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ABSTRACT

In this note we continue the study of the stress event model, a simple and intuitive dynamic model for credit risky portfolios, proposed by Duffie and Singleton (1999). The model is a bottom-up version of the multi-factor portfolio credit model proposed by Longstaff and Rajan (2008). By a novel identification of independence conditions, we are able to decompose the loss distribution into a series expansion which not only provides a clear picture of the characteristics of the loss distribution but also suggests a fast and accurate approximation for it. Our approach has three important features: (i) it is able to match the standard CDS index tranche prices and the underlying CDS spreads, (ii) the computational speed of the loss distribution is very fast, comparable to that of the Gaussian copula, (iii) the computational cost for additional factors is mild, allowing for more flexibility for calibrations and opening the possibility of studying multi-factor default dependence of a portfolio via a bottom-up approach. We demonstrate the tractability and efficiency of our approach by calibrating it to investment grade CDS index tranches.

keywords: credit derivatives, CDO, bottom-up approach, multi-name, intensity-based, risk and portfolio.

# 1. Introduction

The bottom-up stress event model, proposed by Duffie and Singleton (1999), is a simple and intuitive model for portfolio credit risk. The model is seldom applied in practice since it is generally believed that the default times, as well as the loss distribution, of a portfolio under this modeling framework can only be generated by computationally expensive Monte Carlo simulation. In this note an alternative approach is taken, avoiding Monte Carlo simulations, making the model tractable and leading to efficient calibrations to data. The idea of the stress event model is easy to understand. Besides idiosyncratic default, each firm may default if there is a joint credit event (Duffie and Singleton 1999) or alternatively referred to as stress event (Schönbucher 2003). This allows correlation through both changes in stress event intensity as well as through the occurrences of the stress events. The formal definition of the default time of a firm is given in Section 3. In Section 4, we develop a new approach to compute the loss distribution of a portfolio for the stress event model. We first identify independence conditions under which defaults of firms are independent. The loss distribution can then be decomposed into a series expansion for which each term admits a closed form expression. It turns out that only the first few terms of the series are needed to accurately approximate the loss distribution since stress events are infrequent. This leads to a very efficient method to compute the loss distribution of a portfolio.

The multi-factor model, a top-down approach model proposed by Longstaff and Rajan (2008), provides strong empirical evidence that default dependence of a portfolio is necessarily multi-factor. The stress event model, like other bottom-up approach models, faces significant computational challenges when the number of non-idiosyncratic factors is more than one. This curse of dimensionality comes from the rapid increase of the number of unconditional loss distributions needed to compute the loss distribution. For example, if the number of conditional loss distributions needed to compute in a one-factor model is 100, it is expected that the number of conditional loss distributions needed in a $L$-factor model would be $100^L$. This is not the case for our new approach due to the novel identification of independence conditions which result in important simplifications to the corresponding series expansion of the loss distribution for the stress event model. It turns out that the number of conditional loss distributions needed in our approach only increases mildly with the number of non-idiosyncratic factors. Hence, the increase in computational time due to the additional non-idiosyncratic factors in the stress event model is much smaller than that in other bottom-up approach models. This extra flexibility for adding additional non-idiosyncratic factors in the stress event model leads to a better fit to market data.

We demonstrate the tractability and efficiency of our approach by two calibration examples in Section 5. In the first example, the model is calibrated to the first five tranches of the 5-year CDX.NA.IG series 13 and the 125 underlying CDS spreads simultaneously. All the CDS spreads are matched exactly and the model implied tranche prices are within the bid-ask spread. In the second calibration example, we regard the stress event model as a top-down model and calibrate it to the term structure of the iTraxx Europe series 7 on four different days simultaneously. The 26 parameters of the model are calibrated to the 60 data and the root-mean-square relative error of the model implied tranche prices is 4.25%.

## 2. Related Literature

There are two approaches in multi-name credit risk modeling. In the bottom-up approach, individual losses of names are modeled and then aggregated over the portfolio. This approach is pursued by Duffie and Singleton (1999), Duffie and Gârleanu (2001), Mortensen (2006), Joshi and Stacey (2006), Papageorgiou and Sircar (2007), Peng and Kou (2008), Eckner (2009) and others. On the other hand, the top-down approach, which models the dynamics of a portfolio loss distribution directly, is also an active research area. Top-down models are investigated by Errais et al. (2006), Brigo et al. (2007), Cont and Minca (2008), Longstaff and Rajan (2008), Arnsdorf and Halperin (2008), Bayraktar and Yang (2009), Giesecke et al. (2010) and others.

# 3. Model Formulation

For notational consistency, we reserve the subscript index $i$ for specifying a firm and the superscript for indexing a sector in the rest of this section. In a portfolio which consists of credit risky securities issued by $N$ firms, the default time of firm $i$ under the stress event model framework is defined as follows:

$$\tau_i = \inf\left\{ s \geq 0 : \bar{N}_i(s) + \sum_{l=1}^{L}\sum_{j=1}^{\infty} \mathbf{1}_{\{s>t_j^l\}} X_{i,j}^l > 0 \right\}, \tag{1}$$

for $i = 1, .., N$, where

- $t_j^l$ is the $j$-th jump time of a Poisson process $N^l(s)$ associated with sector $l$,
- all $\bar{N}_i$ and $N^l$ are independent Poisson processes with intensities $\bar{\lambda}_i(s)$ and $\lambda^l(s)$ respectively,
- $\mathbf{1}_{\{s>t_j^l\}}$ is an indicator function that equals one if $s > t_j^l$ and zero otherwise,
- $X_{i,j}^l$ are Bernoulli random variables indicating if a stress event at time $t_j^l$ has killed the $i$-th firm or not, independent of the Poisson processes,
- $L$ is the number of non-idiosyncratic factors(sectors) affecting a portfolio; we will interchangeably use the terms "non-idiosyncratic factor" and "sector" since firms affected by a common non-idiosyncratic factor can be considered belonging to a common sector.

$\bar{N}_i$ is an idiosyncratic Poisson process associated with firm $i$ which is driven by firm-specific factors. Once there is a jump in $\bar{N}_i$, firm $i$ defaults immediately. In addition, if $N^l$ has a jump at $t^l_j$, firm $i$ may default with a probability $\mathrm{P}(X^l_{i,j} = 1) = p^l_i$. We say that firm $i$'s default is caused by the $l$-th sector if $p^l_i > 0$. It is worth noting that only the first jump in $\bar{N}_i$ is relevant for default triggering of the $i$-th firm and later jumps are irrelevant, whereas each jump in $N^l$ could be the default triggering event.

The Poisson processes $\bar{N}_i$ and $N^l$ considered in this note are doubly stochastic processes, i.e. the intensities $\bar{\lambda}_i$ and $\lambda^l$ may also be stochastic. In the general exposition of the model, it is not necessary to specify the processes followed by the intensities. In Section 5, where the model is calibrated to data, the intensities will be taken to be constant in one case and follow an affine-jump diffusion process in another.

# 4. Loss Distribution

The loss distribution of a portfolio is a dynamic process which evolves stochastically over time. A common approach in calculating the loss distribution of a credit risky portfolio for bottom-up approaches is by computing the loss under conditional independence. The unconditional default distribution is then the weighted sum of the conditional ones, i.e.

$$\mathrm{P}(D(t) = n) = \int_{\Omega} \mathrm{P}(D(t) = n|\omega)\mathrm{P}(d\omega), \quad n = 1, ..., N, \tag{2}$$

where $D(t)$ is the number of defaults by time $t$ and $\omega$ is a condition under which defaults of firms are independent. We assume that the recovery rate of each security is a constant $R$

and a uniform notional amount $\delta$ for all firms in the portfolio, thus the loss of a portfolio is

$$L_t = \sum_{i=1}^{N} \delta_i (1 - R_i) \mathbf{1}_{\{\tau_i \le t\}} = \delta(1 - R) \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \le t\}} = \delta(1 - R) D(t). \tag{3}$$

Therefore, modeling the loss distribution is equivalent to modeling the default distribution. The first challenge of evaluating Eq.(2) is to find a computationally efficient scheme to calculate the conditional loss distribution $\mathrm{P}(D(t) = n|\omega)$. To this end, we adopted the recursive algorithm suggested by Andersen et al. (2003). In fact, the recursive algorithm can also compute the loss distribution of a portfolio with different recovery rate and notional for each name. The computational cost for each conditional loss distribution is relatively expensive for a large portfolio, thus the number of conditional loss distribution needed to compute the unconditional loss distribution for each time $t$ would significantly affect the efficiency of the overall calculation. It turns out that only a moderate number of conditional loss distributions are need to accurately approximate the full loss distribution in our approach. The second challenge lies in the evaluation of $\mathrm{P}(d\omega)$. This is in fact a threefold challenge. One needs to identify conditions under which defaults are independent, choose a partition for the probability space $\Omega$ in order to enhance calculation, and evaluate the probabilities of these independence conditions. We present a novel identification of independence conditions which arises naturally from the formulation of the stress event model. We also introduce a systematic way of choosing a countable partitions of $\Omega$ which automatically arranges the sizes of $\mathrm{P}(d\omega)$ in descending order. In addition, we provide explicit formulas for the probabilities of independence conditions for a wide class of stochastic intensities.

*a. Independence Conditions*

For intensity-based models, like the correlated intensity model by Duffie and Gârleanu (2001), a realization of the non-idiosyncratic part of the firms' default intensities is usually employed as an independence condition for defaults. We believe that this framework can provide a similar set of independence conditions for the stress event model. However, this approach may not be very efficient in the present situation and will not be pursued here. Instead, we follow a different approach to identifying independence conditions for the stress event model, which will prove to lead to more efficient analysis and calculations. These independence conditions arise naturally from the definition of the individual firm's default times in the stress event model as they are related to the occurrences of the non-idiosyncratic events in the model. Consider a scenario characterized by non-idiosyncratic events

$$\omega^u = \omega(u(L, \vec{m}_L, t)) = \left\{\omega : t_j^l(\omega) = u_j^l \in (0, t],\ j = 1, ..., m_l,\ l = 1, ..., L\right\}, \tag{4}$$

where

$$u(L, \vec{m}_L, t) = \begin{pmatrix} u_1^1 & u_2^1 & \dots & \dots & u_{m_1}^1 & \\ u_1^2 & u_2^2 & \dots & u_{m_2}^2 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^L & u_2^L & \dots & \dots & \dots & u_{m_L}^L \end{pmatrix} \quad \text{and} \quad \vec{m}_L = (m_1, m_2, ..., m_L). \tag{5}$$

$u(L, \vec{m}_L, t)$ is an array of $L$ rows and each row has $m_l$ entries which specifies the jump times of $N^l$ up to time $t$. This is the scenario that there are $m_l$ stress events occurring at $u_1^l, u_2^l, ..., u_{m_l}^l$ all before time $t$ in the $l$-th sector for $l = 1, ..., L$. For a given $\omega^u$, Eq.(1)

becomes

$$\tau_i(\omega^u) = \inf_{s\geq 0}\left\{\bar{N}_i(s) + \sum_{l=1}^{L}\left(\sum_{j=1}^{m_l}\mathbf{1}_{\{s>u_j^l\}}X_{i,j}^l + \sum_{j=m_{l+1}}^{\infty}\mathbf{1}_{\{s>t_j^l\}}X_{i,j}^l\right) > 0\right\}. \tag{6}$$

Eq.(6) is just a splitting of the terms in the defining Eq.(1), under the condition $\omega^u$, into the terms before $t$ where the occurrence times of the stress events are known and after $t$ where they are random. Define

$$\tilde{\tau}_i(\omega^u) = \inf_{s\geq 0}\left\{\bar{N}_i(s) + \sum_{l=1}^{L}\sum_{j=1}^{m_l}\mathbf{1}_{\{s>u_j^l\}}X_{i,j}^l > 0\right\}, \tag{7}$$

which is almost identical to Eq.(6) except that the last sum inside the brackets is deleted. Note that if $\tau_i(\omega^u) \leq t$, then

$$\tau_i(\omega^u) = \tilde{\tau}_i(\omega^u), \tag{8}$$

since the default must be triggered by a jump of $\bar{N}_i$ or $N_i^l$ before $t$ and is irrelevant to anything that happens after $t$. The default indicators under $\omega^u$ are

$$\mathbf{1}_{\{\tau_i(\omega^u)\leq t\}} = \mathbf{1}_{\{\tilde{\tau}_i(\omega^u)\leq t\}}, \tag{9}$$

for $i = 1, ..., N$. The key observation leading to the independence condition is that $\mathbf{1}_{\{\tau_i(\omega^u)\leq t\}}$ are independent since all $\tilde{\tau}_i(\omega^u)$ are defined by independent Poisson processes and Bernoulli random variables as indicated by Eq.(7). Consequently, we can apply the recursive algorithm of Andersen et al. (2003) to compute the conditional loss distribution of a portfolio.

*b. Conditional Individual Survival Probability and Conditional Loss Distribution*

In order to build the conditional loss distribution, we have to compute the individual survival probability for each firm under the independence condition. The conditional survival probability of firm $i$ for a given $\omega^u$ as specified by Eq.(4) is

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{P}\left(\bar{\tau}_i > t|\omega^u\right) \prod_{l=1}^{L} \mathrm{P}\left(\tau_i^l > t|\omega^u\right), \tag{10}$$

where $\bar{\tau}_i$ is the first jump time of the idiosyncratic Poisson process $\bar{N}_i$ and $\tau_i^l$ is a jump time $t_j^l$ of $N^l$ such that it is the first time the Bernoulli random variable $X_{i,j}^l = 1$ among $j = 1, 2, \ldots$. The seeming notational inconsistency between the $i$ and $j$ in the random stopping times $\tau_i^l$ and $t_j^l$ arises from the effort to make precise that among the stress events in the $l$-th sector, $\tau_i^l$ is the first time that affects the $i$-th firm through the random variable $X_{i,j}^l$. Since the idiosyncratic default intensity $\bar{\tau}_i$ does not depend on the occurrences of the stress events in the sectors,

$$\mathrm{P}\left(\bar{\tau}_i > t|\omega^u\right) = \mathrm{P}\left(\bar{\tau}_i > t\right) \tag{11}$$

$$= \mathrm{E}\left[e^{-\int_0^t \bar{\lambda}_i(s)ds} \middle| \bar{\lambda}_i(0)\right]. \tag{12}$$

On the other hand,

$$\mathrm{P}\left(\tau_i^l > t|\omega^u\right) = (1 - p_i^l)^{m_l} \tag{13}$$

is the conditional survival probability that firm $i$ is not killed by the $m_l$ stress events in the $l$-th sector before $t$. As a result,

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{E}\left[e^{-\int_0^t \bar{\lambda}_i(s)ds} \middle| \bar{\lambda}_i(0)\right] \prod_{l=1}^{L} (1 - p_i^l)^{m_l}. \tag{14}$$

It is important to note that this conditional survival probability as well as the corresponding conditional loss distribution depend only on the idiosyncratic intensities and, most crucially, the number of stress events in each sector by time $t$, but NOT the occurrence times of the stress events. Consequently,

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{P}\left(\tau_i > t|\vec{m}_L\right), \tag{15}$$

and the conditional loss distribution becomes

$$\mathrm{P}(D(t) = n|\omega^u) = \mathrm{P}(D(t) = n|\vec{m}_L), \tag{16}$$

which can be computed by using the conditional survival probabilities given by Eq.(14). Eq.(16) makes a crucial point that although there are uncountable independence conditions $\omega^u$, the number of conditional loss distributions is countable since the number of possible scenarios of stress events, specified by $\vec{m}_L = (m_1, m_2, ..., m_L)$, is countable.

*c. Unconditional Loss Distribution for Deterministic Intensities*

In this subsection, we derive explicitly that the sum of probabilities of the independence conditions equals one and present the closed form expression of the probability of the condition given by Eq.(4). Then, we aggregate the conditional loss distributions and the density $\mathrm{P}(d\omega^u)$ to form the unconditional loss distribution of a portfolio. Finally, we provide a series expansion for the unconditional loss distribution such that the terms of the series are enumerated in descending order of their 'sizes'. We present our calculations explicitly for the case where the intensities $\lambda^l$ are deterministic, and the corresponding stochastic version is provided in the next subsection.

For a deterministic intensity $\lambda^l$, the probability of $m_l$ stress events occuring by time $t$ in the $l$-th sector is

$$q^l_{m_l} = e^{-\Lambda^l(t)} \frac{(\Lambda^l(t))^{m_l}}{m_l!}, \tag{17}$$

where

$$\Lambda^l(t) = \int_0^t \lambda^l(s) ds \tag{18}$$

is the cumulative intensity. It is clear that

$$\prod_{l=1}^{L}\left(\sum_{m_l=0}^{\infty} q_{m_l}^{l}\right) = 1. \tag{19}$$

Furthermore, the probability that the $m_l > 0$ stress events occur at $u_1^l$,...,$u_{m_l-1}^l$ and $u_{m_l}^l$ is

$$e^{-\Lambda^l(t)}\frac{(\Lambda^l(t))^{m_l}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\left(\frac{\lambda^l(u_{j_{m_l}}^l)du_{j_{m_l}}^l}{\Lambda^l(t)}\right) = \frac{e^{-\Lambda^l(t)}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\lambda^l(u_{j_{m_l}}^l)du_{j_{m_l}}^l. \tag{20}$$

It is possible that $m_l = 0$ which is the scenario that there is no stress event in the $l$-th sector. The probability of this case is simply

$$q_0^l = e^{-\Lambda^l(t)}. \tag{21}$$

For the sake of notational brevity, when $m_l = 0$, define

$$\prod_{j_0=1}^{0}\frac{\lambda^l(u_{j_0}^l)du_{j_0}^l}{0!} = 1, \tag{22}$$

and

$$\int_0^t\prod_{j_0=1}^{0}\frac{\lambda^l(u_{j_0}^l)du_{j_0}^l}{0!} = 1. \tag{23}$$

Hence the probability that there are $m_l$ stress events occurring in the $l$-th sector for $l = 1, ..., L$ at times given by an array $u(L, \vec{m}_L, t)$ is

$$\mathrm{P}(d\omega^u) = \prod_{l=1}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u_{j_{m_l}}^l)du_{j_{m_l}}^l, \tag{24}$$

where $\omega^u$ is defined by Eq.(4). We can rearrange Eq.(19) as a sum of products

$$\prod_{l=1}^{L}\left(\sum_{m_l=1}^{\infty} q^l_{m_l}\right) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(q^1_{m_1}\cdots q^L_{m_L}\right) \tag{25}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=1}^{L} q^l_{m_l}\right) \tag{26}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=1}^{L} e^{-\Lambda^l(t)}\frac{(\Lambda^l(t))^{m_l}}{m_l!}\right) \tag{27}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=1}^{L} e^{-\Lambda^l(t)}\frac{(\Lambda^l(t))^{m_l}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\left(\frac{\int_0^t \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}}{\Lambda^l(t)}\right)\right) \tag{28}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!}\prod_{j_{m_l}=1}^{m_l}\left(\int_0^t \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right)\right) \tag{29}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!}\int_{(0,t]^{m_l}}\prod_{j_{m_l}=1}^{m_l} \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) \tag{30}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}}\prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!}\prod_{j_{m_l}=1}^{m_l} \lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) \tag{31}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}}\prod_{l=1}^{L}\prod_{j_{m_l}=1}^{m_l} \frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) \tag{32}$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}} \mathrm{P}(d\omega^u)\right). \tag{33}$$

This derivation explicitly shows how the probabilities $\mathrm{P}(d\omega^u)$ aggregate to one. Since under each $\omega^u$, default indicators $\mathbf{1}_{\{\tau_i(\omega^u)\le t\}}$ are independent as shown in the previous subsection, the unconditional loss distribution is then

$$\mathrm{P}(D(t)=n) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_L=0}^{\infty}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}} \mathrm{P}(D(t)=n|\omega^u)\mathrm{P}(d\omega^u)\right). \tag{34}$$

The big summation above is not a very useful expression for computing the unconditional loss distribution. We can rearrange the summation in ascending order of the total number of stress events occurring by $t$ in all sectors, thus

$$\mathrm{P}(D(t)=n)=\sum_{k=0}^{\infty}\left(\sum_{\sum m_l=k}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}}\mathrm{P}(D(t)=n|\omega^u)\mathrm{P}(d\omega^u)\right)\right). \tag{35}$$

Define $\phi_k(t;n)$ as the loss distribution generated by exactly $k$ stress events. It is easy to see that, for $k=0,1,2,...,$

$$\phi_k(t;n)=\sum_{\sum m_l=k}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}}\mathrm{P}(D(t)=n|\omega^u)\mathrm{P}(d\omega^u)\right) \tag{36}$$

$$=\sum_{\sum m_l=k}\left(\int_{(0,t]^{m_1\times\cdots\times m_L}}\mathrm{P}(D(t)=n|\omega^u)\prod_{l=1}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) \tag{37}$$

$$=\sum_{\sum m_l=k}\left(\mathrm{P}(D(t)=n|\vec{m}_L)\int_{(0,t]^{m_1\times\cdots\times m_L}}\prod_{l=1}^{L}\prod_{j_{m_l}=1}^{m_l}\frac{e^{-\Lambda^l(t)}}{m_l!}\lambda^l(u^l_{j_{m_l}})du^l_{j_{m_l}}\right) \tag{38}$$

$$=\sum_{\sum m_l=k}\left(\mathrm{P}(D(t)=n|\vec{m}_L)\prod_{l=1}^{L}\frac{e^{-\Lambda^l(t)}}{m_l!}\left(\Lambda^l(t)\right)^{m_l}\right). \tag{39}$$

In the above derivation, we utilize the property of the conditional loss distribution that it does not depend on the occurrence times of the stress events but depends only on the number of stress events in each sector (see Eq.(16)). We call $\phi_k(t;n)$ the $k$-th order term of the unconditional loss distribution. It is important to notice that $\phi_k(t;n)$, as a countable sum in Eq.(39), is a significant simplification of its original form Eq.(37) which is a sum of multi-dimensional integrals. This simplification is the crux leading to an efficient algorithm for the unconditional loss distribution.

*d. Unconditional Loss Distribution for Stochastic Intensities*

For stochastic intensities $\lambda^l$, the loss distribution contributed by exactly $k$ stress events altogether by $t$ can be computed by taking the expectation of Eq.(39) over all possible intensities $\lambda^l$, then

$$\phi_k(t;n) = \sum_{\sum m_l = k} \left( \mathrm{E}\left[ \mathrm{P}(D(t)=n|\vec{m}_L) \prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^1(0),...,\lambda^L(0) \right] \right) \tag{40}$$

$$= \sum_{\sum m_l = k} \left( \mathrm{P}(D(t)=n|\vec{m}_L) \prod_{l=1}^{L} \frac{1}{m!} \mathrm{E}\left[ e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \right). \tag{41}$$

The conditional loss distribution $\mathrm{P}(D(t) = n|\vec{m}_L)$ is independent of the intensities $\lambda^l$ and can be constructed by the conditional survival probabilities Eq.(14). The expectation

$$\mathrm{E}\left[ e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \tag{42}$$

admits a closed form expression for a wide class of stochastic processes. We provide an explicitly expression of Eq.(42) when $\lambda^l$ is an affine-jump diffusion process in Appendix A. Consequently, introducing stochastic intensities $\lambda^l$ in the stress event model does not undermine the tractability of the model and Eq.(41) can be computed as easily as Eq.(39). It is easy to see that the full loss distribution is then

$$\mathrm{P}(D(t) = n) = \sum_{k=0}^{\infty} \phi_k(t;n). \tag{43}$$

Furthermore, define

$$|\phi_k(t)| := \mathrm{P}(\text{total number of stress events occurs by } t = k) \tag{44}$$

$$= \sum_{n=0}^{N} \phi_k(t;n) \tag{45}$$

$$= \sum_{\sum m_l = k} \left( \prod_{l=1}^{L} \frac{1}{m_l!} \mathrm{E}\left[ e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \right), \tag{46}$$

which measures the 'size' of the $k$-th order term of the loss distribution $\phi_k(t;n)$. Since the intensity $\lambda^l$ of each sector is generally quite small, $|\phi_k(t)|$ decreases for increasing $k$ and is negligible for large $k$. Consequently, only the first few terms of the loss distribution

$\phi_k(t;n)$ are necessary to construct the full loss distribution and this leads to an efficient approximation for it. Furthermore, define

$$\epsilon_K(t) = \mathrm{P}(\text{total number of stress events by } t > K) \tag{47}$$

$$= 1 - \sum_{k=0}^{K} |\phi_k(t)|, \tag{48}$$

which is a measure of the error of the $K$-th order approximation for the loss distribution. $\epsilon_K(t)$ is the probability of scenarios that are not considered in the $K$-th order approximation. The closer the value $\epsilon_K$ is to zero, the more accurate the approximation. Finally, it is easy to verify that

$$\sum_{k=0}^{\infty} |\phi_k(t)| = 1. \tag{49}$$

*e. Approximation of the Loss Distribution*

In order to prevent the leak of probability over time due to the finite order approximation to the loss distribution, we can also include the unaccounted probability to the highest order term in the calculation such that the updated unconditional loss distribution of the $K$-th order term is

$$\tilde{\phi}_K(t;n) = \left( \frac{1 - \sum_{k=0}^{K-1} |\phi_k(t)|}{|\phi_K(t)|} \right) \phi_K(t;n), \tag{50}$$

and approximate the full loss distribution as

$$\mathrm{P}(D(t) = n) \approx \sum_{k=0}^{K-1} \phi_k(t;n) + \tilde{\phi}_K(t;n). \tag{51}$$

Hence, the total probability of the loss distribution is one for all $t$.

On the other hand, under each scenario $\vec{m}_L$ the conditional loss distribution is a multinomial distribution with individual default probability $q_i(\vec{m}_L, t) := 1 - \mathrm{P}(\tau_i > t|\vec{m}_L)$, where the conditional survival probability is given by Eq.(14). It is important to notice that $\sum_{i=1}^{N} q_i(\vec{m}_L, t)/N$ is usually not close to 0 or 1. Hence the conditional loss distributions can be approximated by a normal distribution as suggested by Shelton (2004) in which the first two moments of the conditional loss distribution are fitted. To fit the exact distribution, choose the mean and variance of the normal distribution as follows:

$$\mu(\vec{m}_L, t) = \sum_{i=1}^{N} q_i(\vec{m}_L, t), \tag{52}$$

$$\sigma^2(\vec{m}_L, t) = \sum_{i=1}^{N} q_i(\vec{m}_L, t)\left(1 - q_i(\vec{m}_L, t)\right). \tag{53}$$

In fact, except for the scenario that there is no stress event at all (since $q_i(\vec{m}_L, 0) = 0$) or all $p_i^l$ are close to 1, each conditional loss distribution can be well approximated by a normal distribution. The Gaussian approximation for the conditional loss distribution not only provides an efficient scheme to compute the loss distribution, but also delineates the evolution of the loss distribution in terms of Gaussian packets. Basically, the loss distribution is a weighted sum of Gaussian packets with locations and spreads given by Eq.(52) and Eq.(53) respectively. Since each $q_i(\vec{m}_L, t)$ increases with $t$, the Gaussian packets move to the tail of the loss distribution over time. Furthermore, the probability of each scenario specified by $\vec{m}_L$ is

$$\mathrm{P}\left(\bigcap_{l=1}^{L}\{m_l \text{ crises in the } l\text{-th sector}\}\right) = \mathrm{E}\left[\prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}}{m_l!}\Big|\lambda^1(0), ..., \lambda^L(0)\right] \tag{54}$$

$$= \prod_{l=1}^{L} \frac{1}{m_l!}\mathrm{E}\left[e^{-\Lambda^l(t)}(\Lambda^l(t))^{m_l}|\lambda^l(0)\right] \tag{55}$$

which quantifies the size of each Gaussian packet.

*f. Efficiency Analysis*

The bottleneck of the computation of the loss distribution for bottom-up approaches is usually the calculation of the conditional loss distribution. In the recursive algorithm proposed by Andersen et al. (2003), the number of calculations needed to compute the conditional loss distribution, $\mathrm{P}(D(t) = n|\omega)$ where $n = 0, 1, ..., N$, for a portfolio with $N$ names is roughly $N^2/2$ . The typical number of firms $N$ is quite large (usually over 100), this makes the computation of the conditional loss distribution relatively expensive. Although the Gaussian approximation discussed in the previous subsection can reduce the computational time, the computation of the conditional loss distributions still contributes a significant amount to the overall run time.

In the stress event model, the number of unconditional loss distributions, $\mathrm{P}(D(t) = n|\vec{m}_L)$, needed to compute the loss distribution in the $K$-th order approximation with $L$ sectors is equivalent to the number of solutions to the following Diophantine inequality

$$m_1 + m_2 + \cdots + m_L \leq K, \tag{56}$$

or equivalently the total number of solutions to the following $K + 1$ Diophantine equalities

$$m_1 + m_2 + \cdots + m_L = k, \quad k = 0, 1, ..., K, \tag{57}$$

which is ${}_{L+K}C_K$ (see Appendix B for the proof). Table 1 shows the number of unconditional loss distribution needed for different $L$ and $K$. In most situations, fifth order approximation is very accurate in approximating the loss distribution. In the two calibrations that we will discuss later, we approximate the loss distributions using up to fifth order term in both the two-sector and three-sector models which has 21 and 56 independence conditions respectively.

For the one-factor Gaussian copula, the typical number of independence conditions is 50, so our approach has a computational speed similar to that of the Gaussian copula and is very efficient.

On the other hand, Table 1 shows that the number of unconditional loss distribution needed increases mildly with $L$, so the additional computational cost for more non-idiosyncratic factors is low. This is not the case for other bottom-up approach models in general where the number of unconditional loss distribution needed to compute the loss distribution increases rapidly with the number of non-idiosyncratic factors. The low cost for additional non-idiosyncratic factors gives our model more flexibility to match the market data. In addition, it opens up the possibility to study the multi-factor default dependence of a portfolio via a bottom-up approach.

# 5. Calibration

*a. Calibration to CDX.NA.IG series 13 and the Underlying CDS Spreads Simultaneously*

We first calibrate the three-sector stress event model to market data using the fifth order approximation, i.e. $L = 3$ and $K = 5$. The data set contains the first five index tranche prices of CDX.NA.IG series 13 and the 125 underlying CDS spreads on April 15 2010 which are obtained from Bloomberg terminal. The maturities of the tranches and the CDS are all 5-year. The quotes of the index tranches and the statistics of the CDS spreads are shown in Table 2 and Table 3 respectively. We assume a constant recovery rate $R = 0.35$ which is consistent with empirical evidence for senior unsecured bonds reported by Hamilton et al.

(2004). Furthermore, we assume a constant interest rate $r = 0.94\%$, which is the 12-month Libor rate, for cash flow discounting. For the model parameters, we assume the simplest time-independent intensities for all the Poisson processes. Consequently, the default intensity for each firm $i$ can be computed by the so-called credit triangle (O'Kane 2008), i.e.

$$\lambda_i = \frac{S_i}{(1-R)}, \quad i = 1, ..., N, \tag{58}$$

where $S_i$ is the 5-year CDS spread of firm $i$. Hence, $S_i/(1-R)$ imposes a constraint for other parameters in the default intensity of firm $i$ as follows:

$$\frac{S_i}{(1-R)} = \lambda_i = \bar{\lambda}_i + p_i^1\lambda^1 + p_i^2\lambda^2 + p_i^3\lambda^3. \tag{59}$$

This model specification has $4N+3$ parameters and $N$ constraints. We favor a parsimonious model which is flexible to match tranche spreads. Therefore, we choose a parameter set of six members

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{60}$$

for the calibration, where $\lambda^l$ are the stress event intensities and $p^l$ are representative impact factors. The detailed specifications of $p_i^l$ and $\bar{\lambda}_i$ in terms of the parameters in $\Theta$ are provided in Appendix C. We use the root-mean-square error

$$\mathrm{RMSE} = \sqrt{\frac{1}{5}\sum_{j=1}^{5}\left(\frac{\tilde{S}_{tr,j} - S_{tr,j}}{S_{tr,j}^{Bid} - S_{tr,j}^{Ask}}\right)^2} \tag{61}$$

as the objective function in this calibration, where $S_{tr,j}$, $S_{tr,j}^{Bid}$ and $S_{tr,j}^{Ask}$ are the market mid, bid and ask of the $j$-th tranche respectively, and $\tilde{S}_{tr,j}$ are the model implied tranche prices. The parameter set $\Theta$ which minimizes Eq.(61) is presented in Table 4. In Table 5, the model implied tranche prices $\tilde{S}_{tr,j}$ are shown and each of them is within the bid-ask spread. We implement the calibration using MATLAB and it takes about 0.2 second for each pricing (compute five tranche prices for each set of parameters) on a personal laptop computer[1]. It is interesting to note that even in this toy-model specification (all intensities are constant) for the stress event model, it is able to match both the index tranche prices and the underlying CDS spreads very well.

*b. Calibration to the Term Structure of iTraxx Europe Tranches on Multiple days*

One of the main merits of our approach is that introducing stochastic intensities to the model does not undermine the tractability and efficiency. We will apply the stress event model as a top-down model in this subsection, i.e. the model is calibrated to index tranches only. The data that we are using for the calibration are obtained from the Monthly iTraxx Tranche Fixings (see www.creditfixings.com). They consists of four days of market data of the iTraxx Europe series 7 observed on March 30, April 30, May 31 and June 29 in 2007. On each day, there are five standard tranches with maturities 5, 7 and 10 years. There are altogether 60 data point and they are shown in Table 6. We employ the two-sector stress event model using fifth order approximation in the current calibration, i.e. $L = 2$ and $K = 5$. Since a parsimonious parameter set is favored, we assume that each firm in the portfolio follows the same stochastic idiosyncratic default intensity and the probabilities of default given a stress event are the same for all firms, i.e. $p_i^1 = p^1$ and $p_i^2 = p^2$ for all $i$. There are altogether three intensities processes in this specification, one for the idiosyncratic factor and two for the non-idiosyncratic factors. We further assume that each of these intensities

[1]Intel(R) CPU T2050 1.60 GHz, 1.49 GB RAM.

follows the affine jump-diffusion process (Duffie et al. 2000)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t, \quad \lambda_t = \lambda_0, \tag{62}$$

with the mean reverting level $\theta = 0$. A brief discussion on the affine-jump diffusion process is presented in Appendix A. Recall that $\Lambda(t) = \int_0^t \lambda_s ds$ and

$$\mathrm{E}\left[ e^{-\Lambda(t)} \frac{(\Lambda^k(t))^k}{k!} \Big| \lambda_0 \right], \tag{63}$$

which is the probability that there are $k$ stress events in a sector, admits a closed form expression for an affine jump-diffusion process $\lambda_s$. We report the closed form expression of Eq.(63) for $k = 0$ and derive expressions for positive integer $k$ in Appendix A.

There are four parameters in each intensity process and two constant impact factors $p^1$ and $p^2$, thus the current model specification has 14 fixed parameters:

$$\Theta_{\text{fix}} = \{\bar{\kappa}, \bar{\sigma}, \bar{l}, \bar{\mu}, \kappa^1, \sigma^1, l^1, \mu^1, p^1, \kappa^2, \sigma^2, l^2, \mu^2, p^2\}. \tag{64}$$

Besides, there are three initial intensities for the affine-jump diffusion processes on each of the four days. Consequently, we are calibrating 26 model parameters to the 60 data. Similar to the previous calibration example, a constant recovery rate $R = 0.35$ and a constant interest rate $r = 5.35\%$ are used in the calibration.

The model parameters are calibrated by minimizing the root-mean-square of the relative error

$$\text{RMSE} = \sqrt{\frac{1}{60} \sum_{l=1}^{4} \sum_{k=1}^{3} \sum_{j=1}^{5} \left( \frac{\tilde{S}_{tr,j}^{T_k,t_l} - S_{tr,j}^{T_k,t_l}}{S_{tr,j}^{T_k,t_l}} \right)^2}, \tag{65}$$

where $T_1 = 5$, $T_2 = 7$ and $T_3 = 10$ are the maturities of the tranches, $t_l$ is the index for the observing date and $j$ is the index for the tranche. Thus, $S_{tr,j}^{T_k,t_l}$ is the price of the $j$-th tranche with maturity $T_k$ observed on $t_l$ and $\tilde{S}_{tr,j}^{T_k,t_l}$ is the corresponding model implied tranche price.

We perform a fifth order calculation and the calibrated parameters and model implied tranche prices are presented in Table 7 and Table 8 respectively. The model implied tranche prices match quite well with the market mid prices in general with the root-mean-square relative error RMSE = 4.25% and the maximum relative error 10.82%. It is worth noting that we use the same $\Theta_{\text{fix}}$ for all days while changing three initial intensities $\bar{\lambda}_0$, $\lambda_0^1$ and $\lambda_0^2$ to obtain a reasonably good fit to the 15 data on each day. We also implement the calibration using MATLAB and the same computer as indicated in the first calibration example. It takes about 0.4 second for each pricing (compute 60 tranche prices for each set of parameters). We see that all the default intensities are explosive, i.e. the risk-neutral mean reverting rates $\bar{\kappa}$, $\kappa^1$ and $\kappa^2$ are negative. It appears that the negative mean reverting rates are necessary to give enough upward sloping of the default intensities when we are trying to match the term structure of tranche spreads. In the calibrations of the correlated intensity model, Eckner (2009) also finds negative mean reverting rates of the default intensities. Besides, the calibrations of the Generalized-Poisson loss model performed by Brigo et al. (2007) also indicate upward sloping of the default intensities. The upward sloping of default intensities may suggest that investors take a more pessimistic view about the future default intensities and expect an increase of default intensities over time. The volatility of the idiosyncratic intensity $\bar{\sigma}$ is about double that of the non-idiosyncratic intensities. Jump rates of the intensities ranges from two to ten per hundred years. Jump sizes of the intensities are moderate, ranging from 33 bps to 398 bps. These are significantly lower than the jump size found by Eckner (2009) which is around 3000 bps. The jump size in the correlated intensity model needs to be high in order to give enough default correlation among firms, while jumps of intensities in the stress event model have only minor effect on the correlation among firms.

# 6. Conclusion

In this note, we provide an efficient methodology to compute the loss distribution of a large portfolio in the stress event model. A new approach to independence conditions is proposed. This leads to significant simplifications in computing the loss distribution. We perform calibrations to market data and the results are very promising. In addition, the computational cost for additional common factors, unlike other bottom-up approaches, is mild.

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APPENDIX A

# 7. Basic Affine Jump Diffusions

A stochastic process $\lambda_t$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathrm{P})$ is called a basic Affine Jump Diffusions (AJD) if it satisfies the following SDE:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t, \tag{A1}$$

where $B$ is a standard Brownian motion, and $J$ is an independent compound Poisson process with jump intensity $l$ and exponentially distributed jump sized with mean $\mu$. Duffie et al. (2000) show that the moment generating function of the cumulative intensity $\Lambda(t) = \int_0^t \lambda_s ds$ admits a closed form solution

$$\mathrm{E}\left[e^{q\Lambda(t)}|\lambda_0\right] = e^{\alpha(t)+\beta(t)\lambda_0}, \tag{A2}$$

where

$$\alpha(t) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{c_1 + d_1 e^{-\gamma t}}{c_1 + d_1}\right) + \frac{\kappa\theta t}{c_1} \tag{A3}$$

$$+ l\left(\frac{d_1/c_1 - d_2/c_2}{-\gamma d_2}\right)\log\left(\frac{c_2 + d_2 e^{-\gamma t}}{c_2 + d_2}\right) + \frac{l(1-c_2)t}{c_2} \tag{A4}$$

$$\beta(t) = \frac{1 - e^{-\gamma t}}{c_1 + d_1 e^{-\gamma t}}, \tag{A5}$$

and

$$\gamma = \sqrt{\kappa^2 - 2\sigma^2 q} \tag{A6}$$

$$c_1 = (\kappa + \gamma)/(2q) \tag{A7}$$

$$c_2 = 1 - \mu/c_1 \tag{A8}$$

$$d_1 = (-\kappa + \gamma)/(2q) \tag{A9}$$

$$d_2 = (d_1 + \mu)/c_1. \tag{A10}$$

With the help of the closed form expression of the moment generating function, we can compute the expectation

$$\mathrm{E}\left[e^{-\Lambda(t)}|\lambda_0\right], \tag{A11}$$

which is the form of the probability that there is no stress event by time $t$, by plugging $q = -1$ in Eq.(A2)-Eq.(A10). Longstaff and Rajan (2008) derive a recursive system of ordinary differential equation to compute

$$\mathrm{E}\left[e^{-\Lambda(t)}(\Lambda(t))^k|\lambda_0\right]. \tag{A12}$$

Their approach, though it works, is not very appealing since it is quite time consuming in solving the system of ODE numerically. Besides, it is hard to control the error propagation in the recursive ODE.

In fact, Eq.(A12) can be computed easily by differentiating Eq.(A2) $k$ times with respect to $q$, then

$$\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right) = \mathrm{E}\left[e^{q\Lambda(t)}(\Lambda(t))^k|\lambda_0\right]. \tag{A13}$$

Plugging $q = -1$ and dividing by $k!$ yields the probability that there are $k$ stress events in the sector, i.e.

$$\mathrm{P}(k \text{ stress events by time } t) = \frac{1}{k!}\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right)\Big|_{q=-1}. \tag{A14}$$

The validity of exchanging the order of differentiation and expectation in Eq.(A13) can be verified if $\Lambda(t) \geq 0$ for all $t$, which is true in our consideration here since $\Lambda(t)$, as a cumulative intensity, is always non-negative. As a result, in order to compute the scenario probability, $\mathrm{P}(k$ stress events by time $t)$, we just need to find the $k$-th derivative of the moment generating function Eq.(A2) at $q = -1$. Although Eq.(A14) admits a closed form expression, its complexity grows tremendously with $k$. For example, the closed form expression of Eq.(A14) for $k = 4$, obtained by the symbolic toolbox of MATLAB, needs 285 letter-size pages (with font size 12) to print the result. Therefore, evaluating Eq.(A14) can be quite time consuming even for moderate $k$ and we need a more efficient way to calculate the derivatives. To this end, we adopt the exact numerical differentiation algorithm developed by Tsui (2010), which is very efficient in evaluating high order derivatives.

# APPENDIX B

## 8. The Number of Ways of Allocation

We first prove that the number of different ways of allocating $k$ stress events in $L$ sectors is ${}_{L+k-1}C_k$. Denote "$\circ$" as a stress event and "$|$" as a wall dividing two sectors. Therefore, there are $L-1$ walls and $k$ stress events for each scenario. For example, naming the sectors from right to left starting from the first sector,

$$\circ \,|\, \circ \circ \,||\, \circ \circ \circ \,|\, \circ \,|| \tag{B1}$$

represents a scenario for which there are no stress event in the first, second and fifth sectors, one stress event in both the third and seventh sectors, two stress events in the sixth sector and three stress events in the fourth sector. Note that there are altogether $L+k-1$ objects in each representation. With these notations, the number of ways of having $k$ stress events in $L$ sectors is equivalent to the number of ways of choosing $k$ objects (the stress events) from $L+k-1$ objects which is ${}_{L+k-1}C_k$. Thus, the total number of scenario such that there are $K$ stress event or less in $L$ sectors is

$$\sum_{k=0}^{K} {}_{L+k-1}C_k = {}_{L+K}C_K, \tag{B2}$$

which can be proved by the recursive formula for binomial coefficients.

APPENDIX C

# 9. Determination of $\bar{\lambda}_i$ and $p_i^l$ from CDS spreads

We will fix $\bar{\lambda}_i$ and $p_i^l$ for each name of the portfolio by using the 5-year CDS spreads with the constraints

$$\bar{\lambda}_i \geq 0 \quad i = 1, ..., N, \tag{C1}$$

$$0 \leq p_i^l \leq, 1 \quad l = 1, 2, 3, \quad i = 1, ..., N. \tag{C2}$$

We start by defining a relative credit quality in terms of the 5-year CDS spreads as follows:

$$c_i = \frac{S_i}{\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} S_j}, \tag{C3}$$

Then, for $l = 1, 2$, define an auxiliary impact parameter

$$\tilde{p}_i^l = \min\{c_i p^l, 1\}, \quad i = 1, ..., N \tag{C4}$$

where $p^l$ is a representative impact parameter of the $l$-sector which is to be calibrated to the tranche quotes. For $l = 3$, choose

$$0 \leq \tilde{p}_i^3 = p^3 \leq 1 \tag{C5}$$

for all $i$. For most of the situations, we can choose $p_i^l = \tilde{p}_i^l$. Recall that $\lambda_i = S_i/(1-R)$ and $\lambda^l \geq 0$ are parameters to be calibrated from the tranches, so the idiosyncratic default intensity is

$$\bar{\lambda}_i = \lambda_i - \tilde{p}_i^1 \lambda^1 - \tilde{p}_i^2 \lambda^2 - \tilde{p}_i^3 \lambda^3. \tag{C6}$$

However, $\bar{\lambda}_i$ computed as above could be negative for some cases. For those cases, we lower the values of $p_i^l$ proportionally, so

$$p_i^l = \begin{cases} \tilde{p}_i^l, & \text{if } \lambda_i - \tilde{p}_i^1 \lambda^1 - \tilde{p}_i^2 \lambda^2 - \tilde{p}_i^3 \lambda^3 \geq 0; \\ \dfrac{\lambda_i \tilde{p}_i^l}{\tilde{p}_i^1 \lambda^1 + \tilde{p}_i^2 \lambda^2 + \tilde{p}_i^3 \lambda^3}, & \text{otherwise,} \end{cases} \tag{C7}$$

for all $l$ and $i$, and

$$\bar{\lambda}_i = \lambda_i - p_i^1 \lambda^1 - p_i^2 \lambda^2 - p_i^3 \lambda^3. \tag{C8}$$

With a fixed set of parameters

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{C9}$$

the CDS spreads $S_i$ can be matched exactly by choosing $p_i^l$ and $\bar{\lambda}_i$ by Eq.(C7) and Eq.(C8) respectively. For $l = 1, 2$, the specification of $p_i^l$ basically follows the idea of Eckner (2009) where the dependence on a factor is proportional to the relative credit quality $c_i$. For $l = 3$, we choose $p_i^3$ to be the same if possible to include the possibility of some catastrophic events that have a high probability to kill many firms.

# REFERENCES


Andersen, L., J. Sidenius, and S. Basu, 2003: All your hedges in one basket. *Risk*, **November**.

Arnsdorf, M. and I. Halperin, 2008: Bslp: Markovian bivariate spread-loss model for portfolio credit derivatives. *Journal of Computational Finance*, **12**, 77–100.

Bayraktar, E. and B. Yang, 2009: Multi-scale time-changed birth processes for pricing multi-name credit derivatives. *Applied Mathematical Finance*, **16(5)**, 429–449.

Brigo, D., A. Pallavicini, and R. Torresetti, 2007: CDO calibration with the dynamical generalized Poisson loss model. *Risk Magazine*, **June**.

Cont, R. and A. Minca, 2008: Extracting portfolio default rates from cdo spreads. Working paper, Columbia University.

Duffie, D. and N. Gârleanu, 2001: Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, **57**, 41–59.

Duffie, D., J. Pan, and J. Singleton, 2000: Transform analysis and asset pricing for affine jump diffusions. *Econometrica*, **68, No. 6 (November)**.

Duffie, D. and K. Singleton, 1999: Simulating correlated defaults, graduate School of Business, Stanford Univeristy.

Eckner, A., 2009: Computational techniques for basic affine models of portfolio credit risk. *Journal of Computational Finance*, **13**.

Errais, E., K. Giesecke, and L. Goldberg, 2006: Pricing credit from top down with affine point processes. Working paper, Stanford University.

Giesecke, K., L. Goldberg, and X. Ding, 2010: A top-down approach to multi-name credit. *Operations Research*, **forthcoming**.

Hamilton, D., P. Varma, S. Ou, and R. Cantor, 2004: Default and recovery rates of corporate bond issuers. *Moody's Investors Service, New York*, **January**.

Joshi, M. and A. Stacey, 2006: Intensity gamma: A new approach to pricing portfolio credit derivatives. *RISK*, **19**, July 78–83.

Longstaff, F. and A. Rajan, 2008: An empirical analysis of the pricing of collateralized debt obligations. *Journal of Finance*, **63**, 529–563.

Mortensen, A., 2006: Semi-analytical valuation of basket credit derivatives in intensity-based models. *Journal of Derivatives*, **13, No. 4**, 8–26.

O'Kane, D., 2008: *Modelling Single-name and Multi-name Credit Derivatives*. The Wiley Finance Series.

Papageorgiou, E. and R. Sircar, 2007: Multiscale intensity models and name grouping for valuation of multi-name credit derivatives. *Applied Mathematical Finance*, **15(1)**, 73–105.

Peng, X. and S. S. G. Kou, 2008: Default clustering and valuation of collateralized debt obligations. Working paper, Columbia University.

Schönbucher, P., 2003: *Credit Derivatives Pricing Models*. John Wiley & Son Ltd.

Shelton, D., 2004: Back to normal, proxy integration: A fast accurate method for cdo and cdo-squared pricing. *Citigroup Structured Credit Research*, **August**.

Tsui, L. K., 2010: Exact numerical algorithm for n-th order derivative of a single variable function. Working paper, University of Pittsburgh.

# List of Tables

TABLE 1. Number of unconditional loss distributions required for $K$-th order approximation with $L$ sectors.

| $K \setminus L$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 6 | 10 | 15 | 21 |
| 3 | 4 | 10 | 20 | 35 | 56 |
| 4 | 5 | 15 | 35 | 70 | 126 |
| 5 | 6 | 21 | 56 | 126 | 252 |

TABLE 2. Tranche spreads of CDX.NA.IG series 13 on April 15 2010. All quotes are upfronts in percentage with fixed 100bps running spread.

| CDX | 0-3% | $3-7\%$ | $7-10\%$ | $10-15\%$ | $15-30\%$ |
|---|---|---|---|---|---|
| Bid | 51.530 | 16.000 | 4.888 | -1.210 | -3.100 |
| Mid | 52.185 | 16.605 | 5.345 | -0.855 | -2.880 |
| Ask | 52.840 | 17.210 | 5.810 | -0.500 | -2.660 |

Table 3. Summary of the closing data of the mid 5-year CDS spreads of the 125 names in CDX.NA.IG series 13 on April 15 2010.

| Statistics | bps |
|---|---:|
| Min | 25.39 |
| Max | 349.62 |
| Median | 74.36 |
| Mean | 87.76 |
| Standard Deviation | 47.04 |

Table 4. Model parameters calibrated from tranche spreads of CDX.NA.IG series 13 and the underlying CDS spreads.

| | $\lambda^1$ | $p^1$ | $\lambda^2$ | $p^2$ | $\lambda^3$ | $p^3$ |
|---|---|---|---|---|---|---|
| CDX | 0.0427828 | 0.0883129 | 0.0122258 | 0.122791 | 0.00391431 | 1.000000 |

TABLE 5. Model implied tranche spreads of CDX.NA.IG series 13 on April 15 2010 using fifth order calculation. Bid-ask spreads are included for comparison.

| CDX | 0-3% | $3-7\%$ | $7-10\%$ | $10-15\%$ | $15-30\%$ |
|---|---|---|---|---|---|
| Bid | 51.530 | 16.000 | 4.888 | -1.210 | -3.100 |
| Model | 52.274 | 16.703 | 5.386 | -0.836 | -2.669 |
| Ask | 52.840 | 17.210 | 5.810 | -0.500 | -2.660 |

Table 6. Market mid prices of iTraxx Europe series 7 version 1. 0-3% tranche is quoted in percentage as an upfront with a fixed 500bps and all the other tranches are spreads in bps without upfront.

| Maturity | Tranche | Mar 30, 07 | Apr 30, 07 | May 31, 07 | Jun 29, 07 |
|---|---|---|---|---|---|
| 5-year | 0-3% | 11.23% | 9.94% | 6.33% | 11.75% |
| | 3-6% | 57.75 | 49.82 | 39.90 | 62.05 |
| | 6-9% | 14.28 | 12.45 | 10.33 | 16.29 |
| | 9-12% | 6.24 | 5.53 | 4.39 | 7.48 |
| | 12-22% | 2.58 | 2.54 | 1.93 | 3.10 |
| 7-year | 0-3% | 25.77% | 24.84% | 20.61% | 26.38% |
| | 3-6% | 133.79 | 121.2 | 105.08 | 137.13 |
| | 6-9% | 37.25 | 31.99 | 27.04 | 37.39 |
| | 9-12% | 17.33 | 15.75 | 13.05 | 17.00 |
| | 12-22% | 5.85 | 5.67 | 5.20 | 7.50 |
| 10-year | 0-3% | 40.51% | 38.95% | 35.00% | 40.53% |
| | 3-6% | 338.96 | 322.20 | 294.21 | 368.60 |
| | 6-9% | 98.59 | 93.48 | 85.17 | 108.55 |
| | 9-12% | 46.91 | 43.59 | 38.98 | 50.33 |
| | 12-22% | 14.38 | 14.50 | 12.20 | 15.95 |

Table 7. Model parameters calibrated from tranche spreads of iTraxx Europe series 7 version 1.

| $\overline{\kappa}$ | $\overline{\sigma}$ | $\overline{l}$ | $\overline{\mu}$ |
|---|---|---|---|
| -0.20401 | 0.22716 | 0.07650 | 0.01170 |

| $\kappa^1$ | $\sigma^1$ | $l^1$ | $\mu^1$ | $p^1$ |
|---|---|---|---|---|
| -0.13713 | 0.11516 | 0.10890 | 0.03977 | 0.04287 |

| $\kappa^2$ | $\sigma^2$ | $l^2$ | $\mu^2$ | $p^2$ |
|---|---|---|---|---|
| -0.55844 | 0.13363 | 0.01813 | 0.00326 | 0.24261 |

| | Mar 30, 07 | Apr 30, 07 | May 31, 07 | Jun 29, 07 |
|---|---|---|---|---|
| $\overline{\lambda}_0$ | 0.00021853 | 0.00021325 | 0.00000000 | 0.00035351 |
| $\lambda_0^1$ | 0.00226915 | 0.00085552 | 0.00000000 | 0.00388838 |
| $\lambda_0^2$ | 0.00001968 | 0.00001919 | 0.00000000 | 0.00006339 |

Table 8. Model implied tranche prices of iTraxx Europe series 7 version 1.

| Maturity | Tranche | Mar 30, 07 | Apr 30, 07 | May 31, 07 | Jun 29, 07 |
|---|---|---|---|---|---|
| 5-year | 0-3% | 11.03% | 9.82% | 6.22% | 11.49% |
| | 3-6% | 59.32 | 50.90 | 42.47 | 63.78 |
| | 6-9% | 13.85 | 12.40 | 10.33 | 14.81 |
| | 9-12% | 6.20 | 5.82 | 4.73 | 7.27 |
| | 12-22% | 2.58 | 2.45 | 1.90 | 3.24 |
| 7-year | 0-3% | 26.55% | 25.29% | 21.16% | 27.24% |
| | 3-6% | 131.53 | 117.32 | 98.65 | 138.88 |
| | 6-9% | 36.93 | 33.17 | 28.47 | 39.03 |
| | 9-12% | 15.72 | 14.58 | 12.48 | 17.19 |
| | 12-22% | 6.11 | 5.85 | 4.83 | 7.35 |
| 10-year | 0-3% | 42.46% | 41.75% | 38.79% | 43.62% |
| | 3-6% | 332.12 | 312.68 | 275.68 | 344.06 |
| | 6-9% | 102.41 | 93.42 | 82.76 | 108.14 |
| | 9-12% | 47.27 | 43.50 | 38.80 | 49.85 |
| | 12-22% | 14.77 | 14.09 | 12.58 | 16.22 |
| $\epsilon_5(10)$ | | 0.0090 | 0.0081 | 0.0070 | 0.0092 |