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MULTI-INDEX LE ROY FUNCTIONS OF MITTAG-LEFFLER-PRABHAKAR TYPE

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Dedicated to Professor Francesco Mainardi
on the occasion of his 80th anniversary

Abstract: The so-called Special Functions of Fractional Calculus (SF of FC) became important tools of FC, as solutions of fractional order differential and integral equations and systems that model various phenomena from natural and applied sciences, and social events. Among them, recently we have introduced and studied the multi-index analogues of the Mittag-Leffler function that include a very long list of SF of FC.

As a next step, here we introduce multi-index analogues of the Mittag-Leffler-Prabhakar type Le Roy functions (abbr. as multi-MLPR) with $4m$-indices. We emphasize on the relations between the multi-index Mittag-Leffler functions, Prabhakar function, Le Roy type functions and multi-index Mittag-Lefler functions of Le Roy type, and also with the hyper-Bessel functions and other SF, appearing as eigenfunctions of fractional analogues of the hyper-Bessel operators of Dimovski.

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1. Introduction: SF of FC and multi-index Mittag-Leffler functions

The developments in theoretical and applied science have always required a knowledge of the properties of the "mathematical functions" (in terms of H. Bateman, [5]), from elementary trigonometric functions to the variety of Special Functions (SF), appearing in studies of natural and social phenomena, in formulation of engineering problems, and numerical simulations. The well known "Classical SF" (SF of Mathematical Physics, or Named SF) appear as solutions of differential and integral equations of integer order, mainly of 2nd order, but also of higher (integer) ones. Most of them are representable as particular case of the Meijer G-functions (see e.g. [5, Vol.1], [31]).

With the revival of the FC as not only an exotic theory and the recognition that the fractional order models can describe better the fractal nature or the world, the solutions of the fractional order differential and integral equations and systems also gained their important place and became unavoidable tools. These are the so-called "Special Functions of Fractional Calculus" (SF of FC), in the general case as Fox H-functions (details in [31], [14], [13], etc.).

Among the SF of FC (see Kiryakova [19]), the basic role have the generalized Wright hypergeometric functions ${}_p\Psi_q$,

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{array}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\,\frac{z^k}{k!} \tag{1}$$

$$ = H^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{array}{c}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{array}\right]. \tag{2}$$

Let us denote: $\rho=\prod\limits_{i=1}^{p}A_i^{-A_i}\prod\limits_{j=1}^{q}B_j^{B_j}$, $\Delta=\sum\limits_{k=1}^{j}B_j-\sum\limits_{i=1}^{p}A_i$. If $\Delta>-1$, the ${}_p\Psi_q$-function is an entire function of z, $z\in\mathbb{C}$, and if $\Delta=-1$, this series is absolutely convergent in the disk $\{|z|<\rho\}$, while for $|z|=\rho$ if $\Re(\mu)=\Re\left\{\sum\limits_{j=1}^{q}b_j-\sum\limits_{i=1}^{p}a_i+\frac{p-q}{2}\right\}>1/2$. For more details on special functions (1) and the Fox H-function, see in [31], [13], [14], etc. In particular, for $A_1=\dots=A_p=1, B_1=\dots=B_q=1$ in (1) and (2), the Wright g.h.f. reduces to the more popular generalized hypergeometric ${}_pF_q$-function, which is a "classical" SF representable as a Meijer's G-function (for details, see e.g. [5, Vol.1]):

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,1)\end{array}\middle| z\right]=c\,{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;\,z)$$

$$ =c\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\,\frac{z^k}{k!} \tag{3}$$

$$= G^{1,p}_{p,q+1}\left[-z\,\middle|\, \begin{array}{c} 1-a_1,\ldots,1-a_p \\ 0,1-b_1,\ldots,1-b_q \end{array}\right];$$

$$\text{with} \quad c=\left[\prod_{i=1}^{p}\Gamma(a_i)\;/\;\prod_{j=1}^{q}\Gamma(b_j)\right],\ (a)_k := \Gamma(a+k)/\Gamma(a).$$

Among the most popular examples of generalized hypergeometric functions that are not "classical" but SF of FC (with irrational parameters A_j, B_k in (1)), is the so-called Queen function of FC, the Mittag-Leffler (ML) function E_α (Mittag-Leffler, 1902-1905), resp. the more general (2-parametric) one $E_{\alpha,\beta}(z)$ (Wiman 1905, Agarwal, 1953), studied later also by Dzrbashjan (1954, 1960):

$$E_{\alpha,\beta}(z)=\sum_{0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad E_{\alpha,1}(z):=E_\alpha(z),\quad \alpha>0,\,\beta>0. \tag{4}$$

This is an entire function of order $\rho=1/\alpha$ (and the simplest entire one with this order) and type $\sigma=1$. The ML functions (4) have been almost ignored, for a long time, in the handbooks on SF and in existing tables of Laplace transforms. They only appeared shortly in a chapter "Miscellaneous functions" in [5, Vol.3]. Nowadays, with the flourishing of the FC tools and models, there exist a very vast range of literature on the theory of ML functions and their applications as solutions of fractional order equations and systems, just to mention a few, as the books [29], [13], [10], [23], a lot of surveys as [33], [19], etc.

These "fractional exponential functions" are natural extensions of the exponential and trigonometric functions ($\alpha=1$, $\alpha=2$), for example of $E_1(z)=\exp(z)$ and $E_2(-z^2)=\cos z$ satisfying integer (1st and 2nd) order differential equations. However, the "true" ML functions (with irrational α) are solutions of fractional order DEs. Such a simplest example is with the α-exponential (Rabotnov) function:

$$D^\alpha y(\lambda z)=\lambda y(\lambda z)\ \text{ with }\ y(z)=z^{\alpha-1}E_{\alpha,\alpha}(z^\alpha).$$

Other typical examples where the solutions appear in terms of combinations of ML functions (4) are the Cauchy problem for R-L fractional differential equation: $D^\alpha y(z)-\lambda y(z)=f(z)$, with either Riemann-Liuoville D^α or Caputo ${}^C D_\alpha$ fractional derivatives and initial conditions of the corresponding (R-L or C-) type.

A ML type function with 3 parameters, known as the Prabhakar function (T.R. Prabhakar, [30]) is also often studied and used as a FC tool, for $\alpha,\beta,\tau\in\mathbb{C}$, $\operatorname{Re}\alpha>0$ and with the Pochhammer symbol $(\tau)_0=1,(\tau)_k=\Gamma(\tau+k)/\Gamma(\tau)$:

$$E^{\tau}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\tau)_k}{\Gamma(\alpha k+\beta)}\,\frac{z^k}{k!}\,. \tag{5}$$

It is again an entire function, of order and type, which are the same as for $E_{\alpha,\beta}$ and E_α. For $\tau=1$ we get the ML function $E_{\alpha,\beta}$, and if also $\beta=1$, it is E_α.

These ML type functions are simple cases of the Wright g.h.f. (1) and of the H-function, namely:

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\middle| z\right] = H^{1,1}_{1,2}\left[-z\middle|\begin{array}{c}(0,1)\\(0,1),(1-\beta,\alpha)\end{array}\right],$$

$$E^{\tau}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)}{}_1\Psi_1\left[\begin{array}{c}(\tau,1)\\(\beta,\alpha)\end{array}\middle| z\right] = H^{1,1}_{1,2}\left[-z\middle|\begin{array}{c}(1-\tau,1)\\(0,1),(1-\beta,\alpha)\end{array}\right].$$

Here we are focusing first on the multi-index (multi-parametric) Mittag-Leffler (multi-ML) functions introduced almost simultaneously (1994-1996) by Luchko et al. [38], [1] and Kiryakova [15], then studied by Kiryakova [16], [17], [18], Kilbas-Koroleva-Rogosin [12], Paneva-Konovska [23], [24], etc. They form indeed a very large class of the Wright generalized hypergeometric functions ${}_p\Psi_q$, incorporating long list of SF of FC (see e.g. [18], [19]).

Namely, these extensions of the classical ML functions (4), have been introduced by replacing the 2 parameters α and β by 2 sets (vector indices, $2m$ indices): $(\alpha_1,...,\alpha_m)$ and $(\beta_1,...,\beta_m)$, $\alpha_i>0$, $\beta_i\in\mathbb{R}$, with integer $m\geq 1$. These multi-index ML functions include many of the SF of FC as particular cases and appear as solutions of fractional order problems of multi-order $(\alpha_i)_1^m$, $m\geq 1$:

$$E_{(\alpha_i),(\beta_i)}(z):=E^{(m)}_{(\alpha_i),(\beta_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha_1k+\beta_1)\ldots\Gamma(\alpha_mk+\beta_m)} \tag{6}$$
$$={}_1\Psi_m\left[\begin{array}{c}(1,1)_1^m\\(\beta_i,\alpha_i)_1^m\end{array}\middle| z\right]=H^{1,1}_{1,m+1}\left[-z\middle|\begin{array}{c}(0,1)_1^m\\(0,1)_1^m,(1-\beta_i,\alpha_i)_1^m\end{array}\right].$$

Later, Paneva-Konovska [24], [23] extended (6) to multi-index analogues of the Prabhakar function (5), by $3m$ parameters, with additional $(\tau_1,...,\tau_m)$ instead of τ:

$$E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z)=\sum_{k=0}^{\infty}\frac{(\tau_1)_k...(\tau_m)_k}{\Gamma(\alpha_1k+\beta_1)...\Gamma(\alpha_mk+\beta_m)}\frac{z^k}{(k!)^m} \tag{7}$$
$$=\frac{1}{A}\,{}_m\Psi_{2m-1}\left[\begin{array}{c}(\tau_i,1)_1^m\\(\beta_i,\alpha_i)_1^m,(1,1),...,(1,1)\end{array}\middle| z\right]=\frac{1}{A}\,H^{1,m}_{m,2m}(-z),$$

where $(\tau_i)_k=\Gamma(\tau_i+k)/\Gamma(\tau_i)$ are the Pochhammer symbols, $A=\left[\prod_{i=1}^{m}\Gamma(\tau_i)\right]$. Evidently, the case $\forall\ \tau_i=1$, $i=1,...,m$, leads to the above $2m$ parameter ML type function (6), while for $m=1$ we get the Prabhakar function (5).

We note also that Kilbas-Rogosin-Koroleva [12] extended the $2m$ index ML functions $E^{(m)}_{(\alpha_i),(\beta_i)}(z)$ with all $\alpha_i > 0$ to the case when all α_i, $i = 1, ..., m$ can be real and different from zero, even negative.

The studies on multi-index (vector) ML functions have also a trend including a multivariable case (with $\beta > 0, \alpha_i > 0$, integers $l_i \geq 0$, $i = 1, ..., m$), introduced by Luchko et al. (see e.g. [38]) for the purposes of operational calculus; and later applied in many other works to present solutions of BVPs for fractional order PDEs:

$$E_{(\alpha_1,...,\alpha_m),\beta}(z_1, ..., z_m) = \sum_{k=0}^{\infty} \sum_{l_1+...+l_m=k} (k; l_1, ..., l_m) \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma(\beta + \sum_{i=1}^{m} \alpha_i l_i)}.$$

Here $(k; l_1, ..., l_m) = k!/(l_1)!...(l_m!)$ are the multinomial coefficients.

Some short facts of the basic theory of the multi-index ML functions (6) and (7), from Kiryakova [16], [17], [18], Paneva-Konovska [24], [23], [25] and Paneva-Konovska and Kiryakova [27], include the following.

Theorem 1. *The multi-index ML functions* (6) *and* (7) (*with real parameters* $\alpha_i > 0$) *are entire functions of order* ρ *with*

$$\frac{1}{\rho} = \alpha_1 + ... + \alpha_m, \tag{8}$$

and type σ*:*

$$1/\sigma = (\rho\alpha_1)^{\rho\alpha_1} \dots (\rho\alpha_m)^{\rho\alpha_m} \quad (\sigma > 1 \text{ for } m > 1). \tag{9}$$

Moreover, for each $\varepsilon > 0$ *we have an asymptotic estimate, as:*

$$|E_{(\alpha_i),(\beta_i)}(z)| \leq \exp\left((\sigma + \varepsilon)|z|^{\rho}\right), \ |z| \geq r_0 > 0,$$

with ρ, σ *as above,* $r_0(\varepsilon)$ *sufficiently large.*

We have shown that the multi-ML ($2m$-parametric) functions (6) are eigenfunctions of the Gelfond-Leontiev (G-L, 1951) operators of generalized differentiation that we generated by means of the coefficients of these entire functions. In fact, this was the inspiring idea to introduce this class of special functions in Kiryakova [15]. Namely, the multi-index ML function $E_{(\alpha_i),(\beta_i)}(z)$ satisfies a differential equation of multi-order ($\alpha_1 > 0, ..., \alpha_m > 0$) of a general form as:

$$D_{(\alpha_i),(\beta_i)} E_{(\alpha_i),(\beta_i)}(\lambda z) = \lambda E_{(\alpha_i),(\beta_i)}(\lambda z), \quad \lambda \neq 0,$$

see [16], etc. We note that the G-L operator of generalized differentiation $D_{(\alpha_i),(\beta_i)}$ with respect to (generated by) $E_{(\alpha_i),(\beta_i)}(z)$ is practically a generalized fractional derivative of the form $z^{-1}\, D^{(\beta_i-\alpha_i-1),(\alpha_i)}_{(1/\alpha_i),m}$ (in sense of Kiryakova [14]), and also a fractional order extension of the hyper-Bessel operators of Dimovski (see [3], [14, Ch.3]).

Other useful properties of the multi-ML functions, aside from their representations in terms of the Wright g.h.f. and H-functions (6), (7) (see Kiryakova [18], Paneva-Konovska [24], Paneva-Konovska and Kiryakova [27] for proofs and conditions on the parameters), include results related to Mellin-Barnes type integral representation and Mellin transform, namely:

Lemma 1. *The following Mellin-Barnes type integral representation holds,* $z \neq 0$*:*

$$E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z) = \frac{1}{2\pi i \prod\limits_{i=1}^{m} \Gamma(\tau_i)} \int\limits_{\mathcal{L}} \frac{\Gamma(s) \prod\limits_{i=1}^{m} \Gamma(\tau_i - s)}{[\Gamma(1-s)]^{m-1} \prod\limits_{i=1}^{m} \Gamma(\beta_i - s\alpha_i)} (-z)^{-s} ds, \quad (10)$$

where the path $\mathcal{L}$ is suitably chosen to separate all the poles of the Gamma-functions in denominator. This is based on the Mellin transforms of the multi-index ML functions, see [13, p.48], [18], and in the general $3m$-case, [27], for $0 < \Re(s) < \min \tau_i$*:*

$$\mathcal{M}\left\{ E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(-z); s \right\} = \frac{1}{\prod\limits_{i=1}^{m} \Gamma(\tau_i)} \frac{\Gamma(s) \prod\limits_{i=1}^{m} \Gamma(\tau_i - s)}{[\Gamma(1-s)]^{m-1} \prod\limits_{i=1}^{m} \Gamma(\beta_i - s\alpha_i)}. \quad (11)$$

Note that the choice of the contour $\mathcal{L}$ and the inclusion of the additional "Pochhammer parameters" τ_i, $i = 1, ..., m$ cause the difference observed in the same kind of formulas for the $2m$-parametric case (6) in Kiryakova [17], [18], as well as in the recent work by Rogosin-Dubatovskaya [34] on multi-index Mittag-Leffler analogues of Le Roy type.

Many of the elementary and special functions are particular cases of $E_{\alpha,\beta}$, $E^{\tau}_{\alpha,\beta}$, and much more – of the multi-index ML functions $E_{(\alpha_i),(\beta_i)}(z)$ and $E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}$. All details and long lists of examples can be found in the mentioned our previous works.

2. Mittag-Leffler functions of Le Roy type

However, recently, there appeared several works on further extensions of the Mittag-Leffler type functions, as "fractional indices" analogues of the Le Roy function [21], [22]. These special functions have been introduced almost simultaneously by Gerhold [9], Garra-Polito [7], and are referred to as "Gerhold" or "Garra-Polito" or "Mittag-Leffler type Le Roy functions" (abbr. MLR-functions). Extensive studies on the basic properties of the MLR-functions and some of their applications are presented next also by Garrappa-Rogosin-Mainardi [8], Tomovski-Mehrez [36], Garra-Orsingher-Polito [6], Pogany [28], Gorska-Horzela-Garrappa [11], Kolokoltsov [20], Simon [35], etc.

Namely, Gerhold [9] and Garra–Polito [7] have introduced a new function related to the special functions of the Mittag-Leffler family, as a generalization of the function

$$F_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma} = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^\gamma}, \quad \gamma > 0, \tag{12}$$

studied by É. Le Roy in the period 1895-1905 ([21], [22]) in connection with the problem of analytic continuation of power series with a finite radius of convergence. In the mentioned recent works [9], [7], [8], the so-called Mittag-Leffler type Le Roy function has been introduced and studied:

$$F^{(\gamma)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^\gamma}, \quad z \in \mathbb{C},\ \alpha, \beta, \gamma \in \mathbb{C}. \tag{13}$$

These authors have proved that (13) is an entire function for all $\Re\alpha > 0, \beta \in \mathbb{R}, \gamma > 0$ and studied some properties, then some next deeper results on (13) have been extended also in [6], [11], [35].

While developing the idea and some results on the special functions (16) from the next section, and revising this final version, we have been informed by private communication for a next step done towards a multi-index extension of Le Roy function (13) in the sense of multi-index Mittag-Leffler functions (6). Namely, in [34], Rogosin and Dubatovskaya considered the following function (we may refer to it shortly as multi-MLR function):

$$F^{(\gamma)_m}_{(\alpha,\beta)_m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod\limits_{j=1}^{m} [\Gamma(\alpha_j k+\beta_j)]^{\gamma_j}}, \quad z \in \mathbb{C}, \tag{14}$$

with vector parameters (multi-indices) $(\alpha, \beta)_m = (\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)$, $(\gamma)_m = (\gamma_1, ..., \gamma_m)$, supposing $\forall j : \alpha_j \neq 0$, and some other conditions are specified along with their study. As expected, it is proven that (14) is an entire function and its order ρ and type σ are evaluated. Supposing $\Re(\alpha_j \cdot \gamma_j) > 0$, $\beta_j \in \mathbb{C}$, $j = 1, 2, ..., m$, the results in [34] read as:

$$1/\rho = \sum_{j=1}^{m} \Re(\alpha_j \cdot \gamma_j), \quad 1/\sigma = \rho \cdot \exp\left\{ \rho \sum_{j=1}^{m} \Re(\alpha_j \cdot \gamma_j \cdot \log \alpha_j) \right\}. \tag{15}$$

Some particular cases clarifying the nature of these multi-MLR functions and Mellin-Barnes type integral representations are also provided in [34].

3. Multi-index analogues of Mittag-Leffler-Prabhakar functions of Le Roy type

Now we consider the following extension of the functions (13) (as well as of (14)), to combine the multi-index Mittag-Leffler functions (6) (with $2m$ parameters instead of the two: α and β), their Prabhakar variant (7) (with $3m$ parameters, replacing τ by m-set), and the fractional Le Roy type functions (by taking m different parameters γ_i instead of the γ in (13)).

Definition. We introduce the following class of multi-index Mittag-Leffler-Prabhakar functions of Le Roy type (abbrev. as multi-MLPR):

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(z)$$

$$= \sum_{k=0}^{\infty} \frac{(\tau_1)_k \dots (\tau_m)_k}{(k!)^m} \cdot \frac{z^k}{[\Gamma(\alpha_1 k + \beta_1)]^{\gamma_1} \dots [\Gamma(\alpha_m k + \beta_m)]^{\gamma_m}} \tag{16}$$

$$= \sum_{k=0}^{\infty} c_k \, z^k, \quad \text{with } c_k = \prod_{i=1}^{m} \left\{ \frac{\Gamma(k+\tau_i)}{\Gamma(k+1)} \cdot \frac{1}{\Gamma(\tau_i)} \cdot \frac{1}{[\Gamma(\alpha_i k + \beta_i)]^{\gamma_i}} \right\},$$

with $4m$ different parameters. For simplicity of expressions, in this paper we suppose $\forall i = 1, ..., m$: $\alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0$, although a case for $\Re(\alpha_i) > 0, \Re(\tau_i) > 0$, $\beta_i \in \mathbb{C}$ can be also considered.

As already mentioned, almost in parallel, Rogosin and Dubatovskaya [34] have studied the functions (14), that can be considered as the above functions (16) when $\forall i = 1, ..., m : \tau_i = 1$.

3.1. Basic properties of the multi-MLPR functions

Theorem 2. *The multi-index MLPR-function* (16) *is an entire function of the complex variable z of order ρ and type σ, evaluated as follows:*

$$\rho = \frac{1}{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}, \quad \frac{1}{\rho} = \alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m, \tag{17}$$

and

$$\sigma = \frac{1}{\rho}\left(\prod_{i=1}^{m}(\alpha_i)^{-\alpha_i\gamma_i}\right)^{\rho}, \quad \frac{1}{\sigma} = \prod_{i=1}^{m}\left((\rho\alpha_i)^{\rho\alpha_i}\right)^{\gamma_i}, \tag{18}$$

that is,

$$\sigma = \frac{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}{(\alpha_1^{\alpha_1\gamma_1}\cdots\alpha_m^{\alpha_m\gamma_m})^{1/(\alpha_1\gamma_1+\cdots+\alpha_m\gamma_m)}}, \tag{19}$$

$$\frac{1}{\sigma} = \frac{(\alpha_1^{\alpha_1\gamma_1}\cdots\alpha_m^{\alpha_m\gamma_m})^{1/(\alpha_1\gamma_1+\cdots+\alpha_m\gamma_m)}}{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}.$$

Moreover, for each positive ε the asymptotic estimate

$$|\,\mathbb{F}_m(z)\,| < \exp\left((\sigma+\varepsilon)|z|^{\rho}\right), \quad |z| \geq r_0 > 0, \tag{20}$$

holds, with ρ and σ like in (17) *and* (19), *for $|z| \geq r_0(\varepsilon)$, and $r_0(\varepsilon)$ being sufficiently large.*

Let us remind that for simplicity, here we have limited ourselves to the case of real positive parameters $\alpha_i, \beta_i, \tau_i, \gamma_i,\ i = 1, 2, ..., m$.

Proof. Since the proof goes analogously to this one for the Prabhakar function of Le Roy type in Paneva-Konovska [26], only the idea is given here. The details are omitted and will be exposed elsewhere, also for not so restrictive conditions on the parameters.

The theorem is proved using mainly Stirling's asymptotic formula for the Γ-function along with Γ-functions quotient property (see e.g. [23, Remark 6.5]). By the Cauchy-Hadamard formula, the radius of convergence of the series (16) is obtained to be ∞, i.e. the function (16) is an entire function. The order ρ and type σ are obtained by the formulae

$$\rho = \limsup_{k\to\infty}\frac{k\ln k}{\ln(1/|c_k|)} \quad \text{and} \quad (\sigma e\rho)^{1/\rho} = \limsup_{k\to\infty}\left(k^{1/\rho}|c_k|^{1/k}\right), \tag{21}$$

taking the coefficients c_k from (16).

The asymptotic estimate (20) follows from the definitions of order and type of an entire function. □

3.2. Integral representations

In the previous section we have discussed that the multi-index MLPR-function $\mathbb{F}_m(z) = \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(z)$, defined by (16), is an entire function, and its order and type have been given. Separating our considerations in two cases, we begin with this one, when the parameters γ_i are all positive integers, i.e. $\gamma_i \in \mathbb{N} = \{1, 2, \dots\}$, for $i = 1, \dots, m$. Then, all the functions $\Gamma(\beta_i - s\alpha_i)$ are meromorphic with simple poles $s_{i,k} = (\beta_i + k)/\alpha_i$ ($k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$). Further, an integral representation of (16) is given by means of Mellin–Barnes integral, extending the integral formula obtained by Paneva-Konovska [26].

For this purpose, we first consider the sets

$$S_l = \{s : s = -k \quad (k \in \mathbb{N}_0)\},$$

and

$$S_r = \{s : s = l + \tau_i, \quad l \in \mathbb{N}_0,\, \tau_i > 0 \ (i = 1, \dots, m)\}.$$

Remark 1. *The intersection of the sets S_l and S_r is empty, i.e. $S_l \cap S_r = \varnothing$. Moreover, the set S_l lies on the left hand side of the strip*

$$S = \{s : s \in \mathbb{C}, \quad 0 < \Re(s) < \min_{i=1\div m} (\tau_i)\}, \tag{22}$$

while the set S_r lies on its right.

Theorem 3. *Let $\alpha_i > 0$, $\beta_i > 0, \tau_i > 0$ and $\gamma_i \in \mathbb{N}$, $i = 1, ..., m$. Then the multi-index MLPR-function* (16) *is expressed by the following Mellin–Barnes-type contour integral representation:*

$$\mathbb{F}_m(z) = \frac{1}{2\pi i\, A} \int_{\mathcal{L}} \mathcal{H}_m(-s)(-z)^{-s} ds, \quad A = \prod_{i=1}^{m} \Gamma(\tau_i),\ |arg(-z)| < \pi, \tag{23}$$

where

$$\mathcal{H}_m(-s) = \mathcal{H}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(-s) = \frac{\Gamma(s) \prod\limits_{i=1}^{m} \Gamma(\tau_i - s)}{[\Gamma(1-s)]^{m-1} \prod\limits_{i=1}^{m} [\Gamma(\beta_i - s\alpha_i)]^{\gamma_i}}, \tag{24}$$

and $\mathcal{L}$ is an arbitrary contour in $\mathbb{C}$ running from $-i\infty$ to $+i\infty$ in a way that the poles $s = -k$ ($k \in \mathbb{N}_0$) of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles $s = l + \tau_i$ ($l \in \mathbb{N}_0$) of $\Gamma(\tau_i - s)$ to the right of it.

Proof. According to Remark 1, none of the poles of $\Gamma(s)$ and $\Gamma(\tau_i - s)$ are in the strip S, defined by (22). Moreover, the poles $s = -k$ ($k \in \mathbb{N}_0$) of $\Gamma(s)$ lie to the left of this strip and the poles $s = l + \tau_i$ ($l \in \mathbb{N}_0$) of $\Gamma(\tau_i - s)$ to its right.

Let us consider the right hand side of (23)

$$I(z) = \frac{1}{2\pi i\, A} \int_{\mathcal{L}} \mathcal{H}_m(-s)(-z)^{-s} ds. \tag{25}$$

Taking into account the asymptotic formula (see e.g. [5, Vol.1, 1.1.(8)]):

$$\Gamma(s) = \frac{(-1)^k}{k!(s+k)}[1 + O(s+k)] \quad (s \to -k;\ k = 0, 1, 2, \dots), \tag{26}$$

and calculating the residues of the integrand of (25) at the simple poles $s_k = -k$, $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} I(z) &= \frac{1}{A}\sum_{k=0}^{\infty} \mathrm{Res}_{s=-k}\left\{\mathcal{H}_m(-s)(-z)^{-s}\right\} \\ &= \frac{1}{A}\sum_{k=0}^{\infty} \lim_{s\to -k}\left\{(s+k)\,\mathcal{H}_m(-s)(-z)^{-s}\right\} \\ &= \frac{1}{A}\sum_{k=0}^{\infty} \frac{(-1)^k(-z)^k \prod_{i=1}^{m}\Gamma(\tau_i + k)}{k!\,[\Gamma(k+1)]^{m-1}[\Gamma(\beta_i + k\alpha_i)]^{\gamma_i}} \\ &= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{m}(\tau_i)_k}{\prod_{i=1}^{m}[\Gamma(\beta_i + k\alpha_i)]^{\gamma_i}}\, \frac{z^k}{(k!)^m}, \end{aligned}$$

that proves (23). □

Remark 2. *Let us note that in the particular case* $m = 1$ *and* $\tau_1 = \tau = 1$, *Theorem 3 gives the result referring to Le Roy type functions* (13) *from the mentioned articles, and for arbitrary* τ *it is the result from Paneva-Konovska [26]. If additionally* $\gamma = 1$, *the result obtained is related to the Mittag-Leffler function* $E^{\tau}_{\alpha,\beta}$, *see for example [13]. The case* $\forall \tau_i = \gamma_i = 1$ *leads to the result, obtained in Kiryakova [17] for* $E_{(\alpha_i),(\beta_i)}(z)$, (6).

The Mellin transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by

$$(\mathcal{M}f)(s) = \mathcal{M}[f(t)](s) = F(s) = \int_0^\infty f(t)t^{s-1}dt \quad (s \in S \subset \mathbb{C}), \tag{27}$$

S is a suitable vertical strip (see, for example, the book by Titchmarsh [37]), and the inverse Mellin transform is given for $t \in \mathbb{R}^+$ by the formula ($\nu = Re(s)$):

$$(\mathcal{M}^{-1}F)(t) = \mathcal{M}^{-1}[F(s)](t) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} F(s)t^{-s}ds, \quad 0 < t < \infty, \tag{28}$$

where the integral is understood in the sense of the Cauchy principal value. For more detailed information on the Mellin integral transform, its properties and applications, we refer the reader to the classical books with tables of integral transforms.

In what follows we use the results of Theorem 3. By setting $-z = t$ and having in mind that $0 < t < \infty$, we see that $|\arg(-z)| = |\arg(t)| = 0 < \pi$. That is why the representation (23) holds true with $-z = t$.

In this way, we can formulate the results for the Mellin transform image of the Le Roy-type function (16).

Theorem 4. *Let the parameters α_i, β_i, and τ_i be positive, and let γ_i be positive integers, i.e. $\gamma_i \in \mathbb{N}$, for $i = 1, 2, \ldots, m$. Then the Mellin transform of the multi-index MLPR-function* (16) *is expressed as follows*

$$\mathcal{M}[\mathbb{F}_m(-t)](s) = \frac{\mathcal{H}_m(-s)}{A} \quad (0 < \Re(s) < \min_{i=1\div m}(\tau_i)), \tag{29}$$

with $t > 0$.

Proof. In particular, if $\mathcal{L}$ is the straight line $Re(s) = \nu$, lying down the stripe (22), and taking $-z = t \in (0, \infty)$, then the relation (23) leads to

$$\mathbb{F}_m(-t) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{\mathcal{H}_m(-s)}{A}t^{-s}ds, \quad 0 < t < \infty. \tag{30}$$

The relation (30) means that the function $\mathbb{F}_m(-t)$ is the inverse Mellin transform of the function $\frac{\mathcal{H}_m(-s)}{A}$. Therefore the direct Mellin transform of the function $\mathbb{F}_m(-t)$ is given by (29), which is the desired result. □

Remark 3. *Let us note that in the particular case when all $\gamma_i = 1$, Theorem 4 gives the result for the multi-index Mittag-Leffler function of Prabhakar type $E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z)$. If additionally, all $\tau_i = 1$, this result concerns the*

$2m$-parametric function $E_{(\alpha_i),(\beta_i)}(z)$. Both results are presented in [27]. In the case $m = 1$, Theorem 4 gives the result for the Prabhakar function of Le Roy type (35), *obtained recently in [26].*

Now, we consider the other case, when at least one γ_i is not necessarily an integer. Then we establish two more integral representations, using the above techniques. For the sake of simplicity, further in this section we consider again the parameters α_i, β_i, γ_i, τ_i to be positive. In this case the function $\Gamma(\alpha_i s + \beta_i)$ is a meromorphic function of the complex variable s with simple poles at the points $s = -\frac{\beta_i + k}{\alpha_i}$ $(k = 0, 1, 2, \dots)$. When γ_i is not integer, the function $[\Gamma(\alpha_i s + \beta_i)]^{\gamma_i}$ is a multi-valued function of s. We fix its principal branch by drawing the cut along the negative semi-axis, starting from $-\beta_i/\alpha_i$, ending at $-\infty$, and supposing that $[\Gamma(\alpha_i \xi + \beta_i)]^{\gamma_i}$ is positive for all the positive values of ξ. The principal branch of $\prod_{i=1}^{m}[\Gamma(\alpha_i \xi + \beta_i)]^{\gamma_i}$ is chosen by cutting along the negative semi-axis, starting from $-\min_{\gamma_i \notin \mathbb{N}}(\beta_i/\alpha_i)$ to the $-\infty$, and supposing that $[\Gamma(\alpha_i \xi + \beta_i)]^{\gamma_i}$ are all positive for the positive values of ξ.

Theorem 5. *Let α_i, β_i, τ_i, $\gamma_i > 0$ for $i = 1, ..., m$, and let $\gamma_1, \dots, \gamma_{i_0} \notin \mathbb{N}$ $(i_0 \in \{1, \dots, m\})$. Let additionally $[\Gamma(\alpha_i s + \beta_i)]^{\gamma_i}$ be the described branches of these multi-valued functions $(i = 1, ..., i_0)$. Then the multi-index MLPR-function* (16) *can be expressed by the following $\mathcal{L}_{+\infty}$-contour integral representation:*

$$\mathbb{F}_m(z) = \frac{1}{2\pi i\, A} \int\limits_{\mathcal{L}} \mathcal{H}_m(s)(-z)^s ds + \frac{1}{\prod\limits_{i=1}^{m}[\Gamma(\beta_i)]^{\gamma_i}}, \qquad |arg(-z)| < \pi, \tag{31}$$

with $A = \prod\limits_{i=1}^{m} \Gamma(\tau_i)$,

$$\mathcal{H}_m(s) = \mathcal{H}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(s) = \frac{\Gamma(-s) \prod\limits_{i=1}^{m} \Gamma(\tau_i + s)}{[\Gamma(1+s)]^{m-1} \prod\limits_{i=1}^{m} [\Gamma(\beta_i + s\alpha_i)]^{\gamma_i}}, \tag{32}$$

and where $\mathcal{L}_{+\infty}$ is a right loop lying in a horizontal stripe, starting at the point $+\infty + i\varphi_1$, terminating at the point $+\infty + i\varphi_2$ $(-\infty < \varphi_1 < 0 < \varphi_2 < +\infty)$, and crossing the real line at a point c, $0 < c < 1$.

Proof. The chosen contour $\mathcal{L}_{+\infty}$ is negatively oriented. It separates the poles $s = k$ ($k \in \mathbb{N}$) of the function $\Gamma(-s)$ from the poles $s_{i,l} = -\tau_i - l$ ($i = 1, \dots, m$, $l \in \mathbb{N}$) of the functions $\Gamma(\tau_i + s)$, along with the pole $s = 0$ of the function $\Gamma(-s)$. Let us consider the integral of the right hand side of (31), denoting for convenience

$$I(z) = \frac{1}{2\pi i\, A} \int_{\mathcal{L}} \mathcal{H}_m(s)(-z)^s ds. \tag{33}$$

Calculating the residues of the integrand of (32) at the simple poles $s_k = k$, $k = 0, 1, 2, \dots$, and taking into account the asymptotic formula (26), applied for $\Gamma(-s)$:

$$\Gamma(-s) = \frac{(-1)^k}{k!(-s+k)}[1 + O(-s+k)] \quad (s \to k;\ k = 0, 1, 2, \dots),$$

we have

$$\begin{aligned} I(z) &= \frac{-1}{A} \sum_{k=1}^{\infty} \mathrm{Res}_{s=k} \left\{ \mathcal{H}_m(s)(-z)^s \right\} \\ &= \frac{-1}{A} \sum_{k=1}^{\infty} \lim_{s \to k} \left\{ (s-k)\mathcal{H}_m(s)(-z)^s \right\} \\ &= \frac{1}{A} \sum_{k=1}^{\infty} \frac{(-1)^k (-z)^k \prod_{i=1}^{m} \Gamma(\tau_i + k)}{k!\, [\Gamma(k+1)]^{m-1} [\Gamma(\beta_i + k\alpha_i)]^{\gamma_i}} \\ &= \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{m} (\tau_i)_k}{\prod_{i=1}^{m} [\Gamma(\beta_i + k\alpha_i)]^{\gamma_i}} \frac{z^k}{(k!)^m}, \end{aligned}$$

that proves (31). □

Now we get another form of the representation of the Le Roy type function (16). We consider the multi-valued function $[\Gamma(\alpha_i(-s) + \beta_i)]^{\gamma_i}$ and fix its principal branch by drawing the cut along the positive semi-axis, starting from β_i/α_i, ending at $+\infty$, and supposing that $[\Gamma(\alpha_i(-\xi) + \beta_i)]^{\gamma_i}$ is positive for all the negative values of ξ. The principal branch of $\prod_{i=1}^{m} [\Gamma(\alpha_i(-\xi) + \beta_i)]^{\gamma_i}$ is chosen by cutting along the positive semi-axis, starting from $\min_{\gamma_i \notin \mathbb{N}} (\beta_i/\alpha_i)$ to the $+\infty$, and supposing that $[\Gamma(\alpha_i(-\xi) + \beta_i)]^{\gamma_i}$ are all positive for the negative values of ξ.

Theorem 6. *Let α_i, β_i, τ_i, $\gamma_i > 0$ for $i = 1, ..., m$, and let $\gamma_1, \dots, \gamma_{i_0} \notin \mathbb{N}$ ($i_0 \in \{1, \dots, m\}$). Let additionally $[\Gamma(\beta_i - \alpha_i s)]^{\gamma_i}$ be the described branches of these multi-valued functions ($i = 1, ..., i_0$). Then the multi-index MLPR-function* (16) *can be expressed by the following $\mathcal{L}_{-\infty}$-contour integral representation:*

$$\mathbb{F}_m(z) = \frac{1}{2\pi i\, A} \int_{\mathcal{L}} \mathcal{H}_m(-s)(-z)^{-s} ds + \frac{1}{\prod_{i=1}^{m} [\Gamma(\beta_i)]^{\gamma_i}}, \quad |arg(-z)| < \pi, \qquad (34)$$

with $A = \prod_{i=1}^{m} \Gamma(\tau_i)$, $\mathcal{H}_m(s)$ defined by (32)*, and where $\mathcal{L}_{-\infty}$ is a left loop lying in a horizontal stripe, starting at the point $-\infty + i\varphi_1$, terminating at the point $-\infty + i\varphi_2$ ($-\infty < \varphi_1 < 0 < \varphi_2 < +\infty$), and crossing the real line at a point c, $-1 < c < 0$.*

Proof. It follows the lines of the previous one. The details are omitted. □

Remark 4. *Let us note that in both representations* (31) *and* (34)*, we cannot include the term corresponding to the pole at $s = 0$, since in this case either $\mathcal{L}_{-\infty}$ or $\mathcal{L}_{+\infty}$ should cross the branch cut of the corresponding multi-valued function.*

Several open problems can be further handled for the multi-MLPR functions (16), concerning for example, conditions of complete monotonicity, and other properties similar to that of the multi-index ML, multi-index ML-Prabhakar functions ($m > 1$) and the generalized ML functions of Le Roy type as $F^{(\gamma)}_{\alpha,\beta}$ and $F^{(\gamma)}_{\alpha,\beta;\tau}$ ($m = 1$).

Let us note that on contrary to the cases of the Mittag-Leffler type functions (4), (5) (and all their particular cases) and multi-index Mittag-Leffler functions (6), (7), the recently introduced Le Roy functions of Mittag-Leffler type (13) and the multi-MLPR functions (14) cannot be presented in terms of Wright generalized hypergeometric functions ${}_p\Psi_q$ nor either as cases of the Fox H-functions. Nevertheless, we may consider these Le Roy type functions as members of the family of Special Functions of Fractional Calculus.

4. Examples of the multi-index MLPR-function

• Let us take $\underline{m=1}$ in (16). A case of such MLPR type special function of the form

$$F_{\alpha,\beta;\gamma}^{\tau;1}(z)=\sum_{k=0}^{\infty}\frac{(\tau)_k}{[\Gamma(\alpha k+\beta)]^{\gamma}}\,, \tag{35}$$

has been mentioned in Tomovski and Mehrez [36]. This function has been studied in details by Paneva-Konovska [26]. There, it is proved that (35) has the order $\rho=1/\alpha\gamma$, and type $\sigma=\gamma$, that do not depend on “Prabhakar” index τ.

For $\tau=1$, it is the Mittag-Leffler function of Le Roy type (13), $F_{\alpha,\beta}^{(\gamma)}(z)$, introduced and studied in the mentioned works [9], [7], [8], [36], [6], [28], [11], [35], as entire function with order ρ and type σ that are in agreement with the more general results (17)-(18), namely:

$$\rho=\frac{1}{\alpha\gamma},\quad \sigma=\gamma.$$

Naturally, if $m=1, \alpha_1=\beta_1=\tau_1=1, \gamma_1=\gamma>0$, the original Roy function (12) appears:

$$F_{1,1}^{(\gamma)}(z)=F_{\gamma}(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k!)^{\gamma}}$$

with order $\rho=1/\gamma$ and type $\sigma=\gamma$.

And, for $m=1$, if $\gamma_1=\tau_1=1$, it is the famous Mittag-Leffler function $E_{\alpha,\beta}$ of order $\rho=1/\alpha$ and type $\sigma=1$; while for $\tau_1=\tau$ we have the Prabhakar function $E_{\alpha,\beta}^{\tau}$. We skip the comments for their important role as SF of FC and all long lists of particular cases, presented in [29, pp.17-20], [18]-[19], [33], [10].

• The case $\underline{m=2}$. In the paper [28] Pogany mentioned as an example only, a special function of the form (we keep the denotations from [28]):

$$F_{(p,q;r,s)}^{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(pk+q)]^{\alpha}\;[\Gamma(rk+s)]^{\beta}}\,.$$

This function illustrates the case $m=2$ of the new special functions we consider here, namely it can be written as:

$$\mathbb{F}_{(p,r),(q,s);(\alpha,\beta)}^{(1,1);2}=\sum_{k=0}^{\infty}\frac{(1)_k(1)_k\;z^k}{(k!)^2\;\;[\Gamma(pk+q)]^{\alpha}\;[\Gamma(rk+s)]^{\beta}}.$$

So, in this case, we obtain that this is an entire function with:

$$\rho=\frac{1}{p\alpha+r\beta}\,,\quad \sigma=\frac{p\alpha+r\beta}{\left(p^{p\alpha}\cdot r^{r\beta}\right)^{1/(p\alpha+r\beta)}},$$

and as expected, the order and type do not depend on 2nd parameters q and s in the Gamma functions.

We may continue the examples for $m = 2$ with the case of the not enough popular ML type function of Dzrbashjan [4] (1960, in Russian only), with 2×2 indices, which he denoted alternatively by (below we may set $1/\rho_i := \alpha_i, \mu_i := \beta_i$, $i = 1, 2$):

$$\Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1})\Gamma(\mu_2 + \frac{k}{\rho_2})}$$
$$:= E_{(\frac{1}{\rho_1},\frac{1}{\rho_2}),(\mu_1,\mu_2)}(z) = E_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)}(z). \tag{36}$$

Dzrbashjan found the order and type of this entire function, claimed on few simple particular cases, and considered some integral relations between (36) and Mellin transforms on a set of axes. Then, he developed a theory of integral transforms in the class L_2, involving kernel close to functions (36) and further, proposed approximations of entire functions in L_2 for an arbitrary finite system of axes in complex plane starting from the origin. The 2×2-indices ML type functions (36) are studied in details also by Luchko in recent works (as one of 2020). He allows the parameters ρ_1, ρ_2 to be also negative or zero, and called them “4-parameters Wright functions of second kind”.

Simple cases of (36) as mentioned by Dzrbashjan himself, were: the ML function (itself): $E_{\frac{1}{\rho},\mu}(z) = E_{(\frac{1}{\rho},0),(\mu,1)}(z) = \Phi_{\rho,\infty}(z;\mu,1)$; also:

$$\frac{1}{1-z} = E_{(0,0),(1,1)}(z) = \Phi_{\infty,\infty}(z;1,1);$$

the Bessel function:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu E_{(1,1),(\nu+1,1)}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \Phi_{1,1}\left(-\frac{z^2}{4};1,\nu+1\right).$$

To these examples, we have added ([18], [19]) also: the Struve and Lommel functions:

$$s_{\mu,\nu}(z) = \frac{1}{4} z^{\mu+1} E_{(1,1),((3-\nu+\mu)/2,(3+\nu+\mu)/2)}\left(-\frac{z^2}{4}\right),$$

$H_\nu(z) = \frac{1}{\pi 2^{\nu-1}(1/2)_\nu} s_{\nu,\nu}(z)$.

The (classical) Wright function studied by Fox (1928), Wright (1933), Humbert and Agarwal (1953), is also a case of the multi-index M-L function with $m = 2$:

$$\varphi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}={}_0\Psi_1\left[\begin{array}{c}-\\(\beta,\alpha)\end{array}\Big|z\right]=E^{(2)}_{(\alpha,1),(\beta,1)}(z), \tag{37}$$

and thus, it also falls in the case of multi-index ($m = 2$) ML functions of Le Roy type!

The Wright function $\varphi(\alpha,\beta;z)$ plays important role in the solution of linear PFDEs as the fractional diffusion-wave equation studied by Nigmatullin (1984-1986, to describe the diffusion process in media with fractal geometry, $0 < \alpha < 1$) and by Mainardi et al. (1994 -), for propagation of mechanical diffusive waves in viscoelastic media, $1 < \alpha < 2$). In the form $M(z;\beta) = \varphi(-\beta, 1-\beta; -z)$, $\beta := \alpha/2$, it is called also as the Mainardi function. In our denotations, this is: $M(z;\beta) = E^{(2)}_{(-\beta,1),(1-\beta)}(-z)$, $m = 2$ and has its examples like: $M(z;1/2) = 1/\sqrt{\pi}\exp(-z^2/4)$ and the Airy function: $M(z;1/3) = 3^{2/3}\,Ai(z/3^{1/3})$. In other form and denotation, the same Wright function (37) is known as Wright-Bessel, or misnamed as Bessel-Maitland function:

$$J^{\mu}_{\nu}(z) = \varphi(\mu,\nu+1;-z) = {}_0\Psi_1\left[\begin{array}{c}-\\(\nu+1,\mu)\end{array}\Big|-z\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-z)^k}{\Gamma(\nu+k\mu+1)\,k!} = E^{(2)}_{(1/\mu,1),(\nu+1,1)}(-z)\,, \tag{38}$$

again as an example of the Dzrbashjan function. But it is an obvious "fractional index" analogue of the classical Bessel function $J_\nu(z)$, and example of multi-index M-L function of Le Roy type. Several other "fractional-indices" generalizations of $J_\nu(z)$ have been also exploited as SF of FC, and we can present them as multi-index M-L functions and as Le Roy type functions (16): the generalized Wright-Bessel-Lommel functions (Pathak, 1966-1967); the generalized Lommel-Wright function with 4 indices, introduced by de Oteiza, Kalla and Conde (1986), etc. (see in [18], [19]).

• The case of arbitrary $\underline{m \geq 2}$.

As mentioned, while this work was in progress, we have been informed by Rogosin and Dubatovskaya about their studies on a kind of multi-index Mittag-Leffler functions of Le Roy type, defined by (14),

$$F^{(\gamma)_m}_{(\alpha,\beta)_m}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\prod_{j=1}^{m}\left[\Gamma(\alpha_j k+\beta_j)\right]^{\gamma_j}}, \quad z\in\mathbb{C}.$$

In [34] they have provided the order and type of these entire functions as in (15), Mellin-Barnes type integral representations, some interpretations and particular cases. Evidently, (14) appear as functions (16) with $\forall \tau_i = 1$, $i = 1, ..., m$.

For the case of $\forall \gamma_i = 1, \tau_i = 1$, $i = 1, ..., m$, we have the $2m$ multi-index M-L functions (6), and if $\forall \gamma_i = 1$, but τ_i are chosen arbitrary, these are the $3m$ multi-index ML functions (7), and our formulas (17) and (18) give their order and type.

• Some other special cases, that are examples of multi-index ML functions and so, also of the multi-index MLPR-functions (for more examples, see in [18],[19]):

Consider the case $m \geq 2$, with $\forall \alpha_i = 1$, $i = 1, \ldots, m$. Then:

$$E^{(m)}_{(1,...,1),(\beta_i+1)}(z) = {}_1\Psi_m\left[\begin{array}{c}(1,1)\\(\beta_i,1)_1^m\end{array}\middle| z\right] = \text{const}\ {}_1F_m\left(1; \mu_1, \mu_2, ..., \mu_m; z\right)$$

reduces to ${}_1F_m$, and also to a Meijer's $G^{1,1}_{1,m+1}$-function. Denote $\beta_i = \nu_i + 1, i = 1, \ldots, m$, and let additionally one of the β_i to be 1, e.g.: $\beta_m = 1$, i.e. $\nu_m = 0$. Then the multi-index ML function becomes a hyper-Bessel function of Delerue ([2], [14, Ch.3]):

$$J^{(m-1)}_{\nu_i,...,\nu_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1}\nu_i} E^{(m)}_{(1,1,...,1),(\nu_1+1,\nu_2+1,...,\nu_{m-1}+1,1)}\left(-\left(\frac{z}{m}\right)^m\right). \tag{39}$$

In view of the above relation, the multi-index ML functions with arbitrary $(\alpha_1, \ldots, \alpha_m) \neq (1, \ldots, 1)$ can be seen as fractional-indices analogues of the hyper-Bessel functions. The hyper-Bessel functions (39) themselves are multi-index analogues of the Bessel function. These are closely related to the theory of the hyper-Bessel differential operators of Dimovski [3]

$$\begin{aligned} Bf(t) &= t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\cdots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m}f(t) = t^{-\beta}\,P_m\left(t\frac{d}{dt}\right)f(t) \\ &= t^{-\beta}\prod_{k=1}^{m}\left(t\frac{d}{dt} + \beta\nu_k\right)f(t),\ \ t > 0, \end{aligned} \tag{40}$$

and form a fundamental system of solutions of the differential equations of the form $By(z) = \lambda\, y(z)$ (see Kiryakova [14, Th.3.4.3]). For example, if $\beta = m$, $\nu_1 < \nu_2 < ... < \nu_m = 0 < \nu_1 + 1$ in (40), the solution of the Cauchy problem $By(z) = -y(z)$, $y(0) = 1$, $y^{(j)}(0) = 0$, $j = 1, ..., m-1$, is given by the normalized hyper-Bessel function: $y(z) = j^{(m-1)}_{\nu_1,..,\nu_{m-1}}(-z)$, related to the Bessel-Clifford function of m-th order:

$$C_{\nu_1,\dots,\nu_m}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{\Gamma(\nu_1+k+1)\dots\Gamma(\nu_m+k+1)\,k!}$$
$$=E^{(m+1)}_{(1,\dots,1),(\nu_1+1,\dots,\nu_m+1,1)}(-z).$$

Now, let us mention the special functions appearing in some recent papers by Ricci (say, in [32]). He considers the so-called Laguerre derivative $D_L = \frac{d}{dz}z\frac{d}{dz}$ and its iterates $D_{mL} = \frac{d}{dz}z\frac{d}{dz}z\dots\frac{d}{dz}z$. But these are the same as the particular hyper-Bessel differential operators considered in operational calculus by Ditkin and Prudnikov (1963). Then, the L-exponentials $e_1(z), e_2(z), ..., e_m(z), ...$ which are eigenfunctions of D_{mL}, that is, $D_{mL}\, e_m(\lambda z) = \lambda\, e_m(\lambda z)$, are shown to have the form $e_m(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k!)^{m+1}}$. Observe that these are exactly Le Roy functions with integer $\gamma = m+1$ and can be seen also to be

$$e_m(z)={}_0F_m(-;1,1,...,1;z)={}_1\Psi_{m+1}\left[\begin{array}{c}(1,1)\\(1,1),(1,1),...,(1,1)\end{array}\bigg|\, z\right].$$

Thus, these are examples of both hyper-Bessel functions and multi-index M-L functions $E^{(m+1)}_{(1,\dots,1),(1,\dots,1)}(z)$. Ricci applied these SF and the related Laguerre-type generalized hypergeometric functions as solutions in population dynamics. Namely, again in a recent work, Ricci considered some Laguerre-type (L-) Bessel functions, L-type Gauss hypergeometric functions, and the Laguerre-type generalized hypergeometric functions ${}_{Lp}F_q$ which can be shown to be representable by ${}_pF_{q+1}$:

$$\begin{aligned}{}_{Lp}F_q(a_1,..,a_p;b_1,...,b_q;z)&=\sum_{k=0}^{\infty}\frac{a_1^{(k)}...a_p^{(k)}}{b_1^{(k)}...b_q^{(k)}}\cdot\frac{z^k}{(k!)^2}\\ =\sum_{k=0}^{\infty}\frac{a_1^{(k)}...a_p^{(k)}}{b_1^{(k)}...b_q^{(k)}\,(1)^{(k)}}\cdot\frac{z^k}{k!}&={}_pF_{q+1}(a_1,...,a_p;b_1,...,b_q,1;z).\end{aligned} \tag{41}$$

These special functions fall again as examples of the hyper-Bessel, multi-index ML and multi-index MLPR-functions.

Finally, one may consider multi-index analogues of the Rabotnov (α-exponential function), with all $\alpha_i = \beta_i = \alpha > 0, i = 1, ..., m$:

$$y_\alpha^{(m)}(z)=z^{\alpha-1}E^{(m)}_{(\alpha,.,\alpha),(\alpha,.,\alpha)}(z^\alpha)=z^{\alpha-1}\sum_{k=0}^{\infty}\frac{z^{\alpha k}}{[\Gamma(\alpha+\alpha k)]^m}, \tag{42}$$

and for $\alpha = 1$ we have the original Le Roy function: $\sum_{k=0}^{\infty}\frac{z^k}{[k!]^m}$.

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