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Materials Science and Engineering A285 (2000) 239–245

MATERIALS SCIENCE & ENGINEERING A

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Multi-inclusion method for finite deformations: exact results and applications

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Abstract

For linearly elastic heterogeneous solids, averaging theorems are developed by Nemat-Nasser and Hori (S. Nemat-Nasser, M. Hori, Micromechanics: Overall Properties of Heterogeneous Solids, Elsevier, Amsterdam, 1993; and S. Nemat-Nasser, M. Hori, J. Eng. Mater. Technol. 117 (1995) 412), using a multi-inclusion model. This model is based on the observation that the average strain and stress in any annulus of a nested sequence of ellipsoids embedded in an infinite homogeneous solid, can be computed *exactly* when each annulus undergoes a uniform transformation with an associated transformation strain different from the others; the central ellipsoid may have a non-uniform transformation strain. This result generalizes the pioneering observation of Tanaka and Mori (K. Tanaka, T. Mori, J. Elast. 2 (1972) 199; and T. Mori, K. Tanaka, Acta Met. 21 (1973) 571) who showed that the average strain and stress in the region between two similar and coaxial ellipsoids in an infinite uniform elastic solid, are zero for any transformation strain within the inner ellipsoid; see also Benveniste, and, and Mori and Wakashima (T. Benveniste, Mech. Mater. 6 (1987) 147; and T. Mori, K. Wakashima, Successive iteration method in the evaluation of average fields in elastically inhomogeneous materials, Micromechanics and Inhomogeneity — The T. Mura 65th Anniversary Volume, Springer, New York, 1990, pp. 269–282). Similar results apply to the finite deformation problems, provided that the nominal stress rate and the rate of change of the deformation gradient, (measured relative to any arbitrary state) are used as the dynamical and kinematical variables; see Nemat-Nasser (S. Nemat-Nasser, Mech. Mater. 31, (1999) 493) for a comprehensive account of a rigorous treatment of the transition from micro- to macro-variables of a representative volume element of a finitely deformed aggregate. An *exact method* for homogenization of an ellipsoidal heterogeneity in an unbounded finitely deformed homogeneous solid, is developed, using the generalized Eshelby tensor. It is shown that many results for single-, double-, and multi-inclusion problems in linear elasticity (see S. Nemat-Nasser, M. Hori, Micromechanics: Overall Properties of Heterogeneous Solids, Elsevier, Amsterdam, 1993; and S. Nemat-Nasser, M. Hori, J. Eng. Mater. Technol. 117 (1995) 412), also apply to the finite-deformation rate problems, provided suitable kinematical and dynamical variables are used. The problem of the double inclusion is considered and *exact* expressions are given for the average field quantities, taken over the region between the two, as well as within each ellipsoidal domains, one containing the other, when arbitrary eigenvelocity gradients are prescribed within an arbitrary region contained in the inner ellipsoid. This generalizes to the fully nonlinear, finitely-deformed, elastoplastic case, the Tanaka–Mori (K. Tanaka, T. Mori, J. Elast. 2 (1972) 199) result, and the double inclusion result of Nemat-Nasser and Hori (S. Nemat-Nasser, M. Hori, Micromechanics: Overall Properties of Heterogeneous Solids, Elsevier, Amsterdam, 1993; and S. Nemat-Nasser, M. Hori, J. Eng. Mater. Technol. 117 (1995) 412), which have been developed for linearly elastic solids. The application of the exact results to the problem of estimating the overall mechanical response of a finitely deformed heterogeneous representative volume element (RVE) is outlined and the overall effective pseudo-modulus tensor of the RVE is calculated for rate-independent elastoplastic materials.

Keywords: Finite deformation; Stress; Strain

1. Introduction

Let a deformable body B be in configuration C_0 at a reference time t_0. Denote by C its configuration at the current instant $t > t_0$. The mapping from C_0 to C is denoted by $x_i = x_i(X_1, X_2, X_3, t) = x_i(X_A, t)$, i, $A =$ 1, 2, 3, or $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ which defines particle positions $\mathbf{x}$ of components x_i in C in terms of their reference positions $\mathbf{X}$ of components X_A, both taken with respect to the background rectangular Cartesian coordinate system. Denote the deformation gradient by $\mathbf{F} = (\partial \mathbf{x}/\partial \mathbf{X})^{\mathrm{T}}$.

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PII: S0921-5093(00)00643-2

With $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}$ defining the velocity gradient in reference to the current state, the *Kirchhoff stress*, $\boldsymbol{\tau}$, the *Cauchy stress*, $\boldsymbol{\sigma} = (\rho/\rho_0)\boldsymbol{\tau}$, and the *nominal stress*, $\boldsymbol{S}^{\mathrm{N}}$, are defined such that, Hill [1],

$$\frac{1}{\rho_0}\,\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \frac{1}{\rho}\,\mathrm{tr}(\boldsymbol{\sigma}\mathbf{D}) = \frac{1}{\rho_0}\,\mathrm{tr}(\mathbf{S}^{\mathrm{N}}\dot{\mathbf{F}}) \tag{1.1}$$

represents the rate of stress work per unit mass; here ρ_0 and ρ are the initial and current mass densities. Hence, the nominal stress, $\mathbf{S}^{\mathrm{N}}$, is given by

$$\mathbf{S}^{\mathrm{N}} = \mathbf{F}^{-1}\boldsymbol{\tau} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \quad J = \det\mathbf{F}. \tag{1.2a,b}$$

Here and in the sequel, *matrix operation rules are used*, so that $\mathbf{S}^{\mathrm{N}}$ has the following component representation, $S^{\mathrm{N}}_{Ai} = F^{-1}_{Aj}\tau_{ji}$, where j is summed. Similarly, $\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \tau_{ij}D_{ji} = S_{Aj}\dot{F}_{jA}$, where all repeated indices are always summed.

At finite strains and rotations, relations among kinematical and dynamical variables are *nonlinear*. These relations hold within each grain in an representative volume element (RVE). It is desirable to define the *overall average* kinematical and dynamical quantities for the RVE such that similar relations remain valid at the *macrolevel*. The nonlinearity, however, precludes representation of all deformation and stress measures and their rates, as unweighted volume averages over the RVE. Thus, for finite deformation, there is an inherent arbitrariness in the selection of suitable kinematical and dynamical quantities whose overall measures are defined in terms of unweighted volume averages of the corresponding micromeasures, and which are then employed to define other overall quantities, using the usual continuum mechanics relations. Therefore, the selection must be made with care and with due regard for the physical basis of the basic phenomenon and the experimental procedures and results.

It turns out that, for finite deformation, the deformation gradient, $\mathbf{F}$, its rate, $\dot{\mathbf{F}}$, and the nominal stress, $\mathbf{S}^{\mathrm{N}}$, and its rate $\dot{\mathbf{S}}^{\mathrm{N}}$, are suitable deformation and stress measures for the purpose of averaging. Their unweighted volume averages are completely defined in terms of the surface data (whether uniform or not), and for either the uniform traction or the linear displacement boundary data, they lead to many useful relations which can be employed to effectively characterize the overall aggregate response. Indeed, the following exact results hold in general for any finitely deformed heterogeneous solid of a reference volume V, bounded by ∂V:

$$\langle\mathbf{F}\rangle \equiv \frac{1}{V}\int_V \mathbf{F}(\mathbf{X}, t)\mathrm{d}V = \frac{1}{V}\int_{\partial V} \mathbf{x}\otimes\mathbf{N}\,\mathrm{d}A,$$

$$\langle\dot{\mathbf{F}}\rangle \equiv \frac{1}{V}\int_V \dot{\mathbf{F}}(\mathbf{X}, t)\mathrm{d}V = \frac{1}{V}\int_{\partial V} \dot{\mathbf{x}}\otimes\mathbf{N}\,\mathrm{d}A, \tag{1.3a,b}$$

where $\mathbf{x}$ and $\dot{\mathbf{x}}$ are arbitrarily defined compatible current boundary positions and their rates, respectively, and $\mathbf{N}$ is the exterior unit normal to ∂V in the reference state; $\mathbf{x}\otimes\mathbf{N}$ has the following components, x_iN_{A}.

In a similar way, equilibrium in the absence of body forces, leads to

$$\langle\mathbf{S}^{N}\rangle \equiv \frac{1}{V}\int_V \mathbf{S}^{\mathrm{N}}(\mathbf{X}, t)\mathrm{d}V = \frac{1}{V}\int_{\partial V} \mathbf{X}\otimes\mathbf{T}\,\mathrm{d}A.$$

$$\langle\dot{\mathbf{S}}^{N}\rangle \equiv \frac{1}{V}\int_V \dot{\mathbf{S}}^{\mathrm{N}}(\mathbf{X}, t)\mathrm{d}V = \frac{1}{V}\int_{\partial V} \mathbf{X}\otimes\dot{\mathbf{T}}\,\mathrm{d}A \equiv \frac{\partial}{\partial t}\langle\mathbf{S}^{N}\rangle \tag{1.4a,b}$$

where $\mathbf{T}$ and $\dot{\mathbf{T}}$ are the traction and traction-rate boundary data, respectively. These equations are valid whether or not the reference configuration coincides with the current configuration.

Finally, note the exact identity,

$$\langle\dot{\mathbf{F}}\mathbf{S}^{\mathrm{N}}\rangle - \langle\dot{\mathbf{F}}\rangle\langle\mathbf{S}^{\mathrm{N}}\rangle = \frac{1}{V}\int_{\partial V} (\dot{\mathbf{x}} - \langle\dot{\mathbf{F}}\rangle\mathbf{X})\otimes\{\mathbf{N}\cdot(\mathbf{S}^{\mathrm{N}} - \langle\mathbf{S}^{\mathrm{N}}\rangle)\}\mathrm{d}A. \tag{1.5}$$

from which other identities can easily be obtained.

2. Constitutive model and phase transformation problem

The nominal stress rate and the velocity gradient, denoted by $\dot{\mathbf{S}}^{\mathrm{N}}$ and $\dot{\mathbf{F}}$, with components $\dot{S}^{\mathrm{N}}_{Ai}$ and $\dot{F}_{iA}$, are used as the basic stress-rate and deformation-rate measures. These quantities are assumed to be connected through certain instantaneous pseudo-moduli, $\mathscr{F}$, of components $\mathscr{F}_{AiBj}$, as follows:

$$\dot{\mathbf{S}}^{\mathrm{N}} = \mathscr{F}{:}\dot{\mathbf{F}} \quad \text{or} \quad S^{\mathrm{N}}_{Ai} = \mathscr{F}_{AiBj}\dot{F}_{jB}. \tag{2.1a}$$

The pseudo-modulus tensor $\mathscr{F}$ is symmetric with respect to the exchange of Ai and Bj, but not with respect to the exchange of A and i or B and j. Assume that the pseudo-modulus $\mathscr{F}$ admits an inverse, $\mathscr{G}$, so that

$$\mathscr{F}{:}\mathscr{G} = \mathbf{1}^{(4)}, \quad \text{or} \quad \mathscr{F}_{AiBj}\mathscr{G}_{jBkC} = \delta_{AC}\delta_{ik} \tag{2.1b}$$

where $\mathbf{1}^{(4)}$ with components $\delta_{ij}\delta_{AB}$, is the general, fourth-order identity tensor. Then, it follows that

$$\dot{\mathbf{F}} = \mathscr{G}{:}\dot{\mathbf{S}}^{\mathrm{N}}, \quad \text{or} \quad \dot{F}_{iA} = \mathscr{G}_{iAjB}S^{\mathrm{N}}_{Bj} \tag{2.1c}$$

The tensor $\mathscr{G}$ with components $\mathscr{G}_{iAjB}$ is the *pseudo-compliance*.

2.1. Phase transformation problem

Let a region Ω in V undergo a phase transformation which, if Ω were free from the constraint imposed by the surrounding material[1], it would attain a constant transformation (inelastic) velocity gradient $\dot{\mathbf{F}}^*$ with components $\dot{F}^*_{iA}$. Let the resulting velocity gradient of

[1] I.e. if Ω is cut and allowed to change without any constraints imposed on its boundary $\partial\Omega$.

Ω in the presence of the constraint from the surrounding matrix, be $\dot{\mathbf{F}}$ with components $\dot{F}^*_{iA}$. In general, $\dot{\mathbf{F}}$ is spatially non-uniform. However, when V is *homogeneous and unbounded*, and Ω is *ellipsoidal*, then, following Eshelby's procedure, it is shown by Iwakuma and Nemat-Nasser [2] that the resulting final velocity gradient $\dot{\mathbf{F}}$ in Ω (for any constant transformation velocity gradient $\dot{\mathbf{F}}^*$) is also constant. In this case, $\dot{\mathbf{F}}$ is related linearly to the transformation velocity gradient $\dot{\mathbf{F}}^*$ by

$$\dot{\mathbf{F}} = \mathscr{S}:\dot{\mathbf{F}}^* \quad \text{or} \quad \dot{F}_{iA} = \mathscr{S}_{iABj}\dot{F}^*_{jB}. \tag{2.2}$$

The fourth-order tensor $\mathscr{S}$, in general, has no symmetries. Iwakuma and Nemat-Nasser [2] outline a method for computing $\mathscr{S}$ for a general ellipsoidal Ω. When the operator $\mathscr{F}_{AiBj}\partial^2(...)/(\partial X_A \partial X_B)$ is elliptic, a real-valued tensor $\mathscr{S}$ exists and can be computed in terms of the aspect ratios of the ellipsoid Ω and the pseudo-modulus tensor $\mathscr{F}_{AiBj}$.

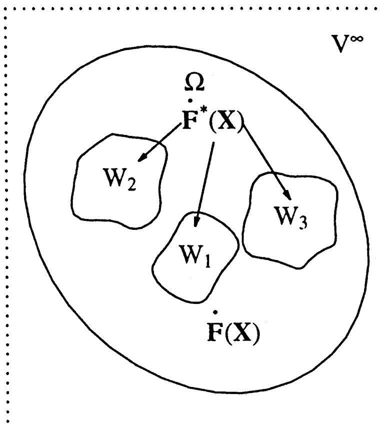

Fig. 1. An ellipsoidal Ω in an unbounded uniform V^∞, contains $W = W_1 + W_2 + W_3$; arbitrary eigenvelocity gradients $\dot{\mathbf{F}}^*(\mathbf{X})$ are distributed within W.

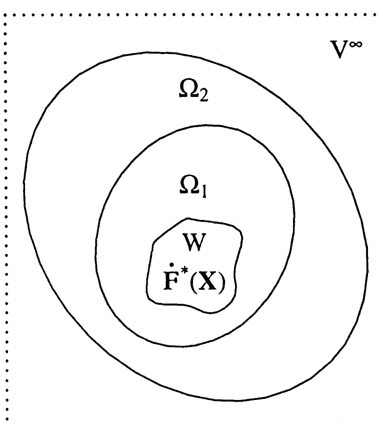

Fig. 2. An ellipsoidal Ω_2 in an unbounded uniform V^∞, contains an ellipsoidal Ω_1 which contains an arbitrary region W; arbitrary eigenvelocity gradients $\dot{\mathbf{F}}^*(\mathbf{X})$ are distributed within W.

Since $\dot{\mathbf{F}}^*$ is the *stress-free inelastic* velocity gradient in Ω the nominal stress rate in this region is produced by the different velocity gradient $(\dot{\mathbf{F}} - \dot{\mathbf{F}}^*)$ and it follows that

$$\dot{\mathbf{S}}^{\mathrm{N}} = \mathscr{F}:(\dot{\mathbf{F}} - \dot{\mathbf{F}}^*) = \mathscr{F}:(\mathscr{S} - \mathbf{1}^{(4)}):\mathbf{F}^* \quad \text{in } \Omega \tag{2.3a}$$

or in component form,

$$\begin{aligned}\dot{S}^{\mathrm{N}}_{Ai} &= \mathscr{F}_{AiBj}(\dot{F}_{jB} - \dot{F}^*_{jB}) \\ &= \mathscr{F}_{AiBj}(\mathscr{S}_{jBCk} - \delta_{BC}\delta_{jk})\dot{F}^*_{kC} \quad \text{in } \Omega\end{aligned} \tag{2.3b}$$

3. Generalized double-inclusion problem

The double-inclusion model of Nemat-Nasser and Hori ([3], p. 351) provides a homogenization method, which includes as special cases, the Mori–Tanaka [4] and the self-consistent methods. Here, we examine its application to finite deformation problems in elastoplasticity. Consider an arbitrary finite region W in an unbounded domain V^∞ of pseudo-modulus tensor $\mathscr{F}$. The region W may consist of several disconnected subregions, say, W_α, $\alpha = 1, 2, \ldots, n$, of arbitrary shapes. Suppose that *an arbitrary eigenvelocity gradient field*, $\dot{\mathbf{F}}^*(\mathbf{X})$, is distributed in W. Let Ω be an arbitrary ellipsoidal domain in V^∞, such that W is totally contained within Ω; see Fig. 1. Then, the *average* velocity gradient and the corresponding *average* nominal stress rate, taken over domain Ω, are completely determined by the generalized Eshelby tensor for Ω.

Indeed, denoting the average value of $\dot{\mathbf{F}}^*(\mathbf{X})$ over W by $\langle\dot{\mathbf{F}}^*\rangle w$, and those of $\dot{\mathbf{F}}(\mathbf{X})$ and $\dot{\mathbf{S}}^{\mathrm{N}}$ over Ω by $\langle\dot{\mathbf{F}}\rangle_\Omega$ and $\langle\dot{\mathbf{S}}^{\mathrm{N}}\rangle_\Omega$, respectively, obtain

$$\langle\dot{\mathbf{F}}\rangle_\Omega = \frac{W}{\Omega}\mathscr{S}^\Omega:\langle\dot{\mathbf{F}}^*\rangle w, \tag{3.1a}$$

$$\langle\dot{\mathbf{S}}^{\mathrm{N}}\rangle_\Omega = \frac{W}{\Omega}\mathscr{F}:(\mathscr{S}^\Omega - \mathbf{1}^{(4)}):\langle\dot{\mathbf{F}}^*\rangle w. \tag{3.1b}$$

These remarkable exact results are valid for any finite W, containing any arbitrary eigenvelocity gradient (transformation velocity gradient) $\dot{\mathbf{F}}^*(\mathbf{X})$. They can be used to homogenize a heterogeneous solid and to obtain the corresponding overall pseudo-moduli.

As an application of these general results, consider an ellipsoid Ω_1 within another ellipsoid Ω_2, embedded in a uniform unbounded domain V^∞ of pseudo-modulus tensor $\mathscr{F}$; see Fig. 2. Let Ω_1 contain an arbitrary region W, in which an arbitrary eigenvelocity gradient $\dot{\mathbf{F}}^*(\mathbf{X})$ is distributed. Under similar minimal assumptions as above, and using a similar procedure which produced (5.1), it can easily be shown that the average value of the velocity gradient over the annulus $\Omega_2 - \Omega_1$, is given by

$$\langle\dot{\mathbf{F}}\rangle_{\Omega_2 - \Omega_1} = \frac{W}{\Omega_2 - \Omega_1}\{\mathscr{S}(\Omega_2) - \mathscr{S}(\Omega_1)\}:\langle\dot{\mathbf{F}}^*\rangle_W. \tag{3.2a}$$

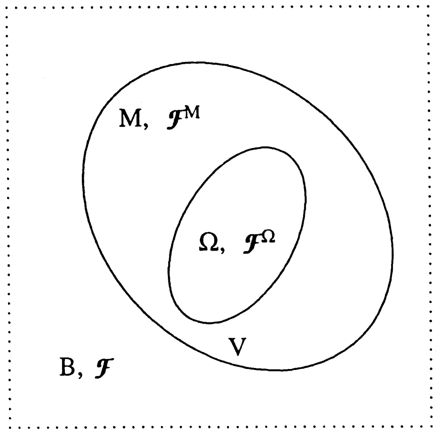

Fig. 3. An unbounded uniform solid B of elasticity $\mathscr{F}$, contains two ellipsoidal regions, V and Ω ($\Omega \subset V$), of elasticity tensors $\mathscr{F}^\Omega$ in Ω and $\mathscr{F}^M$ in $\Gamma = V - \Omega$.

The average stress rate in $\Omega_2 - \Omega_1$, is given by

$$\langle \dot{\mathbf{S}}^N \rangle_{\Omega_2 - \Omega_1} = \frac{W}{\Omega_2 - \Omega_1} \mathscr{F} : \{\mathscr{S}(\Omega_2) - \mathscr{S}(\Omega_1)\} : \langle \dot{\mathbf{F}}^* \rangle_W. \tag{3.2b}$$

These equations are the finite-deformation version of the so-called Tanaka–Mori result (Tanaka and Mori [5]; see also, Nemat-Nasser and Hori [3], section 11, p. 379).

The generalized Eshelby tensor depends on both the shape and the orientation of the ellipsoid. However, only the ratios of the axes enter the components of this tensor in a coordinate system coincident with the directions of the principal axes of the ellipsoid. Therefore, if the two ellipsoidal regions, Ω_1 and Ω_2, have the same shape and orientation, i.e. if the corresponding principal axes have common ratios and directions, and since both are embedded in the same unbounded domain, then

$$\mathscr{S}(\Omega_2) = \mathscr{S}(\Omega_1). \tag{3.3a}$$

Hence, the average velocity gradient and the average nominal stress rate in $\Omega_2 - \Omega_1$ vanish,

$$\langle \dot{\mathbf{F}} \rangle_{\Omega_2 - \Omega_1} = 0, \qquad \langle \dot{\mathbf{S}}^N \rangle_{\Omega_2 - \Omega_1} = 0. \tag{3.3b,c}$$

These exact results hold for any W of any arbitrary shape, containing any arbitrary eigenvelocity gradient, $\dot{\mathbf{F}}^*(\mathbf{X})$.

3.1. Generalized phase-transformation problem

As a generalization of the above result, consider a finitely deformed solid of any constitutive property which can be represented by (Eqs. (2.1a), (2.1b) and (2.1c)) in terms of the nominal stress rate, $\dot{\mathbf{S}}^N$, and velocity gradient, $\dot{\mathbf{F}}$. Assume eigenvelocity gradients (*variable*), $\dot{\mathbf{F}}^{*1}(\mathbf{X})$, are prescribed in an ellipsoidal region Ω which is contained in another ellipsoidal region V. Assume further that constant eigenvelocity gradients, $\dot{\mathbf{F}}^{*2}$, are prescribed in the remaining part, $V - \Omega$. *Ellipsoids Ω and* V *need be neither coaxial nor similar*. Clearly, the resulting velocity gradient and nominal stress rate fields, $\dot{\mathbf{F}}(\mathbf{X})$ and $\dot{\mathbf{S}}^N(\mathbf{X})$, in general are not constant in V and Ω. Nevertheless, the average values of the velocity gradient and the nominal stress rate in Ω, in V, and in $\Gamma = V - \Omega$ can be calculated exactly, based on the results presented above. In Ω, these are:

$$\langle \dot{\mathbf{F}} \rangle_\Omega = \mathscr{S}^\Omega : \langle \dot{\mathbf{F}}^{*1} \rangle_\Omega + (\mathscr{S}^V - \mathscr{S}^\Omega) : \dot{\mathbf{F}}^{*2}, \tag{3.4a}$$

$$\langle \dot{\mathbf{S}}^N \rangle_\Omega = \mathscr{F} : (\mathscr{S}^\Omega - \mathbf{1}^{(4)}) : \langle \dot{\mathbf{F}}^{*1} \rangle_\Omega + \mathscr{F} : (\mathscr{S}^V - \mathscr{S}^\Omega) : \dot{\mathbf{F}}^{*2}, \tag{3.4b}$$

where $\mathscr{S}^V$ and $\mathscr{S}^\Omega$ are the generalized Eshelby's tensors for ellipsoids V and Ω, respectively. Similarly, the average velocity gradient and nominal stress rate over V are *exactly* given by

$$\langle \dot{\mathbf{F}} \rangle_V = \mathscr{S}^V : \{ f \langle \dot{\mathbf{F}}^{*1} \rangle_\Omega + (1 - f) \dot{\mathbf{F}}^{*2} \}, \tag{3.4c}$$

$$\langle \dot{\mathbf{S}}^N \rangle_V = \mathscr{F} : (\mathscr{S}^V - \mathbf{1}^{(4)}) : \{ f \langle \dot{\mathbf{F}}^{*1} \rangle_\Omega + (1 - f) \dot{\mathbf{F}}^{*2} \}, \tag{3.4d}$$

where f is the volume fraction of Ω in V, $f = \Omega / V$. Finally, the average velocity gradient and nominal stress rate over the annulus Γ are given by

$$\langle \dot{\mathbf{F}} \rangle_\Gamma = \mathscr{S}^V : \dot{\mathbf{F}}^{*2} + \frac{f}{1 - f} (\mathscr{S}^V - \mathscr{S}^\Omega) : (\langle \dot{\mathbf{F}}^{*1} \rangle_\Omega - \dot{\mathbf{F}}^{*2}), \tag{3.4e}$$

$$\langle \dot{\mathbf{S}}^N \rangle_\Gamma = \mathscr{F} : (\mathscr{S}^V - \mathbf{1}^{(4)}) : \dot{\mathbf{F}}^{*2} + \frac{f}{1 - f} \mathscr{F} : (\mathscr{S}^V - \mathscr{S}^\Omega) : (\langle \dot{\mathbf{F}}^{*1} \rangle_\Omega - \dot{\mathbf{F}}^{*2}). \tag{3.4f}$$

It is seen that the results can easily be generalized to a nested sequence of ellipsoidal regions, where, in each of the annuli, a distinct but constant eigenvelocity gradient is prescribed; see Nemat-Nasser [6].

3.2. Effective moduli of heterogeneous finitely deformed elastic–plastic RVE

Now consider using the above exact results to estimate the overall pseudo-moduli of a finitely deformed heterogeneous elastoplastic solid; a composite or a polycrystal. We directly generalize the double-inclusion model of Nemat-Nasser and Hori [3] to finite-deformation problems, and then specialize the results and apply them to other cases. Consider two ellipsoids one, Ω, contained in the other, V, in an unbounded region, with the following pseudo-moduli (see Fig. 3):

$$\mathscr{F} = \mathscr{F}(\mathbf{X}) = \begin{cases} \mathscr{F}^{\Omega} & \mathbf{X} \text{ in } \Omega \\ \mathscr{F}^{M} & \mathbf{X} \text{ in } M \\ \mathscr{F} & \mathbf{X} \text{ outside of } V \end{cases} \tag{3.5}$$

In general, the homogenizing eigenvelocity gradient is not constant. Hence, apply the *consistency conditions* to the *average* field quantities. To this end consider (Eqs. (3.4a), (3.4b), (3.4c) and (3.4d)), and identify $\dot{\mathbf{F}}^{*1}$ with $\bar{\dot{\mathbf{F}}}^{*\Omega}$, and $\dot{\mathbf{F}}^{*2}$ with $\bar{\dot{\mathbf{F}}}^{*M}$, respectively. Let the farfield velocity gradient, $\dot{\mathbf{F}}^{\infty}$, be prescribed. Write the consistency conditions for the average field quantities, as follows:

$$\mathscr{F}^{\Omega}:\{\dot{\mathbf{F}}^{\infty} + \mathscr{S}^{\Omega}:\bar{\dot{\mathbf{F}}}^{*\Omega} + (\mathscr{S}^{V} - \mathscr{S}^{\Omega}):\bar{\dot{\mathbf{F}}}^{*M}\}$$
$$= \mathscr{F}:\{\dot{\mathscr{F}}^{\infty} + (\mathscr{S}^{\Omega} - \mathbf{1}^{(4)}):\bar{\dot{\mathbf{F}}}^{*\Omega} + (\mathscr{S}^{V} - \mathscr{S}^{\Omega}):\bar{\dot{\mathbf{F}}}^{*M}\}$$
$$\mathscr{F}^{M}:\left\{\dot{\mathscr{F}}^{\infty} + \mathscr{S}^{V}:\bar{\dot{\mathbf{F}}}^{*M} + \frac{f}{1-f}(\mathscr{S}^{V} - \mathscr{S}^{\Omega}):(\bar{\dot{\mathbf{F}}}^{*\Omega} - \bar{\dot{\mathbf{F}}}^{*M})\right\}$$
$$= \mathscr{F}:\left\{\dot{\mathbf{F}}^{\infty} + (\mathscr{S}^{V} - \mathbf{1}^{(4)}):\bar{\dot{\mathbf{F}}}^{*M} + \frac{f}{1-f}(\mathscr{S}^{V} - \mathscr{S}^{\Omega}):(\bar{\dot{\mathbf{F}}}^{*\Omega} - \bar{\dot{\mathbf{F}}}^{*M})\right\}, \tag{3.6a,b}$$

Solve these equations for $\bar{\dot{\mathbf{F}}}^{*\Omega}$ and $\bar{\dot{\mathbf{F}}}^{*M}$, and substitute the results into Eqs. (3.4c) and (3.4d). The overall pseudo-modulus tensor. $\bar{\mathscr{F}}$, may be defined through the relation $\langle\dot{\mathbf{S}}^{\mathrm{N}}\rangle_V = \bar{\mathscr{F}}:\langle\dot{\mathbf{F}}\rangle_V$ which gives

$$\bar{\mathscr{F}} = \mathscr{F}:\{\mathbf{1}^{(4)} + (\mathscr{S}^{V} - \mathbf{1}^{(4)}):\mathscr{A}\}:\{\mathbf{1}^{(4)} + \mathscr{S}^{V}:\mathscr{A}\}^{-1}, \tag{3.7a}$$

where $\mathscr{A}$ is defined by

$$f\bar{\dot{\mathbf{F}}}^{*\Omega} + (1-f)\bar{\dot{\mathbf{F}}}^{*M} \equiv \mathscr{A}:\dot{\mathbf{F}}^{\infty}. \tag{3.7b}$$

This last expression defines the average of the variable homogenizing eigenvelocity gradient over V. When V and Ω are similar and coaxial, then $\mathscr{A}$ is given exactly by

$$\mathscr{A} = f(\mathscr{C}^{\Omega} - \mathscr{S})^{-1} + (1-f)(\mathscr{C}^{M} - \mathscr{S})^{-1}, \tag{3.7c}$$

where

$$\mathscr{C}^{\Omega} \equiv (\mathscr{F} - \mathscr{F}^{\Omega})^{-1}:\mathscr{F}, \qquad \mathscr{C}^{M} \equiv (\mathscr{F} - \mathscr{F}^{M})^{-1}:\mathscr{F} \tag{3.7d,e}$$

are the concentration tensors. In this case, the average homogenizing eigenvelocity gradient is *exactly* defined by

$$\bar{\dot{\mathbf{F}}}^{*\Omega} = (\mathscr{C}^{\Omega} - \mathscr{S})^{-1}:\dot{\mathbf{F}}^{\infty}, \qquad \bar{\dot{\mathbf{F}}}^{*M} = (\mathscr{C}^{M} - \mathscr{S})^{-1}:\dot{\mathbf{F}}^{\infty}. \tag{3.7f,g}$$

For this case, the overall pseudo-modulus, $\bar{\mathscr{F}}$, is obtained by substituting $\mathscr{A}$ from Eq. (3.7c) into Eq. (3.7a). The final expression encompasses the results of several commonly used models, now generalized to finite deformations. This includes the self-consistent and the two-phase models, as summarized in what follows.

3.3. Special cases

By specializing the model of Fig. 3, and hence the expression for the tensor $\mathscr{A}$, the results of the double-inclusion model can be reduced to those of other models, generalized to finite deformations.

3.3.1. Self-consistent model

In the self-consistent model, the concentration tensor is obtained by embedding an ellipsoidal inclusion in an infinite homogeneous matrix whose pseudo-modulus tensor is the unknown overall modulus tensor, $\bar{\mathscr{F}}$. To show that the self-consistent model is in fact a special case of the double-inclusion model, let in Fig. 3, $\mathscr{F}$ be equal to $\bar{\mathscr{F}}$, the yet unknown overall modulus tensor of the RVE, and from Eq. (3.7a) obtain

$$f(\bar{\mathscr{C}}^{\Omega} - \bar{\mathscr{S}})^{-1} + (1-f)(\bar{\mathscr{C}}^{M} - \bar{\mathscr{S}})^{-1} = 0. \tag{3.8a}$$

where $\bar{\mathscr{S}}$ is the generalized Eshelby tensor for Ω embedded in an infinite homogeneous matrix of the pseudo-modulus $\bar{\mathscr{F}}$, $\bar{\mathscr{C}}^{\mathrm{M}} = (\bar{\mathscr{F}} - \mathscr{F}^{\mathrm{M}})^{-1}:\bar{\mathscr{F}}$, and $\bar{\mathscr{C}}^{\Omega} = (\bar{\mathscr{F}} - \mathscr{F}^{\Omega})^{-1}:\bar{\mathscr{F}}$. As in the linear case discussed by Nemat-Nasser and Hori ([3], p. 352, 353), it is most remarkable that Eq. (3.8a) yields the self-consistent overall pseudo-modulus tensor. The proof is essentially the same as that originally given be Nemat-Nasser and Hori [3] for the linear elasticity case. Starting with Eq. (3.8a), after some manipulations, obtain

$$\bar{\mathscr{F}} = \mathscr{F}^{M} + f(\mathscr{F}^{\Omega} - \mathscr{F}^{M}) :\{\mathbf{1}^{(4)} + \bar{\mathscr{S}}:[(\bar{\mathscr{F}} - \mathscr{F}^{\Omega})^{-1}:\bar{\mathscr{F}} - \bar{\mathscr{S}}]^{-1}\} \tag{3.8b}$$

which is recognized as the self-consistent result,

$$\bar{\mathscr{F}}^{SC} = \mathscr{F}^{M} + f(\mathscr{F}^{\Omega} - \mathscr{F}^{M}):\{\mathbf{1}^{(4)} + \bar{\mathscr{S}}:(\mathscr{C}^{\Omega} - \bar{\mathscr{S}})^{-1}\}; \tag{3.8c}$$

see Iwakuma and Nemat-Nasser [2]. Note that only when it is assumed that V and Ω are coaxial and similar, that the general double-inclusion model Eqs. (3.7a) and (3.7b) reduces to the self-consistent for $\mathscr{F} = \bar{\mathscr{F}}$. Therefore, the model defined by Eqs. (3.7a) and (3.7b) is more general than the self-consistent one.

3.3.2. Two-phase model

Assume that V and Ω are coaxial and similar, and that $\mathscr{F}^{M} \equiv \mathscr{F}$. This model is shown in Fig. 4. The material properties are defined by

$$\mathscr{F} = \mathscr{F}(\mathbf{X}) = \begin{cases} \mathscr{F}^{\Omega} & \mathbf{X} \text{ in } \Omega \\ \mathscr{F} & \mathbf{X} \text{ in } M \\ \mathscr{F} & \mathbf{X} \text{ outside of } V; \end{cases} \tag{3.9}$$

i.e. V is a subdomain of B, within which a coaxial and similar ellipsoid, Ω, is embedded. In this case, $(\mathscr{C}^{M} - \mathscr{S})^{-1}$ vanishes, and Eqs. (3.7a) and (3.7c) yields

$$\bar{\mathscr{F}}^{TP}=\mathscr{F}:\{\mathbf{1}^{(4)}+f(\mathscr{S}^{\Omega}-\mathbf{1}^{(4)}):(\mathscr{C}^{\Omega}-\mathscr{S}^{\Omega})^{-1}\}:\{\mathbf{1}^{(4)}+f\mathscr{S}^{\Omega}:(\mathscr{C}^{\Omega}-\mathscr{S}^{\Omega})^{-1}\}^{-1}. \tag{3.10a}$$

This is the result of the Mori–Tanaka model, generalized and applied to finite deformations. Note that the mathematical derivation is *exact*, in the sense that

$$\langle\dot{\mathbf{S}}^{\mathrm{N}}\rangle_V=\bar{\mathscr{F}}^{TP}:\langle\dot{\mathbf{F}}\rangle_V. \tag{3.10b}$$

exactly.

3.3.3. General comments

With a similar approach which led to the effective pseudo-modulus tensor defined by Eq. (3.7a) for the double-inclusion model, the overall pseudo-compliance tensor, $\bar{\mathscr{G}}$, can be formulated. The result, however, is simply the inverse of $\bar{\mathscr{F}}$. For the linear case, this model is proposed by Nemat-Nasser and Hori ([3], pp. 351–353) who point out that the resulting overall elasticity and compliance tensors:

1. are each other's inverse;
2. do not depend on the surface data on ∂V; and
3. do not depend on the location of Ω relative of V.

These conclusions also apply to the special cases of the two-phase and the self-consistent models; see Nemat-Nasser and Hori ([3], pp. 341–343). They remain valid for the general nonlinear and finite-deformation case presented in this paper.

4. Multi-inclusion model

Similarly to the linear case developed by Nemat-Nasser and Hori ([3], p. 353), the results of the double-inclusion model can be generalized and applied to an ellipsoid, V, which contains a nested series of ellipsoids, Ω_α ($\alpha=1, 2, \ldots, m$), such that $\Omega_1\subset\Omega_2\subset\ldots\subset\Omega_m\equiv V$. The ellipsoid Ω_1 and each annular region $\Gamma_\alpha=\Omega_\alpha-\Omega_{\alpha-1}$ have uniform elastoplastic properties, defined by the corresponding pseudo-modulus tensor, $\mathscr{F}^{\alpha}$, $\alpha=1, 2, \ldots, m$. The multi-inclusion V is embedded in an infinite domain of uniform modulus tensor $\mathscr{F}$. The properties of the composite, thus are defined by

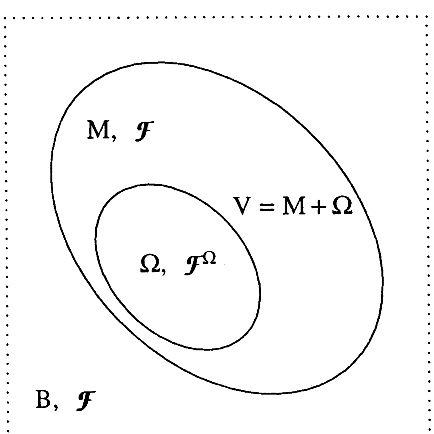

Fig. 4. A double inclusion $V=M+\Omega$ is embedded in a uniform infinite domain B; elastic–plastic mudul;us tensor of Ω is $\mathscr{F}^{\Omega}$, that of $M=V-\Omega$ and the infinite domain is $\mathscr{F}$; this is a two phase model.

$$\mathscr{F}=\mathscr{F}(\mathbf{X})=\begin{cases}\mathscr{F}^{1} & \text{if } \mathbf{X} \text{ in } \Omega_1\\ \mathscr{F}^{\alpha} & \text{if } \mathbf{X} \text{ in } \Gamma_\alpha \quad (\alpha=2, 3, \ldots, m)\\ \mathscr{F} & \text{otherwise,}\end{cases} \tag{4.1}$$

where $\mathbf{X}$ is the reference particle position.

Applying the average consistency condition, as before, it is easy to show that the volume average of the nominal stress rate and the velocity gradient, $\langle\dot{\mathbf{S}}^{\mathrm{N}}\rangle_V$ and $\langle\dot{\mathbf{F}}\rangle_V$, are related by the modulus tensor $\bar{\mathscr{F}}$ which is given by Eq. (3.7a) in which the tensor $\mathscr{A}$ is now defined by

$$\sum_{\alpha=1}^{m} f_\alpha\bar{\dot{\mathbf{F}}}^{*\alpha}\equiv\mathscr{A}:\dot{\mathbf{F}}^{\infty}. \tag{4.2a}$$

The same result is obtained for an ellipsoidal V which contains $m-1$ ellipsoidal heterogeneities Ω_α ($\alpha=1, 2, \ldots, m-1$). The remaining part of V defines the region Γ. The properties of this multi-inclusion system are given by

$$\mathscr{F}(\mathbf{X})=\begin{cases}\mathscr{F}^{\alpha} & \text{if } \mathbf{X} \text{ in } \Omega_\alpha \quad (\alpha=2, 3, \ldots, m-1)\\ \mathscr{F}^{m} & \text{if } \mathbf{X} \text{ in } \Gamma\\ \mathscr{F} & \text{otherwise.}\end{cases} \tag{4.3}$$

The volume fractions of Ω_α and Γ are defined by $f_\alpha=\Omega_\alpha/V$ ($\alpha=1, 2, \ldots, m-1$) and $f_m=\Gamma/V$. For each $\Sigma_{\beta\neq\alpha}^{m}\{f_\beta/(1-f_\alpha)\}\bar{\dot{\mathbf{F}}}^{*\beta}$, are uniformly distributed, and the corresponding average velocity gradient is estimated. From these the required homogenizing eigenvelocity gradients are obtained, leading eventually to the overall pseudo-modulus tensor given by Eqs. (3.7a) and (3.7b).

When in either multi-inclusion model, all ellipsoids are similar and coaxial, then Eq. (4.2a) reduces to

$$\mathscr{A}=\sum_{\alpha=1}^{m} f_\alpha(\mathscr{C}^{\alpha}-\mathscr{S})^{-1}. \tag{4.2b}$$

Hence, the overall effective pseudo-modulus tensor becomes,

$$\bar{\mathscr{F}}=\mathscr{F}:\left\{\mathbf{1}^{(4)}+\sum_{\alpha=1}^{m} f_\alpha(\mathscr{S}-\mathbf{1}^{(4)}):(\mathscr{C}^{\alpha}-\mathscr{S})^{-1}\right\}:\left\{\mathbf{1}^{(4)}+\sum_{\alpha=1}^{m} f_\alpha\mathscr{S}:(\mathscr{C}^{\alpha}-\mathscr{S})^{-1}\right\}^{-1}. \tag{4.4}$$

Acknowledgements

This work was supported in part by the US Army Research Office under contract No. ARO MURI DAAH04-96-1-0376 and in part by NSF CMS 9729053 to the University of California, San Diego.

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