Multi-Instantons Localized at the Origin

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Abstract. We obtain a family of self-dual Yang-Mills fields in an SU(2) gauge theory. Some of them describe pseudoparticles with arbitrary topological numbers and with action densities concentrated around the origin.

1. Introduction

By now, much is known about self-duality equations (SDE) for SU(2) gauge theories. In principle, Atiyah et al. [1] have solved the problem completely, but only a restricted number of solutions are explicitly known and understood as solitons by knowledge of their action or energy density. The most popular one is the k-instanton discovered by 'tHooft [2]. It corresponds essentially to a superposition of k widely separated instantons [3].

Recently, research for multi-monopoles has led to new (time independent) solutions [4]; these have a cylindric symmetry, finite energy and their energy density is maximal on a circle in such a way that most of the energy is concentrated in a torus-like region of space.

In this paper, we exhibit a class of time dependent solutions with finite action and an action density maximal on a circle in Euclidean space-time. To obtain such solutions, we require a particular transformation law of the fields under the subgroup $SO(2) \times SO(2)$ of rotations in the x_1, x_2 plane and in the x_0, x_3 plane. The solutions belong to a large class obtained in [8] with different motivations. In Sect. 2 of the paper, we rapidly explain the ansatz and the construction; then we study in Sect. 3 the physically relevant solutions. Some conclusions are drawn in Sect. 4.

Let us first write the equations to be satisfied; in order to study the SDE

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \tfrac{1}{2} \varepsilon_{\mu\nu\varrho\sigma} F^{\varrho\sigma} \tag{1.1}$$

for the gauge field $F_{\mu\nu}$ defined as usual in terms of a gauge potential A_μ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, we have used the Yang formalism. It works with the light-like coordinates

$$Y = \frac{1}{\sqrt{2}}(x^0 - ix^3), \quad Z = \frac{1}{\sqrt{2}}(x^2 + ix^1), \quad \bar{Y} = Y^*, \quad \bar{Z} = Z^* \tag{1.2}$$

and, in the so called triangular or Yang gauge, the SDE $(1, 1)$ reduce to the Yang equations below (see [6] for more details)

$$\left(\frac{f_{,Y}}{f}\right)_{,\bar{Y}} + \left(\frac{f_{,Z}}{f}\right)_{,\bar{Z}} - \frac{e_{,Y} \cdot g_{,\bar{Y}}}{f^2} - \frac{e_{,Z} \cdot g_{,\bar{Z}}}{f^2} = 0, \tag{1.3}$$

$$\left(\frac{e_{,Y}}{f^2}\right)_{,\bar{Y}} + \left(\frac{e_{,Z}}{f^2}\right)_{,\bar{Z}} = 0, \tag{1.4a}$$

$$\left(\frac{g_{,\bar{Y}}}{f^2}\right)_{,Y} + \left(\frac{g_{,\bar{Z}}}{f^2}\right)_{,Z} = 0. \tag{1.4b}$$

The functions f, e, g are the basic objects; one can show that:
  (i) if

$$f = f^*, \quad e = -g^* \quad (\text{respectively } e = g^*) \tag{1.5a, b}$$

the associated gauge quantities are in the algebra of SU(2) [respectively SU(1, 1)],
  (ii) the new functions defined by

$$(f', e', g') = \frac{1}{\Delta}(f, -g, -e), \quad \Delta = f^2 - eg, \tag{1.6}$$

$$(f', e', g') = (kk^* f, (k^*)^2 e, (k)^2 g) \tag{1.7}$$

still satisfy (1.3) and (1.4) as long as (f, e, g) satisfy them [7].

2. Construction of Solutions

In order to simplify the problem of solving the Yang equations (1.3) and (1.4), we have found it convenient to look for sequences of solutions of the following form

$$S_n \equiv (f_n, e_n, g_n) \quad n \in \mathbb{Z}$$

$$= \frac{1}{K_{n-1}}(K_n, Y^{a_n} Z^{b_n}, \mu_n \bar{Y}^{a_n} \bar{Z}^{b_n}), \tag{2.1}$$

where the K_n's are real functions of $r(= Y\bar{Y})$ and $s(= Z\bar{Z})$ only, μ_n are signs and a_n, b_n are chosen to be integers throughout the paper. The ansatz (2.1) is such that, under any rotations of $SO(2) \times SO(2)$ with angles (ω, φ), each triplet S_n transforms like a dilatation (1.7) with $k = \exp\frac{i}{2}(a_n\omega + b_n\varphi)$.

By substitution of the ansatz (2.1) into Eqs. (1.4) one can easily show that they are satisfied provided two successive K_n are related by

$$\partial_s K_n = a_n K_{n-1} - r\, \partial_r K_{n-1}, \tag{2.2a}$$

$$\partial_r K_n = -b_n K_{n-1} + s\, \partial_s K_{n-1}. \tag{2.2b}$$

Equations (2.2) are compatible with each other iff

$$[r\, \partial_{rr} + s\, \partial_{ss} + (1 - a_n)\, \partial_r + (1 - b_n)\, \partial_s] K_{n-1} = 0. \tag{2.3}$$

After elementary algebraic manipulations, one can show that the infinite system (2.2) and (2.3) is self-consistent iff $a_{n+1} = a_n + 1$ and $b_{n+1} = b_n + 1$. Without losing generality, we will chose in the following

$$a_n = n - 1, \qquad b_n = b + n - 1 \qquad (b \geqq 0). \tag{2.4}$$

Then choosing for K_0 the most general analytic function obeying (2.3) (see appendix) we have

$$K_0(r, s, b) = \sum_{i=0}^{\infty} c_i p_i(r, s, -1, b-1). \tag{2.5}$$

One gets recursively for the K_n's

$$K_n(r, s, b) = \sum_{i=0}^{\infty} c_{i-n} p_i(r, s, n-1, n+b-1). \tag{2.6}$$

In (2.5) the c_i's $(i \geqq 0)$ are arbitrary constants; in the integrations (2.6), we have inserted the new arbitrary constants under the name of c_i with negative values of the index i.

The last and hardest thing we have to do is to calculate (if possible) the constants $c_i (-\infty \leqq i \leqq \infty)$ [see (2.5) and (2.6)] and the signs μ_n [see (2.1)] in such a way that the remaining Yang equation (1.3) will also be satisfied. For the moment the only solutions we have discovered depend on a real parameter λ and can be written as

$$c_i = 0 \qquad \text{if} \quad i \geqq b, \tag{2.7a}$$

$$c_i = \lambda^i c_0(b) \quad \text{if} \quad i < b. \tag{2.7b}$$

With these coefficients the formal series are polynomials, let us call them $K_n(b)$ for simplicity, which obey the identities below:

Linear Identities

$$K_n(b) = \left(\frac{1}{\lambda} + s - r \right) K_{n-1}(b) + rs K_{n-2}(b), \tag{2.8}$$

$$\sqrt{|\lambda|} \, K_n(b+1) = (1 - \lambda(r+s)) K_n(b) + \frac{1}{\sqrt{|\lambda|}} \lambda s K_n(b-1). \tag{2.9}$$

Quadratic Identity

$$(K_n(b))^2 - K_{n+1}(b) K_{n-1}(b) = \mu_n(b) r^{a_n} s^{b_n}, \tag{2.10}$$

provided

$$\mu_n(b) = (-1)^{n-1} (-\lambda)^{b-2} (c_0)^2. \tag{2.11}$$

We see immediately that the hemiticity relation (1.5b) fixes the values of the normalisation factor $c_0(b)$ in (2.7); it must be equal to

$$c_0(b) = |\lambda|^{\frac{2-b}{2}} \tag{2.12}$$

The proofs of the relations (2.8)–(2.11), again with the proof that they imply Eq. (1.3), can be found in [8].

Thus the formulae (2.1)–(2.12) provide a family of self-dual fields indexed by two discrete parameters (n and b) and by a scale factor λ. The hermiticity of each solution is easily analysed knowing (1.5) and (2.1) and the signs $\mu_n(b)$ obtained from (2.11) and (2.12):

$$\mu_n(b) = (-1)^{n-1}(-\varepsilon)^b \qquad (\varepsilon = \mathrm{sgn}(\lambda)). \tag{2.13}$$

3. Finite Action Solutions

Most of the solutions constructed in the preceding section are of no physical interest because they have infinite action. However taking $n=0$ and $\varepsilon = -1$, one obtains a subfamily (indexed by b) of solutions with finite action.

Let us write the relevant solutions transformed by (1.6); using (2.10) one obtains

$$(f, e, g)(b) = \frac{1}{K_1}(K_0, \bar{Y}^{-1}\bar{Z}^{b-1}, -Y^{-1}Z^{b-1}). \tag{3.1}$$

The new f defined in (3.1) are strictly negative and finite functions when restricted to euclidean space-time (i.e., $r \geqq 0$, $s \geqq 0$). Here we write in general the dominant terms of K_0 and K_1 near the origin together with their asymptotic behaviour; the other terms can easily be derived from (A.2) and (2.6)

$$K_0(b) = |\lambda|^{\frac{2-b}{2}}(1 + |\lambda|(s + (b-1)r) + \ldots + |\lambda|^{b-1}(r+s)^{b-1}), \tag{3.2}$$

$$K_1(b) = -|\lambda|^{\frac{2-b}{2}}(1 + |\lambda|br + \ldots + |\lambda|^{b-1}r(r+s)^{b-2}). \tag{3.3}$$

Knowing (3.1), one can explicitly calculate the associated gauge quantities, particularly the action density $\mathscr{A}$ (see [8, formula (3.50)]). After a lengthy calculation, we have verified that the action density of the solutions (3.1) is finite and smooth everywhere over space-time; it depends only on K_0 through the formula

$$\mathscr{A}(b) = -\square\square \ln K_0(b). \tag{3.4}$$

For $b=1$, this function is identically zero, for $b=2,3$ it is respectively equal to

$$\mathscr{A}(2) = \frac{6|\lambda|^2}{K^4}, \qquad K = 1 + |\lambda|(r+s), \tag{3.5}$$

$$\mathscr{A}(3) = \frac{2|\lambda|^2}{K^4}(K^2 + 24|\lambda|sK + 27|\lambda|^2s^2), \quad K = 1 + |\lambda|(s+2r) + |\lambda|^2(s+r)^2. \tag{3.6}$$

The function (3.5) is invariant under SO(4) rotations, it admits a unique maximum at the origin, and decreases monotonically in all directions from the origin. The function (3.6) has neither local maximum nor minimum on euclidean space-time; the absolute maximum on this domain occurs at the points with $r=0$, $s \simeq 0.3$; on the $s=0$ axis, the function decreases monotonically.

For larger values of b, we do not have the explicit dependence of $\mathscr{A}$ on r and s but we have verified numerically that the function $\mathscr{A}$ has the same shape as (3.6).

Table 1. In the Table, we give approximate values of $\mathscr{A}_{\max}$ together with the coordinates $s_{\max}$ for the first few values of b; $r_{\max} = 0$

b	$s_{\max}$ in unit of $|\lambda|^{-1}$	$\mathscr{A}_{\max}$ in unit of $|\lambda|^2$
2	0.0	6.0
3	0.3	7.7
4	0.55	16.7
5	0.68	34.6
6	0.77	65.7

The formula (3.4) together with the asymptotic behaviour of K_0 [see (3.2)] allows us to calculate the action as a function of b. Using Stokes' theorem, one obtains $A = 16\pi^2(b-1)$. Remembering that our solutions are self-dual, we also have the topological number T of our solutions; it is equal to

$$T = \frac{A}{16\pi^2} = b - 1. \tag{3.7}$$

Thus the family (3.1) contains solutions with arbitrary values for the topological number. For $b = 1$, it is a pure gauge, for $b = 2$, it is the famous instanton [3]; for larger values of b our solutions are interesting in that they are different from the k-instanton of 'tHooft. Indeed these last admit an action density with k separated local maxima, precisely attained at the different positions of the instantons.

4. Conclusions

In conclusion let us note several common points between our solutions (we will call them multi-instantons) and the recently discovered multi-monopoles in [4, 5]:

(i) Both solutions describe 5-parameters objects (4 positions + 1 scale factor) located around a single point in space.

(ii) Topologically, multi-monopoles (respectively multi-instantons) behave as several single monopoles (respectively instantons) located at finite distance from the origin. They are characterised by an integer: the magnetic flux (respectively the topological number).

(iii) The multi-object does not have the same symmetry as the object, i.e. the multi-monopole is not spherically symmetric under SO(3) (as is the monopole) and the multi-instanton has not the full symmetry under SO(4) that the instanton does.

(iv) At least for the first few values of b, the energy density of the multi-monopole (plotted against s and x_3) and the action density of the multi-instanton (plotted against r and s) have almost the same shape (see figures in [5]), i.e. with the absolute maximum delocalised from the origin. As a consequence the energy and action densities are maximal on a circle in space (space-time).

For all these reasons, we think that the solutions presented in (3.1) describe particular cases of several instantons superposed at the same point and, consequently, in interaction with each other. The existence of such solutions was already announced in [9] but the authors did not give any explicit form in their paper.

Appendix

The differential equation $[r\,\partial_{rr}+s\,\partial_{ss}-a\,\partial_r-b\,\partial_s]P=0$, where a and b are constants, admits among its solutions homogeneous polynomials of arbitrary degree d; let us call them $p_d(r,s,a,b)$; these can be written

$$p_d(r,s,a,b)=\frac{1}{d!}\sum_{k=0}^{d}A_{d,k}(a,b)r^k s^{d-k}, \tag{A.1}$$

with the coefficients $A_{d,k}(a,b)$ given, up to a multiplicative constant, by (Γ denotes the usual Euler function)

$$A_{d,k}(a,b)=(-1)^k\mathscr{C}_d^k\frac{\Gamma(a-k+1)}{\Gamma(a-d+1)}\cdot\frac{\Gamma(b-d+k+1)}{\Gamma(b-d+1)}. \tag{A.2}$$

Then for K_0 in (2.5), we have chosen a general linear combination of the polynomials (A.1) together with the notation [see (2.4)]

$$K_m(r,s,a_n,b_n)\equiv K_n(r,s,b). \tag{A.3}$$

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