

Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets

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ABSTRACT

In this paper, two optimisation models are established to determine the criterion weights in multi-criteria decision-making situations where knowledge regarding the weight information is incomplete and the criterion values are interval neutrosophic numbers. The proposed approach combines interval neutrosophic sets and TOPSIS, and the closeness coefficients are expressed as interval numbers. Furthermore, the relative likelihood-based comparison relations are constructed to determine the ranking of alternatives. A fuzzy cross-entropy approach is proposed to calculate the discrimination measure between alternatives and the absolute ideal solutions, after a transformation operator has been developed to convert interval neutrosophic numbers into simplified neutrosophic numbers. Finally, an illustrative example is provided, and a comparative analysis is conducted between the approach developed in this paper and other existing methods, to verify the feasibility and effectiveness of the proposed approach.

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1. Introduction

Fuzzy sets (FSs), which were proposed by Zadeh (1965), are regarded as a comprehensive tool for solving multi-criteria decision-making (MCDM) problems (Bellman & Zadeh, 1970). In order to resolve the uncertainty of non-membership degrees, Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs), which are an extension of Zadeh's FSs. IFSs have been widely applied in solving MCDM problems to date (Chen & Chang, 2015; Chen, Cheng, & Chiou, 2016; Wang et al., 2014; Yue, 2014). Moreover, interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov & Gargov, 1989) were proposed, which are an extension of FSs and IFSs. In recent years, MCDM problems with IVIFSs have attracted much attention from researchers (Chen, 2014; Liu, Shen, Zhang, Chen, & Wang, 2015; Tan et al., 2014; Wan & Dong, 2014). Furthermore, the TOPSIS method, proposed by Hwang and Yoon (1981), has also been used for solving MCDM problems (Cao, Wu, & Liang, 2015; Yue, 2014; Zhang & Yu, 2012). Moreover, fuzzy linear programming models have been constructed to address MCDM problems with incomplete criterion weight information (Chen, 2014; Dubey, Chandra, & Mehra, 2012; Wan, Wang, Lin & Dong, 2015; Wan, Xu, Wang, & Dong, 2015; Zhang & Yu, 2012).

Although the theories of FSs and IFSs have been developed and generalised, they cannot deal with all types of uncertainties in real problems. Indeed certain types of uncertainties, such as indeterminate and inconsistent information, cannot be managed. For example, FSs and IFSs cannot effectively deal with a situation where a paper is sent to a reviewer, and he or she says it is 70% acceptable and 60% unacceptable, and his or her statement is 20% uncertain; therefore, some new theories are required.

Since neutrosophic sets (NSs) (Smarandache, 1999) consider the truth membership, indeterminacy membership and

falsity membership simultaneously, it is more practical and flexible than FSs and IFSs in dealing with uncertain, incomplete and inconsistent information. For the aforementioned example, the reviewer's opinion can be presented as $x(0.7, 0.2, 0.6)$ by means of NSs. However, without a specific description, it is hard to apply NSs in actual scientific and engineering situations. Hence, single-valued neutrosophic sets (SVNSs), which are an extension of NSs, were introduced by Wang et al. (2010). Subsequently, the similarity and entropy measures (Majumdar & Samant, 2014), the correlation coefficient (Ye, 2013) and the cross-entropy (Ye, 2014c) of SVNSs have been developed. Additionally, Ye (2014a) introduced simplified neutrosophic sets (SNSs), and Peng, Wang, Wang, Zhang, and Chen (2015) and Peng, Wang, Zhang, and Chen (2014) defined their novel operations and aggregation operators. Finally, further extensions of NSs, such as interval neutrosophic sets (INSSs) (Wang et al., 2005), bipolar neutrosophic sets (BNSs) (Deli, Ali, & Smarandache, 2015) and multivalued neutrosophic sets (MVNSs) (Peng, Wang, Wu, Wang, & Chen, 2015; Wang & Li, 2015), have also been proposed.

In certain real-life situations, the INS, as a particular extension of an NS, can be more flexible in assessing objections than an SNS. Recently, the studies relating to INSSs have been focused on particular areas, which can be roughly classified into two groups.

The first group is based on interval neutrosophic aggregation operators, such as interval neutrosophic number weighted averaging (INNWA) and interval neutrosophic number weighted geometric (INNWG) operators (Zhang, Wang, & Chen, 2014), interval neutrosophic number Choquet integral (INNCI) operator (Sun et al., 2015), interval neutrosophic number ordered weighted averaging (INNOWA) and interval neutrosophic

number ordered weighted geometric (INNWG) operators (Ye, 2015) and interval neutrosophic prioritised ordered weighted aggregation (INPOWA) operator (Liu & Wang, 2015).

The second group is based on interval neutrosophic measures (Broumi & Smarandache, 2014a, 2014b; Chi & Liu, 2013; Ye, 2014b; Zhang, Ji, Wang, & Chen 2015a; Zhang et al. 2015b). Specifically, Chi and Liu (2013) extended TOPSIS to an INS based on a distance measure. Broumi and Smarandache (2014a, 2014b) defined a new cosine similarity measure and a correlation coefficient of an INS. Moreover, Ye (2014b) defined two interval neutrosophic similarity measures based on the Hamming and Euclidean distances. Zhang et al. (2015b) defined several interval neutrosophic outranking relations based on ELECTRE IV. Finally, Zhang et al. (2015a) proposed an improved weighted correlation coefficient based on the integrated weight for an INS.

The aforementioned methods are effective when managing interval neutrosophic MCDM problems; however, they have some drawbacks which are outlined below.

(1) The existing MCDM methods require the criterion weight information to be completely known (Broumi & Smarandache, 2014b; Ye, 2014b, 2015; Zhang et al., 2014, 2015a, 2015b), or are supported by fuzzy measures of criteria (Sun et al., 2015). However, due to the increasing complexity of MCDM problems, it is difficult and subjective to provide exact criterion weight information or fuzzy measures. (2) The MCDM methods (Broumi & Smarandache, 2014b; Chi & Liu, 2013; Ye, 2014b; Zhang et al., 2015a, 2015b) utilise the weighted measures, such as distance, correlation coefficient and similarity measures, to rank alternatives; in these processes, the interval neutrosophic information is measured with crisp real numbers. However, the means of aggregating interval numbers into real numbers may lead to some operational deficiencies and a large amount of information loss. (3) All of the developed aggregation operators (Liu & Wang, 2015; Sun et al., 2015; Ye, 2015; Zhang et al., 2014) directly process the assessment information with INNs. However, these approaches are complex and tedious, and not enough attention is given to reducing the computational complexity when processing the evaluation information.

To overcome these disadvantages, in this paper, a novel and comprehensive approach for managing MCDM with INNs is developed. The main novelties and the significant contributions are summarised below.

(1) In order to derive the criterion weights, two mathematical programming models are constructed to determine the optimal weight values, and decision-makers (DMs) may provide incomplete or inconsistent opinions on the weight information. (2) A novel form of the closeness coefficient with interval values is derived on the basis of TOPSIS and a cross-entropy of INNs. Furthermore, the weighted interval closeness coefficients are employed to determine the ranking of alternatives. (3) To lower the computational complexity, a modified transformation operator is developed for converting INNs into simplified neutrosophic numbers (SNNs). Nevertheless, the parameters of the transformation operator are not artificially set, but obtained through mathematical derivation.

The rest of the paper is organised as follows. In Section 2, interval numbers, as well as the concepts of NSs, SNSs and INNs, are briefly reviewed. In Section 3, on the basis of an operator that

can transform each INN into an SNN, a cross-entropy for SNNs is proposed. Subsequently, a TOPSIS approach in the context of INNs for solving MCDM problems with incompletely known criterion information is developed in Section 4. In Section 5, an illustrative example is provided and a comparative analysis is conducted between the proposed approach and other existing methods. Finally, conclusions are drawn in Section 6.

2. Preliminaries

In this section, some basic concepts and definitions related to INNs, including interval numbers, definitions and operational laws of NSs, SNSs and INNs are introduced; these will be utilised in the latter analysis.

2.1. Interval numbers

Interval numbers and their operations are of utmost significance when exploring the operations of INNs. In the following paragraphs, some definitions and operational laws of interval numbers are provided.

Definition 1 (Sengupta & Pal, 2000; Xu, 2008): Let $\tilde{a} = [a^L, a^U] = \{x | a^L \leq x \leq a^U\}$, and then \tilde{a} is called an interval number. In particular, $\tilde{a} = [a^L, a^U]$ will be degenerated to a real number, if $a^L = a^U$.

Consider two non-negative interval numbers $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$, where $0 \leq a^L \leq x \leq a^U$ and $0 \leq b^L \leq x \leq b^U$. Subsequently, their operations are defined as follows (Sengupta & Pal, 2000; Xu, 2008):

- (1) $\tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U]$,
- (2) $\lambda \tilde{a} = [\lambda a^L, \lambda a^U]$, $\lambda > 0$.

Definition 2 (Xu & Da, 2003): For any two interval numbers $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$, the possibility of $\tilde{a} \geq \tilde{b}$ is formulated by $p(\tilde{a} \geq \tilde{b}) = \max\{1 - \max\{\frac{b^U - a^L}{L(\tilde{a}) + L(\tilde{b})}, 0\}, 0\}$, where $L(\tilde{a}) = a^U - a^L$ and $L(\tilde{b}) = b^U - b^L$.

The possibility degree of $\tilde{a} \geq \tilde{b}$ has the following properties (Xu & Da, 2003):

- (1) $0 \leq p(\tilde{a} \geq \tilde{b}) \leq 1$;
- (2) $p(\tilde{a} \geq \tilde{b}) = p(\tilde{b} \geq \tilde{a}) = 0.5$, if $p(\tilde{a} \geq \tilde{b}) = p(\tilde{b} \geq \tilde{a})$;
- (3) $p(\tilde{a} \geq \tilde{b}) + p(\tilde{b} \geq \tilde{a}) = 1$;
- (4) $p(\tilde{a} \geq \tilde{b}) = 0$ if $a^U \leq b^L$, $p(\tilde{a} \geq \tilde{b}) = 1$, if $a^L \geq b^U$;
- (5) For any interval numbers \tilde{a} , \tilde{b} and \tilde{c} , $p(\tilde{a} \geq \tilde{c}) \geq 0.5$, if $p(\tilde{a} \geq \tilde{b}) \geq 0.5$ and $p(\tilde{b} \geq \tilde{c}) \geq 0.5$; $p(\tilde{a} \geq \tilde{c}) = 0.5$, if and only if $p(\tilde{a} \geq \tilde{b}) = p(\tilde{b} \geq \tilde{c}) = 0.5$.

2.2. NSs and SNSs

Due to the influence of subjective factors, it is difficult for DMs to explicitly express preferences if indeterminacy and inconsistency exist. NS can effectively capture such information (Guo & Şengür, 2014; Mohan, Krishnaveni, & Guo, 2013; Solis & Panoutsos, 2013).

Definition 3 (Smarandache, 1999): Let X be a space of points (objects) with a generic element in X , denoted by x . An NS A in X is characterised by a truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. $t_A(x)$, $i_A(x)$ and $f_A(x)$ are real

standard or non-standard subsets of $]0^-, 1^+[$; that is, $t_A(x) : X \rightarrow]0^-, 1^+[$, $i_A(x) : X \rightarrow]0^-, 1^+[$, and $f_A(x) : X \rightarrow]0^-, 1^+[$. There is no restriction on the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$, thus $0^- \leq \sup t_A(x) + \sup i_A(x) + \sup f_A(x) \leq 3^+$.

Since it is difficult to apply NSs to practical problems, Ye (2014a) reduced NSs of non-standard interval numbers into a type of SNS of standard interval numbers.

Definition 4 (Rivieccio, 2008; Ye, 2014a): Let an NS A in X be characterised by $t_A(x)$, $i_A(x)$ and $f_A(x)$, which are single subintervals/subsets in the real standard $[0, 1]$; that is, $t_A(x) : X \rightarrow [0, 1]$, $i_A(x) : X \rightarrow [0, 1]$, and $f_A(x) : X \rightarrow [0, 1]$. Also, the sum of $t_A(x)$, $i_A(x)$ and $f_A(x)$ satisfies the condition $0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3$. Then, a simplification of A is denoted by $A = \{(x, t_A(x), i_A(x), f_A(x)) | x \in X\}$, which is called an SNS and is a subclass of NSs. If $\|X\| = 1$, an SNS will be degenerated to an SNN.

Definition 5 (Rivieccio, 2008; Ye, 2014a): An SNS A is contained in the other SNS B , denoted by $A \subseteq B$, if and only if $t_A(x) \leq t_B(x)$, $i_A(x) \geq i_B(x)$ and $f_A(x) \geq f_B(x)$, for any $x \in X$. The complement set of A , denoted by A^c , is defined as $A^c = \{(x, f_A(x), i_A(x), t_A(x)) | x \in X\}$.

2.3. INSS

In actual applications, sometimes it is not easy to express the truth membership, indeterminacy membership and falsity membership by crisp values, but they may be easily described by interval numbers. Wang et al. (2005) further defined INSS.

Definition 6 (Rivieccio, 2008; Wang et al., 2005): Let X be a space of points (objects) with generic elements in X , denoted by x . An INS \tilde{A} in X is characterised by a truth-membership function $t_{\tilde{A}}(x)$, an indeterminacy-membership function $i_{\tilde{A}}(x)$ and a falsity-membership function $f_{\tilde{A}}(x)$. For each point x in X , $t_{\tilde{A}}(x) = [t_{\tilde{A}}^L, t_{\tilde{A}}^U]$, $i_{\tilde{A}}(x) = [i_{\tilde{A}}^L, i_{\tilde{A}}^U]$ and $f_{\tilde{A}}(x) = [f_{\tilde{A}}^L, f_{\tilde{A}}^U]$, and $0 \leq t_{\tilde{A}}^U + i_{\tilde{A}}^U + f_{\tilde{A}}^U \leq 3$. In particular, an INS will be reduced to an SNS, if $t_{\tilde{A}}^L = t_{\tilde{A}}^U = t_{\tilde{A}}^L$, $i_{\tilde{A}}^L = i_{\tilde{A}}^U = i_{\tilde{A}}^L$ and $f_{\tilde{A}}^L = f_{\tilde{A}}^U = f_{\tilde{A}}^L$. In addition, if $\|X\| = 1$, an INS will be degenerated to an INN.

Definition 7 (Rivieccio, 2008; Wang et al., 2005): An INS \tilde{A} is contained in the other INS \tilde{B} , denoted by $\tilde{A} \subseteq \tilde{B}$, if and only if $t_{\tilde{A}}^L \leq t_{\tilde{B}}^L$, $t_{\tilde{A}}^U \leq t_{\tilde{B}}^U$, $i_{\tilde{A}}^L \geq i_{\tilde{B}}^L$, $i_{\tilde{A}}^U \geq i_{\tilde{B}}^U$, $f_{\tilde{A}}^L \geq f_{\tilde{B}}^L$ and $f_{\tilde{A}}^U \geq f_{\tilde{B}}^U$, for any $x \in X$. In particular, $\tilde{A} = \tilde{B}$, if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{A} \supseteq \tilde{B}$. The complement set of \tilde{A} , denoted by \tilde{A}^c , is defined as $\tilde{A}^c = \{(x, [f_{\tilde{A}}^L, f_{\tilde{A}}^U], [i_{\tilde{A}}^L, i_{\tilde{A}}^U], [t_{\tilde{A}}^L, t_{\tilde{A}}^U]) | x \in X\}$.

3. A transformation operator and a cross-entropy measure of SNNs

In this section, a transformation operator is developed to convert INNs into SNNs. Moreover, a simplified neutrosophic cross-entropy measure is defined.

3.1. A transformation operator between INNs and SNNs

Bustince and Burillo (1995) proposed an operator $H_{p,q}$, which can transform each IVFS into an IFS. As an improved extension of $H_{p,q}$, an operator $H_{p,q,r}$ is defined for converting each INN into an SNN.

Definition 8 Let $p, q, r \in [0, 1]$ be three fixed numbers; any INN can be transformed into an SNN through the operator $H_{p,q,r}$:

$$H_{p,q,r}(\tilde{A}) = \left(t_{\tilde{A}}^L + pW_{t_{\tilde{A}}}, i_{\tilde{A}}^L + qW_{i_{\tilde{A}}}, f_{\tilde{A}}^L + rW_{f_{\tilde{A}}} \right),$$

where $W_{t_{\tilde{A}}} = t_{\tilde{A}}^U - t_{\tilde{A}}^L$, $W_{i_{\tilde{A}}} = i_{\tilde{A}}^U - i_{\tilde{A}}^L$ and $W_{f_{\tilde{A}}} = f_{\tilde{A}}^U - f_{\tilde{A}}^L$. Obviously, $H_{p,q,r}(\tilde{A})$ is an SNN, determined with respect to p, q and r ; that is, $H_{p,q,r}(\tilde{A})$ is well defined in all value ranges of p, q and r .

Example 1: Assume two INNs $\tilde{A} = ([0.7, 0.8], [0, 0.1], [0.1, 0.2])$ and $\tilde{B} = ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])$. Then, the following transformation results can be obtained utilising the operator $H_{p,q,r}$:

$$H_{p,q,r}(\tilde{A}) = (0.7 + p \times 0.1, 0 + q \times 0.1, 0.1 + r \times 0.1) \text{ and}$$

$$H_{p,q,r}(\tilde{B}) = (0.4 + p \times 0.1, 0.2 + q \times 0.1, 0.3 + r \times 0.1).$$

It is shown that $H_{p,q,r}(\tilde{A})$ and $H_{p,q,r}(\tilde{B})$ will be two specific SNNs if the values of p, q and r are given; the method to determine the parameter values will be discussed in detail in Section 4.

3.2. A cross-entropy measure of SNNs

Shang and Jiang (1997) proposed a fuzzy cross-entropy and a symmetric discrimination information measure between two FSs. Vlachos and Sergiadis (2007) then proposed an intuitionistic fuzzy cross-entropy for IFSs, and Ye (2011) defined a fuzzy cross-entropy for IVIFSs. Furthermore, the fuzzy cross-entropy has been employed for various purposes including deriving criterion weights by constructing mathematical programming models (Zhang & Yu 2012); technical efficiency analysis (Macedo & Scotto, 2014); multi-objective optimisation (Caballero, Hernández-Díaz, Laguna, & Molina, 2015); and MCDM problems (Meng & Chen, 2015; Peng, Wang, Wu, et al., 2014; Ye, 2014c; Zhao et al., 2013).

In a similar manner to the proposals of Ye (2014c) and Vlachos and Sergiadis (2007), the following definition of a fuzzy cross-entropy for SNNs is proposed.

Definition 9 Let $A, B \in \text{SNS}$, and then the cross-entropy $I_{\text{NS}}(A, B)$ between A and B should satisfy the following conditions:

- (1) $I_{\text{NS}}(A, B) = I_{\text{NS}}(B, A)$;
- (2) $I_{\text{NS}}(A, B) = I_{\text{NS}}(A^c, B^c)$, where A^c and B^c are the complement sets of A and B , respectively, as defined in Definition 5; and
- (3) $I_{\text{NS}}(A, B) \geq 0$ and $I_{\text{NS}}(A, B) = 0$, if and only if $A = B$.

Definition 10 Let $A = (t_A, i_A, f_A)$ and $B = (t_B, i_B, f_B)$ be two SNNs, and then the cross-entropy of A and B can be defined as follows:

$$I_{\text{NS}}(A, B) = t_A \ln \frac{2t_A}{t_A + t_B} + i_A \ln \frac{2i_A}{i_A + i_B} + f_A \ln \frac{2f_A}{f_A + f_B}. \quad (1)$$

Equation (1) can indicate the degree of discrimination of A from B . It is obvious that $I_{\text{NS}}(A, B)$ is not symmetrical with

respect to its arguments. Therefore, a modified symmetric discrimination information measure based on $I_{NS}(A, B)$ can be defined as

$$D_{NS}(A, B) = I_{NS}(A, B) + I_{NS}(B, A). \quad (2)$$

The larger $D_{NS}(A, B)$ is, the larger the difference between A and B will be, and vice versa.

Proposition 1: *The measures defined in Equations (1) and (2) are the simplified neutrosophic cross-entropy, and satisfy conditions (1)–(3) given in Definition 9.*

Proof: Clearly, conditions (1) and (2) are obvious. The proof of condition (3) is shown below.

Consider the function $f(x) = x \ln x$, where $x \in (0, 1]$. Then, $f'(x) = 1 + \ln x$ and $f''(x) = 1/x > 0$, where $x \in (0, 1]$. Accordingly, $f(x) = x \ln x$ is a convex function. Therefore, for any two points $x_1, x_2 \in (0, 1]$, the inequality $\frac{f(x_1)+f(x_2)}{2} \geq f(\frac{x_1+x_2}{2})$ holds.

Utilise $f(x) = x \ln x$ in the above inequality and $x_1 \ln x_1 + x_2 \ln x_2 - (x_1 + x_2) \ln \frac{x_1+x_2}{2} \geq 0$ can be obtained; in this case, the equality holds only if $x_1 = x_2$. Similarly, the following equation can be obtained:

$$\begin{aligned} D_{NS}(A, B) &= I_{NS}(A, B) + I_{NS}(B, A) \\ &= \underbrace{(t_A \ln t_A + t_B \ln t_B) - (t_A + t_B) \ln \frac{t_A + t_B}{2}}_T \\ &\quad + \underbrace{(i_A \ln i_A + i_B \ln i_B) - (i_A + i_B) \ln \frac{i_A + i_B}{2}}_I \\ &\quad + \underbrace{(f_A \ln f_A + f_B \ln f_B) - (f_A + f_B) \ln \frac{f_A + f_B}{2}}_F. \end{aligned}$$

Because $T \geq 0$, $I \geq 0$ and $F \geq 0$, $D_{NS}(A, B) \geq 0$ holds; $D_{NS}(A, B) = 0$ holds only if $t_A = t_B$, $i_A = i_B$ and $f_A = f_B$, namely $A = B$.

Example 2: Assume two SNNs $A = (0.8, 0.1, 0.2)$ and $B = (0.5, 0.3, 0.4)$. Then, the following result can be obtained by applying Equations (1) and (2):

$$D_{NS}(A, B) = I_{NS}(A, B) + I_{NS}(B, A) = 0.1212.$$

4. An MCDM approach based on the cross-entropy and TOPSIS

This section presents an approach that is based on the cross-entropy and TOPSIS for solving interval neutrosophic MCDM problems with incomplete weight information.

For an MCDM problem, let $A = \{a_1, a_2, \dots, a_m\}$ be a set consisting of m alternatives, and let $C = \{c_1, c_2, \dots, c_n\}$ be a set consisting of n criteria. Assume that $w = (w_1, w_2, \dots, w_n)$ is the weight vector of criteria, where $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Let $\tilde{B} = [\tilde{b}_{ij}] = [([t_{ij}^L, t_{ij}^U], [i_{ij}^L, i_{ij}^U], [f_{ij}^L, f_{ij}^U])]_{m \times n}$ be the decision matrix, where $[\tilde{b}_{ij}] = [([t_{ij}^L, t_{ij}^U], [i_{ij}^L, i_{ij}^U], [f_{ij}^L, f_{ij}^U])]_{m \times n}$ is the evaluation

information of each alternative $a_i (i = 1, 2, \dots, m)$ on the criterion $c_j (j = 1, 2, \dots, n)$ in the form of INN.

In general, there are two types of criterion, namely maximising and minimising criteria. In order to make the criterion types constant, the minimising criteria need to be transformed into maximising ones. Suppose the standardised matrix is expressed as $\tilde{R} = [\tilde{r}_{ij}]$. The original decision matrix \tilde{B} can then be converted into \tilde{R} based on the primary transformation principle of Xu and Hu (2010), where

$$\tilde{r}_{ij} = \begin{cases} \tilde{b}_{ij} = ([t_{ij}^L, t_{ij}^U], [i_{ij}^L, i_{ij}^U], [f_{ij}^L, f_{ij}^U]), & \text{for maximising criterion } c_j \\ \tilde{b}_{ij}^c = ([f_{ij}^L, f_{ij}^U], [i_{ij}^L, i_{ij}^U], [t_{ij}^L, t_{ij}^U]), & \text{for minimising criterion } c_j \end{cases}, \quad (3)$$

in which \tilde{b}_{ij}^c is the complement set of \tilde{b}_{ij} , defined in Definition 7.

4.1. A fuzzy cross-entropy based on TOPSIS

The absolute positive ideal solution (PIS) and the absolute negative ideal solution (NIS) of INNs are, respectively, denoted by a^+ and a^- , and can be expressed as follows (Chi & Liu, 2013):

$$a^+ = ([1, 1], [0, 0], [0, 0]) \text{ and } a^- = ([0, 0], [1, 1], [1, 1]).$$

In order to obtain the cross-entropy or degree of discrimination between $a_i (i = 1, 2, \dots, m)$ and the ideal solutions, each INN is transformed into an SNN based on the operator $H_{p,q,r}$, which was introduced in Definition 8. Let $p_{ij}, q_{ij}, r_{ij} \in [0, 1]$, and then any INN, denoted by $([t_{ij}^L, t_{ij}^U], [i_{ij}^L, i_{ij}^U], [f_{ij}^L, f_{ij}^U])$, can be transformed into the following form:

$$H_{p,q,r}(\tilde{r}_{ij}) = (t_{ij}^L + p_{ij}W_{t_{ij}}, i_{ij}^L + q_{ij}W_{i_{ij}}, f_{ij}^L + r_{ij}W_{f_{ij}}), \quad (4)$$

where $W_{t_{ij}} = t_{ij}^U - t_{ij}^L$, $W_{i_{ij}} = i_{ij}^U - i_{ij}^L$ and $W_{f_{ij}} = f_{ij}^U - f_{ij}^L$.

Using Equations (1), (2) and (4), the degree of discrimination of $a_i (i = 1, 2, \dots, m)$ from the ideal solutions a^+ and a^- with respect to $c_j (j = 1, 2, \dots, n)$ can be, respectively, calculated as follows:

$$\begin{aligned} D_{ij}^+ &= (t_{ij}^L + p_{ij}W_{t_{ij}}) \ln \frac{2(t_{ij}^L + p_{ij}W_{t_{ij}})}{(t_{ij}^L + p_{ij}W_{t_{ij}} + 1)} \\ &\quad + \ln \frac{2}{(t_{ij}^L + p_{ij}W_{t_{ij}} + 1)} + (i_{ij}^L + q_{ij}W_{i_{ij}}) \ln 2 \\ &\quad + (f_{ij}^L + r_{ij}W_{f_{ij}}) \ln 2, \end{aligned} \quad (5)$$

$$\begin{aligned} D_{ij}^- &= (t_{ij}^L + p_{ij}W_{t_{ij}}) \ln 2 + (i_{ij}^L + q_{ij}W_{i_{ij}}) \ln \frac{2(i_{ij}^L + q_{ij}W_{i_{ij}})}{(i_{ij}^L + q_{ij}W_{i_{ij}} + 1)} \\ &\quad + \ln \frac{2}{(i_{ij}^L + q_{ij}W_{i_{ij}} + 1)} \\ &\quad + (f_{ij}^L + r_{ij}W_{f_{ij}}) \ln \frac{2(f_{ij}^L + r_{ij}W_{f_{ij}})}{(f_{ij}^L + r_{ij}W_{f_{ij}} + 1)} \\ &\quad + \ln \frac{2}{(f_{ij}^L + r_{ij}W_{f_{ij}} + 1)}. \end{aligned} \quad (6)$$

In the TOPSIS method, the closeness coefficient $D_{ij} = D_{ij}^- / (D_{ij}^- + D_{ij}^+)$ denotes the performance of $a_i (i = 1, 2, \dots, m)$ on $c_j (j = 1, 2, \dots, n)$. Then, the performance of a_i , denoted by D_i , can be obtained by using Equations (5) and (6):

$$D_i = \sum_{j=1}^n w_j D_{ij} = \sum_{j=1}^n w_j \frac{D_{ij}^-}{D_{ij}^- + D_{ij}^+}, \quad (7)$$

where w_j represents the weight of c_j . Therefore, the larger D_i is, the better a_i will be.

$$D_{ij}^L = \frac{t_{ij}^L \ln 2 + i_{ij}^U \ln \frac{2i_{ij}^U}{(i_{ij}^U+1)} + f_{ij}^U \ln \frac{2f_{ij}^U}{(f_{ij}^U+1)} + \ln \frac{2}{(t_{ij}^U+1)} + \ln \frac{2}{(f_{ij}^U+1)}}{(t_{ij}^L + i_{ij}^U + f_{ij}^U) \ln 2 + t_{ij}^L \ln \frac{2t_{ij}^L}{(t_{ij}^L+1)} + i_{ij}^U \ln \frac{2i_{ij}^U}{(i_{ij}^U+1)} + f_{ij}^U \ln \frac{2f_{ij}^U}{(f_{ij}^U+1)} + \ln \frac{2}{(t_{ij}^L+1)} + \ln \frac{2}{(i_{ij}^U+1)} + \ln \frac{2}{(f_{ij}^U+1)}}, \quad (8)$$

$$D_{ij}^U = \frac{t_{ij}^U \ln 2 + t_{ij}^L \ln \frac{2t_{ij}^L}{(t_{ij}^L+1)} + f_{ij}^L \ln \frac{2f_{ij}^L}{(f_{ij}^L+1)} + \ln \frac{2}{(t_{ij}^U+1)} + \ln \frac{2}{(f_{ij}^L+1)}}{(t_{ij}^U + i_{ij}^L + f_{ij}^L) \ln 2 + t_{ij}^U \ln \frac{2t_{ij}^U}{(t_{ij}^U+1)} + i_{ij}^L \ln \frac{2i_{ij}^L}{(i_{ij}^L+1)} + f_{ij}^L \ln \frac{2f_{ij}^L}{(f_{ij}^L+1)} + \ln \frac{2}{(t_{ij}^U+1)} + \ln \frac{2}{(i_{ij}^L+1)} + \ln \frac{2}{(f_{ij}^L+1)}}, \quad (9)$$

In view of the fact that an INS is characterised by a truth-membership function, an indeterminacy-membership function and a falsity-membership function, whose values are intervals rather than specific numbers, it is infeasible to designate an SNN for the given INN by artificially choosing only certain p_{ij} , q_{ij} and r_{ij} in Equation (4) to indicate the evaluation information. This is because it may lead to the distortion or loss of the original information (Zhang & Yu, 2012).

By combining Equations (5) and (6), D_{ij} becomes a composite function and may take different values as the numerical variations p_{ij} , q_{ij} and r_{ij} change. In order to avoid information loss, an interval $\tilde{D}_{ij} = [D_{ij}^L, D_{ij}^U]$ is applied to represent the performance of $a_i (i = 1, 2, \dots, m)$ on $c_j (j = 1, 2, \dots, n)$, where D_{ij}^L and D_{ij}^U are the lower and upper bounds, respectively.

In the following paragraphs, D_{ij}^L and D_{ij}^U are determined through mathematical derivation.

As shown in Equations (5) and (6), both D_{ij}^+ and D_{ij}^- are multivariate continuous functions with respect to p_{ij} , q_{ij} and r_{ij} . They can also reach the maximum and minimum in the domain $p_{ij}, q_{ij}, r_{ij} \in [0, 1]$. Then, calculate the partial derivative of D_{ij}^+ and D_{ij}^- on p_{ij} , q_{ij} and r_{ij} :

$$\frac{\partial D_{ij}^+}{\partial p_{ij}} = W_{t_{ij}} \ln \frac{2(t_{ij}^L + p_{ij}W_{t_{ij}})}{(t_{ij}^L + p_{ij}W_{t_{ij}} + 1)}, \quad \frac{\partial D_{ij}^+}{\partial q_{ij}} = W_{i_{ij}} \ln 2 \geq 0, \quad \text{and}$$

$$\frac{\partial D_{ij}^+}{\partial r_{ij}} = W_{f_{ij}} \ln 2 \geq 0.$$

$$\text{Since } 2(t_{ij}^L + p_{ij}W_{t_{ij}}) \leq t_{ij}^L + p_{ij}W_{t_{ij}} + 1, \quad \frac{\partial D_{ij}^+}{\partial p_{ij}} \leq 0.$$

$$\text{Similarly, } \frac{\partial D_{ij}^-}{\partial p_{ij}} = W_{t_{ij}} \ln 2 \geq 0, \quad \frac{\partial D_{ij}^-}{\partial q_{ij}} = W_{i_{ij}} \ln \frac{2(t_{ij}^L + q_{ij}W_{i_{ij}})}{(t_{ij}^L + q_{ij}W_{i_{ij}} + 1)} \leq 0, \quad \text{and } \frac{\partial D_{ij}^-}{\partial r_{ij}} = W_{f_{ij}} \ln \frac{2(f_{ij}^L + r_{ij}W_{f_{ij}})}{(f_{ij}^L + r_{ij}W_{f_{ij}} + 1)} \leq 0.$$

This means that D_{ij}^+ is a monotone decreasing function with respect to p_{ij} and a monotone increasing function with respect to q_{ij} or r_{ij} . Moreover, D_{ij}^- is a monotone increasing function with respect to p_{ij} and a monotone decreasing func-

tion with respect to q_{ij} or r_{ij} . Thus, D_{ij}^+ can reach its maximum and D_{ij}^- can reach the minimum if $p_{ij} = 0$ and $q_{ij} = r_{ij} = 1$. Likewise, D_{ij}^+ reaches its minimum and D_{ij}^- reaches the maximum if $p_{ij} = 1$ and $q_{ij} = r_{ij} = 0$. As a result, it is easy to understand that D_{ij} reaches its minimum when $p_{ij} = 0$ and $q_{ij} = r_{ij} = 1$, and reaches its maximum when $p_{ij} = 1$ and $q_{ij} = r_{ij} = 0$.

Then, D_{ij}^L and D_{ij}^U can be obtained by integrating Equations (5) and (6) into D_{ij} , respectively:

where D_{ij}^L and D_{ij}^U are also the minimum and maximum of $D_{ij} = D_{ij}^- / (D_{ij}^- + D_{ij}^+)$, respectively.

Based on the analysis above, Equation (7) can be rewritten as

$$\tilde{D}_i = \sum_{j=1}^n w_j D_{ij} = \sum_{j=1}^n w_j [D_{ij}^L, D_{ij}^U], \quad (10)$$

where $\tilde{D}_i \geq \tilde{D}_j$ means that $a_i (i = 1, 2, \dots, m)$ is not inferior to $a_j (j = 1, 2, \dots, m)$, and the weight information is incompletely known. Obviously, \tilde{D}_i represents the comprehensive evaluation values, and stands for the preference of a_i ; that is, the larger \tilde{D}_i is, the better a_i will be.

4.2. Fuzzy linear programming models for determining criterion weights

In the decision-making process, the importance of different criteria should be taken into consideration. Suppose Γ_0 denotes the set of all the weight vectors, and

$$\Gamma_0 = \left\{ (w_1, w_2, \dots, w_n) \mid w_j \geq 0 (j = 1, 2, \dots, n), \sum_{j=1}^n w_j = 1 \right\}.$$

In some actual decision-making situations, the incomplete information regarding the criterion weights provided by DMs can usually be constructed using several basic ranking forms (Dubey et al., 2012; Li, 2011). These weight information structures may be expressed in the following five basic relations, which are denoted by the subsets $\Gamma_s (s = 1, 2, \dots, 5)$ in Γ_0 , respectively (Chen, 2014).

(1) A weak ranking:

$$\Gamma_1 = \{ (w_1, w_2, \dots, w_n) \in \Gamma_0 \mid w_{j_1} \geq w_{j_2} \text{ for all } j_1 \in \gamma_1 \text{ and } j_2 \in \Lambda_1 \},$$

where γ_1 and Λ_1 are two disjoint subsets of the subscript index set $N = \{1, 2, \dots, n\}$ of all criteria.

(2) A strict ranking:

$$\Gamma_2 = \{(w_1, w_2, \dots, w_n) \in \Gamma_0 | w_{j_1} - w_{j_2} \geq \delta_{j_1 j_2} \text{ for all } j_1 \in \gamma_2 \text{ and } j_2 \in \Lambda_2\},$$

where $\delta_{j_1 j_2} > 0$ is a constant, and γ_2 and Λ_2 are two disjoint subsets of N .

(3) A ranking of differences:

$$\Gamma_3 = \{(w_1, w_2, \dots, w_n) \in \Gamma_0 | w_{j_1} - w_{j_2} \geq w_{j_3} - w_{j_4} \text{ for all } j_1 \in \gamma_3, j_2 \in \Lambda_3, j_3 \in \Omega_3 \text{ and } j_4 \in \Psi_3\},$$

where $\gamma_3, \Lambda_3, \Omega_3$ and Ψ_3 are four disjoint subsets of N .

(4) An interval form:

$$\Gamma_4 = \{(w_1, w_2, \dots, w_n) \in \Gamma_0 | \eta_{j_1} \geq w_{j_1} \geq \delta_{j_1} \text{ for all } j_1 \in \gamma_4\},$$

where $\eta_{j_1} > 0$ and $\delta_{j_1} > 0$ are constants, satisfying $\eta_{j_1} > \delta_{j_1}$, and γ_4 is a subset of N .

(5) A ranking with multiples:

$$\Gamma_5 = \{(w_1, w_2, \dots, w_n) \in \Gamma_0 | w_{j_1} \geq \delta_{j_1 j_2} w_{j_2} \text{ for all } j_1 \in \gamma_5 \text{ and } j_2 \in \Lambda_5\},$$

where $\delta_{j_1 j_2} > 0$ is a constant, and γ_5 and Λ_5 are two disjoint subsets of N .

Let Γ denote a set of the known information on the criterion weights, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$. Given the conditions in Γ , the optimal weight values of the criteria in Equation (10) can be determined via the following linear programming models:

$$\begin{aligned} \max D &= \sum_{i=1}^n \sum_{j=1}^n w_j D_{ij}^M \\ \text{s.t. } (w_1, w_2, \dots, w_n) &\in \Gamma \end{aligned} \quad (\text{M1})$$

where $D_{ij}^M = (D_{ij}^L + D_{ij}^U)/2$.

The DMs might express inconsistent opinions about the preferences and weight information in the case of contingency. Under these circumstances, a multi-objective nonlinear programming model using a goal programming technique can be used to tackle problems that involve inconsistent weight information. For $j_1 \neq j_2 \neq j_3 \neq j_4$, Γ is revised as Γ' by introducing several non-negative deviation variables:

$$\begin{aligned} \Gamma' &= \{(w_1, w_2, \dots, w_n) \in \Gamma_0 | w_{j_1} + e_{(1)j_1 j_2}^- \geq w_{j_2} \text{ for all } j_1 \in \gamma_1 \text{ and } j_2 \in \Lambda_1; \\ &w_{j_1} - w_{j_2} + e_{(2)j_1 j_2}^- \geq \delta_{j_1 j_2} \text{ for all } j_1 \in \gamma_2 \text{ and } j_2 \in \Lambda_2; \\ &w_{j_1} - w_{j_2} - w_{j_3} + w_{j_4} + e_{(3)j_1 j_2 j_3 j_4}^- \geq 0 \text{ for all } j_1 \in \gamma_3, j_2 \in \Lambda_3, j_3 \in \Omega_3 \text{ and } j_4 \in \Psi_3; \\ &w_{j_1} + e_{(4)j_1}^- \geq \delta_{j_1}, w_{j_1} - e_{(4)j_1}^+ \leq \eta_{j_1} \text{ for all } j_1 \in \gamma_4; \\ &w_{j_1}/w_{j_2} + e_{(5)j_1 j_2}^- \geq \delta_{j_1 j_2} \text{ for all } j_1 \in \gamma_5 \text{ and } j_2 \in \Lambda_5\}. \end{aligned}$$

Furthermore, based on Model (M1), the following bi-objective nonlinear programming model can be established in

case of inconsistent preference information:

$$\begin{aligned} \max D &= \sum_{i=1}^n \sum_{j=1}^n w_j D_{ij}^M \\ \min E &= \sum_{j_1, j_2, j_3, j_4 \in N} (e_{(1)j_1 j_2}^- + e_{(2)j_1 j_2}^- + e_{(3)j_1 j_2 j_3 j_4}^- + e_{(4)j_1}^- + e_{(4)j_1}^+ + e_{(5)j_1 j_2}^-) \\ \text{s.t. } &\begin{cases} (w_1, w_2, \dots, w_n) \in \Gamma' \\ e_{(1)j_1 j_2}^- \geq 0 & j_1 \in \gamma_1 \text{ and } j_2 \in \Lambda_1 \\ e_{(2)j_1 j_2}^- \geq 0 & j_1 \in \gamma_2 \text{ and } j_2 \in \Lambda_2 \\ e_{(3)j_1 j_2 j_3 j_4}^- \geq 0 & j_1 \in \gamma_3, j_2 \in \Lambda_3, j_3 \in \Omega_3 \text{ and } j_4 \in \Psi_3 \\ e_{(4)j_1}^- \geq 0, e_{(4)j_1}^+ \geq 0 & j_1 \in \gamma_4 \\ e_{(5)j_1 j_2}^- \geq 0 & j_1 \in \gamma_5 \text{ and } j_2 \in \Lambda_5 \end{cases} \end{aligned} \quad (\text{M2})$$

where $D_{ij}^M = (D_{ij}^L + D_{ij}^U)/2$. By solving Model (M2), the optimal weight vector $w = (w_1, w_2, \dots, w_n)$ and the optimal deviation values $e_{(1)j_1 j_2}^-$, $e_{(2)j_1 j_2}^-$, $e_{(3)j_1 j_2 j_3 j_4}^-$, $e_{(4)j_1}^-$, $e_{(4)j_1}^+$ and $e_{(5)j_1 j_2}^-$ can be derived.

Considering an MCDM problem that contains incomplete and consistent preference information, Model (M1) can be applied to obtain the best weight vector, and the optimisation model can be easily solved by using the simplex method. For an MCDM problem that contains incomplete and inconsistent preference information, Model (M2) can be employed to determine the optimal solution.

4.3. The proposed algorithm with incomplete criteria information

In view of the determination of the weight vector of criteria, the comprehensive preference of each alternative a_i can be denoted by an interval value; that is, Equation (10) is revised as $\tilde{D}_i = \sum_{j=1}^n w_j D_{ij} = [\sum_{j=1}^n w_j D_{ij}^L, \sum_{j=1}^n w_j D_{ij}^U]$ by using the operational laws of interval values given in Definition 1. Then, a pairwise comparison must be made between the alternatives, and subsequently the pairwise comparison matrix (likelihood matrix) P can be constructed as follows:

$$P = (p_{ij})_{m \times m} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}, \quad (11)$$

where $p_{ij} = p(a_i \geq a_j) = p(\tilde{D}_i \geq \tilde{D}_j) = \max\{1 - \max\{\frac{D_{ij}^U - D_{ij}^L}{L(\tilde{D}_i) + L(\tilde{D}_j)}, 0\}, 0\}$, and $L(\tilde{D}_i) = D_{ij}^U - D_{ij}^L$.

According to Xu and Da (2003), the ranking vector of the likelihood matrix can be defined as follows:

$$\omega_i = \frac{\sum_{j=1}^n p_{ij} + \frac{m}{2} - 1}{m(m-1)}, \quad i = 1, 2, \dots, m. \quad (12)$$

Consequently, the ranking of all alternatives can be obtained according to the descending order of ω_i ($i = 1, 2, \dots, m$). That is, the larger ω_i is, the better the alternative a_i will be.

Based on the above analysis, an approach can be developed for an MCDM problem that contains three key stages: (1) collection and normalisation stage, (2) determination stage and (3) selection stage. A conceptual model of the proposed approach is shown in Figure 1.

The main steps are outlined as follows.

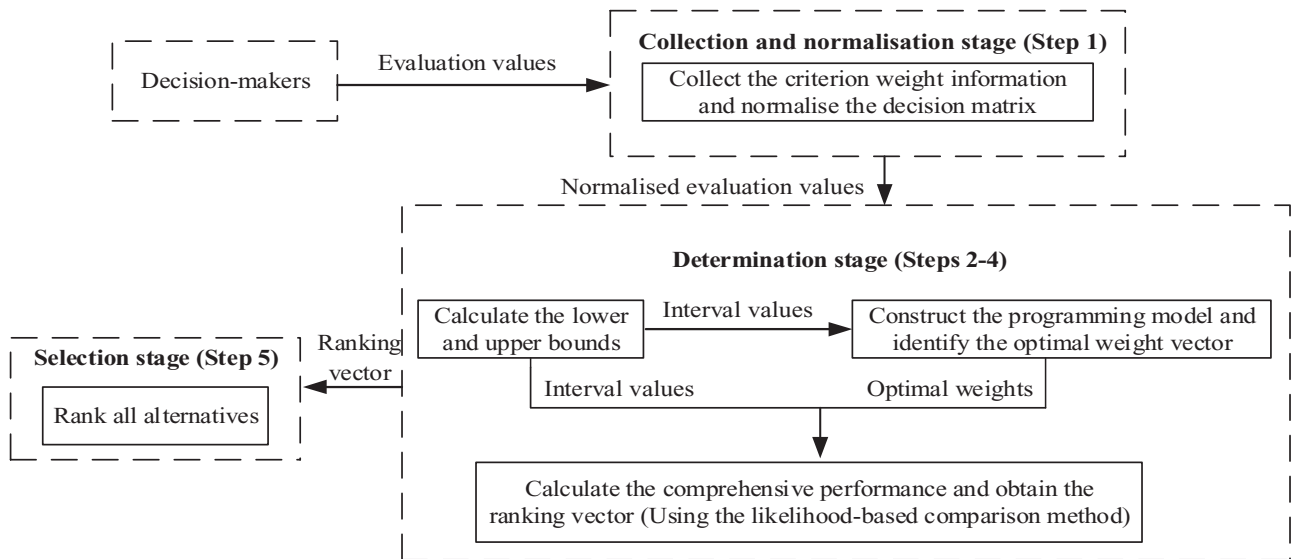


Figure 1. A flow chart of the proposed approach.

Step 1: Normalise the decision matrix.

Use Equation (3) to transform \tilde{B} into \tilde{R} . For convenience, the normalised values of $a_i (i = 1, 2, \dots, m)$ with respect to $c_j (j = 1, 2, \dots, n)$ are also expressed as $([t_{ij}^L, t_{ij}^U], [i_{ij}^L, i_{ij}^U], [f_{ij}^L, f_{ij}^U])$.

Step 2: Calculate the lower and upper bounds of \tilde{D}_{ij} .

Use Equations (8) and (9) to derive the lower bound D_{ij}^L and the upper bound D_{ij}^U , respectively.

Step 3: Identify the optimal weight vector and calculate the preference of each alternative.

Solve Model (M1) or (M2) to identify the optimal weight vector, and calculate the comprehensive performance $\tilde{D}_i (i = 1, 2, \dots, m)$ by using the operational laws of the interval values in Definition 1.

Step 4: Construct the likelihood matrix and obtain the ranking vector.

Construct the likelihood matrix P by using Equation (11) and obtain the ranking vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ based on Equation (12).

Step 5: Determine the ranking of all alternatives.

Determine the ranking of all alternatives according to the descending order of $\omega_i = (1, 2, \dots, m)$ and select the optimal one(s).

5. An illustrative example

In this section, the example of an investment appraisal project is used to demonstrate the application of the proposed MCDM approach; then its validity and effectiveness will be tested through a comparative analysis.

The following case is adapted from Wang et al. (2015).

ABC Nonferrous Metals Co. Ltd. is a large state-owned company whose main business is the deep processing of non-ferrous

metals. It is also the largest manufacturer of multi-species non-ferrous metals, with the exception of aluminium, in China. To expand its main business, the company regularly engages in overseas investment and a department consisting of executive managers and several experts in the field has been established to make decisions regarding global mineral investment. This overseas investment department recently decided to select a pool of alternatives from several foreign countries based on preliminary surveys. After a thorough investigation, five countries were taken into consideration, denoted by $\{a_1, a_2, \dots, a_5\}$. There are many factors that affect the investment environment, but four were chosen based on the experience of the department's personnel, namely c_1 : resources; c_2 : politics and policy; c_3 : economy; and c_4 : infrastructure.

The members of the overseas investment department have met to determine the evaluation information. Consequently, following a heated discussion, they came to a consensus on the final evaluations which were expressed by INNs shown in Table 1. Moreover, they were only able to provide incomplete information on the weights, that is, $\Gamma = \{0.15 \leq w_1 \leq 0.3, 0.15 \leq w_2 \leq 0.25, 0.25 \leq w_3 \leq 0.4, 0.3 \leq w_4 \leq 0.45, 2.5w_1 \leq w_3\}$.

5.1. Illustration of the proposed approach

In the following steps, the main procedures of obtaining the optimal ranking of alternatives are presented.

Step 1: Normalise the decision matrix.

As all the criteria are maximising type, the matrix does not need to be normalised, i.e., $\tilde{R} = \tilde{B}$.

Step 2: Calculate the lower and upper bounds of \tilde{D}_{ij} .

Derive the lower bound D_{ij}^L and the upper bound D_{ij}^U by using Equations (8) and (9), respectively:

$$(\tilde{D}_{ij})_{5 \times 4} = \begin{bmatrix} [0.5525, 0.7141] & [0.5363, 0.6410] & [0.6130, 0.7571] & [0.6726, 0.7613] \\ [0.4966, 0.7571] & [0.4964, 0.7853] & [0.5363, 0.6429] & [0.5297, 0.6962] \\ [0.5610, 0.7528] & [0.4462, 0.6148] & [0.5525, 0.6753] & [0.4568, 0.5965] \\ [0.3580, 0.4433] & [0.4433, 0.5825] & [0.4350, 0.5525] & [0.6962, 0.8171] \\ [0.4406, 0.5637] & [0.6962, 0.8171] & [0.5873, 0.7141] & [0.4964, 0.6272] \end{bmatrix}.$$

Table 1. The evaluation information.

	c_1	c_2	c_3	c_4
a_1	[[0.7,0.8],[0.5,0.7],[0.1,0.2]]	[[0.6,0.8],[0.4,0.5],[0.3,0.3]]	[[0.8,0.8],[0.4,0.6],[0.1,0.2]]	[[0.7,0.9],[0.3,0.4],[0.2,0.2]]
a_2	[[0.6,0.8],[0.4,0.6],[0.1,0.3]]	[[0.5,0.7],[0.3,0.5],[0.1,0.3]]	[[0.6,0.6],[0.2,0.3],[0.4,0.5]]	[[0.6,0.8],[0.4,0.4],[0.2,0.4]]
a_3	[[0.4,0.6],[0.2,0.2],[0.2,0.4]]	[[0.6,0.7],[0.4,0.6],[0.3,0.4]]	[[0.7,0.8],[0.6,0.7],[0.1,0.2]]	[[0.5,0.6],[0.5,0.6],[0.2,0.3]]
a_4	[[0.4,0.5],[0.5,0.6],[0.4,0.4]]	[[0.5,0.6],[0.3,0.4],[0.4,0.5]]	[[0.6,0.7],[0.7,0.8],[0.2,0.3]]	[[0.8,0.9],[0.3,0.4],[0.1,0.2]]
a_5	[[0.6,0.7],[0.4,0.5],[0.4,0.5]]	[[0.8,0.9],[0.3,0.4],[0.1,0.2]]	[[0.7,0.8],[0.5,0.6],[0.1,0.2]]	[[0.5,0.7],[0.5,0.5],[0.2,0.3]]

Step 3: Identify the optimal weight vector and calculate the preference of each alternative.

Because no inconsistent weight information exists in the evaluation, Model (M1) can be applied to identify the optimal weight vector:

$$\begin{aligned} \max D &= 2.8198w_1 + 3.0295w_2 + 3.033w_3 + 3.175w_4 \\ \text{s.t.} \quad &\begin{cases} 0.15 \leq w_1 \leq 0.3 \\ 0.15 \leq w_2 \leq 0.24 \\ 0.25 \leq w_3 \leq 0.4 \\ 0.3 \leq w_4 \leq 0.45 \\ 2.5w_1 \leq w_3 \\ \sum_{j=1}^4 w_j = 1 \end{cases} \end{aligned}$$

The optimal weight vector can be obtained as $w = (0.15, 0.15, 0.375, 0.325)$. Then, \tilde{D}_i ($i = 1, 2, \dots, 5$) can be calculated by referring to the operational laws of interval values in Definition 1:

$$\begin{aligned} \tilde{D}_1 &= [0.6118, 0.7346], \quad \tilde{D}_2 = [0.5222, 0.6987], \\ \tilde{D}_3 &= [0.5067, 0.6522], \end{aligned}$$

$$\tilde{D}_4 = [0.5096, 0.6266] \quad \text{and} \quad \tilde{D}_5 = [0.5521, 0.6788].$$

Step 4: Construct the likelihood matrix and obtain the ranking vector.

Use Equation (11) to construct the likelihood matrix P and obtain the ranking vector $\omega = (\omega_1, \omega_2, \dots, \omega_5)$ based on Equation (12):

$$\begin{aligned} P &= (p_{ij})_{5 \times 5} \\ &= \begin{bmatrix} 0.5 & 0.7096 & 0.8493 & 0.9384 & 0.7316 \\ 0.2904 & 0.5 & 0.5962 & 0.6444 & 0.4836 \\ 0.1507 & 0.4038 & 0.5 & 0.5434 & 0.3679 \\ 0.0616 & 0.3556 & 0.4566 & 0.5 & 0.3057 \\ 0.2684 & 0.5164 & 0.6321 & 0.6943 & 0.5 \end{bmatrix}, \\ \omega_1 &= 0.2614, \quad \omega_2 = 0.2007, \quad \omega_3 = 0.1733, \\ \omega_4 &= 0.1590 \quad \text{and} \quad \omega_5 = 0.2056. \end{aligned}$$

Step 5: Determine the ranking of all alternatives.

According to the descending order of $\omega_i = (1, 2, \dots, 5)$, the ranking of all alternatives is $a_1 > a_5 > a_2 > a_3 > a_4$ and the best one is a_1 .

5.2. Comparative analysis and discussion

In order to further verify the feasibility and effectiveness of the proposed approach, a comparative analysis is now conducted using six existing methods with the analysis being based on the same illustrative example.

- (1) Chi and Liu's method (2013) contains two major phases (criterion weights determination and ranking obtainment with TOPSIS). First, the maximising deviation method is developed to determine the criterion weights and then an extended TOPSIS method is employed to rank the alternatives.
- (2) In Ye's method (2014b), the distance-based similarity measures are employed, which involves aggregating the weighted similarity measures between each alternative and the PIS.
- (3) In Zhang et al.'s methods (2014), first, the comprehensive INNs are aggregated by using the INNWA or INNWG operators, and then the ranking vectors can be obtained by constructing the likelihood matrices based on the score function. Moreover, Zhang et al. (2015b) developed an outranking method for MCDM on the basis of the score function constructing the outranking relation matrix. Since Zhang et al.'s method (2015b) does not take the criterion weights into consideration but the illustrative example does, it is necessary to construct the outranking relation matrix after calculating the weighted evaluation values.

Considering the criterion weights obtained using the proposed programming model, the results obtained by different methods are summarised in Table 2.

It can be seen that there are some differences between them. The reasons for the inconsistency of the rankings are explained as follows.

- (1) The difference in the ranking results of the proposed approach and that of Chi and Liu (2013) is the sequence

Table 2. The ranking results of the different methods.

Methods	Ranking results
Chi and Liu's method (2013)	$a_1 > a_5 > a_2 > a_4 > a_3$
Ye's methods (2014b)	
Similarity measure based on the Hamming distance	$a_1 > a_5 > a_2 > a_4 > a_3$
Similarity measure based on the Euclidean distance	$a_1 > a_5 > a_2 > a_3 > a_4$
Zhang et al.'s methods (2014)	
Method based on the INNWA operator	$a_1 > a_5 > a_2 > a_4 > a_3$
Method based on the INNWG operator	$a_1 > a_5 > a_2 > a_3 > a_4$
Zhang et al.'s method (2015b) ($p = 0.2$ and $q = 0.1$)	$a_1 > a_2 > \{a_3, a_4, a_5\}$
The proposed approach	$a_1 > a_5 > a_2 > a_3 > a_4$

Table 3. The ranking results obtained by revising the methods of Ye (2014b).

Methods	Ranking results
Similarity measure based on the Hamming distance	$a_1 > a_5 > a_2 > a_3 > a_4$
Similarity measure based on the Euclidean distance	$a_1 > a_5 > a_2 > a_3 > a_4$

of a_3 and a_4 . In Chi and Liu's method (2013), the relative closeness coefficients (performance of each alternative) are conducted based on the relative ideal solutions, which have certain drawbacks. First, it is not easy to choose the appropriate PIS and NIS with INNs because each INN has three interval elements. Second, the PIS and NIS are closely related to the number of alternatives as well as the evaluation values. Thus, they may vary as the original information changes. Third, the relative closeness coefficients are in the form of crisp real numbers, which may cause information loss and affect the ranking results. Finally, suppose that there are m alternatives and n criteria to be evaluated. In order to determine the criterion weights, Chi and Liu's method (2013) needs to derive $m \times m \times n$ distance measures, whereas the proposed approach only needs to calculate $m \times n$ cross-entropy measures. Thus, it takes less time than that of Chi and Liu (2013).

- (2) Similarly, the order of a_3 and a_4 is the only difference between the proposed approach and the first method of Ye (2014b). This is because the comparison methods in Ye (2014b) only consider the weighted similarity measures between each alternative and the PIS. If only the PIS is taken into account and the NIS is ignored, the ranking of alternatives may be incorrectly reversed and this may be amplified in the final results. However, the ranking result in this method will be identical to that of the proposed approach in a situation where, simultaneously, the PIS is replaced with the absolute one and the absolute NIS is not ignored; moreover, the closeness coefficient would have to be utilised to determine the ranking of alternatives. The updated results are shown in Table 3. Therefore, the methods in Ye (2014b) are not reliable enough.
- (3) The positions of a_3 and a_4 obtained by the method based on the INNWA operator (Zhang et al., 2014) are not consistent with either those obtained by the method based on the INNWG operator (Zhang et al., 2014) or the proposed approach. This may be caused by the inherent characteristics of aggregation operators, as the INNWA operator focuses on the impact of the overall criterion values, while the INNWG operator emphasises the impact of a single item. Additionally, the outranking method of Zhang et al. (2015b) can only yield partial orders of alternatives, in which a_3 , a_4 and a_5 are indistinguishable. This method has to convert the INNs into real numbers and artificially set both the threshold p and indifference threshold q before constructing the dominance relations. It is by nature inappropriate to replace the INNs with real numbers, and it may lead to information loss in the transformation process. Furthermore,

when manually providing the parameters p and q , it is difficult to avoid subjective randomness. Therefore, the result obtained by the outranking method is not always reliable.

According to the comparative analysis, the following advantages over the other methods can be outlined.

- (1) The calculations required for the proposed approach are relatively straightforward and time-saving, and the burden of computation can be greatly decreased with the help of the proven mathematical derivation.
- (2) In the proposed approach, the interval closeness coefficients are conducted to rank alternatives. In this way, the fuzziness of the original information can be maintained and fully utilised. Therefore, the proposed approach is more competent in interval neutrosophic MCDM than the other methods considered.
- (3) In the proposed approach, the transformation operator is employed to convert INNs into SNNs, which can avoid various kinds of aggregation operators processing directly with INNs. Furthermore, the parameters of the transformation operator are determined through mathematical derivation and not artificially produced. Thereby, the final ranking obtained by the proposed approach is more conclusive than those produced by the other methods, and it is evident that the proposed approach is accurate and reliable.

6. Conclusions

INNs are flexible at expressing the uncertain, imprecise, incomplete and inconsistent information that is very common in scientific and engineering situations; therefore, the study of MCDM methods with INNs is highly significant. In this paper, a transformation operator and cross-entropy were defined. Consequently, an MCDM method was established based on cross-entropy and TOPSIS, which calculated the cross-entropy after transforming INNs into SNNs on the basis of the transformation operator. Furthermore, it aggregated the performances of alternatives into interval numbers, from which two mathematical programming models were constructed to identify the criterion weights. Finally, a ranking result was obtained by comparing these weighted interval numbers with a possibility degree method.

The advantages of this study are that the approach is both simple and convenient to compute and effective at decreasing the loss of evaluation information. The feasibility and validity of the proposed approach have been verified through the illustrative example and comparative analysis. The comparison results demonstrated that the proposed approach can provide more reliable and precise outcomes than other methods. Therefore, this approach has great application potential in solving MCDM problems in an interval neutrosophic environment, in which criterion values with respect to alternatives are evaluated by the form of INNs and the criterion weights are incomplete.

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