# Multi-component Painlevé ODE's and related non-autonomous KdV stationary hierarchies 

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#### Abstract

First, starting from two hierarchies of autonomous Stäckel ODE's, we reconstruct the hierarchy of KdV stationary systems. Next, we deform considered autonomous Stäckel systems to non-autonomous Painlevé hierarchies of ODE's. Finally, we reconstruct the related non-autonomous KdV stationary hierarchies from respective Painlevé systems.


## 1 Introduction

Two particular classes of second order nonlinear ordinary differential equations (ODE's) playing important role in a variety of branches of modern mathematics and physics. The first class is represented by separable (Stäckel) equations with autonomous Hamiltonian representations. The second class is represented by Painlevè equations with non-autonomous Hamiltonian representations. Thus, both types of ODE's can be alternatively considered as respective autonomous and non-autonomous Hamiltonian dynamical systems. The Stäckel equations can be written in the so-called Lax representation in the form of isospectral deformation equations while the Painlevè equations can be written in the Lax representation in the form of isomonodromic deformation equations. Both, separable and Painlevè equations, appear in a wide range of applications in physics and mathematics, so are definitely worth of investigation.

A significant progress in construction of new Stäckel and Painlevé equations took place since the modern theory of nonlinear integrable PDE's has been born (the so-called soliton theory). It was found that both type of equations are inseparably connected with the soliton systems with whom they share many properties. Actually, they have been constructed under particular reductions of soliton PDE's.

The systematic construction of Stäckel systems of arbitrary degrees of freedom from stationary flows and restricted flows of soliton hierarchies as well as constrained flows of respective Lax hierarchies is nowadays well developed [14, 2, 3, 15] (see also review of these methods in [5] and references therein). A bit less is known about similar constructions of Painlevé systems with arbitrary number of degrees of freedom. Nevertheless, many interesting results the reader can find in [1, 16, 17, 18, 21, 23, 26, 27, 30, 31, 32,

In this article we present an inverse approach to the relation between Painlevé ODE's and soliton PDE's on the example of the KdV family. Actually, from particular hierarchies of Painlevé systems we construct the related hierarchies of non-autonomous and non-homogeneous deformations of the KdV PDE's.

Let me briefly sketch this idea. Recently, we have developed a deformation theory of autonomous Stäckel equations to non-autonomous Painlevé equations [10, 11, 12. To be more precise, in the literature was presented so far several constructions of Painlevé hierarchies with increasing number of degrees of freedom. What important, each number of degrees of freedom was related with a single equation. In that sense, our deformation approach contains more. We start from a hierarchy of autonomous separable systems with increasing number of degrees of freedom, where a system of $n$ degrees of freedom consists $n$ commuting (i.e. Frobenious integrable) evolution equations. After deformation we obtain a hierarchy
of non-autonomous Painlevé systems with increasing number of degrees of freedom, where system of $n$ degrees of freedom consists of $n$, Frobenious integrable, non-autonomous Painlevé evolution equations. So, from now on, I will use the phrase a hierarchy of Painlevé systems rather then a Painlevé hierarchy known from the literature.

On the other hand, in articles [6, 29] we have made an interesting observation that the complete soliton hierarchies can be reconstructed from particular finite dimensional Stäckel systems, representing their stationary flows.

In the following article we take the advantages from that observation and taking the Painleve deformations of Stäckel systems related to KdV stationary flows, we construct related hierarchies of nonautonomous and non-homogeneous deformations of the KdV hierarchy. Such new hierarchies seems to be interesting objects for further investigation, as their stationary flows reconstruct a particular hierarchies of Painlevé systems.

This paper is organized as follows. In Section2we briefly collect the necessary information on the KdV hierarchy. In Section 3 we reconstruct the KdV hierarchy from the hierarchy of Stäckel systems, which are representation of the KdV stationary systems related the first KdV Hamiltonian structure. In Section 4 we do the same from the hierarchy of Stäckel systems which are representation of the KdV stationary systems related the second KdV Hamiltonian structure. In Section 5, applying recently developed theory [10, 11, 12, we deform autonomous Stäckel hierarchies from Sections 3and 4 to non-autonomous Painlevé hierarchies. Finally, in Section 6, we reconstruct, like in the autonomous case, two non-autonomous KdV hierarchies of stationary systems, related to particular non-homogenous KdV hierarchies.

## 2 KdV hierarchy

Let us remind some elementary facts about the KdV hierarchy, important for our further considerations. The KdV equation

$$
\begin{equation*}
u_{t}=\frac{1}{4} u_{x x x}+\frac{3}{2} u u_{x} \tag{2.1}
\end{equation*}
$$

is a member of the following bi-Hamiltonian chain of nonlinear PDE's

$$
\begin{equation*}
u_{t_{n}}=\mathcal{K}_{n}=\pi_{0} d \mathcal{H}_{n}=\pi_{1} d \mathcal{H}_{n-1}, \quad n=1,2, \ldots \tag{2.2}
\end{equation*}
$$

where two Poisson operators are

$$
\begin{equation*}
\pi_{0}=\partial_{x}, \quad \pi_{1}=\frac{1}{4} \partial_{x}^{3}+\frac{1}{2} u \partial_{x}+\frac{1}{2} \partial_{x} u \tag{2.3}
\end{equation*}
$$

The hierarchy (2.2) can be generated by a recursion operator and its adjoint in a following way

$$
\begin{gather*}
N=\pi_{1} \pi_{0}^{-1}=\frac{1}{4} \partial_{x}^{2}+u+\frac{1}{2} u_{x} \partial_{x}^{-1}, \quad N^{\dagger}=\frac{1}{4} \partial_{x}^{2}+u-\frac{1}{2} \partial_{x}^{-1} u_{x}  \tag{2.4}\\
\mathcal{K}_{n+1}=N^{n} \mathcal{K}_{1}, \quad \gamma_{n}=d \mathcal{H}_{n}=\left(N^{\dagger}\right)^{n} \gamma_{0}, \quad n=1,2, \ldots . \tag{2.5}
\end{gather*}
$$

In particular, conserved one-forms are

$$
\begin{align*}
& \gamma_{0}=2 \\
& \gamma_{1}=u \\
& \gamma_{2}=\frac{1}{4} u_{x x}+\frac{3}{4} u^{2}  \tag{2.6}\\
& \gamma_{3}=\frac{1}{16} u_{4 x}+\frac{5}{8} u u_{x x}+\frac{5}{16} u_{x}^{2}+\frac{5}{8} u^{3} \\
& \gamma_{4}=\frac{1}{64} u_{6 x}+\frac{7}{32} u u_{4 x}+\frac{7}{16} u_{x} u_{3 x}+\frac{21}{64} u_{x x}^{2}+\frac{35}{32} u^{2} u_{x x}+\frac{35}{32} u u_{x}^{2}+\frac{35}{64} u^{4},
\end{align*}
$$

and related symmetries are

$$
\begin{aligned}
\mathcal{K}_{1} & =u_{x} \\
\mathcal{K}_{2} & =\frac{1}{4} u_{x x x}+\frac{3}{2} u u_{x} \\
\mathcal{K}_{3} & =\frac{1}{16} u_{5 x}+\frac{5}{8} u u_{3 x}+\frac{5}{4} u_{x} u_{x x}+\frac{15}{8} u^{2} u_{x}, \\
\mathcal{K}_{4} & =\frac{1}{64} u_{7 x}+\frac{7}{32} u u_{5 x}+\frac{21}{32} u_{x} u_{4 x}+\frac{35}{32} u_{x x} u_{3 x}+\frac{35}{32} u_{x}^{3}+\frac{35}{8} u u_{x} u_{x x}+\frac{35}{32} u^{2} u_{3 x}+\frac{35}{16} u^{3} u_{x}, \\
& \vdots
\end{aligned}
$$

As $u$ belongs to the whole hierarchy (2.2) it depends on infinitely many evolution parameters: $u=$ $u\left(t_{1}, t_{2}, t_{3}, \ldots\right)$.

In addition, with the KdV hierarchy of symmetries is related a hierarchy of master symmetries

$$
\begin{equation*}
\sigma_{m}=N^{m+1} \sigma_{-1}, \quad \tau_{-1}=1 \tag{2.8}
\end{equation*}
$$

non-local in general

$$
\begin{align*}
\sigma_{-1} & =1 \\
\sigma_{0} & =u+\frac{1}{2} x u_{x}  \tag{2.9}\\
\sigma_{1} & =\frac{1}{2} u_{x x}+\frac{1}{8} x u_{3 x}+u^{2}+\frac{1}{2} x u u_{x}+\frac{1}{4} u_{x} \partial_{x}^{-1} u
\end{align*}
$$

Both, symmetries $\mathcal{K}_{n}(2.5)$ and master symmetries $\sigma_{m}(2.8)$ constitute so called Virasoro algebra (hereditary algebra)

$$
\begin{equation*}
\left[\mathcal{K}_{m}, \mathcal{K}_{n}\right]=0, \quad\left[\sigma_{m}, \mathcal{K}_{n}\right]=\left(n-\frac{1}{2}\right) \mathcal{K}_{n+m}, \quad\left[\sigma_{m}, \sigma_{n}\right]=(n-m) \sigma_{n+m} \tag{2.10}
\end{equation*}
$$

Alternatively, the hierarchy (2.2) can be reconstructed from the isospectral Lax representation. Actually, consider some eigenvalue problem together with time evolutions of its eigenfunctions

$$
\begin{align*}
& L(u) \psi=\lambda \psi, \quad \lambda_{t_{n}}=0 \\
& \psi_{t_{n}}=B_{n}(u) \psi, \quad n=1,2, \ldots \tag{2.11}
\end{align*}
$$

where $L$ and $B_{r}$ are some differential operators. The compatibility conditions (Frobenius integrability conditions) for (2.11) takes the form

$$
\begin{equation*}
L_{t_{n}}=\left[B_{n}, L\right], \quad n=1,2, \ldots \tag{2.12}
\end{equation*}
$$

known as isospectral deformation equations, as the eigenvalues of the operator $L$ are independent of all times $t_{r}$, and are equivalent with evolutionary hierarchy of PDE's. For the KdV hierarchy

$$
\begin{equation*}
L=\partial_{x}^{2}+u, \quad B_{n}=\left(L^{n-\frac{1}{2}}\right)_{\geq 0}=\sum_{i=0}^{n-1}\left(-\frac{1}{4} \mathcal{K}_{i}+\frac{1}{2} \gamma_{i} \partial_{x}\right) L^{n-i-1} \tag{2.13}
\end{equation*}
$$

where explicitly

$$
\begin{align*}
& B_{1}=\partial_{x} \\
& B_{2}=\partial_{x}^{3}+\frac{3}{2} u \partial_{x}+\frac{3}{4} u_{x}  \tag{2.14}\\
& B_{3}=\partial_{x}^{5}+\frac{5}{2} u \partial_{x}^{3}+\frac{15}{4} u_{x}^{2} \partial_{x}^{2}+\frac{5}{8}\left(3 u^{2}+5 u_{x x}\right) \partial_{x}+\frac{15}{16}\left(u_{3 x}+2 u u_{x}\right)
\end{align*}
$$

Isospectral deformation equations (2.12), (2.13) can be presented in the equivalent form of so called zero curvature equations, more suitable for our further considerations. Rewriting equations (2.11) for the KdV hierarchy in the form

$$
\begin{align*}
\Psi_{x} & =U(\lambda ; u) \Psi, & \Psi=\left(\psi, \psi_{x}\right)^{T}  \tag{2.15}\\
\Psi_{t_{n}} & =V_{n}(\lambda ; u) \Psi, & n=1,2, \ldots \tag{2.16}
\end{align*}
$$

the compatibility conditions for (2.15) and (2.16) takes the form

$$
\begin{equation*}
\frac{d}{d t_{n}} U-\frac{d}{d x} V_{n}+\left[U, V_{n}\right]=0 \Longleftrightarrow u_{t_{n}}=\mathcal{K}_{n} \quad n=1,2, \ldots \tag{2.17}
\end{equation*}
$$

known as zero curvature conditions and reconstruct the KdV hierarchy. In (2.17) $\frac{d}{d x}$ means the total $x$-derivative and $\frac{d}{d t_{n}}$ means the $t_{n}$-evolutionary derivative. Actually, we have

$$
V_{k}=\left(\begin{array}{cc}
-\frac{1}{2} P_{x} & P  \tag{2.18}\\
P(\lambda-u)-\frac{1}{2} P_{x x} & \frac{1}{2} P_{x}
\end{array}\right), \quad P_{k}=\frac{1}{2} \sum_{i=0}^{k-1} \gamma_{i} \lambda^{k-i-1}
$$

In consequence

$$
\begin{gather*}
V_{1}=U=\left(\begin{array}{cc}
0 & 1 \\
\lambda-u & 0
\end{array}\right), \quad V_{2}=\left(\begin{array}{cc}
-\frac{1}{4} u_{x} & \lambda+\frac{1}{2} u \\
\lambda^{2}-\frac{1}{2} u \lambda-\frac{1}{2} u^{2}-\frac{1}{4} u_{x x} & \frac{1}{4} u_{x}
\end{array}\right),  \tag{2.19}\\
V_{3}=\left(\begin{array}{cc}
-\frac{1}{4} u_{x} \lambda-\frac{1}{16}\left(u_{3 x}+6 u u_{x}\right) & \lambda^{2}+\frac{1}{2} u \lambda+\frac{1}{8}\left(u_{x x}+3 u^{2}\right) \\
\lambda^{3}-\frac{1}{2} u \lambda^{2}-\frac{1}{8}\left(u_{x x}+u^{2}\right) \lambda-\left(\frac{1}{16} u_{4 x}+\frac{1}{2} u u_{x x}+\frac{3}{8} u_{x}^{2}+\frac{3}{8} u^{3}\right) & \frac{1}{4} u_{x} \lambda+\frac{1}{16}\left(u_{3 x}+6 u u_{x}\right)
\end{array}\right), \tag{2.20}
\end{gather*}
$$

Finally, the compatibility conditions between equations from (2.16) take the form of zero curvature conditions

$$
\begin{equation*}
\frac{d}{d t_{r}} V_{s}-\frac{d}{d t_{s}} V_{r}+\left[V_{s}, V_{r}\right]=0, \quad r, s=1,2, \ldots \tag{2.21}
\end{equation*}
$$

and are valid as differential consequences of commutativity of $r$-th and $s$-th KdV vector fields.
Let us define the $n$-th KdV stationary systems as 13

$$
\begin{equation*}
u_{t_{r}}=\mathcal{K}_{r}, \quad 0=\mathcal{K}_{n+1}, \quad r=1, \ldots, n \tag{2.22}
\end{equation*}
$$

The stationary restriction $0=\mathcal{K}_{n+1}$ provides constraint on the infinite-dimensional (functional) manifold, on which the KdV hierarchy is defined, reducing it to the finite-dimensional (stationary) submanifold $\mathcal{M}_{n}$ of dimension $(2 n+1)$, parametrized by jet coordinates $u, u_{x}, \ldots, u_{(2 n) x}$. The system (2.22) has two different $2 n$ dimensional representations, related to two different foliations of $\mathcal{M}_{n}$. The first one is of the form

$$
\begin{equation*}
u_{t_{r}}=\mathcal{K}_{r}, \quad 0=\gamma_{n+1}+c, \quad r=1, \ldots, n, \quad c \in \mathbb{R} \tag{2.23}
\end{equation*}
$$

where last equation in (2.23) is integrated an $(n+1)$ stationary flow of the KdV hierarchy (2.2) in the first Hamiltonian representation

$$
\begin{equation*}
0=\mathcal{K}_{n+1}=\partial_{x} \gamma_{n+1} \Longrightarrow 0=\gamma_{n+1}+c \tag{2.24}
\end{equation*}
$$

The second one is

$$
\begin{equation*}
u_{t_{r}}=\mathcal{K}_{r}, \quad 0=\frac{1}{2} \gamma_{n}\left(\gamma_{n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n}\right)_{x}\right]^{2}+u \gamma_{n}^{2}+c, \quad r=1, \ldots, n \tag{2.25}
\end{equation*}
$$

where the last equation in (2.25) is integrated an $(n+1)$ stationary flow of the hierarchy (2.2) in the second Hamiltonian representation. Indeed, differentiating it by $x$ and dividing by $2 \gamma_{n}$ we get

$$
\begin{equation*}
0=\left(\frac{1}{4} \partial_{x}^{3}+\frac{1}{2} u \partial_{x}+\frac{1}{2} \partial_{x} u\right) \gamma_{n}=\mathcal{K}_{n+1} \tag{2.26}
\end{equation*}
$$

Both representations constitute systems of ODE's with $n$ degrees of freedom, on $2 n$-dimensional phase space with jet coordinates $u, u_{x}, \ldots, u_{(2 n-1) x}$, as higher derivatives of $u$ with respect to $x$ are eliminated by the constraint (2.24) and (2.25), respectively.

Moreover, Lax representation of the system (2.23) is given by

$$
\begin{equation*}
\frac{d}{d t_{r}} V_{n+1}=\left[V_{r}, V_{n+1}\right], \quad r=2, \ldots, n, \quad \frac{d}{d x} V_{n+1}=\left[V_{1}, V_{n+1}\right] \tag{2.27}
\end{equation*}
$$

following from (2.21), with constraint $0=\gamma_{n+1}+c$, imposed on $V_{n+1}$, while Lax representation of the system (2.25) is given by (2.27) with constraint $0=\frac{1}{2} \gamma_{n}\left(\gamma_{n}\right)_{x x}-\frac{1}{4}\left(\gamma_{n}\right)_{x}^{2}+u \gamma_{n}^{2}+c$, imposed on $V_{n+1}$.

Stäckel representations of stationary systems (2.23) and (2.25) were derived in [13] and are presented in the next two sections.

## 3 Hamiltonian representation of the first KdV hierarchy of stationary systems

Let us briefly systematize known facts about the hierarchy of Stäckel systems, being the Hamiltonian representation of the KdV hierarchy of stationary systems (2.23) [2, 5, 6, 13.

Consider finite dimensional separable Hamiltonian systems, so called Stäckel systems, generated by the following hyperelliptic separation (spectral) curves on $(\lambda, \mu)$-plane

$$
\begin{equation*}
\lambda^{2 n+1}+c \lambda^{n}+\sum_{r=1}^{n} h_{r} \lambda^{n-r}=\mu^{2}, \quad n \in \mathbb{N}, \quad c=\text { const } . \tag{3.1}
\end{equation*}
$$

For fix $n \in \mathbb{N}$, consider a $2 n$-dimensional Poisson manifold $(M, \pi)$ and a particular set $\xi=\left(\lambda_{1}, \ldots, \lambda_{n}\right.$, $\left.\mu_{1}, \ldots, \mu_{n}\right) \in M$ of Darboux (canonical) coordinates on $M$, i.e. $\left\{\lambda_{i}, \lambda_{j}\right\}_{\pi}=\left\{\mu_{i}, \mu_{j}\right\}_{\pi}=0,\left\{\lambda_{i}, \mu_{j}\right\}_{\pi}=\delta_{i j}$. By taking $n$ copies of (3.1) at points $(\lambda, \mu)=\left(\lambda_{i}, \mu_{i}\right), i=1, \ldots, n$, we obtain a system of $n$ linear equations (separation relations) for $h_{r}$

$$
\begin{equation*}
\lambda_{i}^{2 n+1}+c \lambda_{i}^{n}+\sum_{r=1}^{n} h_{r} \lambda_{i}^{n-k}=\mu_{i}^{2}, \quad i=1, \ldots, n, \quad n \in \mathbb{N} . \tag{3.2}
\end{equation*}
$$

Solving (3.2) with respect to $h_{r}$ yields $n$ functions (Hamiltonians) $h_{r}$ on ( $M, \pi$ )

$$
\begin{align*}
h_{r} & =\sum_{i=1}^{n}(-1)^{r+1} \frac{\partial s_{r}}{\partial \lambda_{i}} \frac{\mu_{i}^{2}}{\Delta_{i}}+\sum_{i=1}^{n}(-1)^{r} \frac{\partial s_{r}}{\partial \lambda_{i}} \frac{\lambda_{i}^{2 n+1}+c \lambda_{i}^{n}}{\Delta_{i}} \\
& =\mu^{T} K_{r} G_{0} \mu+V_{r}^{(2 n+1)}+c V_{r}^{(n)}, \quad r=1, \ldots, n \tag{3.3}
\end{align*}
$$

where

$$
\begin{align*}
G_{0} & =\operatorname{diag}\left(\frac{1}{\Delta_{1}}, \ldots, \frac{1}{\Delta_{n}}\right), \quad \Delta_{j}=\prod_{k \neq j}\left(\lambda_{j}-\lambda_{k}\right)  \tag{3.4}\\
K_{r} & =(-1)^{r+1} \operatorname{diag}\left(\frac{\partial s_{r}}{\partial \lambda_{1}}, \cdots, \frac{\partial s_{r}}{\partial \lambda_{n}}\right), \quad r=1, \ldots, n \tag{3.5}
\end{align*}
$$

$s_{r}$ are elementary symmetric polynomials in $\lambda_{i}$ and $V_{r}^{(k)}, k \in \mathbb{Z}$ are basic separable potentials. Matrix $G_{0}$ can be interpreted as a contravariant metric tensor on an $n$-dimensional manifold $Q$ such that $M=T^{*} Q$. It can be shown that the metric $G_{0}$ is flat. Matrices $K_{r}$ are $(1,1)$-Killing tensors for the metric $G_{0}$.

Thus, we have constructed from scratch infinite hierarchy of separable autonomous Hamiltonian systems

$$
\begin{equation*}
\xi_{t_{r}}=\pi d h_{r}=X_{r}, \quad r=1, \ldots, n, \quad n \in \mathbb{N} \tag{3.6}
\end{equation*}
$$

where each system consists of $n$ evolution ODE's. The first Hamiltonian $h_{1}$ of each system can be interpreted as the Hamiltonian of a particle in the pseudo-Riemannian $n$-dimensional configuration space
$\left(Q, g_{0}=G_{0}^{-1}\right)$, moving under particular potential force and the remaining Hamiltonians are its constants of motion.

By their very construction from separation relations, the Hamiltonians $h_{r}$ Poisson commute for all $r, k=1, \ldots, n$

$$
\begin{equation*}
\left\{h_{r}, h_{k}\right\}_{\pi}=\pi\left(d h_{r}, d h_{k}\right)=0 \tag{3.7}
\end{equation*}
$$

and so that $\left[X_{r}, X_{k}\right]=0$. It means that each system is Liouville integrable and hence is Frobenius integrable, i.e. the system (3.6) has a common, unique solution $\xi\left(t_{1}, \ldots, t_{n}, \xi_{0}\right)$ through each point $\xi_{0} \in M$, depending in general on all the evolution parameters $t_{s}$, constructed from separation relations (3.2) by quadratures.

In what follows we will work in so called Viéte canonical coordinates $(q, p) \in T^{*} Q$

$$
\begin{equation*}
q_{i}=(-1)^{i} s_{i}(\lambda), \quad p_{i}=-\sum_{k=1}^{n} \frac{\lambda_{k}^{n-i} \mu_{k}}{\Delta_{k}}, \quad i=1, \ldots, n \tag{3.8}
\end{equation*}
$$

in which all functions $h_{r}$ are polynomial functions of their arguments. Explicitly

$$
G_{0}^{i j}=q_{i+j-n-1}, \quad\left(K_{r}\right)_{j}^{i}= \begin{cases}q_{i-j+r-1}, & i \leq j \text { and } r \leq j  \tag{3.9}\\ -q_{i-j+r-1}, & i>j \text { and } r>j \\ 0, & \text { otherwise }\end{cases}
$$

where we set $q_{0}=1, q_{k}=0$ for $k<0$ or $k>n$. Elementary separable potentials $V_{r}^{(\alpha)}$ can be explicitly constructed by the recursion formula [7]

$$
V^{(\alpha)}=R^{\alpha} V^{(0)}, \quad V^{(\alpha)}=\left(V_{1}^{(\alpha)}, \ldots, V_{n}^{(\alpha)}\right)^{T}, \quad R=\left(\begin{array}{cccc}
-q_{1} & 1 & 0 & 0  \tag{3.10}\\
\vdots & 0 & \ddots & 0 \\
\vdots & 0 & 0 & 1 \\
-q_{n} & 0 & 0 & 0
\end{array}\right)
$$

with $V^{(0)}=(0, \ldots, 0,-1)^{T}$. The first $n$ basic separable potentials are trivial

$$
V_{k}^{(\alpha)}=-\delta_{k, n-\alpha}, \quad \alpha=0, \ldots, n-1
$$

The first nontrivial positive and negative potentials are

$$
V^{(n)}=\left(q_{1}, \ldots, q_{n}\right)^{T}, \quad V^{(-1)}=\left(\frac{1}{q_{n}}, \ldots, \frac{q_{n-1}}{q_{n}}\right)^{T}
$$

and higher positive and negative potentials are more complicated polynomials and rational functions in all $q_{i}$.

Besides, the autonomous Hamiltonian equations (3.6) are represented by (i.e. are differential consequences of) Lax isospectral deformation equations

$$
\begin{equation*}
\frac{d}{d t_{r}} L(\lambda ; \xi)=\left[U_{r}(\lambda ; \xi), L(\lambda ; \xi)\right], \quad r=1, \ldots, n \tag{3.11}
\end{equation*}
$$

where $\frac{d}{d t_{r}}$ is the evolutionary derivative along the $r$-th flow from (3.6)

$$
\begin{equation*}
\frac{d}{d t_{r}} A=\frac{\partial A}{d t_{r}}+\left\{A, h_{r}\right\}=\frac{\partial A}{\partial t_{r}}+A^{\prime}\left[X_{r}\right] \tag{3.12}
\end{equation*}
$$

with $L(\lambda ; \xi)$ and $U_{r}(\lambda ; \xi)$ being matrices belonging to some Lie algebra and depending rationally on the spectral parameter $\lambda$.

Actually, for the hierarchy generated by Hamiltonians (3.3), there is an infinitely many nonequivalent Lax representations [8]. Here we chose the one compatible with (2.17)-(2.20).

Theorem 1 [8] The $L(\lambda ; \xi)$ matrix for the system (3.6) takes a form

$$
L(\lambda ; \xi)=\left(\begin{array}{cc}
v(\lambda ; \xi) & u(\lambda ; \xi)  \tag{3.13}\\
w(\lambda ; \xi) & -v(\lambda ; \xi)
\end{array}\right)
$$

where, in Viéte coordinates $\xi=(q, p)$, we have

$$
\begin{equation*}
u(\lambda ; q)=\lambda^{n}+\sum_{k=1}^{n} q_{k} \lambda^{n-k}, \quad v(\lambda ; q, p)=-\sum_{k=1}^{n} M_{k}(q, p) \lambda^{n-k}, \quad M_{k}=\sum_{j=1}^{n} G_{0}^{k j} p_{j} \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
w(\lambda ; q, p, t)=-\left[\frac{v(\lambda)^{2}-\lambda^{2 n+1}-c \lambda^{n}}{u(\lambda)}\right]_{+} . \tag{3.15}
\end{equation*}
$$

Here, the operation $[\cdot]_{+}$means the projection on the uniquely defined quotient of the division of an analytic function $A$ over a (pure) polynomial $u(\lambda)$ such that the following decomposition holds:

$$
A=\left[\frac{A}{u}\right]_{+} u+r
$$

where the (unique) remainder $r$ is a lower degree polynomial than the polynomial $u$, see for details [8]. In particular, when $A$ is a Laurent polynomial, we have

$$
\left[\frac{A}{u}\right]_{+} \equiv\left[\frac{[A]_{\geqslant 0}}{u}\right]_{\geqslant 0}+\left[\frac{[A]_{<0}}{u}\right]_{<0}
$$

where $[\cdot]_{\geqslant 0}$ is the projection on the part consisting of non-negative degree terms in the expansion into Laurent series at $\infty$ and $[\cdot]_{<0}$ is the projection on the part consisting of negative degree terms in the expansion into Laurent series at 0. Moreover,

$$
\begin{equation*}
U_{r}(\lambda ; \xi)=\left[\frac{u_{r}(\lambda) L(\lambda)}{u(\lambda)}\right]_{+}, \quad \text { for } r=1, \ldots, n \tag{3.16}
\end{equation*}
$$

where

$$
u_{r}(\lambda)=\lambda^{r-1}+\sum_{k=1}^{r-2} q_{k} \lambda^{r-k-1}
$$

What is interesting, the separation curve (3.1) is reconstructed from Lax matrix $L(\lambda ; \xi)$ through

$$
\operatorname{det}[L(\lambda ; \xi)-\mu I]=0
$$

The relation between the Stäckel hierarchy (3.6) and the KdV stationary hierarchy (2.23) is as follows.
Theorem 2 [13] For fixed $n \in \mathbb{N}$ and identification of $t_{1}$ with $x$ we get the following equivalence between the Stäckel system (3.6) and the $n$-th KdV stationary system (2.23)

$$
\begin{gather*}
\xi_{t_{r}}=X_{r}, \quad r=1, \ldots, n  \tag{3.17}\\
\hat{\Downarrow} \\
u_{t_{r}}=\mathcal{K}_{r}, \quad 0=\gamma_{n+1}+c, \quad r=1, \ldots, n, \tag{3.18}
\end{gather*}
$$

where the transformation between jet and Viéte coordinates is as follows

$$
\begin{equation*}
q_{k}=\frac{1}{2} \gamma_{k}, \quad k=1, \ldots, n, \quad p_{k}=\frac{1}{2} \sum_{j=1}^{n}\left(G_{0}^{-1}\right)_{k j}\left(q_{j}\right)_{x} \tag{3.19}
\end{equation*}
$$

In particular, the first flow $\xi_{x}=X_{1}$ reconstructs (3.19) and the constraint

$$
\begin{equation*}
0=\left(p_{1}\right)_{x}-\left(X_{1}\right)^{n+1} \Leftrightarrow 0=\gamma_{n+1}+c \tag{3.20}
\end{equation*}
$$

Besides, the first component of the r-th flow (3.17) reconstructs the r-th KdV equation

$$
\begin{equation*}
\left(q_{1}\right)_{t_{r}}=\left(X_{r}\right)^{1} \Leftrightarrow u_{t_{r}}=\mathcal{K}_{r}, \quad r=1, \ldots, n \tag{3.21}
\end{equation*}
$$

while the remaining components of systems (3.17) for $r=2, \ldots, n$ are differential consequences of (3.20) and (3.21). On the level of Lax representation of the Stäckel hierarchy (3.11) and the Lax representation (2.27) of the $K d V$ stationary system (2.23), fixing $n$ we have the following identities

$$
\begin{equation*}
U_{i}=V_{i}, \quad i=1, \ldots, n \tag{3.22}
\end{equation*}
$$

and additionally

$$
\begin{equation*}
L=V_{n+1} \quad \text { under constraint } 0=\gamma_{n+1}+c . \tag{3.23}
\end{equation*}
$$

Both systems (3.17) and (3.18) share the same solutions, which are the so called finite gape solutions and rational solutions of related KdV hierarchy [6].

Example 3 Consider the case $n=3$. In ( $q, p$ ) coordinates, three Hamiltonians $h_{r}=p^{T} K_{r} G_{0} p+V_{r}^{(7)}+$ $c V_{r}^{(3)}$, metric tensor $G_{0}$ and its inverse are

$$
\begin{aligned}
h_{1}= & \left(2 p_{1} p_{3}+p_{2}^{2}+2 q_{1} p_{2} p_{3}+q_{2} p_{3}^{2}\right)+\left(q_{1}^{5}-4 q_{1}^{3} q_{2}+3 q_{1}^{2} q_{3}+3 q_{1} q_{2}^{2}-2 q_{2} q_{3}\right)+c q_{1}, \\
h_{2}= & {\left[2 q_{1} p_{1} p_{3}+2 q_{1} p_{2}^{2}+2 p_{1} p_{2}+\left(q_{1} q_{2}-q_{3}\right) p_{3}^{2}+2 q_{1}^{2} p_{2} p_{3}\right]+\left(q_{1}^{4} q_{2}-q_{1}^{3} q_{3}-3 q_{1}^{2} q_{2}^{2}+4 q_{1} q_{2} q_{3}+q_{2}^{3}-q_{3}^{2}\right)+c q_{2}, } \\
h_{3}= & p_{1}^{2}+2 q_{1} p_{1} p_{2}+2 q_{2} p_{1} p_{3}+q_{1}^{2} p_{2}^{2}+\left(q_{2}^{2}-q_{1} q_{3}\right) p_{3}^{2}+2\left(q_{1} q_{2}-q_{3}\right) p_{2} p_{3} \\
& +\left(q_{1}^{4} q_{3}-3 q_{1}^{2} q_{2} q_{3}+2 q_{1} q_{3}^{2}+q_{2}^{2} q_{3}\right)+c q_{3}, \\
& \quad G_{0}=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & q_{1} \\
1 & q_{1} & q_{2}
\end{array}\right), \quad G_{0}^{-1}=\left(\begin{array}{ccc}
q_{1}^{2}-q_{2} & -q_{1} & 1 \\
-q_{1} & 1 & 0 \\
1 & 0 & 0
\end{array}\right)
\end{aligned}
$$

The related autonomous evolution equations are

$$
\begin{gather*}
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3} \\
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right)_{x}=X_{1}=\left(\begin{array}{l}
2 p_{3} \\
2 q_{1} p_{3}+2 p_{2} \\
2 q_{1} p_{2}+2 q_{2} p_{3}+2 p_{1} \\
-2 p_{2} p_{3}-5 q_{1}^{4}+12 q_{1}^{2} q_{2}-6 q_{1} q_{3}-3 q_{2}^{2}-c \\
-p_{3}^{2}+4 q_{1}^{3}-6 q_{1} q_{2}+2 q_{3} \\
-3 q_{1}^{2}+2 q_{2}
\end{array}\right)  \tag{3.24}\\
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3} \\
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right)_{t_{2}}=X_{2}=\left(\begin{array}{l}
2 p_{2}+2 q_{1} p_{3} \\
2 p_{1}+2 q_{1}^{2} p_{3}+4 q_{1} p_{2} \\
2 q_{1} p_{1}+2\left(q_{1} q_{2}-q_{3}\right) p_{3}+2 q_{1}^{2} p_{2} \\
-2 p_{1} p_{3}-2 p_{2}^{2}-4 q_{1} p_{2} p_{3}-q_{2} p_{3}^{2}+6 q_{1} q_{2}^{2}-4 q_{1}^{3} q_{2}+3 q_{1}^{2} q_{3}-4 q_{2} q_{3} \\
-q_{1} p_{3}^{2}-q_{1}^{4}+6 q_{1}^{2} q_{2}-4 q_{1} q_{3}-3 q_{2}^{2}-c \\
p_{3}^{2}+q_{1}^{3}-4 q_{1} q_{2}+2 q_{3}
\end{array}\right.  \tag{3.25}\\
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3}
\end{array}\right)_{p_{1}}=X_{3}=\left(\begin{array}{l}
2 q_{1} p_{2}+2 q_{2} p_{3}+2 p_{1} \\
2 q_{1}^{2} p_{2}+2 q_{1} p_{1}+2\left(q_{1} q_{2}-q_{3}\right) p_{3} \\
2 q_{2} p_{1}+2\left(q_{2}^{2}-q_{1} q_{3}\right) p_{3}+2\left(q_{1} q_{2}-q_{3}\right) p_{2} \\
-2 q_{1} p_{2}^{2}-2 q_{2} p_{2} p_{3}+q_{3} p_{3}^{2}-4 q_{1}^{3} q_{3}+6 q_{1} q_{2} q_{3}-2 p_{1} p_{2}-2 q_{3}^{2} \\
-2 p_{1} p_{3}-2 q_{1} p_{2} p_{3}-2 q_{2} p_{3}^{2}+3 q_{1}^{2} q_{3}-2 q_{2} q_{3} \\
q_{1} p_{3}^{2}+3 q_{1}^{2} q_{2}+2 p_{2} p_{3}-q_{1}^{4}-4 q_{1} q_{3}-q_{2}^{2}-c
\end{array}\right) \tag{3.26}
\end{gather*}
$$

Lax representations (3.11) of considered equations are as follows
$L=\left(\begin{array}{cc}-p_{3} \lambda^{2}-\left(q_{1} p_{3}+p_{2}\right) \lambda-p_{1}-q_{1} p_{2}-q_{2} p_{3} & \lambda^{3}+q_{1} \lambda^{2}+q_{2} \lambda+q_{3} \\ \lambda^{4}-q_{1} \lambda^{3}-\left(q_{2}-q_{1}^{2}\right) \lambda^{2}-\left(p_{3}^{2}+q_{1}^{3}-2 q_{1} q_{2}+q_{3}\right) \lambda & p_{3} \lambda^{2}+\left(q_{1} p_{3}+p_{2}\right) \lambda+p_{1}+q_{1} p_{2}+q_{2} p_{3} \\ -q_{1} p_{3}^{2}-2 p_{2} p_{3}+q_{1}^{4}-3 q_{1}^{2} q_{2}+2 q_{1} q_{3}+q_{2}^{2}+c & \end{array}\right)$,

$$
\begin{align*}
& U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{3} & \lambda+q_{1} \\
\lambda^{2}-q_{1} \lambda+q_{1}^{2}-2 q_{2} & p_{3}
\end{array}\right), \\
& U_{3}=\left(\begin{array}{cc}
-p_{3} \lambda-p_{2}-q_{1} p_{3} & \lambda^{2}+q_{1} \lambda+q_{2} \\
\lambda^{3}-q_{1} \lambda^{2}+\left(q_{1}^{2}-q_{2}\right) \lambda-p_{3}^{2}-q_{1}^{3}+2 q_{1} q_{2}-2 q_{3} & p_{3} \lambda+p_{2}+q_{1} p_{3}
\end{array}\right) . \tag{3.28}
\end{align*}
$$

Substituting $q_{1}=\frac{1}{2} \gamma_{1}=\frac{1}{2} u$ we obtain recursively from (3.24)

$$
\begin{gathered}
\left(q_{1}\right)_{x}=\left(X_{1}\right)^{1} \Longleftrightarrow p_{3}=\frac{1}{4} u_{x} \\
\left(p_{3}\right)_{x}=\left(X_{1}\right)^{6} \Longleftrightarrow q_{2}=\frac{1}{8} u_{x x}+\frac{3}{8} u^{2}=\frac{1}{2} \gamma_{2}, \\
\left(q_{2}\right)_{x}=\left(X_{1}\right)^{2} \Longleftrightarrow p_{2}=\frac{1}{16} u_{3 x}+\frac{1}{4} u u_{x} \\
\left(p_{2}\right)_{x}=\left(X_{1}\right)^{5} \Longleftrightarrow q_{3}=\frac{1}{32} u_{4 x}+\frac{5}{16} u u_{x x}+\frac{5}{32} u_{x}^{2}+\frac{5}{16} u^{3}=\frac{1}{2} \gamma_{3}, \\
\left(q_{3}\right)_{x}=\left(X_{1}\right)^{3} \Longleftrightarrow p_{1}=\frac{1}{64} u_{5 x}+\frac{9}{32} u_{x} u_{x x}+\frac{1}{8} u u_{3 x}+\frac{1}{4} u^{2} u_{x} \\
\left(p_{1}\right)_{x}=\left(X_{1}\right)^{4} \Leftrightarrow \frac{1}{64} u_{6 x}+\frac{7}{32} u u_{4 x}+\frac{7}{16} u_{x} u_{3 x}+\frac{21}{64} u_{x x}^{2}+\frac{35}{32} u^{2} u_{x x}+\frac{35}{32} u u_{x}^{2}+\frac{35}{64} u^{4}+c=\gamma_{4}+c=0,
\end{gathered}
$$

which coincide with (3.19) and the last equation of (3.18). From evolution equation (3.25) we get

$$
\left(q_{1}\right)_{t_{2}}=\left(X_{2}\right)^{1} \Longleftrightarrow u_{t_{2}}=\frac{1}{4} u_{x x x}+\frac{3}{2} u u_{x}=\mathcal{K}_{2}
$$

and the remaining equations are differential consequences of $u_{t_{2}}=\mathcal{K}_{2}$ and $\gamma_{4}+c=0$. For example

$$
\left(q_{3}\right)_{t_{2}}=\left(X_{2}\right)^{3} \Longleftrightarrow\left(\gamma_{4}\right)_{x}=0, \quad\left(p_{1}\right)_{t_{2}}=\left(X_{2}\right)^{4} \Longleftrightarrow\left(\gamma_{4}\right)_{x x}=0
$$

and so on. Finally, from evolution equation (3.26) we get

$$
\left(q_{1}\right)_{t_{3}}=\left(X_{3}\right)^{1} \Longleftrightarrow u_{t_{3}}=\frac{1}{16} u_{5 x}+\frac{5}{8} u u_{3 x}+\frac{5}{4} u_{x} u_{x x}+\frac{15}{8} u^{2} u_{x}=\mathcal{K}_{3},
$$

and again the remaining equations are differential consequences of $u_{t_{3}}=\mathcal{K}_{3}$ and $\gamma_{4}+c=0$.Thus, indeed we have the equivalence between both representations (3.17) and (3.18). Moreover, under above substitution, Stäckel Lax matrices $U_{r}$ (3.28) turn into KdV matrices $V_{r}$ (2.18)-(2.20) while $L$ matrix (3.27) is related to the $K d V$ matrix $V_{4}$ under constraint $0=\gamma_{4}+c$.

Summarizing results of this section, the $n$-th KdV stationary system (2.23) has the $n$-dimensional Stäckel representation (3.6), generated by separation curve (3.1).

## 4 Hamiltonian representation of the second KdV hierarchy of stationary systems

Let us briefly collect known facts about the hierarchy of Stäckel systems being the Hamiltonian representation of the KdV hierarchy of stationary systems (2.25) [5, 6, 13 .

Consider Stäckel systems generated by the following separation curves on $(\lambda, \mu)$-plane

$$
\begin{equation*}
\lambda^{2 n}+c \lambda^{-1}+\sum_{r=1}^{n} h_{r} \lambda^{n-r}=\lambda \mu^{2}, \quad n \in \mathbb{N} \tag{4.1}
\end{equation*}
$$

Following the procedure from the previous section we construct $n$ Hamiltonian functions in involution of the form

$$
\begin{align*}
h_{r} & =\sum_{i=1}^{n}(-1)^{r+1} \frac{\partial s_{r}}{\partial \lambda_{i}} \frac{\lambda_{i} \mu_{i}^{2}}{\Delta_{i}}+\sum_{i=1}^{n}(-1)^{r} \frac{\partial s_{r}}{\partial \lambda_{i}} \frac{\lambda_{i}^{2 n}+c \lambda_{i}^{-1}}{\Delta_{i}} \\
& =\mu^{T} K_{r} G_{1} \mu+V_{r}^{(2 n)}+c V_{r}^{(-1)}, \quad r=1, \ldots, n \tag{4.2}
\end{align*}
$$

where

$$
\begin{equation*}
G_{1}=\operatorname{diag}\left(\frac{\lambda_{1}}{\Delta_{1}}, \ldots, \frac{\lambda_{n}}{\Delta_{n}}\right) \tag{4.3}
\end{equation*}
$$

and Killing tensors $K_{r}$ of $G$ takes again the form (3.5). In Viéte coordinates (3.8)

$$
G_{1}^{i j}= \begin{cases}q_{i+j-n}, & i, j=1, \ldots, n-1  \tag{4.4}\\ -q_{n}, & i=j=n\end{cases}
$$

Consider the hierarchy of Hamiltonian autonomous dynamical systems, where each system consists of $n$ evolution equations

$$
\begin{equation*}
\xi_{t_{r}}=X_{r}=\pi d h_{r}, \quad r=1, \ldots, n, \quad n \in \mathbb{N} \tag{4.5}
\end{equation*}
$$

generated by $n$ Hamiltonian functions $h_{r}$ (4.2). The related Lax isospectral equations are of the form (3.11), where $L(\lambda ; \xi)$ matrix takes the form (3.13), where now, in Viéte coordinates $\xi=(q, p)$, we have [8]

$$
\begin{equation*}
u(\lambda ; q)=\lambda^{n}+\sum_{k=1}^{n} q_{k} \lambda^{n-k}, \quad v(\lambda ; q, p)=-\sum_{k=1}^{n} M_{k}(q, p) \lambda^{n-k}, \quad M_{k}=\sum_{j=1}^{n} G_{1}^{k j} p_{j} \tag{4.6}
\end{equation*}
$$

and

$$
\begin{equation*}
w(\lambda ; q, p)=-\lambda\left[\frac{v(\lambda)^{2} \lambda^{-1}-\lambda^{2 n}-c \lambda^{-1}}{u(\lambda)}\right]_{+} \tag{4.7}
\end{equation*}
$$

Moreover, matrices $U_{r}(\lambda ; \xi)$ are given by the same formula (3.16).
Again, the separation curve (4.1) is reconstructed from Lax matrix $L(\lambda ; \xi)$ through

$$
\operatorname{det}[L(\lambda ; \xi)-\lambda \mu I]=0
$$

The relation between Stäckel hierarchy (4.5) and the second KdV stationary hierarchy (2.25) is as follows.

Theorem 4 [13] For fixed $n \in \mathbb{N}$ and identification $t_{1}$ with $x$ we get the following equivalence between the Stäckel system (4.5) and the $n$-th KdV stationary systems (2.25)

$$
\begin{gather*}
\xi_{t_{r}}=X_{r}, \quad r=1, \ldots, n  \tag{4.8}\\
\hat{\mathbb{1}} \\
u_{t_{r}}=\mathcal{K}_{r}, \quad 0=\frac{1}{2} \gamma_{n}\left(\gamma_{n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n}\right)_{x}\right]^{2}+u \gamma_{n}^{2}+c, \quad r=2, \ldots, n, \tag{4.9}
\end{gather*}
$$

where the transformation between jet and Viéte coordinates is as follows

$$
q_{k}=\frac{1}{2} \gamma_{k}, \quad k=2, \ldots, n, \quad p_{k}=\frac{1}{2} \sum_{j=1}^{n}\left(G_{1}^{-1}\right)_{k j}\left(q_{j}\right)_{x}
$$

with new metric tensor $G_{1}$ (4.4). The constraint is encoded in the last component of the first flow 4.8)

$$
\begin{equation*}
0=\left(p_{n}\right)_{x}-\left(X_{1}\right)^{2 n} \Leftrightarrow 0=\frac{1}{2} \gamma_{n}\left(\gamma_{n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n}\right)_{x}\right]^{2}+u \gamma_{n}^{2}+c \tag{4.10}
\end{equation*}
$$

Besides,

$$
\begin{equation*}
\left(q_{1}\right)_{t_{r}}=X_{r}^{1} \Leftrightarrow u_{t_{r}}=\mathcal{K}_{r}, \quad r=1, \ldots, n \tag{4.11}
\end{equation*}
$$

and remaining components from systems 4.8) for $r=2, \ldots, n$ are differential consequences of 4.10) and 4.11). On the level of Lax representation of the Stäckel hierarchy (3.11) and the Lax representation (2.27) of the KdV stationary system (2.25), fixing $n$ we have the same identities (3.22) and now

$$
\begin{equation*}
L=V_{n+1} \quad \text { under constraint } \quad 0=\frac{1}{2} \gamma_{n}\left(\gamma_{n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n}\right)_{x}\right]^{2}+u \gamma_{n}^{2}+c \tag{4.12}
\end{equation*}
$$

Like in the previous case, both systems (4.8) and (4.9) share the same solutions.

Example 5 Consider once more the case $n=3$. In ( $q, p$ ) coordinates, three Hamiltonians $h_{r}=p^{T} K_{r} G p+$ $V_{r}^{(6)}+c V_{r}^{(-1)}$, metric tensor $G$ and its inverse are

$$
\begin{align*}
& h_{1}=\left(2 p_{1} p_{2}+q_{1} p_{2}^{2}-q_{3} p_{3}^{2}\right)+\left(-q_{1}^{4}+3 q_{1}^{2} q_{2}-2 q_{1} q_{3}-q_{2}^{2}\right)+c q_{3}^{-1} \\
& h_{2}=\left[p_{1}^{2}+2 q_{1} p_{1} p_{2}-2 q_{3} p_{2} p_{3}+\left(q_{1}^{2}-q_{2}\right) p_{2}^{2}-q_{1} q_{3} p_{3}^{2}\right]+\left(-q_{1}^{3} q_{2}+q_{1}^{2} q_{3}+2 q_{1} q_{2}^{2}-2 q_{2} q_{3}\right)+c q_{1} q_{3}^{-1} \\
& h_{3}=\left(-2 q_{3} p_{1} p_{3}-q_{3} p_{2}^{2}-2 q_{1} q_{3} p_{2} p_{3}-q_{2} q_{3} p_{3}^{2}\right)+\left(-q_{1}^{3} q_{3}+2 q_{1} q_{2} q_{3}-q_{3}^{2}\right)+c q_{2} q_{3}^{-1} \\
& G_{1}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & q_{1} & 0 \\
0 & 0 & -q_{3}
\end{array}\right), \quad G_{1}^{-1}=\left(\begin{array}{ccc}
-q_{1} & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -\frac{1}{q_{3}}
\end{array}\right) . \tag{4.13}
\end{align*}
$$

The related autonomous evolution equations are

$$
\left.\begin{array}{c}
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3} \\
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right)_{x}=X_{1}=\left(\begin{array}{l}
2 p_{2} \\
2 q_{1} p_{2}+2 p_{1} \\
-2 q_{3} p_{3} \\
-p_{2}^{2}+4 q_{1}^{3}-6 q_{1} q_{2}+2 q_{3} \\
-3 q_{1}^{2}+2 q_{2} \\
p_{3}^{2}+2 q_{1}+c q_{3}^{-2}
\end{array}\right) \\
\left.\left(\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3} \\
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right)_{t_{3}}^{q_{2}} \begin{array}{l}
q_{3} \\
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right)_{t_{2}}=X_{3}=\left(\begin{array}{l}
2 p_{1}+2 q_{1} p_{2} \\
2 q_{1} p_{1}-2 q_{3} p_{3}+2\left(q_{1}^{2}-q_{2}\right) p_{2} \\
-2 q_{1} q_{3} p_{3}-2 q_{3} p_{2} \\
-2 p_{1} p_{2}-2 q_{1} p_{2}^{2}+q_{3} p_{3}^{2}+3 q_{1}^{2} q_{2}-2 q_{1} q_{3}-2 q_{2}^{2}-c q_{3}^{-1} \\
p_{2}^{2}+q_{1}^{3}-4 q_{1} q_{2}+2 q_{3} \\
2 p_{2} p_{3}+q_{1} p_{3}^{2}-q_{1}^{2}+2 q_{2}+c q_{1} q_{3}^{-2}
\end{array}\right) \\
-2 q_{3} p_{1}-2 q_{1} q_{3} p_{2}-2 q_{2} q_{3} p_{3}  \tag{4.16}\\
2 q_{3} p_{2} p_{3}+3 q_{1}^{2} q_{3}-2 q_{2} q_{3} \\
q_{3} p_{3}^{2}-2 q_{1} q_{3}-c q_{3}^{-1} \\
2 p_{1} p_{3}+p_{2}^{2}+2 q_{1} p_{2} p_{3}+q_{2} p_{3}^{2}+q_{1}^{3}-2 q_{1} q_{2}+2 q_{3}+c q_{2} q_{3}^{-2}
\end{array}\right) .
$$

Lax representations (3.11) of considered equations are as follows

$$
\begin{gather*}
L=\left(\begin{array}{cc}
-p_{2} \lambda^{2}-\left(q_{1} p_{2}+p_{1}\right) \lambda+q_{3} p_{3} & \lambda^{3}+q_{1} \lambda^{2}+q_{2} \lambda+q_{3} \\
\lambda^{4}-q_{1} \lambda^{3}-\left(q_{2}-q_{1}^{2}\right) \lambda^{2}-\left(p_{2}^{2}+q_{1}^{3}-2 q_{1} q_{2}+q_{3}\right) \lambda & p_{2} \lambda^{2}+\left(q_{1} p_{2}+p_{1}\right) \lambda-q_{3} p_{3} \\
-q_{3} p_{3}^{2}+c q_{3}^{-1}
\end{array}\right)  \tag{4.17}\\
U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{2} & \lambda+q_{1} \\
\lambda^{2}-q_{1} \lambda+q_{1}^{2}-2 q_{2} & p_{2}
\end{array}\right) \\
U_{3}=\left(\begin{array}{cc}
-p_{2} \lambda-p_{1}-q_{1} p_{2} & \lambda^{2}+q_{1} \lambda+q_{2} \\
\lambda^{3}-q_{1} \lambda^{2}+\left(q_{1}^{2}-q_{2}\right) \lambda-p_{2}^{2}-q_{1}^{3}+2 q_{1} q_{2}-2 q_{3} & p_{2} \lambda+p_{1}+q_{1} p_{2}
\end{array}\right) \tag{4.18}
\end{gather*}
$$

Substituting $q_{1}=\frac{1}{2} \gamma_{1}=\frac{1}{2} u$ we obtain recursively from 4.14)

$$
\begin{gathered}
\left(q_{1}\right)_{x}=\left(X_{1}\right)^{1} \Longleftrightarrow p_{2}=\frac{1}{4} u_{x}, \\
\left(p_{2}\right)_{x}=\left(X_{1}\right)^{5} \Longleftrightarrow q_{2}=\frac{1}{8} u_{x x}+\frac{3}{8} u^{2}=\frac{1}{2} \gamma_{2}, \\
\left(q_{2}\right)_{x}=\left(X_{1}\right)^{2} \Longleftrightarrow p_{1}=\frac{1}{16} u_{3 x}+\frac{1}{4} u u_{x} \\
\left(p_{1}\right)_{x}=\left(X_{1}\right)^{4} \Longleftrightarrow q_{3}=\frac{1}{32} u_{4 x}+\frac{5}{16} u u_{x x}+\frac{5}{32} u_{x}^{2}+\frac{5}{16} u^{3}=\frac{1}{2} \gamma_{3}
\end{gathered}
$$

$$
\begin{gather*}
\left(q_{3}\right)_{x}=\left(X_{1}\right)^{3} \Longleftrightarrow p_{3}=-\frac{1}{2} \frac{\left(q_{3}\right)_{x}}{q_{3}}=-\frac{1}{2} \frac{\left(\gamma_{3}\right)_{x}}{\gamma_{3}} \\
\left(p_{3}\right)_{x}=\left(X_{1}\right)^{6} \Longleftrightarrow 0=\frac{1}{2} q_{3}\left(q_{3}\right)_{x x}-\frac{1}{4}\left(q_{3}^{2}\right)_{x}+u q_{3}^{2}+c \Rightarrow 0=\pi_{1} \gamma_{3}=\mathcal{K}_{4} \tag{4.19}
\end{gather*}
$$

which coincide with (3.19) with new metric tensor 4.13) and the last equation of (4.9). From evolution equation (3.25) we get

$$
\begin{equation*}
\left(q_{1}\right)_{t_{2}}=\left(X_{2}\right)^{1} \Longleftrightarrow u_{t_{2}}=\frac{1}{4} u_{x x x}+\frac{3}{2} u u_{x}=\mathcal{K}_{2} \tag{4.20}
\end{equation*}
$$

and the remaining equations are differential consequences of 4.19) and 4.20. Finally, from evolution equation 4.16 we get

$$
\begin{equation*}
\left(q_{1}\right)_{t_{3}}=\left(X_{3}\right)^{1} \Longleftrightarrow u_{t_{3}}=\frac{1}{16} u_{5 x}+\frac{5}{8} u u_{3 x}+\frac{5}{4} u_{x} u_{x x}+\frac{15}{8} u^{2} u_{x}=\mathcal{K}_{3}, \tag{4.21}
\end{equation*}
$$

and again the remaining equations are differential consequences of (4.19) and 4.21). Moreover, under above substitution, Stäckel Lax matrices $U_{r}$ (4.18) turn into KdV matrices $V_{r}$ (2.18)-(2.20) while L matrix (4.17) is the KdV matrix $V_{4}$ under the constraint $0=\frac{1}{2} \gamma_{3}\left(\gamma_{3}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n}\right)_{x}\right]^{2}+u \gamma_{3}^{2}+c$. Thus, indeed we have the equivalence between both representations 4.8) and (4.9), 4.10).

Summarizing results of this section, the $n$-th KdV stationary system (2.25) has the $n$-dimensional Stäckel representation (4.5), generated by the separation curve (4.1).

What is interesting, both Stäckel representations (3.6) and (4.5) of KdV stationary systems (3.18) and (4.9) are related by a Miura transformation [13] on the extended phase space (stationary manifold) $\mathcal{M}_{n}=T^{*} Q \times R \ni(q, p, c)$ and in consequence both systems have bi-Hamiltonian representation on $\mathcal{M}_{n}$ [5].

## 5 The Painlevé deformation of Stäckel systems

In the sequence of papers [10, 11, 12] we have presented a systematic deformation of an autonomous Stäckel systems generated by separation curves of the form

$$
\begin{equation*}
\sum_{k=-m}^{2 n+2-m} \lambda^{k}+\sum_{r=1}^{n} h_{r} \lambda^{n-r}=\lambda^{m} \mu^{2}, \quad m \in\{0,1, \ldots, n+1\} \tag{5.1}
\end{equation*}
$$

to non-autonomous Painlevé-type systems

$$
\begin{equation*}
\frac{d \xi}{d t_{r}}=Y_{r}(\xi, t)=\pi d H_{r}(\xi, t), \quad r=1, \ldots, n \tag{5.2}
\end{equation*}
$$

In consequence new Hamiltonian functions $H_{r}$ fulfill the Frobenius integrability conditions

$$
\begin{equation*}
\left\{H_{r}, H_{s}\right\}+\frac{\partial H_{r}}{\partial t_{s}}-\frac{\partial H_{s}}{\partial t_{r}}=f_{r s}\left(t_{1}, \ldots, t_{n}\right), \quad r, s=1, \ldots, n \tag{5.3}
\end{equation*}
$$

where $f_{r s}$ are some functions not depending on the phase-space variables $\xi$, only on the parameters $t_{j}$, and related Hamiltonian vector fields $Y_{k}(\xi, t)$ satisfy

$$
\begin{equation*}
\left[Y_{s}, Y_{r}\right]+\frac{\partial Y_{r}}{\partial t_{s}}-\frac{\partial Y_{s}}{\partial t_{r}}=0, \quad r, s=1, \ldots, n \tag{5.4}
\end{equation*}
$$

as $\pi d\left\{H_{r}, H_{s}\right\}=-\left[Y_{r}, Y_{s}\right]$. Therefore, the set of non-autonomous evolution equations (5.2) has again common (at least local) solutions $\xi\left(t_{1}, \ldots, t_{n}, \xi_{0}\right)$ through each point $\xi_{0}$ of $M$ [20, 25, 28.

Observe, that it is always possible to choose another Hamiltonians $H_{r} \rightarrow H_{r}+\varphi_{r}\left(t_{1}, \ldots, t_{n}\right)$, defining the same dynamical systems (5.2), which satisfy

$$
\begin{equation*}
\left\{H_{r}, H_{s}\right\}+\frac{\partial H_{r}}{\partial t_{s}}-\frac{\partial H_{s}}{\partial t_{r}}=0, \quad r, s=1, \ldots, n \tag{5.5}
\end{equation*}
$$

Besides, the non-autonomous Hamiltonian equations (5.2) are represented by the so-called isomonodromic Lax representation

$$
\begin{equation*}
\frac{d}{d t_{k}} L(\lambda ; \xi, t)=\left[U_{k}(\lambda ; \xi, t), L(\lambda ; \xi, t)\right]+\lambda^{m} \frac{\partial U_{k}(\lambda ; \xi, t)}{\partial \lambda} \tag{5.6}
\end{equation*}
$$

being the compatibility condition for a system of linear equations

$$
\begin{equation*}
\lambda^{m} \frac{\partial \Psi}{\partial \lambda}=L(\lambda ; \xi, t) \Psi, \quad \frac{d}{d t_{k}} \Psi=U_{k}(\lambda ; \xi, t) \Psi \tag{5.7}
\end{equation*}
$$

The deformation procedure for Stäckel systems considered in previous sections, is as follows [11]. First, we deform geodesic constants of motion $E_{r}=\mu^{T} K_{r} G \mu$ by terms linear in momenta

$$
\begin{equation*}
\mathcal{E}_{r}=E_{r}+W_{r}=\mu^{T} K_{r} G \mu+\mu^{T} J_{r}, \quad r=2, \ldots, n, \quad W_{1}=0, \tag{5.8}
\end{equation*}
$$

generated by Killing vectors $J_{r}$ of metric tensors $G_{0}$ (3.9) and $G_{1}$ (4.4). Actually, in $(q, p)$ coordinates,

$$
\begin{equation*}
W_{r}=\sum_{k=n-r+2}^{n}(n+1-k) q_{r+k-n-2} p_{k}, \tag{5.9}
\end{equation*}
$$

for metric $G_{0}$ and

$$
\begin{equation*}
W_{r}=\sum_{k=n-r+1}^{n-1}(n-k) q_{r+k-n-1} p_{k} \tag{5.10}
\end{equation*}
$$

for metric $G_{1}$, respectively.
The Hamiltonians $\mathcal{E}_{r}$ in (5.8) span a Lie algebra $\mathfrak{g}=\operatorname{span}\left\{\mathcal{E}_{r} \in C^{\infty}(M): r=1, \ldots, n\right\}$ with the following commutation relations

$$
\begin{equation*}
\left\{\mathcal{E}_{1}, \mathcal{E}_{r}\right\}=0, \quad r=2, \ldots, n \tag{5.11}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\{\mathcal{E}_{r}, \mathcal{E}_{s}\right\}=(s-r) \mathcal{E}_{r+s-n-2}, \quad r, s=2, \ldots, n \tag{5.12}
\end{equation*}
$$

in the first case and

$$
\begin{equation*}
\left\{\mathcal{E}_{r}, \mathcal{E}_{s}\right\}=(s-r) \mathcal{E}_{r+s-n-1}, \quad r, s=2, \ldots, n \tag{5.13}
\end{equation*}
$$

in the second case. We use the convention that $\mathcal{E}_{r}=0$ as soon as $r \leq 0$ or $r>n$. The algebra $\mathfrak{g}$ has an Abelian subalgebra

$$
\begin{equation*}
\mathfrak{a}=\operatorname{span}\left\{\mathcal{E}_{1}, \ldots, \mathcal{E}_{\kappa}\right\}, \quad \kappa=\left[\frac{n+3}{2}\right] \text { and } \kappa=\left[\frac{n+2}{2}\right] \text { respectively. } \tag{5.14}
\end{equation*}
$$

As the Hamiltonians $\mathcal{E}_{r}$ in (5.8) do not commute, they do not constitute a Liouville integrable system. In particular, there is no reason to expect that they will possess a common, multi-time solution for a given initial data $\xi_{0}$. However, in [10] we found polynomial-in-times deformations $H_{r}\left(t_{1}, \ldots, t_{n}\right)$ of the Hamiltonians $\mathcal{E}_{r}$ such that the Hamiltonians $H_{r}$ satisfy the Frobenius integrability condition (5.3). More specifically, the deformed Hamiltonians $H_{r}$ are given by

$$
\begin{align*}
H_{r} & =\mathcal{E}_{r}, \quad \text { for } r=1, \ldots, \kappa \\
H_{r} & =\sum_{j=1}^{r} \zeta_{r, j}\left(t_{1}, \ldots, t_{r-1}\right) \mathcal{E}_{j}, \quad \zeta_{r, r}=1, \quad \text { for } r=\kappa+1, \ldots, n \tag{5.15}
\end{align*}
$$

where $\zeta_{r, j}(t)$ are some polynomial functions of evolution parameters, determined uniquely from Frobenius conditions (5.5).

In the second step of deformation process we include the appropriate potentials from (3.3) and (4.2). In order to preserve Frobenius conditions, we have supplement these potentials by extra lower order
potentials with time dependent coefficients. Actually, the deformation of Hamiltonians (3.3) takes the form

$$
\begin{equation*}
h_{r}=E_{r}+V_{r}^{(2 n+1)}+c V_{r}^{(n)} \longrightarrow h_{r}^{W}=h_{r}+W_{r}+\sum_{k=n}^{2 n-1} c_{k}(t) V_{r}^{(k)}, \quad r=1, \ldots, n \tag{5.16}
\end{equation*}
$$

and Hamiltonians (4.2) the respective form

$$
\begin{equation*}
h_{r}=E_{r}+V_{r}^{(2 n)}+c V_{r}^{(-1)} \longrightarrow h_{r}^{W}=h_{r}+W_{r}+\sum_{k=n}^{2 n-1} d_{k}(t) V_{r}^{(k)}, \quad r=1, \ldots, n \tag{5.17}
\end{equation*}
$$

where $c_{k}(t)$ and $d_{k}(t)$ are again some polynomial functions of evolution parameters, determined uniquely from Frobenius conditions for functions $H_{r}$

$$
\begin{align*}
H_{r} & =h_{r}^{W}, \quad \text { for } r=1, \ldots, \kappa \\
H_{r} & =\sum_{j=1}^{r} \zeta_{r, j}\left(t_{1}, \ldots, t_{r-1}\right) h_{j}^{W}, \quad \zeta_{r, r}=1, \quad \text { for } r=\kappa+1, \ldots, n \tag{5.18}
\end{align*}
$$

with the same $\zeta_{r, j}(t)$ coefficients as in (5.15). The details of the deformation procedure as well as an appropriate values of coefficients $\zeta_{r, j}(t), c_{k}(t)$ and $d_{k}(t)$ the reader can find in [11].

Both hierarchies of non-autonomous Frobenius integrable systems have isomonodromic Lax representations [12], so are represented by a Painlevé-type equations.

Theorem 6 [12] For the first hierarchy, generated by Hamiltonians (5.18) and (5.16), isomonodromic Lax representations are of the form

$$
\begin{equation*}
\frac{d}{d t_{k}} L(\lambda ; \xi, t)=\left[U_{k}(\lambda ; \xi, t), L(\lambda ; \xi, t)\right]+\frac{\partial U_{k}(\lambda ; \xi, t)}{\partial \lambda}, \quad k=1, \ldots, n \tag{5.19}
\end{equation*}
$$

The $L(\lambda ; \xi, t)$ matrix takes the form (3.13), 3.14) where now

$$
\begin{equation*}
w(\lambda ; q, p, t)=-\left[\frac{v(\lambda)^{2}-\lambda^{2 n+1}-\sum_{k=n}^{2 n-1} c_{k}(t) \lambda^{k}}{u(\lambda)}\right]_{+} \tag{5.20}
\end{equation*}
$$

Moreover,

$$
\begin{align*}
& U_{r}(\lambda ; \xi, t)=\left[\frac{u_{r}(\lambda) L(\lambda)}{u(\lambda)}\right]_{+}, \quad \text { for } r=1, \ldots, \kappa \\
& U_{r}(\lambda ; \xi, t)=\sum_{j=1}^{r} \zeta_{r, j}\left(t_{1}, \ldots, t_{r-1}\right)\left[\frac{u_{j}(\lambda) L(\lambda)}{u(\lambda)}\right]_{+}, \quad \zeta_{r, r}=1, \quad \text { for } r=\kappa+1, \ldots, n \tag{5.21}
\end{align*}
$$

For the second hierarchy, generated by Hamiltonians (5.18) and (5.17), isomonodromic Lax representations are of the form

$$
\begin{equation*}
\frac{d}{d t_{k}} L(\lambda ; \xi, t)=\left[U_{k}(\lambda ; \xi, t), L(\lambda ; \xi, t)\right]+\lambda \frac{\partial U_{k}(\lambda ; \xi, t)}{\partial \lambda} \tag{5.22}
\end{equation*}
$$

The $L(\lambda ; \xi, t)$ matrix takes the form (3.13) with entries $u(\lambda ; q)$ and $v(\lambda ; q, p)$ given by 4.6), where now

$$
\begin{equation*}
w(\lambda ; q, p, t)=-\lambda\left[\frac{v(\lambda)^{2} \lambda^{-1}-\lambda^{2 n}-\sum_{k=n}^{2 n-1} c_{k}(t) \lambda^{k}-c \lambda^{-1}}{u(\lambda)}\right]_{+} \tag{5.23}
\end{equation*}
$$

$U_{r}(\lambda ; \xi, t)$ matrices are again of the form (5.21).

The first members of the hierarchy (5.16), (5.18) are determined by the following Hamiltonians

$$
\begin{array}{ll}
n=1: & h_{1}^{W}=E_{1}+V_{1}^{(3)}+\left(t_{1}+c\right) V_{1}^{(1)}, \quad H_{1}=h_{1}^{W} \\
n=2: & h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(5)}+3 t_{2} V_{r}^{(4)}+\left(t_{1}+c\right) V_{r}^{(3)}, \quad H_{r}=h_{r}^{W}, \quad r=1,2, \\
n=3: & h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(7)}+5 t_{3} V_{r}^{(5)}+3 t_{2} V_{r}^{(4)}+\left(t_{1}+\frac{15}{2} t_{3}^{2}+c\right) V_{r}^{(3)}, \quad H_{r}=h_{r}^{W}, \quad r=1,2,3, \\
n=4: & h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(9)}+7 t_{4} V_{r}^{(7)}+5 t_{3} V_{r}^{(6)}+\left(3 t_{2}+\frac{35}{2} t_{4}^{2}\right) V_{r}^{(5)}+\left(t_{1}+21 t_{3} t_{4}+c\right) V_{r}^{(4)}, \\
& H_{r}=h_{r}^{W}, \quad r=1,2,3, \quad H_{4}=h_{4}^{W}+t_{3} h_{1}^{W} \\
n=5: & h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(11)}+9 t_{5} V_{r}^{(9)}+7 t_{4} V_{r}^{(8)}+5 t_{3} V_{r}^{(7)}+\left(3 t_{2}+45 t_{4} t_{5}\right) V_{r}^{(6)} \\
& \quad+\left(t_{1}+27 t_{3} t_{5}+14 t_{4}^{2}+\frac{105}{2} t_{5}^{3}+c\right) V_{r}^{(5)}, \\
& \\
& H_{r}=h_{r}^{W}, \quad r=1, \ldots, 4, \quad H_{5}=h_{5}^{W}+t_{4} h_{2}^{W}+2 t_{3} h_{1}^{W},  \tag{5.24}\\
\vdots &
\end{array}
$$

while the first members of the second hierarchy (5.17), (5.18) are determined by Hamiltonians

$$
\begin{equation*}
\vdots \tag{5.25}
\end{equation*}
$$

The first hierarchy (5.24) is the complete hierarchy of Painlevé I $\left(P_{I}\right)$ systems, as its first, one dimensional, system is a famous $P_{I}$ equation. Indeed, denoted $q_{1}=q, p_{1}=p$ and $t_{1}=t$ we have

$$
\begin{gathered}
H=h_{1}^{W}=p^{2}+q^{3}+(t+c) q \\
\Downarrow \\
q_{t}=2 p, \quad p_{t}=-3 q^{2}-t-c
\end{gathered}
$$

$$
\begin{aligned}
& n=1: \quad h_{1}^{W}=E_{1}+V_{1}^{(2)}+t_{1} V_{1}^{(1)}+c V_{1}^{(-1)}, \quad H_{1}=h_{1}^{W}, \\
& n=2: \quad h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(4)}+3 t_{2} V_{r}^{(3)}+\left(t_{1}+3 t_{2}^{2}\right) V_{r}^{(2)}+c V_{r}^{(-1)}, \quad H_{r}=h_{r}^{W}, \quad r=1,2, \\
& n=3: \quad h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(6)}+5 t_{3} V_{r}^{(5)}+\left(3 t_{2}+10 t_{3}^{2}\right) V_{r}^{(4)}+\left(t_{1}+10 t_{2} t_{3}+10 t_{3}^{3}\right) V_{r}^{(3)}+c V_{r}^{(-1)}, \\
& H_{r}=h_{r}^{W}, \quad r=1,2, \quad H_{3}=h_{3}^{W}+t_{2} h_{1}^{W}, \\
& n=4: \quad h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(8)}+7 t_{4} V_{r}^{(7)}+\left(5 t_{3}+21 t_{4}^{2}\right) V_{r}^{(6)}+\left(3 t_{2}+28 t_{3} t_{4}+35 t_{4}^{2}\right) V_{r}^{(5)} \\
& +\left(t_{1}+14 t_{2} t_{4}+\frac{15}{2} t_{3}^{2}+63 t_{3} t_{4}^{2}+35 t_{4}^{4}\right) V_{r}^{(4)}+c V_{r}^{(-1)}, \\
& H_{r}=h_{r}^{W}, \quad r=1,2,3, \quad H_{4}=h_{4}^{W}+t_{3} h_{2}^{W}+2 t_{2} h_{1}^{W}, \\
& n=5: \quad h_{r}^{W}=\mathcal{E}_{r}+V_{r}^{(10)}+9 t_{5} V_{r}^{(9)}+\left(7 t_{4}+36 t_{5}^{2}\right) V_{r}^{(8)}+\left(5 t_{3}+54 t_{4} t_{5}+84 t_{5}^{3}\right) V_{r}^{(7)} \\
& +\left(3 t_{2}+36 t_{3} t_{5}+\frac{35}{2} t_{4}^{2}+180 t_{4} t_{5}^{2}+126 t_{5}^{4}\right) V_{r}^{(6)} \\
& +\left(t_{1}+21 t_{3} t_{4}+18 t_{2} t_{5}+108 t_{4}^{2} t_{5}+108 t_{3} t_{5}^{2}+336 t_{4} t_{5}^{3}+126 t_{5}^{5}\right) V_{r}^{(5)}+c V_{r}^{(-1)}, \\
& H_{r}=h_{r}^{W}, \quad r=1,2,3, \quad H_{4}=h_{4}^{W}+t_{3} h_{1}^{W}, \quad H_{5}=h_{5}^{W}+t_{4} h_{3}^{W}+2 t_{3} h_{2}^{W}+\left(3 t_{2}-\frac{1}{2} t_{4}^{2}\right) h_{1}^{W},
\end{aligned}
$$

$$
\begin{gather*}
\Uparrow \\
q_{t t}+6 q^{2}+2(t+c)=0 \tag{5.26}
\end{gather*}
$$

The isomonodromic Lax representation (5.6) is as follows

$$
L=\left(\begin{array}{cc}
-p & \lambda+q  \tag{5.27}\\
\lambda^{2}-q \lambda+q^{2}+t+c & p
\end{array}\right), \quad U=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q & 0
\end{array}\right)
$$

On the other hand, equation (5.26) is equivalent to $P_{I}$ equation

$$
\begin{equation*}
q_{t t}=3 q^{2}+t \tag{5.28}
\end{equation*}
$$

after rescaling $q \rightarrow-2^{\frac{1}{5}} q, t \rightarrow 2^{-\frac{2}{5}} t-c$.
The second, two dimensional system, from that hierarchy is as follows

$$
\begin{gathered}
H_{1}=h_{1}^{W}=2 p_{1} p_{2}+q_{1} p_{2}^{2}-q_{1}^{4}+3 q_{1}^{2} q_{2}-q_{2}^{2}+3 t_{2}\left(q_{2}-q_{1}^{2}\right)+\left(t_{1}+c\right) q_{1} \\
H_{2}=h_{2}^{W}=p_{1}^{2}+2 q_{1} p_{1} p_{2}+\left(q_{1}^{2}-q_{2}\right) p_{2}^{2}+p_{2}-q_{1}^{3} q_{2}+2 q_{1} q_{2}^{2}-3 t_{2} q_{1} q_{2}+\left(t_{1}+c\right) q_{2}, \\
\left\{H_{1}, H_{2}\right\}+\frac{\partial H_{1}}{\partial t_{2}}-\frac{\partial H_{2}}{\partial t_{1}}=3 t_{2} .
\end{gathered}
$$

Thus,

$$
\begin{gather*}
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
p_{1} \\
p_{2}
\end{array}\right)_{t_{1}}=\left(\begin{array}{c}
2 p_{2} \\
2 p_{1}+2 q_{1} p_{2} \\
-p_{2}^{2}+4 q_{1}^{3}-6 q_{1} q_{2}+6 t_{2} q_{1}-t_{1}-c \\
-3 q_{1}^{2}+2 q_{2}-3 t_{2}
\end{array}\right)=Y_{1} \\
\mathbb{\imath}
\end{gather*} \overbrace{1} q_{t_{1} t_{1}=-6 q_{1}^{2}+4 q_{2}-6 t_{2}, \quad\left(q_{2}\right)_{t_{1} t_{1}}=\frac{1}{2}\left[\left(q_{1}\right)_{t_{1}}\right]^{2}+2 q_{1}^{3}-8 q_{1} q_{2}+6 t_{2} q_{1}-2 t_{1}-2 c,}^{\left(\begin{array}{c}
q_{1} \\
q_{2}  \tag{5.29}\\
p_{1} \\
p_{2}
\end{array}\right)_{t_{2}}=\left(\begin{array}{c}
2 q_{1} p_{2}+2 p_{1} \\
2 q_{1} p_{1}+2\left(q_{1}^{2}-q_{2}\right) p_{2}+1 \\
-2 q_{1} p_{2}^{2}-2 p_{1} p_{2}+3 q_{1}^{2} q_{2}-2 q_{2}^{2}+3 t_{2} q_{2} \\
p_{2}^{2}+q_{1}^{3}-4 q_{1} q_{2}+3 t_{2} q_{1}-t_{1}-c
\end{array}\right)=Y_{2}} .
$$

and

$$
\left[Y_{2}, Y_{1}\right]+\frac{\partial Y_{1}}{\partial t_{2}}-\frac{\partial Y_{2}}{\partial t_{1}}=0
$$

Notice that eliminating $q_{2}$ from (5.29) we get the forth order equation for $q \equiv q_{1}$

$$
\frac{1}{4} q_{t_{1} t_{1} t_{1} t_{1}}+5 q q_{t_{1} t_{1}}+\frac{5}{2}\left(q_{t_{1}}\right)^{2}+10 q^{3}+6 t_{2} q+2 t_{1}+2 c=0
$$

which is the second equation from the standard $P_{I}$ hierarchy.
The isomonodromic Lax representation (5.6) is of the form

$$
\begin{gathered}
L=\left(\begin{array}{cc}
-p_{2} \lambda-p_{1}-q_{1} p_{2} & \lambda^{2}+q_{1} \lambda+q_{2} \\
\lambda^{3}-q_{1} \lambda^{2}+\left(q_{1}^{2}-q_{2}+3 t_{2}\right) \lambda-p_{2}^{2}+2 q_{1} q_{2}-3 t_{2} q_{1}+t_{1}+c & p_{2} \lambda+p_{1}+q_{1} p_{2}
\end{array}\right) \\
U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{2} & \lambda+q_{1} \\
\lambda^{2}-q_{1} \lambda+q_{1}^{2}-2 q_{2}+3 t_{2} & p_{2}
\end{array}\right) .
\end{gathered}
$$

The third, three dimensional system, from that hierarchy (5.24), i.e. the deformed system from Example 3, is generated by Hamiltonians

$$
\begin{align*}
& H_{1}=h_{1}^{W}=h_{1}+5 t_{3} V_{1}^{(5)}+3 t_{2} V_{1}^{(4)}+\left(t_{1}+\frac{15}{2} t_{3}^{2}\right) V_{1}^{(3)} \\
& H_{2}=h_{2}^{W}=h_{2}+p_{3}+5 t_{3} V_{2}^{(5)}+3 t_{2} V_{2}^{(4)}+\left(t_{1}+\frac{15}{2} t_{3}^{2}\right) V_{2}^{(3)}  \tag{5.30}\\
& H_{3}=h_{3}^{W}=h_{3}+2 p_{2}+q_{1} p_{3}+5 t_{3} V_{3}^{(5)}+3 t_{2} V_{3}^{(4)}+\left(t_{1}+\frac{15}{2} t_{3}^{2}\right) V_{3}^{(3)}
\end{align*}
$$

where functions $h_{1}, h_{2}$ and $h_{3}$ are given in Example 3 and

$$
\begin{gathered}
V_{1}^{(4)}=q_{2}-q_{1}^{2}, \quad V_{2}^{(4)}=q_{3}-q_{1} q_{2}, \quad V_{3}^{(4)}=-q_{1} q_{3} \\
V_{1}^{(5)}=q_{1}^{3}-2 q_{1} q_{2}+q_{3}, \quad V_{2}^{(5)}=q_{1}^{2} q_{2}-q_{1} q_{3}-q_{2}^{2}, \quad V_{3}^{(5)}=q_{1}^{2} q_{3}-q_{2} q_{3} .
\end{gathered}
$$

The related Hamiltonian vector fields $Y_{r}=\pi d H_{r}, r=1,2,3$ fulfill Frobenius conditions (5.4) and evolution equations $\xi_{t_{r}}=Y_{r}$ have the following isomonodromic Lax representations

$$
\begin{gathered}
L=\left(\begin{array}{cc}
-p_{3} \lambda^{2}-\left(p_{2}+q_{1} p_{3}\right) \lambda-\left(p_{1}+q_{1} p_{2}+q_{2} p_{3}\right) & \lambda^{3}+q_{1} \lambda^{2}+q_{2} \lambda+q_{3} \\
L_{21} & -L_{11}
\end{array}\right) \\
L_{21}=\lambda^{4}-q_{1} \lambda^{3}-\left(V_{1}^{(4)}-5 t_{3}\right) \lambda^{2}-\left(V_{1}^{(5)}+p_{3}^{2}-5 q_{1} t_{3}-3 t_{2}\right) \lambda-V_{1}^{(6)}-q_{1} p_{3}^{2}-2 p_{2} p_{3}-5 t_{3} V_{1}^{(4)}-3 t_{2} q_{1}+t_{1}+\frac{15}{2} t_{3}^{2}+c, \\
V_{1}^{(6)}=-q_{1}^{4}+3 q_{1}^{2} q_{2}-2 q_{1} q_{3}-q_{2}^{2} . \\
U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{3} & \lambda+q_{1} \\
\lambda^{2}-q_{1} \lambda+q_{1}^{2}-2 q_{2}+5 t_{3} & p_{3}
\end{array}\right) \\
U_{3}=\left(\begin{array}{cc}
\lambda^{2}+q_{1} \lambda+q_{2} \\
\lambda^{3}-q_{1} \lambda^{2}-\left(V_{1}^{(4)}-5 t_{3}\right) \lambda-V_{1}^{(5)}-q_{3}-p_{3}^{2}-5 q_{1} t_{3}+3 t_{2} & p_{3} \lambda+q_{1} p_{3}+p_{2}
\end{array}\right)
\end{gathered}
$$

The explicit form of isomonodromic Lax representations of higher dimensional systems from the $P_{I}$ hierarchy (5.24) can be constructed with the help of (5.20) and (5.21).

The first attempt to the hierarchy of $P_{I}$ systems was done in 32 by Takasaki, where the author started from the opposite side, i.e. from string equations of KP (KdV in particular). Using such formalism he was able to construct, for each $n$, only the first Painlevé equation

$$
\frac{d \xi}{d t_{1}}=Y_{1}(\xi, t)=\pi d H_{1}(\xi, t)
$$

from the system (5.2), together with its isomonodromic Lax representation. He failed in construction the remaining equations from the $P_{I}$ system (5.2), with $k=2, \ldots, n$, as he did not control the perturbation terms $W_{k}$ (5.9) in remaining Hamiltonians. Now we know that they are generated by Killing vectors (5.9) of the metric tensor $G_{0}$.

Now, let us turn to the second hierarchy of Painlevé systems (5.25). The first, one dimensional, system is as follows:

$$
\begin{gather*}
H=h_{1}^{W}=-q p^{2}-q^{2}+t q+c q^{-1} \\
\Downarrow \\
q_{t}=-2 q p, \quad p_{t}=p^{2}+2 q-t+c q^{-2} \\
\Uparrow \\
q q_{t t}=\frac{1}{2} q_{t}^{2}-4 q^{3}+2 t q^{2}-2 c \tag{5.31}
\end{gather*}
$$

The isomonodromic Lax representation takes the form (5.22), where

$$
L=\left(\begin{array}{cc}
q p & \lambda+q \\
\lambda^{2}+(-q+t) \lambda+q p^{2}+c q^{-1} & -q p
\end{array}\right), \quad U=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q+t & 0
\end{array}\right)
$$

One can verify that equation (5.31) is the thirty-fourth Painlevé $\left(P_{34}\right)$ equation from the Gambier list.
The second, two dimensional system from the hierarchy, is generated by Hamiltonians

$$
\begin{aligned}
& H_{1}=h_{1}^{W}=p_{1}^{2}-q_{2} p_{2}^{2}+q_{1}^{3}-2 q_{1} q_{2}+3 t_{2}\left(q_{2}-q_{1}^{2}\right)+\left(3 t_{2}^{2}+t_{1}\right) q_{1}+c q_{2}^{-1} \\
& H_{2}=h_{2}^{W}=-2 q_{2} p_{1} p_{2}-q_{1} q_{2} p_{2}^{2}+p_{1}+q_{1}^{2} q_{2}-q_{2}^{2}-3 t_{2} q_{1} q_{2}+\left(3 t_{2}^{2}+t_{1}\right) q_{2}+c q_{1} q_{2}^{-1}
\end{aligned}
$$

$$
\left\{H_{1}, H_{2}\right\}+\frac{\partial H_{1}}{\partial t_{2}}-\frac{\partial H_{2}}{\partial t_{1}}=t_{1}+3 t_{2}^{2}
$$

Thus,

$$
\begin{gather*}
\left(\begin{array}{l}
q_{1} \\
q_{2} \\
p_{1} \\
p_{2}
\end{array}\right)_{t_{1}}=\left(\begin{array}{c}
2 p_{1} \\
-2 q_{2} p_{2} \\
-3 q_{1}^{2}+2 q_{2}+6 t_{2} q_{1}-3 t_{2}^{2}-t_{1} \\
p_{2}^{2}+2 q_{1}-3 t_{2}+c q_{2}^{-2}
\end{array}\right)=Y_{1} \\
\hat{\Downarrow}  \tag{5.32}\\
\left(q_{1}\right)_{t_{1} t_{1}}=-6 q_{1}^{2}+4 q_{2}+12 t_{2} q_{1}-6 t_{2}^{2}-2 t_{1}, \quad q_{2}\left(q_{2}\right)_{t_{1} t_{1}}=\frac{1}{2}\left[\left(q_{2}\right)_{t_{1}}\right]^{2}-4 q_{1} q_{2}^{2}+6 t_{2} q_{2}^{2}-2 c, \\
\left(\begin{array}{c}
q_{1} \\
q_{2} \\
p_{1} \\
p_{2}
\end{array}\right)_{t_{2}}=\left(\begin{array}{l}
-2 q_{2} p_{2}+1 \\
-2 q_{2} p_{1}-2 q_{1} q_{2} p_{2} \\
q_{2} p_{2}^{2}-2 q_{1} q_{2}+3 t_{2} q_{2}-c q_{2}^{-1} \\
2 p_{1} p_{2}+q_{1} p_{2}^{2}-q_{1}^{2}+2 q_{2}+3 t_{2} q_{1}-3 t_{2}^{2}-t_{1}+c q_{1} q_{2}^{-2}
\end{array}\right)=Y_{2}
\end{gather*}
$$

Notice that eliminating $q_{2}$ from (5.32) we get the forth order equation for $q \equiv q_{1}$

$$
\begin{aligned}
0= & -\frac{1}{32} q_{t_{1} t_{1}} q_{t_{1} t_{1} t_{1} t_{1}}+\frac{1}{64}\left(q_{t_{1} t_{1} t_{1}}\right)^{2}-\frac{1}{16}\left(3 q^{2}-6 t_{2} q+3 t_{2}^{2}+t_{1}\right) q_{t_{1} t_{1} t_{1} t_{1}}+\frac{3}{8}\left(q-t_{2}\right) q_{t_{1}} q_{t_{1} t_{1} t_{1}} \\
& -\frac{1}{2}\left(q-\frac{9}{8} t_{2}\right)\left(q_{t_{1} t_{1}}\right)^{2}-\frac{3}{8}\left(q_{t_{1}}\right)^{2} q_{t_{1} t_{1}}-\left[\frac{15}{4} q^{3}-12 t_{2} q^{2}+\left(\frac{51}{4} t_{2}^{2}+\frac{5}{4} t_{1}\right) q-\frac{3}{2} t_{1} t_{2}-\frac{9}{2} t_{2}^{3}\right] q_{t_{1} t_{1}} \\
& -\frac{3}{4} t_{1}\left(q_{t_{1}}\right)^{2}-\frac{9}{2} q^{5}+\frac{99}{4} t_{2} q^{4}-\left(54 t_{2}^{2}+3 t_{1}\right) q^{3}+\left(\frac{21}{2} t_{1} t_{2}+\frac{117}{2} t_{2}^{2}\right) q^{2}-\left(12 t_{1} t_{2}^{2}+\frac{63}{2} t_{2}^{4}+\frac{1}{2} t_{1}^{2}\right) q \\
& +\frac{27}{4} t_{2}^{5}+\frac{3}{4} t_{1}^{2} t_{2}+\frac{9}{2} t_{1} t_{2}^{3}+\frac{1}{16}-c
\end{aligned}
$$

which can be considered as the second equation from the standard $P_{34}$ hierarchy.
The isomonodromic Lax representation (5.22) is as follows

$$
\begin{gathered}
L=\left(\begin{array}{cc}
-p_{1} \lambda+q_{2} p_{2} & \lambda^{2}+q_{1} \lambda+q_{2} \\
\lambda^{3}+\left(-q_{1}+3 t_{2}\right) \lambda^{2}+\left(q_{1}^{2}-3 q_{1} t_{2}-q_{2}+3 t_{2}^{2}+t_{1}\right) \lambda+-q_{2} p_{2}^{2}+c q_{2}^{-1} & p_{1} \lambda-q_{2} p_{2}
\end{array}\right) \\
U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1}+3 t_{2} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{1} & \lambda+q_{1} \\
\lambda^{2}+\left(-q_{1}+3 t_{2}\right) \lambda+q_{1}^{2}-2 q_{2}-3 t_{2} q_{1}+t_{1}+3 t_{2}^{2} & p_{1}
\end{array}\right) .
\end{gathered}
$$

What is interesting, this system in flat coordinates $\left(x_{1}, x_{2}, p_{x_{1}}, p_{x_{2}}\right)$ on $R^{4}$, related with $(q, p)$ coordinates by a point transformation

$$
q_{1}=-x_{1}, \quad q_{2}=-\frac{1}{4} x_{2}^{2}
$$

is the non-autonomous deformation of the famous Henon-Heiles system [22] consider in [24] and in the complete version in 9 .

The third, three dimensional system from that hierarchy (5.25), i.e. the deformed system from Example 5 is generated by Hamiltonians
$H_{1}=h_{1}^{W}=h_{1}+5 t_{3} V_{1}^{(5)}+\left(3 t_{2}+10 t_{3}^{2}\right) V_{1}^{(4)}+\left(t_{1}+10 t_{2} t_{3}+10 t_{3}^{3}\right) V_{1}^{(3)}$,
$H_{2}=h_{2}^{W}=h_{2}+p_{2}+5 t_{3} V_{2}^{(5)}+\left(3 t_{2}+10 t_{3}^{2}\right) V_{2}^{(4)}+\left(t_{1}+10 t_{2} t_{3}+10 t_{3}^{3}\right) V_{2}^{(3)}$,
$H_{3}=h_{3}^{W}+t_{2} h_{1}^{W}=h_{3}+2 p_{1}+q_{1} p_{2}+5 t_{3} V_{3}^{(5)}+\left(3 t_{2}+10 t_{3}^{2}\right) V_{3}^{(4)}+\left(t_{1}+10 t_{2} t_{3}+10 t_{3}^{3}\right) V_{3}^{(3)}+t_{2} h_{1}^{W}$,
where functions $h_{1}, h_{2}$ and $h_{3}$ are given in Example 5. The related Hamiltonian vector fields $Y_{r}=$ $\pi d H_{r}, r=1,2,3$ fulfill Frobenius conditions (5.4) and evolution equations $\xi_{t_{r}}=Y_{r}$ have the following isomonodromic Lax representations

$$
L=\left(\begin{array}{cc}
-p_{2} \lambda^{2}-\left(p_{1}+q_{1} p_{2}\right) \lambda+q_{3} p_{3} & \lambda^{3}+q_{1} \lambda^{2}+q_{2} \lambda+q_{3} \\
L_{21} & -L_{11}
\end{array}\right)
$$

$$
\begin{gathered}
L_{21}=\left(\begin{array}{l}
\lambda^{4}-\left(q_{1}-5 t_{3}\right) \lambda^{3}-\left(V_{1}^{(4)}+5 t_{3} q_{1}-3 t_{2}-10 t_{3}^{2}\right) \lambda^{2}-\left[V_{1}^{(5)}+p_{3}^{2}+5 t_{3} V_{1}^{(4)}+\left(3 t_{2}+10 t_{3}^{2}\right) q_{1}\right. \\
\\
\left.-\left(t_{1}+10 t_{2} t_{3}+10 t_{3}^{3}\right)\right] \lambda-q_{3} p_{3}^{2}+c q_{3}^{-1}, \\
U_{1}=\left(\begin{array}{cc}
0 & 1 \\
\lambda-2 q_{1}+5 t_{3} & 0
\end{array}\right), \quad U_{2}=\left(\begin{array}{cc}
-p_{2} \\
\lambda^{2}-\left(q_{1}-5 t_{3}\right) \lambda+V_{1}^{(4)}-q_{2}-5 t_{3} q_{1}+3 t_{2}+10 t_{3}^{2} & p_{2}
\end{array}\right), \\
U_{3}=\left(\begin{array}{cc}
\lambda^{2}+q_{1} \lambda+q_{2}+t_{2} \\
-p_{2} \lambda-q_{1} p_{2}-p_{1} & \\
-p_{2}^{2}-V_{1}^{(5)}-q_{3}-5 t_{3} V_{1}^{(4)}-\left(5 t_{2}+10 t_{3}^{2}\right) q_{1}+t_{1}+15 t_{2} t_{3}+10 t_{3}^{3} & p_{2} \lambda+q_{1} p_{2}+p_{1}
\end{array}\right)
\end{array}\right.
\end{gathered}
$$

The systematic construction of isomonodromic Lax representations for higher dimensional members (5.25) is described by (5.21)-(5.23).

Also in that case, some elements of similar $P_{34}$-hierarchy appeared in [23], where stationary flows of equations like these from (6.12), but with time independent coefficients, was derived.

## 6 Non-homogenous KdV hierarchies and related non-autonomous stationary systems

In Sections 3 and 4 we have demonstrated how to reconstruct the KdV hierarchy and its stationary systems form the hierarchies of Stäckel systems (3.3) and (4.2), respectively. Here, by the same method, we will construct two different non-homogeneous KdV hierarchies and related non-autonomous stationary systems directly from Painlevé deformations (5.24) and (5.25) of considered Stäckel systems.

We begin from $P_{I}$ hierarchy (5.24). For one-dimensional equation (5.26) $\left(P_{I}\right)$, after identification $t_{1}=x$ and substitution $q=\frac{1}{2} u$ we get $p=\frac{1}{4} u_{x}$ and

$$
\begin{equation*}
0=\frac{1}{4} u_{x x}+\frac{3}{4} u^{2}+x+c, \tag{6.1}
\end{equation*}
$$

which is the integrated stationary flow of the following PDE

$$
\begin{equation*}
u_{t_{2}}=\mathcal{K}_{2}+\sigma_{-1}=\partial_{x}\left(\gamma_{2}+x+c\right) \equiv \pi_{0}\left(\gamma_{1,2}+x+c\right)=\mathcal{K}_{1,2} \tag{6.2}
\end{equation*}
$$

from the KdV family. By the same substitution, the Painlevé-type equations generated by Hamiltonians $H_{1}$ from family (5.24), for $n=2,3,4, \ldots$, are integrated stationary flows of the following hierarchy of non-homogeneous KdV equations

$$
\begin{align*}
& u_{t_{3}}=\mathcal{K}_{3}+\frac{3}{2} t_{2} \mathcal{K}_{1}+\sigma_{-1} \equiv \mathcal{K}_{2,3}=\pi_{0}\left(\gamma_{2,3}+x+c\right) \\
& u_{t_{4}}=\mathcal{K}_{4}+\frac{5}{2} t_{3} \mathcal{K}_{2}+\frac{3}{2} t_{2} \mathcal{K}_{1}+\sigma_{-1} \equiv \mathcal{K}_{3,4}=\pi_{0}\left(\gamma_{3,4}+x+c\right) \\
& u_{t_{5}}=\mathcal{K}_{5}+\frac{7}{2} t_{4} \mathcal{K}_{3}+\frac{5}{2} t_{3} \mathcal{K}_{2}+\frac{3}{2}\left(t_{2}+\frac{7}{4} t_{4}^{2}\right) \mathcal{K}_{1}+\sigma_{-1} \equiv \mathcal{K}_{4,5}=\pi_{0}\left(\gamma_{4,5}+x+c\right)  \tag{6.3}\\
& u_{t_{6}}=\mathcal{K}_{6}+\frac{9}{2} t_{5} \mathcal{K}_{4}+\frac{7}{2} t_{4} \mathcal{K}_{3}+\frac{5}{2}\left(t_{3}+\frac{9}{4} t_{5}^{2}\right) \mathcal{K}_{2}+\frac{3}{2}\left(t_{2}+\frac{9}{2} t_{4} t_{5}\right) \mathcal{K}_{1}+\sigma_{-1} \equiv \mathcal{K}_{5,6}=\pi_{0}\left(\gamma_{5,6}+x+c\right)
\end{align*}
$$

Contrary to the autonomous case, for fixed $n$, the remaining Painlevé-type equations generated by Hamiltonians $H_{2}, \ldots, H_{n}$ from (5.24) do not reconstruct the lower order equations from the hierarchy
(6.3). Actually, the hierarchy of Painlevé-type systems generated by Hamiltonians (5.24) is equivalent to the following hierarchy of KdV non-autonomous stationary systems

$$
\begin{align*}
& n=1: \quad 0=\gamma_{2}+x+c \equiv \gamma_{1,2}+x+c \\
& n=2: \quad u_{t_{2}}=\mathcal{K}_{2} \equiv \mathcal{K}_{2,2}=\pi_{0} \gamma_{2,2} \\
& 0=\gamma_{3}+\frac{3}{2} t_{2} \gamma_{1}+x+c \equiv \gamma_{2,3}+x+c \\
& u_{t_{2}}=\mathcal{K}_{2} \equiv \mathcal{K}_{3,2}=\pi_{0} \gamma_{3,2} \\
& n=3: \quad u_{t_{3}}=\mathcal{K}_{3}+\frac{5}{2} t_{3} \mathcal{K}_{1} \equiv \mathcal{K}_{3,3}=\pi_{0} \gamma_{3,3} \\
& 0=\gamma_{4}+\frac{5}{2} t_{3} \gamma_{2}+\frac{3}{2} t_{2} \gamma_{1}+\frac{5}{4} t_{3}^{2}+x+c \equiv \gamma_{3,4}+x+c  \tag{6.4}\\
& u_{t_{2}}=\mathcal{K}_{2} \equiv \mathcal{K}_{4,2}=\pi_{0} \gamma_{4,2} \\
& u_{t_{3}}=\mathcal{K}_{3}+\frac{7}{2} t_{4} \mathcal{K}_{1} \equiv \mathcal{K}_{4,3}=\pi_{0} \gamma_{4,3} \\
& n=4: \\
& u_{t_{4}}=\mathcal{K}_{4}+\frac{7}{2} t_{4} \mathcal{K}_{2}+\frac{7}{2} t_{3} \mathcal{K}_{1} \equiv \mathcal{K}_{4,4}=\pi_{0} \gamma_{4,4} \\
& 0=\gamma_{5}+\frac{7}{2} t_{4} \gamma_{3}+\frac{5}{2} t_{3} \gamma_{2}+\frac{3}{2}\left(t_{2}+\frac{7}{4} t_{4}^{2}\right) \gamma_{1}+\frac{7}{2} t_{3} t_{4}+x+c \equiv \gamma_{4,5}+x+c \\
& u_{t_{2}}=\mathcal{K}_{2} \equiv \mathcal{K}_{5,2}=\pi_{0} \gamma_{5,2} \\
& u_{t_{3}}=\mathcal{K}_{3}+\frac{9}{2} t_{5} \mathcal{K}_{1} \equiv \mathcal{K}_{5,3}=\pi_{0} \gamma_{5,3} \\
& n=5: \\
& u_{t_{4}}=\mathcal{K}_{4}+\frac{9}{2} t_{5} \mathcal{K}_{2}+\frac{7}{2} t_{4} \mathcal{K}_{1} \equiv \mathcal{K}_{5,4}=\pi_{0} \gamma_{5,4} \\
& u_{t_{5}}=\mathcal{K}_{5}+\frac{9}{2} t_{5} \mathcal{K}_{3}+\frac{9}{2} t_{4} \mathcal{K}_{2}+\frac{9}{2}\left(t_{3}+\frac{5}{4} t_{5}^{2}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{5,5}=\pi_{0} \gamma_{5,5} \\
& 0=\gamma_{6}+\frac{9}{2} t_{5} \gamma_{4}+\frac{7}{2} t_{4} \gamma_{3}+\frac{5}{2}\left(t_{3}+\frac{9}{4} t_{5}^{2}\right) \gamma_{2}+\frac{3}{2}\left(t_{2}+\frac{9}{2} t_{4} t_{5}\right) \gamma_{1} \\
& +\frac{15}{8} t_{5}^{3}+\frac{9}{2} t_{3} t_{5}+\frac{7}{4} t_{4}^{2}+x+c \equiv \gamma_{5,6}+x+c
\end{align*}
$$

Lemma 7 For arbitrary $n \in \mathbb{N}$, non-autonomous vector fields $\mathcal{K}_{n, r}$ fulfill Frobenius conditions

$$
\begin{equation*}
\left[\mathcal{K}_{n, s}, \mathcal{K}_{n, r}\right]+\frac{\partial \mathcal{K}_{n, r}}{\partial t_{s}}-\frac{\partial \mathcal{K}_{n, s}}{\partial t_{r}}=0, \quad r, s=2, \ldots, n+1 \tag{6.5}
\end{equation*}
$$

Theorem 8 The non-autonomous stationary system

$$
\begin{equation*}
u_{t_{r}}=\mathcal{K}_{n, r}=\pi_{0} \gamma_{n, r}, \quad 0=\gamma_{n, n+1}+x+c, \quad r=2, \ldots, n \tag{6.6}
\end{equation*}
$$

has isomonodromic Lax representation

$$
\begin{equation*}
\frac{d}{d t_{r}} U_{n, n+1}=\left[U_{n, r}, U_{n, n+1}\right]+\frac{\partial U_{n, r}}{\partial \lambda}, \quad r=1, \ldots, n \tag{6.7}
\end{equation*}
$$

where $\frac{d}{d t_{r}}$ is the evolutionary derivative (3.12) along the $r$-th flow $\mathcal{K}_{n, r}$,

$$
\begin{gather*}
U_{n, r}=V_{n, r} \quad r=1, \ldots, \kappa, \quad U_{n, r}=\sum_{j=1}^{r} \zeta_{r, j}\left(t_{1}, \ldots, t_{r-1}\right) V_{n, j}, \quad \zeta_{r, r}=1, \quad \text { for } r=\kappa+1, \ldots, n  \tag{6.8}\\
V_{n, r}=\left(\begin{array}{cc}
-\frac{1}{2}\left(P_{n, r}\right)_{x} & P_{n, r} \\
P_{n, r}(\lambda-u)-\frac{1}{2}\left(P_{n, r}\right)_{x x} & \frac{1}{2}\left(P_{n, r}\right)_{x}
\end{array}\right), \quad P_{n, r}=\frac{1}{2} \sum_{i=0}^{r-1} \gamma_{n, i} \lambda^{r-i-1}, \quad \gamma_{n, 0}=\gamma_{0} \tag{6.9}
\end{gather*}
$$

and

$$
U_{n, n+1}=V_{n, n+1} \quad \text { under constraint } \quad 0=\gamma_{n, n+1}+x+c
$$

The proof follows from the isomonodromic Lax representation (5.19), (5.21) of Painlevé representation of (6.6) and relations (3.22) and (3.23) for their autonomous counterparts.

The hierarchy of non-homogeneous KdV equations (6.3) has the following non-isospectral zero curvature representation

$$
\begin{equation*}
\frac{d}{d t_{n}} V_{1}+\frac{\partial}{\partial \lambda} V_{1}-\frac{d}{d x} V_{n, n+1}+\left[V_{1}, V_{n, n+1}\right]=0 \tag{6.10}
\end{equation*}
$$

as $V_{n, 1}=V_{1}$.
Now, let us pass to the second non-autonomous KdV hierarchy of stationary systems, constructed from the Painlevé-type systems (5.25). Again for $n=1$, differentiation of (5.31) by $t$, division by $2 q$ and substitution $t=x, q=\frac{1}{2} u+\frac{1}{2} x$ we get the stationary flow of the following PDE

$$
\begin{equation*}
u_{t_{2}}=\left(\frac{1}{4} \partial_{x}^{3}+\frac{1}{2} u \partial_{x}+\frac{1}{2} \partial_{x}\right)\left(\gamma_{1}+x\right)=\mathcal{K}_{2}+\sigma_{0} \equiv \pi_{1} \gamma_{1,1}=\mathcal{K}_{1,2} \tag{6.11}
\end{equation*}
$$

from the KdV family. For $n=2,3,4, \ldots$, by the substitution $t_{1}=x, q_{1}=\frac{1}{2} u+\frac{2 n-1}{2} t_{n}$, the Painlevé-type equations, generated by Hamiltonians $H_{1}$ from (5.25) hierarchy, are integrated stationary flows (with respect to $\pi_{1}$ ) of the following hierarchy of non-homogeneous KdV equations

$$
\begin{align*}
u_{t_{3}} & =\mathcal{K}_{3}+\frac{3}{2} t_{2} \mathcal{K}_{2}+\frac{3}{8} t_{2}^{2} \mathcal{K}_{1}+\sigma_{0} \equiv \mathcal{K}_{2,3}=\pi_{1}\left(\gamma_{2}+\frac{3}{2} t_{2} \gamma_{1}+\frac{3}{8} t_{2}^{2} \gamma_{0}+x\right)=\pi_{1} \gamma_{2,2} \\
u_{t_{4}} & =\mathcal{K}_{4}+\frac{5}{2} t_{3} \mathcal{K}_{3}+\frac{3}{2}\left(t_{2}+\frac{5}{4} t_{3}^{2}\right) \mathcal{K}_{2}+\frac{1}{2}\left(\frac{5}{2} t_{2} t_{3}+\frac{5}{8} t_{3}^{2}\right) \mathcal{K}_{1}+\sigma_{0} \equiv \mathcal{K}_{3,4} \\
& =\pi_{1}\left[\gamma_{3}+\frac{5}{2} t_{3} \gamma_{2}+\frac{3}{2}\left(t_{2}+\frac{5}{4} t_{3}^{2}\right) \gamma_{1}+\frac{1}{2}\left(\frac{5}{2} t_{2} t_{3}+\frac{5}{8} t_{3}^{2}\right) \gamma_{0}+x\right]=\pi_{1} \gamma_{3,3} \\
u_{t_{5}} & =\mathcal{K}_{5}+\frac{7}{2} t_{4} \mathcal{K}_{4}+\frac{5}{2}\left(t_{3}+\frac{7}{4} t_{4}^{2}\right) \mathcal{K}_{3}+\left(\frac{3}{2} t_{2}+\frac{21}{4} t_{3} t_{4}+\frac{35}{16} t_{4}^{3}\right) \mathcal{K}_{2}+\left(\frac{5}{8} t_{3}^{2}+\frac{7}{4} t_{2} t_{4}+\frac{35}{128} t_{4}^{4}\right) \mathcal{K}_{1}+\sigma_{0} \equiv \mathcal{K}_{4,5} \\
& =\pi_{1}\left[\gamma_{4}+\frac{7}{2} t_{4} \gamma_{3}+\frac{5}{2}\left(t_{3}+\frac{7}{4} t_{4}^{2}\right) \gamma_{2}+\left(\frac{3}{2} t_{2}+\frac{21}{4} t_{3} t_{4}+\frac{35}{16} t_{4}^{3}\right) \gamma_{1}+\left(\frac{5}{8} t_{3}^{2}+\frac{7}{4} t_{2} t_{4}+\frac{35}{128} t_{4}^{4}\right) \gamma_{0}+x\right]=\pi_{1} \gamma_{4,4} \tag{6.12}
\end{align*}
$$

$$
\begin{aligned}
u_{t_{6}}= & \mathcal{K}_{6}+\frac{9}{2} t_{5} \mathcal{K}_{5}+\left(\frac{7}{2} t_{4}+\frac{63}{8} t_{5}^{2}\right) \mathcal{K}_{4}+\left(\frac{5}{2} t_{3}+\frac{45}{4} t_{4} t_{5}+\frac{105}{16} t_{5}^{3}\right) \mathcal{K}_{3}+\left(\frac{3}{2} t_{2}+\frac{21}{8} t_{4}^{2}+\frac{27}{4} t_{3} t_{5}\right. \\
& \left.+\frac{189}{16} t_{4} t_{5}^{2}+\frac{315}{128} t_{5}^{4}\right) \mathcal{K}_{2}+\left(\frac{7}{4} t_{3} t_{4}+\frac{9}{4} t_{2} t_{5}+\frac{45}{16} t_{4} t_{5}+\frac{63}{16} t_{3} t_{5}^{2}+\frac{105}{32} t_{4} t_{5}^{3}+\frac{62}{256} t_{5}^{5}\right) \mathcal{K}_{1}+\sigma_{0} \equiv \mathcal{K}_{5,6} \\
= & \pi_{1}\left[\gamma_{5}+\frac{9}{2} t_{5} \gamma_{4}+\left(\frac{7}{2} t_{4}+\frac{63}{8} t_{5}^{2}\right) \gamma_{3}+\left(\frac{5}{2} t_{3}+\frac{45}{4} t_{4} t_{5}+\frac{105}{16} t_{5}^{3}\right) \gamma_{2}+\left(\frac{3}{2} t_{2}+\frac{21}{8} t_{4}^{2}+\frac{27}{4} t_{3} t_{5}\right.\right. \\
& \left.\left.\quad+\frac{189}{16} t_{4} t_{5}^{2}+\frac{315}{128} t_{5}^{4}\right) \gamma_{1}+\left(\frac{7}{4} t_{3} t_{4}+\frac{9}{4} t_{2} t_{5}+\frac{45}{16} t_{4}^{2} t_{5}+\frac{63}{16} t_{3} t_{5}^{2}+\frac{105}{32} t_{4} t_{5}^{3}+\frac{63}{256} t_{5}^{5}\right) \gamma_{0}+x\right]=\pi_{1} \gamma_{5,5}
\end{aligned}
$$

Again, contrary to the autonomous case, for fixed $n$, the remaining Painlevé-type equations generated by Hamiltonians $H_{2}, \ldots, H_{n}$ from (5.25) do not reconstruct the lower order equations from the hierarchy (6.12). Actually, the hierarchy of Painlevé-type systems generated by Hamiltonians (5.25) is equivalent
to the following hierarchy of non-autonomous KdV stationary systems

$$
\begin{align*}
& n=1: \quad 0=\frac{1}{2} \gamma_{1,1}\left(\gamma_{1,1}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{1,1}\right)_{x}\right]^{2}+u \gamma_{1,1}^{2}+c \\
& n=2: \quad u_{t_{2}}=\mathcal{K}_{2}+\frac{3}{2} t_{2} \mathcal{K}_{1} \equiv \mathcal{K}_{2,2}=\pi_{1} \gamma_{2,1} \\
& 0=\frac{1}{2} \gamma_{2,2}\left(\gamma_{2,2}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{2,2}\right)_{x}\right]^{2}+u \gamma_{2,2}^{2}+c \\
& u_{t_{2}}=\mathcal{K}_{2}+\frac{5}{2} t_{3} \mathcal{K}_{1} \equiv \mathcal{K}_{3,2}=\pi_{1} \gamma_{3,1} \\
& n=3: \quad u_{t_{3}}=\mathcal{K}_{3}+\frac{5}{2} t_{3} \mathcal{K}_{2}+\left(\frac{5}{2} t_{2}+\frac{15}{8} t_{3}^{2}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{3,3}=\pi_{1} \gamma_{3,2} \\
& 0=\frac{1}{2} \gamma_{3,3}\left(\gamma_{3,3}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{3,3}\right)_{x}\right]^{2}+u \gamma_{3,3}^{2}+c \\
& u_{t_{2}}=\mathcal{K}_{2}+\frac{7}{2} t_{4} \mathcal{K}_{1} \equiv \mathcal{K}_{4,2}=\pi_{1} \gamma_{4,1} \\
& n=4: \\
& u_{t_{3}}=\mathcal{K}_{3}+\frac{7}{2} t_{4} \mathcal{K}_{2}+\left(\frac{5}{2} t_{3}+\frac{35}{8} t_{4}^{2}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{4,3}=\pi_{1} \gamma_{4,2}  \tag{6.13}\\
& u_{t_{4}}=\mathcal{K}_{4}+\frac{7}{2} t_{4} \mathcal{K}_{3}+\left(\frac{7}{2} t_{3}+\frac{35}{8} t_{4}^{2}\right) \mathcal{K}_{2}+\left(\frac{7}{2} t_{2}+\frac{35}{4} t_{3} t_{4}+\frac{35}{16} t_{4}^{3}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{4,4}=\pi_{1} \gamma_{4,3} \\
& 0=\frac{1}{2} \gamma_{4,4}\left(\gamma_{4,4}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{4,4}\right)_{x}\right]^{2}+u \gamma_{4,4}^{2}+c \\
& u_{t_{2}}=\mathcal{K}_{2}+\frac{9}{2} t_{5} \mathcal{K}_{1} \equiv \mathcal{K}_{5,2}=\pi_{1} \gamma_{5,1} \\
& u_{t_{3}}=\mathcal{K}_{3}+\frac{9}{2} t_{5} \mathcal{K}_{2}+\left(\frac{7}{2} t_{4}+\frac{63}{8} t_{5}^{2}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{5,3}=\pi_{1} \gamma_{5,2} \\
& n=5: \\
& u_{t_{4}}=\mathcal{K}_{4}+\frac{9}{2} t_{5} \mathcal{K}_{3}+\left(\frac{7}{2} t_{4}+\frac{63}{8} t_{5}^{2}\right) \mathcal{K}_{2}+\left(\frac{7}{2} t_{3}+\frac{45}{4} t_{4} t_{5}+\frac{105}{16} t_{5}^{3}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{5,4}=\pi_{1} \gamma_{5,3} \\
& u_{t_{5}}=\mathcal{K}_{5}+\frac{9}{2} t_{5} \mathcal{K}_{4}+\left(\frac{9}{2} t_{4}+\frac{63}{8} t_{5}^{2}\right) \mathcal{K}_{3}+\left(\frac{9}{2} t_{3}+\frac{63}{4} t_{4} t_{5}+\frac{105}{16} t_{5}^{3}\right) \mathcal{K}_{2} \\
& +\left(\frac{9}{2} t_{2}+\frac{45}{8} t_{4}^{2}+\frac{63}{4} t_{3} t_{5}+\frac{315}{16} t_{4} t_{5}^{2}+\frac{315}{128} t_{5}^{4}\right) \mathcal{K}_{1} \equiv \mathcal{K}_{5,5}=\pi_{1} \gamma_{5,4} \\
& 0=\frac{1}{2} \gamma_{5,5}\left(\gamma_{5,5}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{5,5}\right)_{x}\right]^{2}+u \gamma_{5,5}^{2}+c
\end{align*}
$$

As in the previous case, for arbitrary $n \in \mathbb{N}$, non-autonomous vector fields $\mathcal{K}_{n, r}, r=2, \ldots, n+1$ fulfill Frobenius conditions (6.5).

The non-autonomous stationary system

$$
u_{t_{r}}=\mathcal{K}_{n, r}=\pi_{1} \gamma_{n, r-1}, \quad 0=\frac{1}{2} \gamma_{n, n}\left(\gamma_{n, n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n, n}\right)_{x}\right]^{2}+u \gamma_{n, n}^{2}+c, \quad r=2, \ldots, n
$$

has isomonodromic Lax representation (6.7)-(6.9), where now

$$
U_{n, n+1}=V_{n, n+1} \quad \text { under constraint } \quad 0=\frac{1}{2} \gamma_{n, n}\left(\gamma_{n, n}\right)_{x x}-\frac{1}{4}\left[\left(\gamma_{n, n}\right)_{x}\right]^{2}+u \gamma_{n, n}^{2}+c
$$

Besides, the hierarchy of non-homogeneous KdV equations (6.12) has again the non-isospectral zero curvature representation in the form (6.10).

## 7 Conclusions

For the KdV hierarchy (2.2) the related stationary systems (2.22) have two different representations (2.24) and (2.25), being particular Stäckel systems. On the other hand, starting from the family of such Stäckel systems, one can reconstruct related stationary systems (2.22) and then the whole KdV hierarchy (2.2). In this article we have performed the same procedure for Painlevé deformations of considered Stäckel systems. In consequence we have constructed two non-autonomous families of KdV hierarchies

$$
u_{t_{n, r}}=\mathcal{K}_{n, r}(t)=\pi_{0} \gamma_{n, r}(t), \quad u_{t_{n, n+1}}=\mathcal{K}_{n, n+1}(t)=\pi_{0}\left(\gamma_{n, n+1}(t)+x\right), \quad r=2, \ldots, n, \quad n \in \mathbb{N}
$$

and

$$
u_{\tau_{n, r}}=\mathcal{K}_{n, r}(\tau)=\pi_{1} \gamma_{n, r-1}(\tau), \quad u_{\tau_{n, n+1}}=\mathcal{K}_{n, n+1}(\tau)=\pi_{1}\left(\gamma_{n, n}(\tau)+x\right), \quad r=2, \ldots, n, \quad n \in \mathbb{N}
$$

with related non-autonomous stationary systems

$$
u_{t_{n, r}}=\mathcal{K}_{n, r}(t)=\pi_{0} \gamma_{n, r}(t), \quad 0=\gamma_{n, n+1}(t)+x+c, \quad r=2, \ldots, n, \quad n \in \mathbb{N}
$$

and

$$
u_{\tau_{n, r}}=\mathcal{K}_{n, r}(\tau)=\pi_{1} \gamma_{n, r-1}(\tau), \quad 0=\frac{1}{2} \bar{\gamma}_{n, n}\left(\bar{\gamma}_{n, n}\right)_{x x}-\frac{1}{4}\left[\left(\bar{\gamma}_{n, n}\right)_{x}\right]^{2}+u \bar{\gamma}_{n, n}^{2}+c, \quad r=2, \ldots, n, \quad n \in \mathbb{N}
$$ where $\bar{\gamma}_{n, n}=\gamma_{n, n}(\tau)+x$, having respective Painlevé representations, considered in Section 5.

## References

[1] Airault H., Rational solutions of Painlevé equations, Stud. Apply Math. 61 (1979) 31-53.
[2] Antonowicz, M.; Fordy, A. P.; Wojciechowski, S., Integrable stationary flows: Miura maps and biHamiltonian structures, Phys. Lett. A 124 (1987) 143-150.
[3] Antonowicz, M.; Rauch-Wojciechowski S., Restricted flows of soliton hierarchies: coupled KdV and Harry Dym case, J. Phys. A 24 (1991), no. 21, 5043-5061.
[4] Błaszak M, On separability of bi-Hamiltonian chain with degenerated Poisson structures, J. Math. Phys. 39 (1998) 3213
[5] Błaszak M, Multi-Hamiltonian theory of dynamical, Texts and Monographs in Physics. SpringerVerlag, Berlin, 1998.
[6] Błaszak M., Marciniak K., Stäckel systems generating coupled KdV hierarchies and their finite-gap and rational solutions, J. Phys. A 41 (2008), no. 48, 485202
[7] M. Błaszak, A. Sergyeyev, Generalized Stäckel systems, Phys. Lett. A 375 (2011), no. 27, 2617-2623.
[8] Błaszak M., Domański Z., Lax Representations for Separable Systems from Benenti Class, SIGMA 15 (2019), 045.
[9] M. Błaszak, Non-autonomous Hénon-Heiles system from Painlevé class, Phys. Lett. A 383 (2019), 2149-2152.
[10] Błaszak M., Marciniak K., Sergyeyev A., Deforming Lie algebras to Frobenius integrable nonautonomous Hamiltonian systems, Rep. Math. Phys. 87 (2021) 249-263
[11] Błaszak M., Marciniak K., Domański Z., Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability, Stud Appl Math. 148 (2022) 1208-1250
[12] Błaszak M., Marciniak K., Domański Z., Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation, Stud Appl Math. 149 (2022) 364415
[13] Błaszak M., Szablikowski B. and Marciniak K., Stäckel representations of stationary KdV systems, arXiv:2204.10632 (2022)
[14] Bogojavlenskii, O. I.; Novikov, S. P., The connectionbetween the Hamiltonian formalisms of stationary and nonstationary problems, (Russian) Funkcional. Anal. i Prilozen. 10 (1976), no. 1,9-13
[15] Cao C., Geng X., Neumann and Bargmann systems associated with the coupled KdV soliton hierarchy, J. Phys. A: Math. Gen. 23 (1990) 4117
[16] Chiba H., Multi-Poisson Approach to the Painlevé Equations: from the Isospectral Deformation to the Isomonodromic Deformation, SIGMA 13 (2017) 025
[17] Clarkson P.A., Joshi N., Pickering A., Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach, Inverse Problems 15 (1999) 175
[18] Clarkson P.A., Joshi N., Mazzocco M., The Lax pair for the mKdV hierarchy, in Theories asymptotiques et Equations de Painlevé, Semin. Congr., Vol. 14, Soc. Math. France, Paris, 2006, 53
[19] Falqui G., Pedroni M., Separation of variables for Bi-Hamiltonian systems, Math. Phys. Anal. Geom. 6 (2003) 139
[20] Fecko M., Differential geometry and Lie groups for physicists, Cambridge University Press, New York, 2006.
[21] Flaschka H., Newell A.C., Monodromy-and spectrum-preserving deformations, Comm. Math. Phys. 76 (1980) 65-116.
[22] Fordy A., The Hénon-Heiles system revisited, Physica D 52 (1991) 204-210
[23] Gordoa P. R., Pickering A., Nonisospectral scattering problems: A key to integrable hierarchies, J. Math. Phys. 40 (1999) 5749-5786
[24] Hone A.N.W., Non-autonomous Hénon-Heiles systems, Physica D 118 (1998) 1-16
[25] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, From Gauss to Painlevé. A Modern Theory of Special Functions, Vieweg \& Sohn Verlagsgesellschaft mbH, Braunschweig, 1991
[26] Koike T., On new expressions of the Painlevé hierarchies, in Algebraic Analysis and the Exact WKB Analysis for Systems of Differential Equations, RIMS KĂ’kyûroku Bessatsu, Vol. B5, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, 153
[27] Kudryashov N.A., The first and second Painlevé equations of higher order and some relations between them, Phys. Lett. A 224 (1997) 353
[28] Lundell A.T., A short proof of the Frobenius theorem, Proc. Amer. Math. Soc. 116 (1992), no. 4, 1131-1133
[29] Marciniak K., Błaszak M., Construction of coupled Harry Dym hierarchy and its solutions from Stäckel systems, Nonlinear Anaysis 73 (2010) 3004-3017.
[30] Noumi M., Yamada Y., Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys. 199 (1998) 281-195.
[31] Shimomura S., A certain expression of the first Painlevé hierarchy, Proc. Japan Acad. Ser. A Math. Sci. 80 (2004) 105
[32] Takasaki K., Hamiltonian Structure of PI Hierarchy, SIGMA 3 (2007) 042

