and δ_{j} is weight vector (WV) of $\mathcal{G}_{\mathrm{j}}$ with $\delta_{\mathrm{j}} > 0$, $\sum_{\mathrm{j}=1}^{\mathfrak{s}} \delta_{\mathrm{j}} = 1$ and $\mathrm{IVFFHWA}: \mathcal{Q}^{\mathfrak{s}} \to \mathcal{Q}$ such that

$$\mathrm{IVFFHWA}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = \delta_1 \cdot_\epsilon \mathcal{G}_1 \oplus_\epsilon \delta_2 \cdot_\epsilon \mathcal{G}_2 \oplus_\epsilon \cdots \oplus_\epsilon \delta_{\mathfrak{s}} \cdot_\epsilon \mathcal{G}_{\mathfrak{s}}. \quad (1)$$

Suppose $\delta_{\mathrm{j}} = 1/\mathfrak{s}$, $\forall$ j, then IVFFHWA operator reduces to IVFFWA operator

$$\mathrm{IVFFWA}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = \frac{1}{\mathfrak{s}}(\mathcal{G}_1 \oplus_\epsilon \mathcal{G}_2 \oplus_\epsilon \cdots \oplus_\epsilon \mathcal{G}_{\mathfrak{s}}).$$

Theorem 5 *Suppose* $\mathcal{G}_{\mathrm{j}} = \langle[\rho_{\mathrm{j}}^{lb}, \rho_{\mathrm{j}}^{ub}], [\varrho_{\mathrm{j}}^{lb}, \varrho_{\mathrm{j}}^{ub}]\rangle$ *are IVFFNs, then aggregated value by applying Eq. 1 is*
$\mathrm{IVFFHWA}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$

$$= \left\langle \left[\sqrt[3]{\frac{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-(\rho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}} + (\gamma-1)\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-(\rho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}}}}, \sqrt[3]{\frac{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-(\rho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}} + (\gamma-1)\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-(\rho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}}}} \right], \left[\frac{\sqrt[3]{\gamma}\prod_{\mathrm{j}=1}^{\mathfrak{s}}(\varrho_{\mathrm{j}}^{lb})^{\delta_{\mathrm{j}}}}{\sqrt[3]{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}} + (\gamma-1)\prod_{\mathrm{j}=1}^{\mathfrak{s}}((\varrho_{\mathrm{j}}^{lb})^3)^{\delta_{\mathrm{j}}}}}, \frac{\sqrt[3]{\gamma}\prod_{\mathrm{j}=1}^{\mathfrak{s}}(\varrho_{\mathrm{j}}^{ub})^{\delta_{\mathrm{j}}}}{\sqrt[3]{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}} + (\gamma-1)\prod_{\mathrm{j}=1}^{\mathfrak{s}}((\varrho_{\mathrm{j}}^{ub})^3)^{\delta_{\mathrm{j}}}}} \right] \right\rangle \quad (2)$$

Proof We will use the mathematical induction to verify Eq. (2).
For $\mathfrak{s} = 1$, $\delta = \delta_1 = 1$, Then

$$\begin{aligned} \mathrm{FFEWA}(\mathcal{G}_1) &= \delta_1 \mathcal{G}_1 \\ &= \mathcal{G}_1 \\ &= \left\langle \left[\sqrt[3]{\frac{(1+(\gamma-1)(\rho^{lb})^3)-(1-(\rho^{lb})^3)}{(1+(\gamma-1)(\rho^{lb})^3)+(\gamma-1)(1-(\rho^{lb})^3)}}, \sqrt[3]{\frac{(1+(\gamma-1)(\rho^{ub})^3)-(1-(\rho^{ub})^3)}{(1+(\gamma-1)(\rho^{ub})^3)+(\gamma-1)(1-(\rho^{ub})^3)}} \right], \right. \\ &\quad \left. \left[\frac{\sqrt[3]{\gamma}(\varrho^{lb})}{\sqrt[3]{(1+(\gamma-1)(1-(\varrho^{lb})^3))+(\gamma-1)((\varrho^{lb})^3)}}, \frac{\sqrt[3]{\gamma}(\varrho^{ub})}{\sqrt[3]{(1+(\gamma-1)(1-(\varrho^{ub})^3))+(\gamma-1)((\varrho^{ub})^3)}} \right] \right\rangle. \end{aligned}$$

So, for $\mathfrak{s} = 1$, Eq. (2) is true.
Suppose, for $\mathfrak{s} = k$, result holds.

IVFFHWA$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k)$

$$= \Bigg\langle \Bigg[\sqrt[3]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} - \prod_{j=1}^{k}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}(1-(\rho_j^{lb})^3)^{\delta_j}}}, \sqrt[3]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{k}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}(1-(\rho_j^{ub})^3)^{\delta_j}}} \Bigg], \Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{k}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}((\varrho_j^{lb})^3)^{\delta_j}}}, \frac{\sqrt[3]{\gamma}\prod_{j=1}^{k}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}((\varrho_j^{ub})^3)^{\delta_j}}} \Bigg] \Bigg\rangle.$$

For $\mathfrak{s} = k+1$,
IVFFEWA$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k+1})$

$$= \Bigg\langle \Bigg[\sqrt[3]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} - \prod_{j=1}^{k}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}(1-(\rho_j^{lb})^3)^{\delta_j}}}, \sqrt[3]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{k}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{k}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}(1-(\rho_j^{ub})^3)^{\delta_j}}} \Bigg], \Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{k}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}((\varrho_j^{lb})^3)^{\delta_j}}},$$

$$\frac{\sqrt[3]{\gamma}\prod_{j=1}^{k}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k}((\varrho_j^{ub})^3)^{\delta_j}}} \Bigg] \Bigg\rangle \oplus \Bigg\langle \Bigg[\sqrt[3]{\frac{(1+(\gamma-1)(\rho_{k+1}^{lb})^3)^{\delta_{k+1}} - (1-(\rho_{k+1}^{lb})^3)^{\delta_{k+1}}}{(1+(\gamma-1)(\rho_{k+1}^{lb})^3)^{\delta_{k+1}} + (\gamma-1)(1-(\rho_{k+1}^{lb})^3)^{\delta_{k+1}}}}, \sqrt[3]{\frac{(1+(\gamma-1)(\rho_{k+1}^{ub})^3)^{\delta_{k+1}} - (1-(\rho_{k+1}^{ub})^3)^{\delta_{k+1}}}{(1+(\gamma-1)(\rho_{k+1}^{ub})^3)^{\delta_{k+1}} + (\gamma-1)(1-(\rho_{k+1}^{ub})^3)^{\delta_{k+1}}}} \Bigg], \Bigg[\frac{\sqrt[3]{\gamma}(\varrho_{k+1}^{lb})^{\delta_{k+1}}}{\sqrt[3]{(1+(\gamma-1)(\varrho_{k+1}^{lb})^3)^{\delta_{k+1}} + (\gamma-1)((\varrho_{k+1}^{lb})^3)^{\delta_{k+1}}}}, \frac{\sqrt[3]{\gamma}(\varrho_{k+1}^{ub})^{\delta_{k+1}}}{\sqrt[3]{(1+(\gamma-1)(\varrho_{k+1}^{ub})^3)^{\delta_{k+1}} + (\gamma-1)((\varrho_{k+1}^{ub})^3)^{\delta_{k+1}}}} \Bigg] \Bigg\rangle$$

$$= \Bigg\langle \Bigg[\sqrt[3]{\frac{\prod_{j=1}^{k+1}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} - \prod_{j=1}^{k+1}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{k+1}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k+1}(1-(\rho_j^{lb})^3)^{\delta_j}}}, \sqrt[3]{\frac{\prod_{j=1}^{k+1}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{k+1}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{k+1}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k+1}(1-(\rho_j^{ub})^3)^{\delta_j}}} \Bigg], \Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{k+1}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k+1}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k+1}((\varrho_j^{lb})^3)^{\delta_j}}}, \frac{\sqrt[3]{\gamma}\prod_{j=1}^{k+1}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{k+1}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{k+1}((\varrho_j^{ub})^3)^{\delta_j}}} \Bigg] \Bigg\rangle.$$

Therefore, result is verified for $\mathfrak{s} = k+1$. So, it holds, $\forall\, \mathfrak{s}$. □

Lemma 1 *Suppose* $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ *and* $\sum_{j=1}^{\mathfrak{s}} \delta_j = 1$, *then*

$$\prod_{j=1}^{s} \mathcal{G}_j^{\delta_j} \le \sum_{j=1}^{\mathfrak{s}} \delta_j \mathcal{G}_j,$$

where equality holds if and only if $\mathcal{G}_1 = \mathcal{G}_2 = \cdots = \mathcal{G}_{\mathfrak{s}}$.

Theorem 6 *Suppose* $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ *are IVFFNs, then IVFFHWA*$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$ *is also an IVFFN.*

Proof As $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ are IVFFNs, therefore $0 \le \rho_j^{lb}, \rho_j^{ub}, \varrho_j^{lb}, \varrho_j^{ub} \le 1$ and $0 \le (\rho_j^{ub})^3 + (\varrho_j^{ub})^3 \le 1$. Now,

$$\begin{aligned}
&\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}} \\
&= 1 - \frac{\gamma\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}} \\
&\le 1 - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j} \le 1.
\end{aligned} \tag{3}$$

Also, $(1+(\gamma-1)(\rho_j^{ub})^3) \ge (1-(\rho_j^{ub})^3) \Rightarrow \prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3) - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3) \ge 0$. Therefore,

$$\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}} \ge 0.$$

$$\Rightarrow \sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}} \ge 0.$$

Hence, $0 \le \rho_{\mathrm{IVFFEWA}}^{ub} \le 1$.
Moreover,

$$\begin{aligned}
&\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}} \\
&\le \frac{\gamma\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}} \\
&\le \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j} \\
&\le 1.
\end{aligned}$$

Also,

$$\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j} \ge 0$$

$$\Leftrightarrow \frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}} \ge 0.$$

Hence,

$$0 \le \frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}} \le 1.$$

Hence, $0 \le \varrho_{\mathrm{IVFFHWA}}^{ub} \le 1$. Moreover,

$$(\rho_{\mathrm{IVFFHWA}}^{ub})^3 + (\varrho_{\mathrm{IVFFHWA}}^{ub})^3$$

$$= \frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} - \prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j} + (\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}$$

$$+\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}$$
$$\leq 1-\frac{\gamma\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}$$
$$+\frac{\gamma\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}$$
$$=1.$$

Hence, IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$ is an IVFFN. □

Corollary 1 *The relation between IVFFHWA and IVFFWA operators is given by:*
IVFFEWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq$ IVFFWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$.

Proof Suppose IVFFEWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\langle[(\rho_{\mathcal{G}}^{ub})^\beta,(\rho_{\mathcal{G}}^{lb})^\beta],[(\varrho_{\mathcal{G}}^{ub})^\beta,(\varrho_{\mathcal{G}}^{lb})^\beta]\rangle=\mathcal{G}^\beta$ and IVFFWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\langle[\rho_{\mathcal{G}}^{ub},\rho_{\mathcal{G}}^{lb}],[\varrho_{\mathcal{G}}^{ub},\varrho_{\mathcal{G}}^{lb}]\rangle=\mathcal{G}$. As $\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}\leq\sum_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\sum_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}=2$, so by Eq. (3), we have

$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}}$$
$$\leq\sqrt[3]{1-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}$$
$$\Leftrightarrow(\rho_{\mathcal{G}}^{ub})^\beta\leq\rho_{\mathcal{G}}^{ub},$$

equality holds iff $\rho_1^{ub}=\rho_2^{ub}=\cdots=\rho_{\mathfrak{s}}^{ub}$.
Similarly, as $\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}\leq\sum_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\sum_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}=2$, so by Eq. (3), we have

$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}}$$
$$\leq\sqrt[3]{1-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}$$
$$\Leftrightarrow(\rho_{\mathcal{G}}^{lb})^\beta\leq\rho_{\mathcal{G}}^{lb},$$

equality holds iff $\rho_1^{lb}=\rho_2^{lb}=\cdots=\rho_{\mathfrak{s}}^{lb}$.
As,

$$\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}$$
$$\geq\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\sum_{j=1}^{\mathfrak{s}}\delta_j(1+(\gamma-1)(\varrho_j^{ub})^3)+(\gamma-1)\sum_{j=1}^{\mathfrak{s}}\delta_j(\varrho_j^{ub})^3}$$
$$\geq\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}$$
$$\Rightarrow\sqrt[3]{\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}}$$
$$\geq\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{ub})^{\delta_j}$$
$$\Rightarrow(\varrho_{\mathcal{G}}^{ub})^\beta\geq\varrho_{\mathcal{G}}^{ub},$$

equality holds iff $\varrho_1^{ub}=\varrho_2^{ub}=\cdots=\varrho_{\mathfrak{s}}^{ub}$.
Similarly, as $\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}$

$$\geq\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}{\sum_{j=1}^{\mathfrak{s}}\delta_j(1+(\gamma-1)(\varrho_j^{lb})^3)+(\gamma-1)\sum_{j=1}^{\mathfrak{s}}\delta_j(\varrho_j^{lb})^3}$$
$$\geq\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}$$
$$\Rightarrow\sqrt[3]{\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}}$$
$$\geq\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{lb})^{\delta_j}$$
$$\Rightarrow(\varrho_{\mathcal{G}}^{lb})^\beta\geq\varrho_{\mathcal{G}}^{lb},$$

equality holds iff $\varrho_1^{lb}=\varrho_2^{lb}=\cdots=\varrho_{\mathfrak{s}}^{lb}$.
Hence,

$$\mathcal{S}(\mathcal{G}^\beta)=\frac{1}{2}\Big(((\rho_{\mathcal{G}}^{ub})^\beta)^3+((\rho_{\mathcal{G}}^{lb})^\beta)^3-((\varrho_{\mathcal{G}}^{ub})^\beta)^3-((\varrho_{\mathcal{G}}^{lb})^\beta)^3\Big)$$
$$\leq\frac{1}{2}\Big((\rho_{\mathcal{G}}^{ub})^3+(\rho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{lb})^3\Big)$$
$$=\mathcal{S}(\mathcal{G}).$$

If $\mathcal{S}(\mathcal{G}^\beta)<\mathcal{S}(\mathcal{G})$, then IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})<$ IVFFWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$. If $\mathcal{S}(\mathcal{G}^\beta)=\mathcal{S}(\mathcal{G})$, that is, $\frac{1}{2}\Big(((\rho_{\mathcal{G}}^{ub})^\beta)^3+((\rho_{\mathcal{G}}^{lb})^\beta)^3-((\varrho_{\mathcal{G}}^{ub})^\beta)^3-((\varrho_{\mathcal{G}}^{lb})^\beta)^3\Big)=\frac{1}{2}\Big((\rho_{\mathcal{G}}^{ub})^3+(\rho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{lb})^3\Big)$, then by condition $(\rho_{\mathcal{G}}^{ub})^\beta\leq\rho_{\mathcal{G}}^{ub}$, $(\rho_{\mathcal{G}}^{lb})^\beta\leq\rho_{\mathcal{G}}^{lb}$ and $(\varrho_{\mathcal{G}}^{ub})^\beta\geq\varrho_{\mathcal{G}}^{ub}$, $(\varrho_{\mathcal{G}}^{lb})^\beta\geq\varrho_{\mathcal{G}}^{lb}$; thus, the accuracy function $\mathcal{A}(\mathcal{G}^\beta)=\frac{1}{2}\Big(((\rho_{\mathcal{G}}^{ub})^\beta)^3+((\rho_{\mathcal{G}}^{lb})^\beta)^3+((\varrho_{\mathcal{G}}^{ub})^\beta)^3+((\varrho_{\mathcal{G}}^{lb})^\beta)^3\Big)=\frac{1}{2}\Big((\rho_{\mathcal{G}}^{ub})^3+(\rho_{\mathcal{G}}^{ub})^3+(\varrho_{\mathcal{G}}^{ub})^3+(\varrho_{\mathcal{G}}^{lb})^3\Big)=\mathcal{A}(\mathcal{G})$. Therefore IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=$ IVFFWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$. This implies, IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq$ IVFFWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$, equality holds iff $\mathcal{G}_1=\mathcal{G}_2=\cdots=\mathcal{G}_{\mathfrak{s}}$. □

Proposition 1 *Suppose* $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle$ *are IVFFNs and* $\delta_j\in[0,1]$ *with* $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$.

(i) *Idempotency: If* $\mathcal{G}_j=\mathcal{G}_o=\langle[\rho_o^{lb},\rho_o^{ub}],[\varrho_o^{lb},\varrho_o^{ub}]\rangle$ *for all* j, *then*
IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\mathcal{G}_o$.
(ii) *Boundedness: Let* $\mathcal{G}^-=([\min_j(\rho_j^{lb}),\min_j(\rho_j^{ub})],[\max_j(\varrho_j^{lb}),\max_j(\varrho_j^{ub})])$, $\mathcal{G}^+=([\max_j(\rho_j^{lb}),\max_j(\rho_j^{ub})],[\min_j(\varrho_j^{lb}),\min_j(\varrho_j^{ub})])$, *then*
$\mathcal{G}^-\leq$ IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq\mathcal{G}^+$.
(iii) *Monotonicity: When* $\mathcal{G}_j\leq\mathcal{P}_j,\forall$ j, *then*
IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq$ IVFFHWA$(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{\mathfrak{s}})$.

Proof (i). As $\mathcal{G}_j=\langle[\rho_o^{lb},\rho_o^{ub}],[\varrho_o^{lb},\varrho_o^{ub}]\rangle$ are IVFFNs, $\forall$ j, then
IVFFHWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_o^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_o^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_o^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_o^{lb})^3)^{\delta_j}}},$$
$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_o^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_o^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_o^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_o^{ub})^3)^{\delta_j}}}\Bigg],$$

$$\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_o^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_o^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_o^{lb})^3)^{\delta_j}}},$$
$$\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_o^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_o^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_o^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle$$
$$=\Bigg\langle\Bigg[\sqrt[3]{\frac{(1+(\gamma-1)(\rho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}-(1-(\rho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}{(1+(\gamma-1)(\rho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}+(\gamma-1)(1-(\rho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}},$$
$$\sqrt[3]{\frac{(1+(\gamma-1)(\rho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}-(1-(\rho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}{(1+(\gamma-1)(\rho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}+(\gamma-1)(1-(\rho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}}\Bigg],$$

$$\Bigg[\frac{\sqrt[3]{\gamma}(\varrho_o^{lb})^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}{\sqrt[3]{(1+(\gamma-1)(\varrho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}+(\gamma-1)((\varrho_o^{lb})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}},$$
$$\frac{\sqrt[3]{\gamma}(\varrho_o^{ub})^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}{\sqrt[3]{(1+(\gamma-1)(\varrho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}+(\gamma-1)((\varrho_o^{ub})^3)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}}}\Bigg]\Bigg\rangle$$
$$=\langle[\rho_o^{lb},\rho_o^{ub}],[\varrho_o^{lb},\varrho_o^{ub}]\rangle.$$

(ii). Let $f(x)=\frac{1-x}{1+(\gamma-1)x}, x\in[0,1]$, then $f'(x)=-\frac{\gamma}{(1+(\gamma-1)x)^2}<0$, so $f(x)$ is a decreasing function (DF). As $(\rho_{j,\min}^{ub})^3\le(\rho_j^{ub})^3\le(\rho_{j,\max}^{ub})^3, \forall\ j=1,2,\ldots,\mathfrak{s}$, then $f((\rho_{j,\max}^{ub})^3)\le f((\rho_j^{ub})^3)\le f((\rho_{j,\min}^{ub})^3), \forall$ j, that is, $\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\le\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\le\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}, \forall$ j. Let $\delta_j\in[0,1]$ and $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$, we have

$$\Big(\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)^{\delta_j}\le\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}\le\Big(\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)^{\delta_j}$$

$$\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)^{\delta_j}\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}$$
$$\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)^{\delta_j}$$
$$\Leftrightarrow\Big(\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)^{\sum_{j=1}^{\mathfrak{s}}\varpi_j}$$

$$\Leftrightarrow\Big(\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)\rho_j^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)$$

$$\Leftrightarrow(\gamma-1)\Big(\frac{1-(\rho_{j,\max}^{ub})^3}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)\le(\gamma-1)\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\varpi_j}$$
$$\le(\gamma-1)\Big(\frac{1-(\rho_{j,\min}^{ub})^3}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)$$

$$\Leftrightarrow\Big(\frac{\gamma}{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}\Big)\le1+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{\gamma}{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}\Big)$$
$$\Leftrightarrow\Big(\frac{1+(\gamma-1)(\rho_{j,\min}^{ub})^3}{\gamma}\Big)\le\frac{1}{1+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}}$$
$$\le\Big(\frac{1+(\gamma-1)(\rho_{j,\max}^{ub})^3}{\gamma}\Big)$$
$$\Leftrightarrow\Big(1+(\gamma-1)(\rho_{j,\min}^{ub})^3\Big)\le\frac{\gamma}{1+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}}$$
$$\le\Big(1+(\gamma-1)(\rho_{j,\max}^{ub})^3\Big)$$
$$\Leftrightarrow(\gamma-1)(\rho_{j,\min}^{ub})^3\le\frac{\gamma}{1+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1-(\rho_j^{ub})^3}{1+(\gamma-1)(\rho_j^{ub})^3}\Big)^{\delta_j}}-1$$
$$\le(\gamma-1)(\rho_{j,\max}^{ub})^3$$
$$\Leftrightarrow(\rho_{j,\min}^{ub})^3\le\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\varpi_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\varpi_j}}$$
$$\le(\rho_{j,\max}^{ub})^3.$$

Thus,

$$(\rho_{j,\min}^{ub})\le\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}}$$
$$\le(\rho_{j,\max}^{ub}).\tag{4}$$

Similarly,

$$(\rho_{j,\min}^{lb})\le\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}}$$
$$\le(\rho_{j,\max}^{lb}).\tag{5}$$

Suppose $k(y)=\frac{1+(\gamma-1)(1-y)}{y}$, $y\in(0,1]$, then

$$\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\max}^{lb})^3)}{(\varrho_{j,\max}^{lb})^3}\Big)\le\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\min}^{lb})^3)}{(\varrho_{j,\min}^{lb})^3}\Big)^{\delta_j}$$
$$\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\max}^{lb})^3)}{(\varrho_{j,\max}^{lb})^3}\Big)^{\delta_j}\le\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\min}^{lb})^3)}{(\varrho_{j,\min}^{lb})^3}\Big)^{\delta_j}$$
$$\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\max}^{lb})^3)}{(\varrho_{j,\max}^{lb})^3}\Big)^{\delta_j}\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}$$
$$\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\min}^{lb})^3)}{(\varrho_{j,\min}^{lb})^3}\Big)^{\delta_j}$$
$$\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\max}^{lb})^3)}{(\varrho_{j,\max}^{lb})^3}\Big)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\min}^{lb})^3)}{(\varrho_{j,\min}^{lb})^3}\Big)^{\sum_{j=1}^{\mathfrak{s}}\delta_j}$$
$$\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\max}^{lb})^3)}{(\varrho_{j,\max}^{lb})^3}\Big)\le\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}$$
$$\le\Big(\frac{1+(\gamma-1)(1-(\varrho_{j,\min}^{lb})^3)}{(\varrho_{j,\min}^{lb})^3}\Big)$$

$$\frac{\gamma}{(\varrho^{lb}_{j,\max})^3}-(\gamma-1)\leq\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}\leq\frac{\gamma}{(\varrho^{lb}_{j,\min})^3}-(\gamma-1)$$

$$(\varrho^{lb}_{j,\min})^3\leq\frac{\gamma}{\prod_{j=1}^{\mathfrak{s}}\Big(\frac{1+(\gamma-1)(1-(\varrho_j^{lb})^3)}{(\varrho_j^{lb})^3}\Big)^{\delta_j}+(\gamma-1)}\leq(\varrho^{lb}_{j,\max})^3$$

$$\Rightarrow(\varrho^{lb}_{j,\min})^3\leq\frac{\gamma\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}\leq(\varrho^{lb}_{j,\max})^3$$

$$\Rightarrow(\varrho^{lb}_{j,\min})\leq\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(1-(\varrho_j^{lb})^3))^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}}\leq(\varrho^{lb}_{j,\max}). \tag{6}$$

Similarly,

$$\Rightarrow(\varrho^{ub}_{j,\min})\leq\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(1-(\varrho_j^{ub})^3))^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}}\leq(\varrho^{ub}_{j,\max}). \tag{7}$$

Suppose $\text{IVFFEWA}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\mathcal{G}=\langle[\rho_{\mathcal{G}}^{lb},\rho_{\mathcal{G}}^{ub}],[\varrho_{\mathcal{G}}^{lb},\varrho_{\mathcal{G}}^{ub}]\rangle$, so from Eqs. (4), (5), (6) and (7),

$$\rho^{lb}_{\min}\leq\rho^{lb}_{\mathcal{G}}\leq\rho^{lb}_{\max},\quad\varrho^{lb}_{\min}\leq\varrho^{lb}_{\mathcal{G}}\leq\varrho^{lb}_{\max},$$
$$\rho^{ub}_{\min}\leq\rho^{ub}_{\mathcal{G}}\leq\rho^{ub}_{\max},\quad\varrho^{ub}_{\min}\leq\varrho^{ub}_{\mathcal{G}}\leq\varrho^{ub}_{\max},$$

where $\rho^{lb}_{\min}=\min_j\{(\rho_j^{lb})\}$, $\rho^{ub}_{\min}=\min_j\{(\rho_j^{ub})\}$, $\rho^{lb}_{\max}=\max_j\{(\rho_j^{lb})\}$, $\rho^{ub}_{\max}=\max_j\{(\rho_j^{ub})\}$, $\varrho^{lb}_{\min}=\min_j\{(\varrho_j^{lb})\}$, $\varrho^{ub}_{\min}=\min_j\{(\varrho_j^{ub})\}$, and $\varrho^{lb}_{\max}=\max_j\{(\varrho_j^{lb})\}$, $\varrho^{ub}_{\max}=\max_j\{(\varrho_j^{ub})\}$. So, $\mathcal{S}(\mathcal{G})=\frac{1}{2}[(\rho_{\mathcal{G}}^{lb})^3+(\rho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{lb})^3-(\varrho_{\mathcal{G}}^{ub})^3]\leq\frac{1}{2}[(\rho^{lb}_{\max})^3+(\rho^{ub}_{\max})^3-(\varrho^{lb}_{\min})^3-(\varrho^{ub}_{\min})^3]=\mathcal{S}(\mathcal{G}^+)$ and $\mathcal{S}(\mathcal{G})=\frac{1}{2}[(\rho_{\mathcal{G}}^{lb})^3+(\rho_{\mathcal{G}}^{ub})^3-(\varrho_{\mathcal{G}}^{lb})^3-(\varrho_{\mathcal{G}}^{ub})^3]\geq\frac{1}{2}[(\rho^{lb}_{\min})^3+(\rho^{ub}_{\min})^3-(\varrho^{lb}_{\max})^3-(\varrho^{ub}_{\max})^3]=\mathcal{S}(\mathcal{G}^-)$. As $S(\mathcal{G})<S(\mathcal{G}^+)$ and $S(\mathcal{G})>S(\mathcal{G}^-)$. Therefore

$$\mathcal{G}^-\leq\text{IVFFEWA}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq\mathcal{G}^+.$$

(iii). Straight forward. □

5.1 Some Hamacher ordered weighted averaging operators based on IVFFSs

Definition 8 Suppose $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle(j=1,2,\ldots,\mathfrak{s})$ is a collection of IVFFNs and δ_j is WV of $\mathcal{G}_j$ with $\delta_j>0$, $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$ and IVFFHOWA : $\mathcal{Q}^{\mathfrak{s}}\to\mathcal{Q}$ such that

$$IVFFHOWA(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\delta_1\mathcal{G}_{\varrho(1)}\oplus\delta_2\mathcal{G}_{\varrho(2)}\oplus\cdots\oplus\delta_{\mathfrak{s}}\mathcal{G}_{\varrho(\mathfrak{s})},$$

where $(\varrho(1),\varrho(2),\ldots,\varrho(\mathfrak{s}))$ is the permutation of $(j=1,2,\ldots,\mathfrak{s})$ such that $\mathcal{G}_{\varrho(j-1)}\geq\mathcal{G}_{\varrho(j)}$, $\forall\, j=1,2,\ldots,\mathfrak{s}$.

Theorem 7 *Suppose* $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle$ $\in$IVFFNs, *aggregated value by applying IVFFHOWA is an IVFFN and* $\text{IVFFHOWA}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho^{lb}_{\varrho(j)})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho^{lb}_{\varrho(j)})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho^{lb}_{\varrho(j)})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho^{lb}_{\varrho(j)})^3)^{\delta_j}}},\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho^{ub}_{\varrho(j)})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho^{ub}_{\varrho(j)})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho^{ub}_{\varrho(j)})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho^{ub}_{\varrho(j)})^3)^{\delta_j}}}\Bigg],$$
$$\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho^{lb}_{\varrho(j)})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)((\varrho^{lb}_{\varrho(j)})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho^{lb}_{\varrho(j)})^3)^{\delta_j}}},\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho^{ub}_{\varrho(j)})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)((\varrho^{ub}_{\varrho(j)})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho^{ub}_{\varrho(j)})^3)^{\delta_j}}}\Bigg]\Bigg\rangle. \tag{8}$$

Proof Similar to Theorem 5. □

Corollary 2 *The IVFFHOWA and IVFFOWA operators have the relation:*
$\text{IVFFHOWA}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})\leq\text{IVFFOWA}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$.

Remark 1 IVFFHOWA operators satisfy the properties of Idempotency, Boundedness and Monotonicity.

6 Hamacher weighted geometric operators based on IVFFSs

The interval valued Fermatean fuzzy Hamacher weighted geometric (IVFFHWG) operators are given as:

Definition 9 Suppose $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle(j=1,2,\ldots,\mathfrak{s})$ is a collection of IVFFNs and IVFFHWG : $\mathcal{Q}^{\mathfrak{s}}\to\mathcal{Q}$ such that

$$\text{IVFFHWG}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\delta_1\cdot_{\epsilon}\mathcal{G}_1\oplus_{\epsilon}\delta_2\cdot_{\epsilon}\mathcal{G}_2\oplus_{\epsilon}\cdots\oplus_{\epsilon}\delta_{\mathfrak{s}}\cdot_{\epsilon}\mathcal{G}_{\mathfrak{s}}. \tag{9}$$

Suppose $\delta_j=1/\mathfrak{s}$, $\forall$ j, then IVFFHWG operator reduces to IVFFWG operator

$$\text{IVFFWG}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\frac{1}{\mathfrak{s}}(\mathcal{G}_1\oplus_{\epsilon}\mathcal{G}_2\oplus_{\epsilon}\cdots\oplus_{\epsilon}\mathcal{G}_{\mathfrak{s}}).$$

Theorem 8 *Suppose* $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle$ *are IVFFNs, then aggregated value by applying Eq. (9) is* $\text{IVFFHWG}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\Bigg\langle\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_j^{lb})^3)^{\delta_j}}},\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_j^{ub})^3)^{\delta_j}}}\Bigg],$$

$$\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{lb})^3)^{\delta_j}}},$$
$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle \tag{10}$$

Proof Straight forward. □

Remark 2 IVFFHWG operators satisfy the properties of Idempotency, Boundedness and Monotonicity.

6.1 Hamacher ordered weighted geometric operators based on IVFFSs

Definition 10 Suppose $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle (j = 1, 2, \ldots, \mathfrak{s})$ is a collection of IVFFNs and δ_j is WV of $\mathcal{G}_j$ with $\delta_j > 0$, $\sum_{j=1}^{\mathfrak{s}} \delta_j = 1$ and IVFFHOWG : $\mathcal{Q}^{\mathfrak{s}} \to \mathcal{Q}$ such that

$$\text{IVFFHOWG}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = \delta_1\mathcal{G}_{\varrho(1)} \oplus \delta_2\mathcal{G}_{\varrho(2)} \oplus \cdots \oplus \delta_{\mathfrak{s}}\mathcal{G}_{\varrho(\mathfrak{s})},$$

where $(\varrho(1), \varrho(2), \ldots, \varrho(\mathfrak{s}))$ is the permutation of $(j = 1, 2, \ldots, \mathfrak{s})$ such that $\mathcal{G}_{\varrho(j-1)} \geq \mathcal{G}_{\varrho(j)}, \forall\, j = 1, 2, \ldots, \mathfrak{s}$.

Theorem 9 *Suppose $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ are IVFFNs, the aggregated value by applying IVFFHOWG is an IVFFN and*

IVFFHOWG$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$

$$= \Bigg\langle\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_{\varrho(j)}^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)((\rho_{\varrho(j)}^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_{\varrho(j)}^{lb})^3)^{\delta_j}}},$$
$$\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_{\varrho(j)}^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)((\rho_{\varrho(j)}^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_{\varrho(j)}^{ub})^3)^{\delta_j}}}\Bigg],$$
$$\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}}},$$
$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle. \tag{11}$$

Proof Similar to Theorem 5. □

Remark 3 IVFFHOWG operators satisfy the properties of Idempotency, Boundedness and Monotonicity.

7 Generalized IVFF Einstein weighted averaging operators

Definition 11 Suppose $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle (j = 1, 2, \ldots, \mathfrak{s})$ is a collection of IVFFNs and δ_j is WV of $\mathcal{G}_j$ with $\delta_j > 0$, $\sum_{j=1}^{\mathfrak{s}} \delta_j = 1$ and GIVFFEWA : $\mathcal{Q}^{\mathfrak{s}} \to \mathcal{Q}$ such that
GIVFFEWA$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = \left(\bigoplus_{j=1}^{\mathfrak{s}}(\delta_j \cdot_{\epsilon} \mathcal{G}_j^{\lambda})\right)^{\frac{1}{\lambda}}$, where $\lambda > 0$.

Particularly,

- If $\lambda = 1$, then GIVFFEWA becomes IVFFEWA.
- If $\delta = (1/\mathfrak{s}, 1/\mathfrak{s}, \ldots, 1/\mathfrak{s})^T$, then
GIVFFEWA$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = (\frac{1}{\mathfrak{s}} \cdot_{\epsilon} \bigoplus_{j=1}^{\mathfrak{s}} \mathcal{G}_j^{\lambda})^{\frac{1}{\lambda}}$.

Theorem 10 *Suppose $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle (j = 1, 2, \ldots, \mathfrak{s})$ are IVFFNs, then aggregated value by utilizing GIVFFEWA operator is an IVFFN and*
GIVFFEWA$(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$

$$= \Bigg\langle\Bigg[\frac{\sqrt[3]{2}\Big\{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}\Big\}^{1/3\lambda}}{\sqrt[3]{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(2-\beta)^{\lambda}-(\beta)^{\lambda}\}^{\delta_j}\Big)^{1/\lambda}\\ +\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-\beta)^{\lambda}-(\beta)^{\lambda}\}^{\delta_j}\Big)^{1/\lambda}\end{array}}},$$
$$\frac{\sqrt[3]{2}\Big\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}\Big\}^{1/3\lambda}}{\sqrt[3]{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(2-\beta_1)^{\lambda}-(\beta_1)^{\lambda}\}^{\delta_j}\Big)^{1/\lambda}\\ +\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-\beta_1)^{\lambda}-(\beta_1)^{\lambda}\}^{\delta_j}\Big)^{1/\lambda}\end{array}}}\Bigg],$$
$$\Bigg[\sqrt[3]{\frac{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\Big)^{1/\lambda}\\ -\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\Big)^{1/\lambda}\end{array}}{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\Big)^{1/\lambda}\\ +\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\Big)^{1/\lambda}\end{array}}},$$
$$\sqrt[3]{\frac{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\Big)^{1/\lambda}\\ -\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\Big)^{1/\lambda}\end{array}}{\begin{array}{l}\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\Big)^{1/\lambda}\\ +\Big(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\Big)^{1/\lambda}\end{array}}}\Bigg]\Bigg\rangle.$$

$(2-(\rho_j^{lb})^3)^{\lambda} = \alpha, (\rho_j^{lb})^3 = \beta, (2-(\rho_j^{ub})^3)^{\lambda} = \alpha_1, (\rho_j^{ub})^3 = \beta_1, (1+(\varrho_j^{lb})^3)^{\lambda} = \alpha_2, (1-(\varrho_j^{lb})^3)^{\lambda} = \beta_2, (1+(\varrho_j^{ub})^3)^{\lambda} = \alpha_3, (1-(\varrho_j^{ub})^3)^{\lambda} = \beta_3$

Proof Since

$$\mathcal{G}_j^{\lambda} = \Bigg\langle\Bigg[\frac{\sqrt[3]{2}(\rho_j^{lb})^{\lambda}}{\sqrt[3]{(2-(\rho_j^{lb})^3)^{\lambda}+((\rho_j^{lb})^3)^{\lambda}}}, \frac{\sqrt[3]{2}(\rho_j^{ub})^{\lambda}}{\sqrt[3]{(2-(\rho_j^{ub})^3)^{\lambda}+((\rho_j^{ub})^3)^{\lambda}}}\Bigg],$$
$$\Bigg[\sqrt[3]{\frac{(1+(\varrho_j^{lb})^3)^{\lambda}-(1-(\varrho_j^{lb})^3)^{\lambda}}{(1+(\varrho_j^{lb})^3)^{\lambda}+(1-(\varrho_j^{lb})^3)^{\lambda}}}, \sqrt[3]{\frac{(1+(\varrho_j^{ub})^3)^{\lambda}-(1-(\varrho_j^{ub})^3)^{\lambda}}{(1+(\varrho_j^{ub})^3)^{\lambda}+(1-(\varrho_j^{ub})^3)^{\lambda}}}\Bigg]\Bigg\rangle$$

$$\Rightarrow \bigoplus_{j=1}^{\mathfrak{s}} \delta_j \cdot_{\epsilon} \mathcal{G}_j^{\lambda}$$

$$= \Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}\big(1+\frac{2\beta^{\lambda}}{\alpha+(\beta)^{\lambda}}\big)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\big(1-\frac{2\beta^{\lambda}}{(2-\beta)^{\lambda}+(\beta)^{\lambda}}\big)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\big(1+\frac{2\beta^{\lambda}}{(2-\beta)^{\lambda}+(\beta)^{\lambda}}\big)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\big(1-\frac{2\beta^{\lambda}}{(2-\beta)^{\lambda}+(\beta)^{\lambda}}\big)^{\delta_j}}},$$
$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}\big(1+\frac{2(\beta_1)^{\lambda}}{\alpha_1+(\beta_1)^{\lambda}}\big)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\big(1-\frac{2(\beta_1)^{\lambda}}{(2-\beta_1)^{\lambda}+(\beta_1)^{\lambda}}\big)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\big(1+\frac{2(\beta_1)^{\lambda}}{(2-\beta_1)^{\lambda}+(\beta_1)^{\lambda}}\big)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\big(1-\frac{2(\beta_1)^{\lambda}}{(2-\beta_1)^{\lambda}+(\beta_1)^{\lambda}}\big)^{\delta_j}}}\Bigg],$$
$$\Bigg[\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}\Big(\sqrt[3]{\frac{\alpha_2-\beta_2}{\alpha_2+\beta_2}}\Big)^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}\Big(2-\frac{\alpha_2-\beta_2}{\alpha_2+\beta_2}\Big)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\Big(\frac{\alpha_2-\beta_2}{\alpha_2+\beta_2}\Big)^{\delta_j}}},$$

$$\left.\left.\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}\left(\sqrt[3]{\frac{\alpha_3-\beta_3}{\alpha_3+\beta_3}}\right)^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}\left(2-\frac{\alpha_3-\beta_3}{\alpha_3+\beta_3}\right)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\left(\frac{\alpha_3-\beta_3}{\alpha_3+\beta_3}\right)^{\delta_j}}}\right]\right\rangle$$

$$= \Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}},$$

$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}}\Bigg],$$

$$\Bigg[\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}\left(\sqrt[3]{\alpha_2-\beta_2}\right)^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}},$$

$$\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}\left(\sqrt[3]{\alpha_3-\beta_3}\right)^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}}\Bigg]\Bigg\rangle.$$

Therefore,
$\left(\bigoplus_{j=1}^{\mathfrak{s}}\delta_j\cdot_{\epsilon}\mathcal{G}_j^{\lambda}\right)^{1/\lambda}$

$$= \Bigg\langle\Bigg[\frac{\sqrt[3]{2\left(\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}}}{\sqrt[3]{\begin{array}{l}\left(2-\frac{\prod_{j=1}^{\mathfrak{s}}\{(2-\beta)^{\lambda}+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}\\ \quad+\left(\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}\end{array}}},$$

$$\frac{\sqrt[3]{2\left(\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}}}{\sqrt[3]{\begin{array}{l}\left(2-\frac{\prod_{j=1}^{\mathfrak{s}}\{(2-\beta_1)^{\lambda}+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}\\ \quad+\left(\frac{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}}\right)^{1/\lambda}\end{array}}}\Bigg],$$

$$\Bigg[\sqrt[3]{\frac{\left(1+\frac{2\prod_{j}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}{\prod_{j}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+\prod_{j}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}\right)^{1/\lambda}-\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}\right)^{1/\lambda}}{\left(1+\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}\right)^{1/\lambda}+\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}}\right)^{1/\lambda}}},$$

$$\sqrt[3]{\frac{\left(1+\frac{2\prod_{j}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}{\prod_{j}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+\prod_{j}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}\right)^{1/\lambda}-\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}\right)^{1/\lambda}}{\left(1+\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}\right)^{1/\lambda}+\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}}\right)^{1/\lambda}}}\Bigg]\Bigg\rangle$$

$$= \Bigg\langle\Bigg[\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha+3(\beta)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha-(\beta)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}}},$$

$$\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_1+3(\beta_1)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_1-(\beta_1)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}}}\Bigg],$$

$$\Bigg[\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{lb})^3)^{\lambda}-\beta_2\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_2+3\beta_2\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_2-\beta_2\}^{\delta_j}\right)^{1/\lambda}}},$$

$$\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+(\varrho_j^{ub})^3)^{\lambda}-\beta_3\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_3+3\beta_3\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_3-\beta_3\}^{\delta_j}\right)^{1/\lambda}}}\Bigg]\Bigg\rangle.$$

When $\lambda = 1$, then
$\mathrm{GIVFFEWA}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$

$$= \Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_j^{lb})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{3})^{lb})^{\delta_j}}},$$

$$\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_j^{ub})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}}\Bigg],$$

$$\Bigg[\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(2-(\varrho_j^{lb})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}},$$

$$\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(2-(\varrho_j^{ub})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle.$$

□

7.1 Generalized IVFF Einstein ordered weighted averaging operators

Definition 12 Let $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ be a collection of IVFFNs and δ_j be the WV of $\mathcal{G}_j$ with $\delta_j > 0$ and $\sum_{j=1}^{\mathfrak{s}}\delta_j = 1$, then GIVFFEOWA operator is a mapping $\mathcal{Q}^{\mathfrak{s}} \to \mathcal{Q}$ such that

$$GIVFFEOWA(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}}) = \Big(\bigoplus_{j=1}^{\mathfrak{s}}(\delta_j\cdot_{\epsilon}\mathcal{G}_{\varrho(j)}^{\lambda})\Big)^{\frac{1}{\lambda}},$$

where $\lambda > 0$.

Theorem 11 *Let $\mathcal{G}_j = \langle[\rho_j^{lb}, \rho_j^{ub}], [\varrho_j^{lb}, \varrho_j^{ub}]\rangle$ be IVFFNs, then aggregated value by applying GIVFFEOWA operator is an IVFFN and*
$\mathrm{GIVFFEOWA}(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{\mathfrak{s}})$

$$= \Bigg\langle\Bigg[\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_4+3((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_4-((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_4+3((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_4-((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_4+3((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_4-((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}}},$$

$$\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_5+3((\rho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_5-((\rho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_5+3((\rho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_5-((\rho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_5+3((\rho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_5-((\rho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}}}\Bigg],$$

$$\left[\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_6+3\beta_6\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_6-\beta_6\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_6+3\beta_6\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_6-\beta_6\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_6+3\beta_6\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_6-\beta_6\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_6+3\beta_6\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_6-\beta_6\}^{\delta_j}\right)^{1/\lambda}}},\right.$$
$$\left.\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_7+3\beta_7\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_7-\beta_7\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_7+3\beta_7\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_7-\beta_7\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_7+3\beta_7\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_7-\beta_7\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_7+3\beta_7\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_7-\beta_7\}^{\delta_j}\right)^{1/\lambda}}}\right]\right\rangle.$$

where $(2-(\rho_{\varrho(j)}^{lb})^3)^\lambda=\alpha_4$, $(2-(\rho_{\varrho(j)}^{ub})^3)^\lambda=\alpha_5$, $(1+(\varrho_{\varrho(j)}^{lb})^3)^\lambda=\alpha_6$, $(1-(\varrho_{\varrho(j)}^{lb})^3)^\lambda=\beta_6$, $(1+(\varrho_{\varrho(j)}^{ub})^3)^\lambda=\alpha_7$, $(1-(\varrho_{\varrho(j)}^{ub})^3)^\lambda=\beta_7$

Proof It is similar to Theorem 10. □

When $\lambda=1$, then
GIVFFEOWA$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\left\langle\left[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_{\varrho(j)}^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_{\varrho(j)}^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_{\varrho(j)}^{lb})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}(1-(\rho_{\varrho(j)}^{lb})^3)^{\delta_j}}},\right.\right.$$
$$\left.\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_{\varrho(j)}^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_{\varrho(j)}^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\rho_{\varrho(j)}^{ub})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}(1-(\rho_{\varrho(j)}^{ub})^3)^{\delta_j}}}\right],$$
$$\left[\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}(\varrho_{\varrho(j)}^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(2-(\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}((\varrho_{\varrho(j)}^{lb})^3)^{\delta_j}}},\right.$$
$$\left.\left.\frac{\sqrt[3]{2}\prod_{j=1}^{\mathfrak{s}}(\varrho_{\varrho(j)}^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(2-(\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}+\prod_{j=1}^{\mathfrak{s}}((\varrho_{\varrho(j)}^{ub})^3)^{\delta_j}}}\right]\right\rangle.$$

8 Generalized IVFF Einstein weighted geometric operators

Definition 13 Suppose $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle(j=1,2,\ldots,\mathfrak{s})$ is a collection of IVFFNs and δ_j is WV of $\mathcal{G}_j$ with $\delta_j>0$, $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$ and GIVFFEWG : $\mathcal{Q}^{\mathfrak{s}}\to\mathcal{Q}$ such that

$$\text{GIVFFEWG}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\Big(\bigoplus_{j=1}^{\mathfrak{s}}(\delta_j\cdot_\epsilon\mathcal{G}_j^\lambda)\Big)^{\frac{1}{\lambda}},$$

where $\lambda>0$.

Particularly,

- If $\lambda=1$, then GIVFFEWG becomes IVFFEWG.
- If $\delta=(1/\mathfrak{s},1/\mathfrak{s},\ldots,1/\mathfrak{s})^T$, then GIVFFEWG $(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=(\frac{1}{\mathfrak{s}}\cdot_\epsilon\bigoplus_{j=1}^{\mathfrak{s}}\mathcal{G}_j^\lambda)^{\frac{1}{\lambda}}$.

Theorem 12 *Suppose* $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle(j=1,2,\ldots,\mathfrak{s})$ *are IVFFNs, then aggregated value by utilizing GIVFFEWG operator is an IVFFN and*

GIVFFEWG$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\left\langle\left[\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_8+3\beta_8\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_8-\beta_8\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_8+3\beta_8\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_8-\beta_8\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_8+3\beta_8\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_8-\beta_8\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_8+3\beta_8\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_8-\beta_8\}^{\delta_j}\right)^{1/\lambda}}},\right.\right.$$
$$\left.\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_9+3\beta_9\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_9-\beta_9\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_9+3\beta_9\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_9-\beta_9\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_9+3\beta_9\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_9-\beta_9\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_9+3\beta_9\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_9-\beta_9\}^{\delta_j}\right)^{1/\lambda}}}\right],$$
$$\left[\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}+3\beta_{10}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}-\beta_{10}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}+3\beta_{10}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}-\beta_{10}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}+3\beta_{10}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{10}-\beta_{10}\}^{\delta_j}\right)^{1/\lambda}}},\right.$$
$$\left.\left.\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}+3\beta_{11}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}-\beta_{11}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}+3\beta_{11}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}-\beta_{11}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}+3\beta_{11}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{11}-\beta_{11}\}^{\delta_j}\right)^{1/\lambda}}}\right]\right\rangle.$$

where $(1+(\rho_j^{lb})^3)^\lambda=\alpha_8$, $(1-(\rho_j^{lb})^3)^\lambda=\beta_8$, $(1+(\rho_j^{ub})^3)^\lambda=\alpha_9$, $(1-(\rho_j^{ub})^3)^\lambda=\beta_9$, $(2-(\varrho_j^{lb})^3)^\lambda=\alpha_{10}$, $((\varrho_j^{lb})^3)^\lambda=\beta_{10}$, $(2-(\varrho_j^{ub})^3)^\lambda=\alpha_{11}$, $((\varrho_j^{ub})^3)^\lambda=\beta_{11}$.

8.1 Generalized IVFF Einstein ordered weighted geometric operators

Definition 14 Let $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle$ be a collection of IVFFNs and δ_j be the WV of $\mathcal{G}_j$ with $\delta_j>0$ and $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$, then GIVFFEOWG operator is a mapping $\mathcal{Q}^{\mathfrak{s}}\to\mathcal{Q}$ such that

$$\text{GIVFFEOWG}(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})=\Big(\bigoplus_{j=1}^{\mathfrak{s}}(\delta_j\cdot_\epsilon\mathcal{G}_{\varrho(j)}^\lambda)\Big)^{\frac{1}{\lambda}},$$

where $\lambda>0$.

Theorem 13 *Let* $\mathcal{G}_j=\langle[\rho_j^{lb},\rho_j^{ub}],[\varrho_j^{lb},\varrho_j^{ub}]\rangle$ *be IVFFNs, then aggregated value by applying GIVFFEOWG operator is an IVFFN and*
GIVFFEOWG$(\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_{\mathfrak{s}})$

$$=\left\langle\left[\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}+3\beta_{12}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}-\beta_{12}\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}+3\beta_{12}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}-\beta_{12}\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}+3\beta_{12}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}-\beta_{12}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}+3\beta_{12}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{12}-\beta_{12}\}^{\delta_j}\right)^{1/\lambda}}},\right.\right.$$
$$\left.\sqrt[3]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}+3\beta_{13}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}-\beta_{13}\}^{\delta_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}+3\beta_{13}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}-\beta_{13}\}^{\delta_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}+3\beta_{13}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}-\beta_{13}\}^{\delta_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}+3\beta_{13}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{13}-\beta_{13}\}^{\delta_j}\right)^{1/\lambda}}}\right],$$

$$\left[\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}+3((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}-((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\begin{array}{l}\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}+3((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}-((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}\\ +\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}+3((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{14}-((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}\end{array}}},\right.$$

$$\left.\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}+3((\varrho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}-((\varrho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}\right\}^{1/3\lambda}}{\sqrt[3]{\begin{array}{l}\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}+3((\varrho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}+3\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}-((\varrho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}\\ +\left(\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}+3((\varrho_{\varrho(j)}^{ub})^3)^{\lambda}\}^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}\{\alpha_{15}-((\varrho_{\varrho(j)}^{lb})^3)^{\lambda}\}^{\delta_j}\right)^{1/\lambda}\end{array}}}\right]\Big\rangle.$$

where $(1+(\rho_{\varrho(j)}^{lb})^3)^{\lambda}=\alpha_{12}$, $(1-(\rho_{\varrho(j)}^{lb})^3)^{\lambda}=\beta_{12}$, $(1+(\rho_{\varrho(j)}^{ub})^3)^{\lambda}=\alpha_{13}$, $(2-(\varrho_{\varrho(j)}^{lb})^3)^{\lambda}=\alpha_{14}$, $(2-(\varrho_{\varrho(j)}^{ub})^3)^{\lambda}=\alpha_{15}$

9 Multi-attribute decision-making under IVFF operators

This section will form a use of IVFFHWA or IVFFHWG operator to develop a procedure to solve MADM problems with IVFFNs.

To deal with MADM problem under IVFF environment, consider $X=\{X_1,X_2,\ldots,X_m\}$ is the set of alternatives and $V=\{V_1,V_2,\ldots,V_{\mathfrak{s}}\}$ is the set of reasonable attributes decided by the decision maker. Suppose $\delta=(\delta_1,\delta_2,\ldots,\delta_{\mathfrak{s}})^T$ is the WV with $\delta_j>0$ and $\sum_{j=1}^{\mathfrak{s}}\delta_j=1$. Suppose that $\tilde{\mathcal{E}}=([\rho_{lj}^{lb},\rho_{lj}^{ub}],[\varrho_{lj}^{lb},\varrho_{lj}^{ub}])_{m\times\mathfrak{s}}$, is the IVFF decision matrix, where $[\rho_{lj}^{lb},\rho_{lj}^{ub}]$ is the positive MD by which alternative X_l fulfills the attribute V_j that has been distributed by the decision-makers, and $[\varrho_{lj}^{lb},\varrho_{lj}^{ub}]$ gave the degree that the alternative does not fulfill the attribute that has been distributed by the decision-maker, where $0\leq(\rho_{lj}^{ub})^3+(\varrho_{lj}^{ub})^3\leq 1$.

Algorithm 1 is used to solve the MADM problem with IVFFN based on using IVFFEWA or IVFFEWG operator.

Algorithm 1

Step 1. First of all, make collection of the assessment information of all experts according to the parameters for each object/option and compose a decision matrix $\tilde{\mathcal{E}}=([\rho_{lj}^{lb},\rho_{lj}^{ub}],[\varrho_{lj}^{lb},\varrho_{lj}^{ub}])_{m\times\mathfrak{s}}$ in the following way:

$$\tilde{\mathcal{E}}=\begin{bmatrix}\gamma_{11}&\gamma_{12}&\cdots&\gamma_{1\mathfrak{s}}\\ \gamma_{21}&\gamma_{22}&\cdots&\gamma_{2\mathfrak{s}}\\ \vdots&\vdots&\ddots&\vdots\\ \gamma_{m1}&\gamma_{m2}&\cdots&\gamma_{m\mathfrak{s}}\end{bmatrix}$$

where $([\rho_{11}^{lb},\rho_{11}^{ub}],[\varrho_{11}^{lb},\varrho_{11}^{ub}])=\gamma_{11}$, $([\rho_{12}^{lb},\rho_{12}^{ub}],[\varrho_{12}^{lb},\varrho_{12}^{ub}])=\gamma_{12}$, $([\rho_{1\mathfrak{s}}^{lb},\rho_{1\mathfrak{s}}^{ub}],[\varrho_{1\mathfrak{s}}^{lb},\varrho_{1\mathfrak{s}}^{ub}])=\gamma_{1\mathfrak{s}}$, $([\rho_{21}^{lb},\rho_{21}^{ub}],[\varrho_{21}^{lb},\varrho_{21}^{ub}])=\gamma_{21}$, $([\rho_{22}^{lb},\rho_{22}^{ub}],[\varrho_{22}^{lb},\varrho_{22}^{ub}])=\gamma_{22}$, $([\rho_{2\mathfrak{s}}^{lb},\rho_{2\mathfrak{s}}^{ub}],[\varrho_{2\mathfrak{s}}^{lb},\varrho_{2\mathfrak{s}}^{ub}])=\gamma_{2\mathfrak{s}}$, $([\rho_{m1}^{lb},\rho_{m1}^{ub}],[\varrho_{m1}^{lb},\varrho_{m1}^{ub}])=\gamma_{m1}$, $([\rho_{m2}^{lb},\rho_{m2}^{ub}],[\varrho_{m2}^{lb},\varrho_{m2}^{ub}])=\gamma_{m2}$, $([\rho_{m\mathfrak{s}}^{lb},\rho_{m\mathfrak{s}}^{ub}],[\varrho_{m\mathfrak{s}}^{lb},\varrho_{m\mathfrak{s}}^{ub}])=\gamma_{m\mathfrak{s}}$.

Step 2. For normalize decision matrix, interchange the decision matrix $\tilde{\mathcal{E}}=([\rho_{lj}^{lb},\rho_{lj}^{ub}],[\varrho_{lj}^{lb},\varrho_{lj}^{ub}])_{m\times\mathfrak{s}}$ by $\overline{\tilde{\mathcal{E}}}=([\varrho_{lj}^{lb},\varrho_{lj}^{ub}],[\rho_{lj}^{lb},\rho_{lj}^{ub}])_{m\times\mathfrak{s}}$

$$\overline{\tilde{\mathcal{E}}}=\begin{cases}([\rho_{lj}^{lb},\rho_{lj}^{ub}],[\varrho_{lj}^{lb},\varrho_{lj}^{ub}]); & \text{for benefit parameter,}\\ ([\varrho_{lj}^{lb},\varrho_{lj}^{ub}],[\rho_{lj}^{lb},\rho_{lj}^{ub}]); & \text{for cost parameter.}\end{cases}$$

If all attributes are of a similar kind, it is not compulsory to normalize the values of the attributes, although if there are benefit attributes and cost attributes in decision making, in these cases, we can change the rating values of the cost parameter into the rating values of the benefit parameter.

Step 3. Use the IVFFDM and IVFFHWA or IVFFHWG operator

$$\begin{aligned}\mathcal{B}_l&=\mathrm{IVFFHWA}(X_{l1},X_{l2},\ldots,X_{l\mathfrak{s}})\\ &=\Bigg\langle\Bigg[\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{lb})^3)^{\delta_j}}},\\ &\quad\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\rho_j^{ub})^3)^{\delta_j}}}\Bigg],\\ &\quad\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{lb})^3)^{\delta_j}}},\\ &\quad\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\varrho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\varrho_j^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle.\end{aligned}$$

OR

$$\begin{aligned}\mathcal{B}_l&=\mathrm{IVFFHWG}(X_{l1},X_{l2},\ldots,X_{l\mathfrak{s}})\\ &=\Bigg\langle\Bigg[\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_j^{lb})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_j^{lb})^3)^{\delta_j}}},\\ &\quad\frac{\sqrt[3]{\gamma}\prod_{j=1}^{\mathfrak{s}}(\rho_j^{ub})^{\delta_j}}{\sqrt[3]{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\rho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}((\rho_j^{ub})^3)^{\delta_j}}}\Bigg],\\ &\quad\Bigg[\sqrt[3]{\frac{\prod\limits_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{lb})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{lb})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{lb})^3)^{\delta_j}}},\\ &\quad\sqrt[3]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}-\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{ub})^3)^{\delta_j}}{\prod_{j=1}^{\mathfrak{s}}(1+(\gamma-1)(\varrho_j^{ub})^3)^{\delta_j}+(\gamma-1)\prod_{j=1}^{\mathfrak{s}}(1-(\varrho_j^{ub})^3)^{\delta_j}}}\Bigg]\Bigg\rangle.\end{aligned}$$

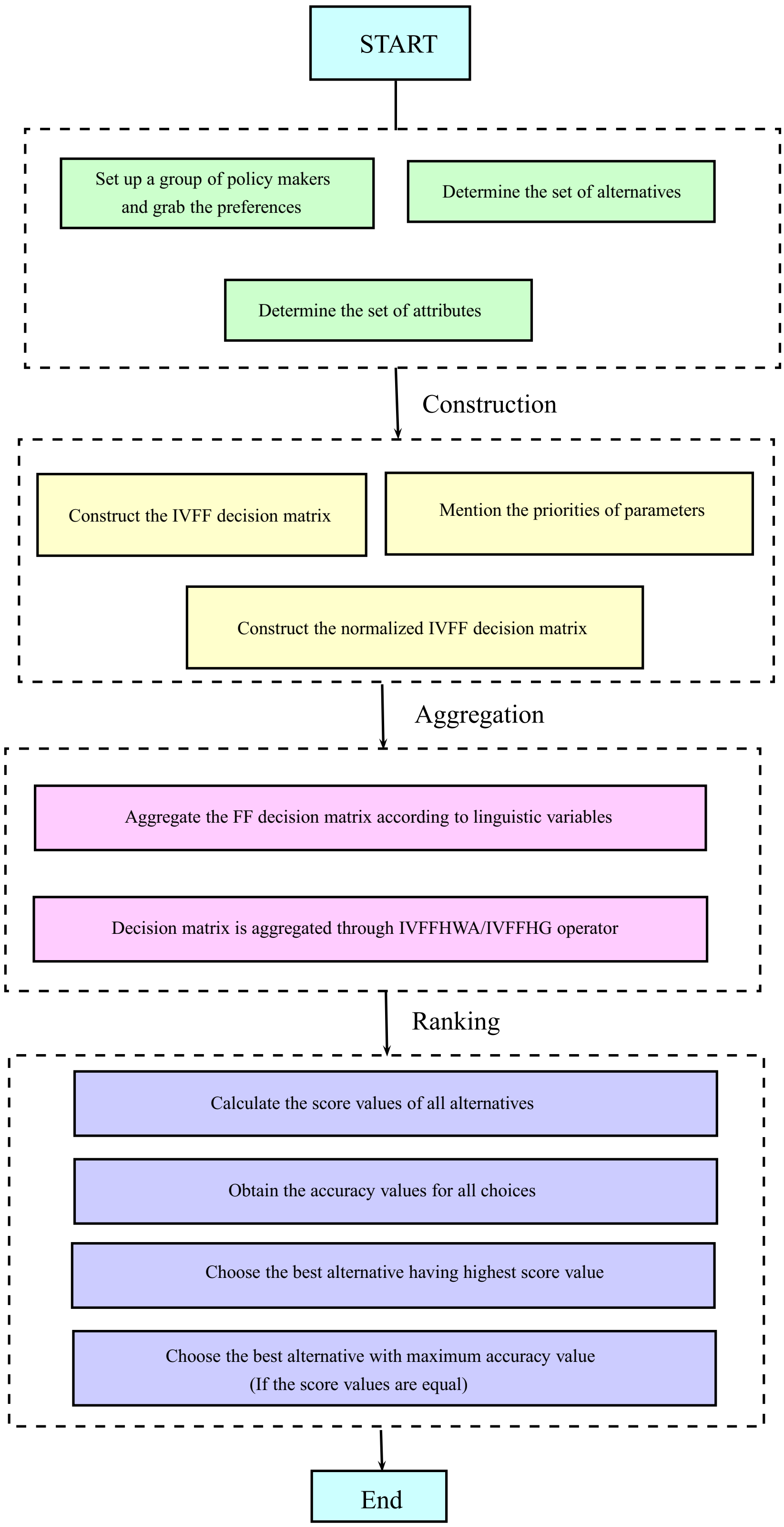
for overall preference values $\mathcal{B}_l(l=1,2,\ldots,m)$ of the alternatives X_l.

Step 4. Apply the score function $\mathcal{S}(\mathcal{B}_l)(l=1,2,\ldots,m)$ for the ranking of alternatives. If score values are equal, then compute the accuracy functions $\mathcal{A}(\mathcal{B}_l)$ and rank according to these values.

Step 5. The alternative containing maximum score value will be the decision.

The flowchart of the proposed model is given in Fig. 2.

Fig. 2 Multi-attribute decision-making under IVFF operators



10 Case study of China: selection of E-waste recycling partner

Due to growth in financial increase, quicker upgradation of electrical and electronic system reasons the increasing amount of waste electric and electronic products (WEEE). At present, China is not simplest a massive consumption nation of electrical merchandise, but additionally a largest importer of e-waste. From studying the modern conditions of e-waste disposal and remedy in China, this section reviews the probable environmental and fitness affects of e-waste, and ambitions to explore the demanding situations and issues of China's e-waste management confronted. On the same time, this can additionally positioned numerous hints on regulating China's e-waste recycling machine.

10.1 Domestic e-waste flows in China

In era, there are mainly three kinds of locations for e-waste generated in China. First off, used Electric and electronic merchandise might input into the second-hand markets. The general public of Chinese language People might favor to save their antique domestic home equipment at domestic or office in place of dispose them. While Given a good sized rate, humans are willing to sell their e-waste (Ongondo et al. 2011). Digital merchandise, such shifting behaviors are explicitly forbidden (2009). Eventually, out of date appliance is probably recycled using non-public corporations for raw substances. Such recyclers generally purchase WEEE from family users at a relative low price, however they do not have expert disposal centers; thus, environmental pollution are specifically because of this channel of e-waste disposal.

10.2 Informal recycling sector of China

Pushed by means of large earnings, the majority of domestic e-waste entered into China's informal recycling sectors, which additionally encompass the disposal of imported WEEE. In 2007, there were more than 700,000 human beings employed in e-waste enterprise, 98% of which were within the casual recycling zone (Ongondo et al. 2011). As the biggest e-waste recycling web page in China even within the world, Guiyu has a population of 150,000, almost 100,000 of which are migrant people engaged in e-waste recycling operations. Normally, those sits have hundreds of individual workshops worried within the WEEE recycling commercial enterprise. But, the disposal procedures contain plenty of primitive techniques, which can be crude. These strategies consist of (1) dismantling digital system; (2) heating and manual elimination of components from printed circuit boards; (3) establishing burning cables and wires for convalescing metals; (4) chipping and melting plastics; (5) toner sweeping; (6) open acid leaching of e-waste to recover valuable metals (Wong et al. 2007; Huo et al. 2007).

10.3 Formal disposal of e-waste in China

For the reason that 2004, the nation development and Reform commission (SDRC) exact Qingdao Haier, Hangzhou Dadi, Beijing Huaxing, Tianjing Datong as countrywide pilot tasks, aiming to set up the WEEE Recycling system and control techniques. But, little progress has been made until now. For example, as the sector's fourth biggest white goods manufacturer, Haier turned into selected because the pilot business enterprise to get better discarded domestic appliances, trying to build a producer-owner recycling version. Moreover, the Haier organization advanced coordination with Tsinghua university to improve recycling technology. However via may also 2007, Haier simplest disposed of 8000 domestic home equipment with the ability of six hundred thousand gadgets in line with year (Yu et al. 2010).

China's private environmental safety agencies also made consequences to improve the recuperation price of e-waste. Shenzhen inexperienced Eco-Manufacture (GEM) excessive-tech Co., Ltd ran his first supermarket for e-waste Recycling in Wuhan (Gem 2011). In keeping with extraordinary products, GEM set exceptional expenses for used or obsolete EEE. Until now, GEM has already signed a strategic cooperation agreement with Wuhan Zhongbai, Gome and Suning. In this manner, GEM hopes to resolve the issues of no e-waste and attain low carbon consumption through marketplace mechanism action. Whatever a hit or now not, GEM furnished a referenced mode for China's e-waste recuperation industry. All in all, that is a critical sport among formal and informal zones in China. At present, informal recyclers are at a more awesome benefit than formal ones. Despite the fact that China has already invested extra inside the e-waste recycling enterprise, formal recycling agencies discover it tough to collect enough WEEE.

10.4 Description of the problem

So as to authenticate the relevancy of the proposed technique, a case have a look at of e-waste recycling associate evaluation of an electronics company tracked in Wuhan, China. On this work, we've got targeted at the usage of an revolutionary approach so that it would oblige the company's shareholders to appraise and choose for the maximum appropriate alternative. First of all, a set of choice-makers is fashioned to assess this technique of decision-making. After the primary screening, we've chosen four recycling Companions, who're worried within the recycling procedures of WEEEs, and shape a set of WRP options $X = X_1, X_2, X_3, X_4$. These four alternatives could be evaluated under the following four attributes: recycling performance and transport records (V_1), environmental management system (V_2), Reduction in GHG emission (V_3) and Recycling cost (V_4). In this study, the attributes V_1, V_2 and V_3 are of benefit criteria and V_4 is of cost criterion. The data used in this application is adopted from Rani and Mishra (2022a, b) and Cui et al. (2021). The procedural steps of the developed framework are given by.

Step 1. The IVFFDM and IVFFNDM is shown in Tables 1 and 2.

Step 2. The weights assigned by decision maker are $\delta_1 = [0.20.0.30]$, $\delta_2 = [0.15.0.25]$, $\delta_3 = [0.18.0.28]$ and $\delta_4 = [0.25.0.35]$.

Step 3. Apply the IVFFHWA or IVFFHWG operator for the selection of e-waste recycling partner. The performance values $\mathcal{B}_l$ of alternatives using the IVFFHWA operator for $\gamma = 1$ is given by

$$\mathcal{B}_1 = ([0.55, 0.78], [0.53, 0.5]),$$
$$\mathcal{B}_2 = ([0.54, 0.71], [0.6, 0.58]),$$
$$\mathcal{B}_3 = ([0.6, 0.75], [0.53, 0.49]),$$
$$\mathcal{B}_4 = ([0.56, 0.74], [0.56, 0.49]).$$

Step 4. Compute the scores $\mathcal{S}(\mathcal{B}_l)$ of IVFFNs $\mathcal{B}_l$.

$$\mathcal{S}(\mathcal{B}_1) = 0.18,$$
$$\mathcal{S}(\mathcal{B}_2) = 0.01,$$
$$\mathcal{S}(\mathcal{B}_3) = 0.19,$$
$$\mathcal{S}(\mathcal{B}_4) = 0.14.$$

Step 5. As $X_3 > X_1 > X_4 > X_2$. Therefore, X_3 is the best alternative.

11 Comparison analysis

In this section, we correlate the developed methodology with existing methods interval valued Fermatean fuzzy weighted sum method (IVFF-WSM), interval valued Fermatean fuzzy weighted product method (IVFF-WPM), interval valued Fermatean fuzzy weighted aggregated sum product assessment(WASPAS) as given in Rani and Mishra (2022a, b), interval valued Pythagorean fuzzy weighted average (IVPFWA) operator (Peng and Yang 2016), interval valued Pythagorean fuzzy weighted geometric (IVPFWG) operator (Peng and Yang 2016) and interval valued Pythagorean fuzzy-TOPSIS method (Garg 2017) for evaluating the best choices. Their corresponding outcomes are represented in Table 3. It is clear from Table 3, the functioning of the relative score values results in the same pattern (increasing or decreasing). Thus, the introduced method always explains the MADM issues on FFSs and IVFFSs framework.

Figure 3 Indicates the score values, correlated with the existing MADM strategies. Lots of exciting styles are accumulated in these outcomes which are adduced by the assessment. Some of these methodologies are as compared with each one and discovered that the opportunity X_3 is the excellent choice, as shown in Fig. 3. In this application, the amount of options is constrained to 4; the end result of the advanced method won't be found as conclusive. Now, if the variety of choices increases, the outcome becomes clearer. Therefore, the advanced idea is valid and can be performed in decision-making issues. There are a few problems of the present theories which may be treated by means of the manner of proposed concept.

11.1 Advantages of the proposed method

Via comparing the proposed operators based totally on IVFFNs with existing methodologies and evaluating the adaptability and efficiency of the proposed operators, we finalize that the proposed scheme has the subsequent advantages:

- The IVFF membership grades generalize the IVIF and IVPF grades as they develop the suitable area of unsure facts and figures.
- IVPFWA and IVPFWG operators are good methodologies for the ranking of alternatives and choosing the most appropriate alternative. As the results obtained from these two methods are given in Table 3 and we get the best option X_3. The results obtained for best choice by our proposed method are also same but IVPFWA and IVPFWG operators are restricted to the domain that the square sum of upper bound of membership and nonmembership degree should be less than equal to 1, so when this boundary is exceeded by 1 then IVPFWA and IVPFWG operators are failed to deal such problems. In such situations, we use the proposed operators for the ranking of alternatives.
- In IVPF-TOPSIS method, alternatives are evaluated by relative closeness degree of the alternatives. It is not appropriate to justify how good or bad an alternative is. In the proposed theory, both benefit and the cost parameter are under considerable with proposed AOs on IVFFSs which results a more accurate outcome compared with simply dealing with benefit or cost criteria.
- The generalization of IVIFNs and IVPFNs to IVFFNs increases the workability of the provided vague data and increase the practicability of operators to the machine where the membership abilities are complicated or not feasible to specify absolutely.

Table 1 Interval valued Fermatean fuzzy decision matrix for e-waste recycling partner selection

	V_1	V_2	V_3	V_4
X_1	([0.45,0.65],[0.55,0.75])	([0.60,0.75],[0.35,0.5])	([0.65,0.75],[0.40,0.55])	([0.40,0.5],[0.65,0.80])
X_2	([0.65,0.7],[0.40,0.65])	([0.5,0.6],[0.65,0.75])	([0.60,0.65],[0.45,0.55])	([0.55,0.65], [0.55,0.70])
X_3	([0.7,0.8],[0.4,0.6])	([0.7,0.75],[0.3,0.45])	([0.55,0.65],[0.45,0.55])	([0.5,0.6],[0.6,0.65])
X_4	([0.68,0.75],[0.45,0.55])	([0.65,0.7],[0.45,0.6])	([0.57,0.65],[0.4,0.55])	([0.5,0.55], [0.5,0.7])

Table 2 Normalized Interval valued Fermatean fuzzy decision matrix for e-waste recycling partner selection

	V_1	V_2	V_3	V_4
X_1	([0.45,0.65],[0.55,0.75])	([0.60,0.75],[0.35,0.5])	([0.65,0.75],[0.40,0.55])	([0.65,0.80],[0.40,0.5])
X_2	([0.65,0.7],[0.40,0.65])	([0.5,0.6],[0.65,0.75])	([0.60,0.65],[0.45,0.55])	([0.55,0.70],[0.55,0.65])
X_3	([0.7,0.8],[0.4,0.6])	([0.7,0.75],[0.3,0.45])	([0.55,0.65],[0.45,0.55])	([0.6,0.65],[0.5,0.6])
X_4	([0.68,0.75],[0.45,0.55])	([0.65,0.7],[0.45,0.6])	([0.57,0.65],[0.4,0.55])	([0.5,0.7],[0.5,0.55])

Table 3 Score values through different methodologies

Methodologies		Score values			Order of preference
Proposed method	0.18	0.01	0.19	0.14	$X_3 > X_1 > X_4 > X_2$
IVFF-WSM (Rani and Mishra 2022a, b)	0.17	0.05	0.22	0.16	$X_3 > X_1 > X_4 > X_2$
IVFF-WPM (Rani and Mishra 2022a, b)	0.12	0.02	0.19	0.15	$X_3 > X_4 > X_1 > X_2$
IVFF-WASPAS (Rani and Mishra 2022a, b)	0.14	0.04	0.21	0.15	$X_3 > X_4 > X_1 > X_2$
IVPFWA operator (Peng and Yang 2016)	0.18	0.06	0.25	0.18	$X_3 > X_1 > X_4 > X_2$
IVPFWG operator (Peng and Yang 2016)	0.14	0.03	0.19	0.15	$X_3 > X_4 > X_1 > X_2$
IVPF-TOPSIS method (Garg 2017)	0.08	0.02	0.31	0.10	$X_3 > X_1 > X_4 > X_2$

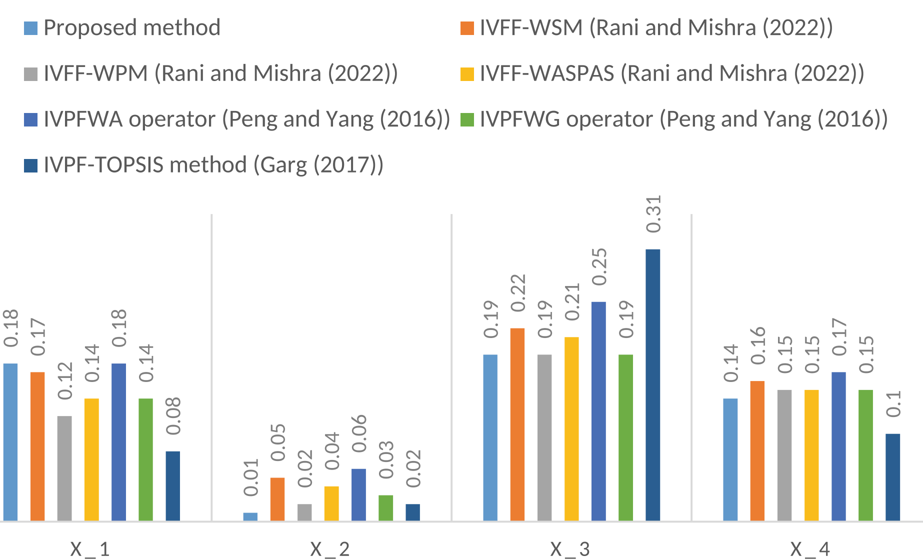


Fig. 3 Ranking of alternatives for different MADM techniques

- Whilst the collection of alternatives and attributes becomes very big, the developed technique has greater workability than the prevailing ones. The results can be attained with the reprocessing of doable records, which allows the evolved operators to making use of greater complicated and practical MADM problems.
- Within the proposed approach, the advantage and the price criteria are each taken into consideration with proposed AOs on IVFFSs which contain a extra precise outcome compared with surely managing advantage or value criteria. Inside the meantime, it increases the practicality of evaluation records and the precision of effects as well.
- The main benefit of the introduced IVFF model is capable of assessing any MCDA issues with uncertainty through IVFFNs as well as IFNs, PFNs, FFNs, IVIFNs and IVPFNs as described in the previous sections.

12 Conclusions and future directions

The objective of this research work is the introduction the AOs for IVFFSs, this theory permits the DM experts to assign the membership and non-membership values of a collection of alternatives in form of the intervals, that is, the domain of vague data and information they can represent is wider. The classical operational laws corresponding to FFSs and IVFFSs has been reviewed. Based on the operations of IVFFSs, we have developed various averaging and geometric operators (IVFFHWA, IVFFHOWA, IVFFHWG, IVFFHOWG operators) to aggregate the IVFF data and discussed their salient features. Furthermore, these operators, i.e., IVFF Einstein averaging and geometric operators and IVFF Hamacher averaging and geometric operators have been utilized in MCGDM. Moreover, certain characteristics of these averaging and geometric operators like idempotency, boundedness and monotonicity have been discussed briefly. Finally, we have considered a case study of e-waste recycling (e-waste recycling case is considered in China) partner selection to manifest the productivity and coherence of the proposed models. Moreover, the reliability and robustness of the obtained results has been demonstrated through sensitivity and comparative analysis. At last, we have conducted a comparison between the developed and some of the extant models, which demonstrated its applicability and advantages.

12.1 Limitations and future directions

The basic limitations of the developed theory are as follows: As the IVFFNs are the generalization of IVIFNs and IVPFNs but it fails in that situations where $(\rho^{ub})^3 + (\varrho^{ub})^3 \geq 1$. The proposed operators are not applicable in that conditions where we assume the parameters for the assessment of anything. Therefore, this work has a lack of parametrization property. To overcome the drawbacks of the proposed and previous studies, the current work can be extended in following ways.

In future, some more AOs for IVFFSs would be developed. At the same time, we will apply these operators for the introduction of new MCDA models and try to investigate several applications including game theory, cluster analysis, medical diagnosis, image processing and MCDA problems. Furthermore, various MAGDM applications in other industries, i.e., agriculture, healthcare, construction companies, etc, can be handled using the proposed technique. Certain new methods, such as, DNMA (double normalization-based multiple aggregation), GLDS (gained and lost dominance score) and MARCOS (Measurement of Alternatives and Ranking according to the Compromise Solution) in the fuzzy context of FFS and the IVFFSs can be used for future development. The proposed method is based on FFS theory, which is a generalization of the PFS and IFS theories. Therefore, there are currently no significant limitations in the application of proposed approach. The only real limitation that is observed is the selection of attributes. Only four attributes were selected, although many other factors, such as groundwater depth, proximity to surface water, elevation, land slope, soil permeability, soil stability, flooding susceptibility, lithology and stratification, faults, land use type, nearby settlements and urbanization, proximity to cultural and protected sites, wind direction, roads, railroads, proximity to building materials, pipelines, powerlines, and proximity to airports are considered while selecting a disposal site. These factors can be considered in future studies.

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Declaration

Conflict of interest The authors declare no conflict of interest.

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