Multi-interval Subfactors and Modularity of Representations in Conformal Field Theory

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Dedicated to John E. Roberts on the occasion of his sixtieth birthday

Abstract

We describe the structure of the inclusions of factors $\mathcal{A}(E) \subset \mathcal{A}(E')'$ associated with multi-intervals $E \subset \mathbb{R}$ for a local irreducible net $\mathcal{A}$ of von Neumann algebras on the real line satisfying the split property and Haag duality. In particular, if the net is conformal and the subfactor has finite index, the inclusion associated with two separated intervals is isomorphic to the Longo-Rehren inclusion, which provides a quantum double construction of the tensor category of superselection sectors of $\mathcal{A}$. As a consequence, the index of $\mathcal{A}(E) \subset \mathcal{A}(E')'$ coincides with the global index associated with all irreducible sectors, the braiding symmetry associated with all sectors is non-degenerate, namely the representations of $\mathcal{A}$ form a modular tensor category, and every sector is a direct sum of sectors with finite dimension. The superselection structure is generated by local data. The same results hold true if conformal invariance is replaced by strong additivity and there exists a modular PCT symmetry.

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1 Introduction

This paper provides the solution to a natural problem in (rational) conformal quantum field theory, the description of the structure of the inclusion of factors associated to two or more separated intervals.

This problem has been considered in the past years, seemingly with different motivations. The most detailed study of this inclusion so far has been done by Xu [50] for the models given by loop group construction for $SU(n)_k$ [47]. In this case Xu has computed the index and the dual principal graph of the inclusions. A suggestion to study this inclusion has been made also in [43, Section 3]. Our analysis is model independent, and will display new structures and a deeper understanding also in these and other models.

Let $\mathcal{A}$ be a local irreducible conformal net of von Neumann algebras on $\mathbb{R}$, i.e. an inclusion preserving map

$$I \mapsto \mathcal{A}(I)$$

from the (connected) open intervals of $\mathbb{R}$ to von Neumann algebras $\mathcal{A}(I)$ on a fixed Hilbert space. One may define $\mathcal{A}(E)$ for an arbitrary set $E \subset \mathbb{R}$ as the von Neumann algebra generated by all the $\mathcal{A}(I)$'s as I varies in the intervals contained in E. By locality $\mathcal{A}(E)$ and $\mathcal{A}(E')$ commute, where E' denotes the interior of $\mathbb{R} \setminus E$, and thus one obtains an inclusion

$$\mathcal{A}(E) \subset \hat{\mathcal{A}}(E),$$

where $\hat{\mathcal{A}}(E) \equiv \mathcal{A}(E')'$. If Haag duality holds, as we shall assume[1], this inclusion is trivial if E is an interval, but it is in general non-trivial for a disconnected region E. We will explain its structure if E is the union of n separated intervals, a situation that can be reduced to the case $n = 2$, namely $E = I_1 \cup I_2$, where I_1 and I_2 are intervals with disjoint closure, as we set for the rest of this introduction.

One can easily realize that the inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is related to the superselection structure of $\mathcal{A}$, i.e. to the representation theory of $\mathcal{A}$, as charge transporters between endomorphisms localized in I_1 and I_2 naturally live in $\hat{\mathcal{A}}(E)$, but not in $\mathcal{A}(E)$.

Assuming the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty$ and the split property[2], namely that $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$, we shall show that indeed $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ contains all the information on the superselection rules.

We shall prove that in this case $\mathcal{A}$ is rational, namely there exist only finitely many different irreducible sectors $\{[\rho_i]\}$ with finite dimension and that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the inclusion considered in [28] (we refer to this as the LR inclusion, cf. Appendix A), which is canonically associated with $\mathcal{A}(I_1)$, $\{[\rho_i]\}$ (with the identi-

[1] As shown in [18], one may always extend $\mathcal{A}$ to the dual net $\mathcal{A}^d$, which is conformal and satisfies Haag duality.

[2] This general property is satisfied, in particular, if $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, where L_0 is the conformal Hamiltonian, cf. [5, 8].

fication $\mathcal{A}(I_2) \simeq \mathcal{A}(I_1)^{\mathrm{opp}}$). In particular,

$$[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \sum_i d(\rho_i)^2,$$

the global index of the superselection sectors. In fact $\mathcal{A}$ will turn out to be rational in an even stronger sense, namely there exist no sectors with infinite dimension, except the ones that are trivially constructed as direct sums of finite-dimensional sectors.

Moreover, we shall exhibit an explicit way to generate the superselection sectors of $\mathcal{A}$ from the local data in E: we consider the canonical endomorphism γ_E of $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$ and its restriction $\lambda_E = \gamma_E|_{A(E)}$; then λ_E extends to a localized endomorphism λ of $\mathcal{A}$ acting identically on $\mathcal{A}(I)$ for all intervals I disjoint from E. We have

$$\lambda = \bigoplus_i \rho_i \bar{\rho}_i, \tag{1}$$

where the ρ_i's are inequivalent irreducible endomorphisms of $\mathcal{A}$ localized in I_1 with conjugates $\bar{\rho}_i$ localized in I_2 and the classes $\{[\rho_i]\}_i$ exhaust all the irreducible sectors.

To understand this structure, consider the symmetric case $I_1 = I$, $I_2 = -I$. Then $\mathcal{A}(-I) = j(\mathcal{A}(I))$, where j is the anti-linear PCT automorphism, hence we may identify $\mathcal{A}(-I)$ with $\mathcal{A}(I)^{\mathrm{opp}}$. Moreover the formula $\bar{\rho}_i = j \cdot \rho_i \cdot j$ holds for the conjugate sector [17], thus by the split property we may identify $\{\mathcal{A}(E), \rho_i\bar{\rho}_i|_{\mathcal{A}(E)}\}$ with $\{\mathcal{A}(I) \otimes \mathcal{A}(I)^{\mathrm{opp}}, \rho_i \otimes \rho_i^{\mathrm{opp}}\}$. Now there is an isometry V_i that intertwines the identity and $\rho_i\bar{\rho}_i$ and belongs to $\hat{\mathcal{A}}(E)$. We then have to show that $\hat{\mathcal{A}}(E)$ is generated by $\mathcal{A}(E)$ and the V_i's and that the V_i's satisfies the (crossed product) relations characteristic of the LR inclusion. This last point is verified by identifying V_i with the standard implementation isometry as in [17], while the generating property follows by the index computation that will follow by the "transportability" of the canonical endomorphism above.

The superselection structure of $\mathcal{A}$ can then be recovered by formula (1) and the split property. Note that the representation tensor category of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ generated by $\{\rho_i \otimes \rho_i^{\mathrm{opp}}\}_i$ corresponds to the connected component of the identity in the fusion graph for $\mathcal{A}$, therefore the associated fusion rules and quantum $6j$-symbols are encoded in the isomorphism class of the inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$, that will be completely determined by a crossed product construction.

A further important consequence is that the braiding symmetry associated with all sectors is always non-degenerate, in other words the localizable representations form a modular tensor category. As shown by Rehren [41], this implies the existence and non-degeneracy of Verlinde's matrices S and T, thus the existence of a unitary representation of the modular group $SL(2, \mathbb{Z})$, which plays a role in topological quantum field theory.

It follows that the net $\mathcal{B} \supset \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ obtained by the LR construction is a field algebra for $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$, namely $\mathcal{B}$ has no superselection sector (localizable in a bounded interval) and there is a generating family of sectors of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ that are implemented by isometries in $\mathcal{B}$. Indeed $\mathcal{B}$ is a the crossed product of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ by the tensor category of all its sectors.

As shown by Masuda [30], Ocneanu's asymptotic inclusion [35] and the Longo-Rehren inclusion in [28] are, from the categorical viewpoint, essentially the same constructions. The construction of the asymptotic inclusion gives a new subfactor $\mathcal{M} \vee (\mathcal{M}' \cap \mathcal{M}_\infty) \subset \mathcal{M}_\infty$ from a hyperfinite II_1 subfactor $\mathcal{N} \subset \mathcal{M}$ with finite index and finite depth and it is a subfactor analogue of the quantum double construction of Drinfel′d [11], as noted by Ocneanu. That is, the tensor category of the $\mathcal{M}_\infty$-$\mathcal{M}_\infty$ bimodules arising from the new subfactor is regarded a "quantum double" of the original category of $\mathcal{M}$-$\mathcal{M}$ (or $\mathcal{N}$-$\mathcal{N}$) bimodules.

On the other hand, as shown in [33], the Longo-Rehren construction gives the quantum double of the original tensor category of endomorphisms. (See also [12, Chapter 12] for a general theory of asymptotic inclusions and their relations to topological quantum field theory.)

Our result thus shows that the inclusion arising from two separated intervals as above gives the quantum double of the tensor category of all localized endomorphisms. However, as the braiding symmetry is non-degenerate, the quantum double will be isomorphic to the subcategory of the trivial doubling of the original tensor category corresponding to the connected component of the identity in the fusion graph. Indeed, in the conformal case, multi-interval inclusions are self-dual.

For our results conformal invariance is not necessary, although conformal nets provide the most interesting situation where they can be applied. We may deal with an arbitrary net on $\mathbb{R}$, provided it is strongly additive (a property equivalent to Haag duality on $\mathbb{R}$ if conformal invariance is assumed) and there exists a cyclic and separating vector for the von Neumann algebras of half-lines (vacuum), such that the corresponding modular conjugations act geometrically as PCT symmetries (automatic in the conformal case). We will deal with this more general context.

Our paper is organized as follows. Then we consider representations. The second section discusses general properties of multi-interval inclusions and in particular gives motivations for the strong additivity assumption. The third section enters the core of our analysis and contains a first inequality between the global index of the sectors and the index of the 2-interval subfactor. In Section 4 we study the structure of sectors associated with the LR net, an analysis mostly based on the braiding symmetry, the work of Izumi [22] and the α-induction, which has been introduced in [28] and further studied in [49, 2, 3]. Section 5 combines and develops the previous analysis to obtain our main results for the 2-interval inclusion. These results are extended to the case of n-interval inclusions in Section 6. We then we illustrate our results in models and examples in Section 7. We collect in Appendix A the results the universal crossed product description of the LR inclusion and of its multiple iterated occurring in our analysis. We include a further appendix concerning the disintegration of locally normal or localizable representations into irreducible ones, that is needed in the paper; these results have however their own interest.

For basic facts concerning conformal nets of von Neumann algebras on $\mathbb{R}$ or S^1, the reader is referred to [17, 28], see also the Appendix B.

2 General properties

In this section we shortly examine a few elementary properties for nets of von Neumann algebras, partly to motivate our strong additivity assumption in the main body of the paper, and partly to examine relations with dual nets. To get our main result, the reader may however skip this part, except for Proposition 5, and get directly to the next section, where we will restrict our study to completely rational nets.

In this section, $\mathcal{A}$ will be a local irreducible net of von Neumann algebras on S^1, namely, $\mathcal{A}$ is an inclusion preserving map

$$\mathcal{I} \ni I \mapsto \mathcal{A}(I)$$

from the set $\mathcal{I}$ of intervals (open, non-empty sets with contractible closure) of S^1 to von Neumann algebras on a fixed Hilbert $\mathcal{H}$ space such that $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute if $I_1 \cap I_2 = \varnothing$ and $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$, where $\vee$ denotes the von Neumann algebra generated.

If $E \subset S^1$ is any set, we put

$$\mathcal{A}(E) \equiv \bigvee\{\mathcal{A}(I) : \ I \in \mathcal{I}, \ I \subset E\}$$

and set

$$\hat{\mathcal{A}}(E) \equiv \mathcal{A}(E')'$$

with $E' \equiv S^1 \smallsetminus E$.[3]

We shall assume Haag *duality* on S^1, which automatically holds if $\mathcal{A}$ is conformal [4], namely,

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I},$$

thus $\hat{\mathcal{A}}(I) = \mathcal{A}(I)$, $I \in \mathcal{I}$, but for a disconnected set $E \subset S^1$,

$$\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$$

is in general a non-trivial inclusion.

We shall say that $E \subset S^1$ is an *n-interval* if both E and E' are unions of n intervals with disjoint closures, namely

$$E = I_1 \cup I_2 \cup \cdots \cup I_n, \quad I_i \in \mathcal{I},$$

where $\bar{I}_i \cap \bar{I}_j = \varnothing$ if $i \neq j$. The set of all n-intervals will be denoted by $\mathcal{I}_n$.

Recall that $\mathcal{A}$ is *n-regular*, if $\mathcal{A}(S^1 \smallsetminus \{p_1, \dots p_n\}) = B(\mathcal{H})$ for any $p_1, \dots p_n \in S^1$.

Notice that $\mathcal{A}$ is 2-regular if and only if the $\mathcal{A}(I)$'s are factors, since we are assuming Haag duality, and that $\mathcal{A}$ is 1-regular if for each point $p \in S^1$

$$\bigcap_n \mathcal{A}(I_n) = \mathbb{C} \tag{2}$$

if $I_n \in \mathcal{I}$ and $\bigcap_n I_n = \{p\}$.

[3]The results in this section are also valid for nets of von Neumann algebras on $\mathbb{R}$, if $\mathcal{I}$ denotes the set of non-empty bounded open intervals of $\mathbb{R}$ and $E' = \mathbb{R} \smallsetminus E$ for $E \subset \mathbb{R}$.

Proposition 1. *The following are equivalent for a fixed $n \in \mathbb{N}$:*

(i) *The inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is irreducible for $E \in \mathcal{I}_n$.*

(ii) *The net $\mathcal{A}$ is $2n$-regular.*

Proof With $E = I_1 \cup \cdots \cup I_n$ and $p_1, \ldots, p_{2n}$ the $2n$ boundary points of E, we have $\mathcal{A}(E)' \cap \hat{\mathcal{A}}(E) = \mathbb{C}$ if and only if $\mathcal{A}(E) \vee \hat{\mathcal{A}}(E)' = B(\mathcal{H})$, which holds if and only if $\mathcal{A}(E) \vee \mathcal{A}(E') = B(\mathcal{H})$, thus if and only if $\mathcal{A}(S^1 \setminus \{p_1, \ldots, p_{2n}\}) = B(\mathcal{H})$, namely $\mathcal{A}$ is $2n$-regular. $\square$

If $\mathcal{A}$ is *strongly additive*, namely,

$$\mathcal{A}(I) = \mathcal{A}(I \setminus \{p\})$$

where $I \in \mathcal{I}$ and p is an interior point of I, then $\mathcal{A}$ is n-regular for all $n \in \mathcal{N}$, thus all $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ are irreducible inclusions of factors, $E \in \mathcal{I}_n$.

A partial converse holds.

If $\mathcal{N} \subset \mathcal{M}$ are von Neumann algebras, we shall say that $\mathcal{N} \subset \mathcal{M}$ has *finite-index* if the Pimsner-Popa inequality [38] holds, namely there exists $\lambda > 0$ and a conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ with $\mathcal{E}(x) \geq \lambda x$, for all $x \in \mathcal{M}_+$, and denote the index by

$$[\mathcal{M} : \mathcal{N}]_{\mathcal{E}} = \lambda^{-1}$$

with λ the best constant for the inequality to hold and

$$[\mathcal{M} : \mathcal{N}] = [\mathcal{M} : \mathcal{N}]_{\min} = \inf_{\mathcal{E}} [\mathcal{M} : \mathcal{N}]_{\mathcal{E}}$$

denotes the minimal index, (see [20] for an overview).

Recall that $\mathcal{A}$ is *split* if there exists an intermediate type I factor between $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ whenever I_1, I_2 are intervals and the closure $\bar{I}_1$ is contained in the interior of I_2. This implies (indeed it is equivalent to e.g. if the $\mathcal{A}(I)$'s are factors) that $\mathcal{A}(I_1) \vee \mathcal{A}(I_2')$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2')$ (cf.[10]) . For a conformal net, the split property holds if $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, cf. [8].

Notice that if $\mathcal{A}$ is split and $\mathcal{A}(I)$ is a factor for $I \in \mathcal{I}$, then $\mathcal{A}(E)$ is a factor for $E \in \mathcal{I}_n$ for any n.

Proposition 2. *Let $\mathcal{A}$ be split and 1-regular. If there exists a constant $C > 0$ such that*

$$[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < C \quad \forall \quad E \in \mathcal{I}_2,$$

then

$$[\mathcal{A}(I) : \mathcal{A}(I \setminus \{p\})] < C \quad \forall I \in \mathcal{I},\ p \in I.$$

Proof With $I \in \mathcal{I}_2$ and $p \in I$ an interior point, let $I_1, I_2 \in \mathcal{I}$ be the connected components of $I \smallsetminus \{p\}$, let $I_2^{(n)} \subset I_2$ be an increasing sequence of intervals with one boundary point in common with I such that $p \notin \overline{I_2^{(n)}}$ and $\bigcup_n I_2^{(n)} = I_2$. Then $E_n \equiv I_1 \cup I_2^{(n)} \in \mathcal{I}_2$ and we have

$$\mathcal{A}(E_n) \nearrow \mathcal{A}(I \smallsetminus \{p\}),$$
$$\hat{\mathcal{A}}(E_n) \nearrow \mathcal{A}(I),$$

where $\mathcal{N}_n \nearrow \mathcal{N}$ means $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots$ and $\mathcal{N} = \bigvee \mathcal{N}_n$, while $\mathcal{N}_n \searrow \mathcal{N}$ will mean $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \cdots$ and $\mathcal{N} = \bigcap \mathcal{N}_n$. The first relation is clear by definition. The second relation follows because

$$\hat{\mathcal{A}}(E_n)' = \mathcal{A}(E_n') = \mathcal{A}(I') \vee \mathcal{A}(L_n),$$

where $E_n' \in \mathcal{I}_2$, $E_n = I' \cup L_n$, and $\bigcap L_n = \{p\}$, therefore $\mathcal{A}(L_n) \searrow \mathcal{C}$. By the split property $\mathcal{A}(I') \vee \mathcal{A}(L_n) \cong \mathcal{A}(I') \otimes \mathcal{A}(L_n)$, hence by eq. (2)

$$\mathcal{A}(E_n') \searrow \mathcal{A}(I'),$$

thus

$$\hat{\mathcal{A}}(E_n) \nearrow \mathcal{A}(I).$$

The rest of the proof is the consequence of the following general proposition. $\square$

Proposition 3. a) *Let*

$$\begin{array}{ccccccc} \mathcal{N}_1 & \subset & \mathcal{N}_2 & \subset & \cdots & \subset & \mathcal{N} \\ \cap & & \cap & & & & \cap \\ \mathcal{M}_1 & \subset & \mathcal{M}_2 & \subset & \cdots & \subset & \mathcal{M} \end{array}$$

be von Neumann algebras, $\mathcal{N} = \bigvee \mathcal{N}_i$, $\mathcal{M} = \bigvee \mathcal{M}_i$,

b) *or let*

$$\begin{array}{ccccccc} \mathcal{N}_1 & \supset & \mathcal{N}_2 & \supset & \cdots & \supset & \mathcal{N} \\ \cap & & \cap & & & & \cap \\ \mathcal{M}_1 & \supset & \mathcal{M}_2 & \supset & \cdots & \supset & \mathcal{M} \end{array}$$

be von Neumann algebras, $\mathcal{N} = \bigcap \mathcal{N}_i$, $\mathcal{M} = \bigcap \mathcal{M}_i$.

Then

$$[\mathcal{M} : \mathcal{N}] \leq \liminf_{i \to \infty} [\mathcal{M}_i : \mathcal{N}_i].$$

Proof It is sufficient to prove the result in the situation b) as the case a) will follow after taking commutants. We may assume $\liminf_{i\to\infty}[\mathcal{M}_i : \mathcal{N}_i] < \infty$.

Let $\mathcal{E}_i : \mathcal{M}_i \to \mathcal{N}_i$ be an expectation and $\lambda > \liminf_{i\to\infty}[\mathcal{M}_i, : \mathcal{N}_i]_{\mathcal{E}_i}$. Then there exists i_0 such that for all $x \in \mathcal{M}_i^+$, $i \geq i_0$,

$$\mathcal{E}_i(x) \geq \lambda^{-1} x.$$

Let $\mathcal{E}_i^{(0)} = \mathcal{E}_i|_{\mathcal{M}}$, considered as a map from $\mathcal{M}$ to $\mathcal{N}_i$, and let $\mathcal{E}$ be a weak limit point of $\mathcal{E}_i^{(0)}$. Then

$$\mathcal{E}(x) \geq \lambda^{-1}x, \qquad x \in \mathcal{M}_+,$$

and $\mathcal{E}(\mathcal{M}) \subset \bigcap_i \mathcal{N}_i = \mathcal{N}$, moreover $\mathcal{E}|_{\mathcal{N}} = \mathrm{id}$, because $\mathcal{E}_i|_{\mathcal{N}} = \mathrm{id}$. Thus $\mathcal{E}$ is an expectation of $\mathcal{M}$ onto $\mathcal{N}$ and

$$[\mathcal{M} : \mathcal{N}] \leq [\mathcal{M} : \mathcal{N}]_{\mathcal{E}} \leq \lambda.$$

As $\mathcal{E}_i$ is arbitrary, we thus have $[\mathcal{M} : \mathcal{N}] \leq \liminf_{i\to\infty}[\mathcal{M}_i, : \mathcal{N}_i]$. $\square$

Recall now that the *dual net* $\mathcal{A}^d$ of $\mathcal{A}$ is the net on the intervals of $\mathbb{R}$ defined by $\mathcal{A}^d(I) \equiv \mathcal{A}(\mathbb{R} \smallsetminus I)'$, where we have chosen a point $\infty \in S^1$ and identified S^1 with $\mathbb{R} \cup \{\infty\}$.

Note that if $\mathcal{A}$ is conformal, then Haag duality automatically holds [18] and the dual net $\mathcal{A}^d$ is also a conformal net which is moreover strongly additive; furthermore $\mathcal{A} = \mathcal{A}^d$, if and only if $\mathcal{A}$ is strongly additive, if and only if Haag duality holds on $\mathbb{R}$.

Corollary 4. *In the hypothesis of Proposition 2, let $\mathcal{A}^d$ be the dual net on $\mathbb{R}$, then*

$$\mathcal{A}(I) \subset \mathcal{A}^d(I)$$

has finite index for all bounded intervals I of $\mathbb{R}$.

Proof Denoting $I_1 = I'$, the complement of I in S^1, the commutant of the inclusion $\mathcal{A}(I) \subset \mathcal{A}^d(I)$ is $\mathcal{A}(I_1 \smallsetminus \{\infty\}) \subset \mathcal{A}(I_1)$, and this has finite index. $\square$

We have no example where $\mathcal{A}(I) \subset \mathcal{A}^d(I)$ is non-trivial with finite index and $\mathcal{A}$ is conformal; therefore the equality $\mathcal{A}(I) = \mathcal{A}^d(I)$, i.e. strong additivity, might follow from the assumptions in Corollary 2 in the conformal case.

Proposition 5. *Let $\mathcal{A}$ be split and strongly additive, then*

(a) *The index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is independent of $E \in \mathcal{I}_2$.*

(b) *The inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is irreducible for $E \in \mathcal{I}_2$.*

Proof Statement (b) is immediate by Proposition 1.

Concerning (a), let $E = I_1 \cup I_2$ and $\tilde{E} = I_1 \cup \tilde{I}_2$ where $\tilde{I}_2 \supset I_2$ are intervals and $I_0 \equiv \tilde{I}_2 \smallsetminus I_2$. Assuming $\lambda^{-1} \equiv [\hat{\mathcal{A}}(\tilde{E}) : \mathcal{A}(\tilde{E})] < \infty$, let $\mathcal{E}_{\tilde{E}}$ be the corresponding expectation with λ-bound. Of course $\mathcal{E}_{\tilde{E}}$ is the identity on $\mathcal{A}(I_0)$, hence

$$\mathcal{E}_{\tilde{E}}(\hat{\mathcal{A}}(E)) \subset \mathcal{A}(I_0)' \cap \mathcal{A}(\tilde{E}) = \mathcal{A}(E)$$

where last equality follows at once by the split property and strong additivity as $\mathcal{A}(I_0)' \cap \mathcal{A}(\tilde{I}_2) = \mathcal{A}(I_2)$.

Therefore $\mathcal{E}_{\tilde{E}} \mid_{\hat{\mathcal{A}}(E)} = \mathcal{E}_E$ showing

$$[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] \leq [\hat{\mathcal{A}}(\tilde{E}) : \mathcal{A}(\tilde{E})],$$

where we omit the symbol "min" as the expectation is unique. Thus the index decreases by decreasing the 2-interval. Taking commutants, it also increases, hence it is constant. $\square$

Corollary 6. *Let $\mathcal{A}$ satisfy the assumption of Proposition 2 and let $\mathcal{A}^d$ be the dual net on $\mathbb{R}$ of $\mathcal{A}$. Then*

$$[\widehat{\mathcal{A}^d}(E) : \mathcal{A}^d(E)] < \infty \quad \forall E \in \mathcal{I}_2.$$

Proof We fix the point ∞ and may assume $E = I_1 \cup I_2$ with $\infty \in I_2$. Set $E' = I_3 \cup I_4$ with $I_3 \ni \infty$. Then $\mathcal{A}^d(I_3) = \mathcal{A}(I_3)$, $\mathcal{A}^d(I_2) = \mathcal{A}(I_2)$ and we have

$$\begin{aligned} \mathcal{A}(E) &\subset \mathcal{A}^d(I_1) \vee \mathcal{A}(I_2) \\ &= \mathcal{A}^d(E) \subset \widehat{\mathcal{A}^d}(E) \\ &= (\mathcal{A}(I_3) \vee \mathcal{A}^d(I_4))' \subset (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))' = \hat{\mathcal{A}}(E). \end{aligned}$$

$\square$

Anticipating results in the following, we have:

Corollary 7. *Let $\mathcal{A}$ be a local irreducible conformal split net on S^1. If $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \mathbf{I}_{\text{global}} < \infty$, $E \in \mathcal{I}_2$, then $\mathcal{A}$ is n-regular for all $n \in \mathbb{N}$.*

Proof If ρ is an irreducible endomorphism of $\mathcal{A}$ localized in an interval I, then $\rho|_{\mathcal{A}(I)}$ is irreducible [17]. Therefore, by Th. 9 (and comments there after) and Prop. 36, the assumptions imply that if $E \in \mathcal{I}_2$ then $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is the LR inclusion associated with the system of all irreducible sectors, which is irreducible. Then $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is irreducible for all $E \in \mathcal{I}_n$ as we shall see in Sect. 6. By Prop. 1 this implies the regularity for all n. $\square$

In view of the above results, it is natural to deal with strongly additive nets, when considering multi-interval inclusions of local algebras and thus to deal with nets of factors on $\mathbb{R}$, as we shall do in the following.

3 Completely rational nets

In this section we will introduce the notion of completely rational net, that will be the main object of our study in this paper, and get a first analysis.

In the following, we shall denote by $\mathcal{I}$ the set of bounded open non-empty intervals of $\mathbb{R}$, set $I' = \mathbb{R} \smallsetminus I$ and define $\mathcal{A}(E) = \bigvee\{\mathcal{A}(I), I \subset E, I \in \mathcal{I}\}$ for $E \subset \mathbb{R}$. We again denote by $\mathcal{I}_n$ the set of unions of n elements of $\mathcal{I}$ with pairwise disjoint closures. [4]

Definition 8. *A local irreducible net $\mathcal{A}$ of von Neumann algebras on the intervals of $\mathbb{R}$ is called completely rational if the following holds:*

(a) *Haag duality on* $\mathbb{R}$: $\mathcal{A}(I') = \mathcal{A}(I)'$, $I \in \mathcal{I}$,

(b) $\mathcal{A}$ *is strongly additive,*

(c) $\mathcal{A}$ *satisfies the split property,*

(d) $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty$, *if* $E \in \mathcal{I}_2$,

Note that, if $\mathcal{A}$ is the restriction to $\mathbb{R}$ of a local conformal net on S^1 (namely a local net which is Möbius covariant with positive energy and cyclic vacuum vector) then (a) is equivalent to (b), cf. [18].

We shall denote by $\mu_{\mathcal{A}} = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ the index of the irreducible inclusion of factors $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ in case $\mu_{\mathcal{A}}$ is independent of $E \in \mathcal{I}_2$, in particular if $\mathcal{A}$ is split, by Proposition 5.

By a *sector* $[\rho]$ of $\mathcal{A}$ we shall mean the equivalence class of a localized endomorphism ρ of $\mathcal{A}$, that will always be assumed to be transportable i.e. localizable in each bounded interval I (see also Appendix B). Unless otherwise specified, a localized endomorphism ρ has finite dimension. If ρ is localized in the interval I, its restriction $\rho|_{\mathcal{A}(I)}$ is an endomorphism of $\mathcal{A}(I)$, thus it gives rise to a *sector* of the factor $\mathcal{A}(I)$ (i.e. a normal unital endomorphism of $\mathcal{A}(I)$ modulo inner automorphisms of $\mathcal{A}(I)$ [25]) and it will be clear from the context which sense will be attributed to the term sector.

The reader unfamiliar with the sector strucure is referres to [25, 28, 17] and to the Appendix B.

Let $E = I_1 \cup I_2 \in \mathcal{I}_2$ and ρ and σ irreducible endomorphisms of $\mathcal{A}$ localized respectively in I_1 and in I_2. Then $\rho\sigma$ restricts to an endomorphism of $\mathcal{A}(E)$, since both ρ and σ restrict.

Denote by γ_E the canonical endomorphism of $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$ and $\lambda_E \equiv \gamma_E|_{\mathcal{A}(E)}$.

Theorem 9. *Let $\mathcal{A}$ be completely rational. With the above notations, $\rho\sigma|_{\mathcal{A}(E)}$ is contained in λ_E if and only if σ is conjugate to ρ. In this case $\rho\sigma|_{\mathcal{A}(E)} \prec \lambda_E$ with multiplicity one.*

Proof By [28] $\rho\sigma|_{\mathcal{A}(E)} \prec \lambda_E$ if and only if there exists an isometry $v \in \hat{\mathcal{A}}(E)$ such that

$$vx = \rho\sigma(x)v \quad \forall x \in \mathcal{A}(E). \tag{3}$$

[4] There will be no conflict with the notations in the previous section as the point ∞ does not contribute to the local algebras and we may extend $\mathcal{A}$ to S^1 setting $\mathcal{A}(I) \equiv \mathcal{A}(I \smallsetminus \{\infty\})$, see Appendix B.

If equation (3) holds, then it holds for $x \in \mathcal{A}(I)$ for all $I \in \mathcal{I}$ by strong additivity, hence $\sigma = \bar{\rho}$.

Conversely, if $\sigma = \bar{\rho}$, then there exists an isometry $v \in \mathcal{A}(I)$ such that $vx = \rho\sigma(x)v$ for all $x \in \mathcal{A}(I)$, where I is the interval $I \supset E$ given by $I = I_1 \cup I_2 \cup \bar{I}_3$ with I_3 the bounded connected component of E'.

Since ρ and σ act trivially on $\mathcal{A}(I_3)$, we have

$$v \in \mathcal{A}(I_3)' \cap \mathcal{A}(I),$$

but

$$\mathcal{A}(I_3)' \cap \mathcal{A}(I) = (\mathcal{A}(I_3) \vee \mathcal{A}(I'))' = \mathcal{A}(E')' = \hat{\mathcal{A}}(E),$$

therefore equation (3) holds true. As the ρ and σ are irreducible, the isometry v in (3) unique up to a phase and this is equivalent to $\rho\bar{\rho}|_{\mathcal{A}(E)} \prec \lambda_E$ with multiplicity one. $\square$

We remark that in the above theorem strong additivity is not necessary for $\rho\bar{\rho} \prec \lambda_E$, as can be replaced by the factoriality of $\mathcal{A}(E)$, equivalently of $\hat{\mathcal{A}}(E)$; this holds e.g. in the conformal case.

Moreover also the split property is unnecessary, it has not been used.

We shall say that the net $\mathcal{A}$ on $\mathbb{R}$ has a *modular PCT* symmetry, if there exists a cyclic separating (vacuum) vector Ω for each $\mathcal{A}(I)$, if I is a half-line (Reeh-Schlieder property), and the modular conjugation J of $\mathcal{A}(a, \infty)$ with respect to Ω has the geometric property

$$J\mathcal{A}(I + a)J = \mathcal{A}(-I + a), \quad I \in \mathcal{I}, \quad a \in \mathbb{R}. \tag{4}$$

This is automatic if $\mathcal{A}$ is conformal, see [4, 15]. It easy to see that the modular PCT property implies translation covariance, where the translation unitaries are products of modular conjugations, but positivity of the energy does not necessarily holds.

Note that eq. (4) implies Haag duality for half-lines

$$\mathcal{A}(-\infty, a)' = \mathcal{A}(a, \infty), \quad a \in \mathbb{R}\ .$$

Setting $j \equiv \mathrm{Ad}J$, the conjugate sector exists and it is given by the formula [16]

$$\bar{\rho} = j \cdot \rho \cdot j.$$

Corollary 10. *If $\mathcal{A}$ is completely rational with modular PCT, then $\mathcal{A}$ is rational, namely there are only finitely many irreducible sectors $[\rho_0], [\rho_1], \ldots, [\rho_n]$ with finite dimension and we have*

$$\sum_{i=0}^{n} d(\rho_i)^2 \leq \mu_{\mathcal{A}}. \tag{5}$$

Proof It is sufficient to show this last inequality. By the split property, the endomorphisms $\rho_i\bar{\rho}_i|_{\mathcal{A}(E)}$ can be identified with the endomorphisms $\rho_i \otimes \bar{\rho}_i$ on $\mathcal{A}(I_1)\otimes\mathcal{A}(I_2)$, hence they are mutually inequivalent.

By Theorem 9,

$$\bigoplus_{i=1}^{n} \rho_i\bar{\rho}_i|_{\mathcal{A}(E)} \prec \lambda_E, \tag{6}$$

hence

$$\mu_{\mathcal{A}} = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = d(\lambda_E) \geq \sum d(\rho_i)^2.$$

□

We now give a partial converse to Theorem 9.

Lemma 11. *Let $\mathcal{A}$ be completely rational and let $\mathcal{E}_E$ be the conditional expectation $\hat{\mathcal{A}}(E) \to \mathcal{A}(E)$.*

(a) *If $E \subset \tilde{E}$ and $E, \tilde{E} \in \mathcal{I}_2$, then*

$$\mathcal{E}_{\tilde{E}}|_{\hat{\mathcal{A}}(E)} = \mathcal{E}_E.$$

(b) *There exists a canonical endomorphism $\gamma_{\tilde{E}}$ of $\hat{\mathcal{A}}(\tilde{E})$ to $\mathcal{A}(\tilde{E})$ such that $\gamma|_{\hat{\mathcal{A}}(E)}$ is a canonical endomorphism of $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$ and satisfies*

$$\gamma|_{\hat{\mathcal{A}}(E)'\cap\mathcal{A}(\tilde{E})} = \mathrm{id}.$$

Proof (a) has been shown in the proof of Proposition 5.

(b) is an immediate variation of [16, Proposition 2.3] and [28, Theorem 3.2].

□

Theorem 12. *Let $\mathcal{A}$ be completely rational. Given $E \in \mathcal{I}_2$, λ_E extends to a localized (transportable) endomorphism λ of $\mathcal{A}$ such that $\lambda|_{\mathcal{A}(I)} = \mathrm{id}$, if $I \subset E'$, $I \in \mathcal{I}$. Moreover, $d(\lambda) = d(\lambda_E) = \mu_{\mathcal{A}}$.*

In particular, if $\mathcal{A}$ is conformal, then λ is Möbius covariant with positive energy.

Proof Let $E = (a, b) \cup (c, d)$ where $a < b < c < d$ and $\tilde{E} = (a', b) \cup (c, d')$ where $a' < a$ and $d' > d$. By Lemma 11 we have a $\gamma_{\tilde{E}}$ with $\lambda_{\tilde{E}}|_{\mathcal{A}(I)} = \mathrm{id}$, if $I \subset \mathcal{I}$, $I \in \tilde{E} \smallsetminus E$.

Analogously there is a canonical endomorphism $\gamma : \hat{\mathcal{A}}(\tilde{E}) \to \mathcal{A}(\tilde{E})$ acting trivially on $\mathcal{A}(E)$. We may write

$$\gamma_{\tilde{E}} = \mathrm{Ad}\ u \cdot \gamma$$

with $u \in \mathcal{A}(\tilde{E})$, hence

$$\lambda_{\tilde{E}} = \mathrm{Ad}u \cdot \lambda, \quad \lambda = \gamma|_{\mathcal{A}(\tilde{E})}.$$

Since $\gamma\mid_{\mathcal{A}(a,b)}= \mathrm{id}$, $\gamma\mid_{\mathcal{A}(c,d)}= \mathrm{id}$, we have

$$\lambda_{\tilde{E}} = \mathrm{Ad}u \quad \text{on } \mathcal{A}(a,b), \mathcal{A}(c,d).$$

Therefore, the formula

$$\tilde{\lambda} = \mathrm{Ad}u$$

defines an endomorphism of $\mathcal{A}(a,d)$ acting trivially an $\mathcal{A}(b,c)$, with

$$\tilde{\lambda}|_{\mathcal{A}((a,b)\cup(c,d))} = \lambda_E.$$

We may also have chosen γ "localized" in $(a',a'')\cup(d'',d')$ with $a' < a'' < a$ and $d < d'' < d'$ so that we may assume $\tilde{\lambda}$ to act trivially on $\mathcal{A}((a'',b)\vee(c,d''))$.

Letting $a', a'' \to -\infty$ and $d'', d' \to +\infty$, we construct, by an inductive limit of the $\tilde{\lambda}$'s, an endomorphism λ of the quasi-local C^*-algebra $\overline{\bigcup_{s>0}\mathcal{A}(-s,s)}$.

Clearly, λ is localized in (a,d), acts trivially on $\mathcal{A}(b,c)$ and is transportable. Moreover, λ has finite index as the operators $R, \bar{R} \in (i, \lambda^2)$ in the standard solution for the conjugate equation [25, 29]

$$\bar{R}^*\bar{\lambda}(R) = 1, \quad R^*\lambda(\bar{R}) = 1,$$

on $\hat{\mathcal{A}}(E)$ give the same relation on $\mathcal{A}(I)$ for any $I \supset E$, $I \in \mathcal{I}$.

If $\mathcal{A}$ is conformal, then ρ is covariant with respect to translations and dilations by [17]. As we may vary the point ∞, λ is covariant with respect to dilations and translations with respect to different point at ∞, hence λ is Möbius covariant. $\square$

Lemma 13. *Let $\mathcal{A}$ be completely rational. Then there are at most $\lfloor \mu_{\mathcal{A}} \rfloor$ mutually different irreducible sectors of $\mathcal{A}$ (with finite or infinite dimension).*

Proof Consider the family $\{[\rho_\lambda]\}$ of all irreducible sectors and let N be the cardinality of this family. With $E = I_1 \cup I_2 \in \mathcal{I}_2$, we may assume that each ρ_λ is localized in I_1 and choose endomorphisms σ_λ equivalent to ρ_λ and localized in I_2. Let then $u_\lambda \in (\rho_\lambda, \sigma_\lambda) \subset \hat{\mathcal{A}}(E)$ be a unitary intertwiner and $\mathcal{E}$ the conditional expectation from $\hat{\mathcal{A}}(E)$ to $\mathcal{A}(E)$. Since

$$u_\lambda \rho_\lambda(x) = \sigma_\lambda(x) u_\lambda = x u_\lambda\ , \quad \forall x \in \mathcal{A}(I_1),$$

we have

$$u^*_{\lambda'} u_\lambda\, \rho_\lambda(x) = \rho_{\lambda'}(x)\, u^*_{\lambda'} u_\lambda\ , \quad \forall x \in \mathcal{A}(I_1),$$

hence $T = \mathcal{E}(u^*_{\lambda'}u_\lambda) \in \mathcal{A}(E)$ intertwines $\rho_\lambda|\mathcal{A}(I_1)$ and $\rho_{\lambda'}|\mathcal{A}(I_1)$. The split property allowing us to identify $\mathcal{A}(E)$ and $\mathcal{A}(I_1)\otimes\mathcal{A}(I_2)$, every state φ in $\mathcal{A}(I_2)_*$ gives rise to a conditional expectation $\mathcal{E}_\varphi : \mathcal{A}(E) \to \mathcal{A}(I_1)$. Then $\mathcal{E}_\varphi(T) \in (\rho_\lambda, \rho_{\lambda'})$, and the inequivalence of $\rho_\lambda|\mathcal{A}(I_1), \rho_{\lambda'}|\mathcal{A}(I_1)$, see above, entails $\mathcal{E}_\varphi(T) = 0$. Since this holds for every $\varphi \in \mathcal{A}(I_2)_*$ we conclude

$$T = \mathcal{E}(u^*_{\lambda'}u_\lambda) = 0, \quad \lambda' \neq \lambda.$$

Let $\mathcal{M}$ be the Jones extension of $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$, $e \in \mathcal{M}$ the Jones projection implementing $\mathcal{E}$ and let $\mathcal{E}_1 : \mathcal{M} \to \hat{\mathcal{A}}(E)$ be the dual conditional expectation. Then $eu^*_{\lambda'}u_\lambda e = 0$ if $\lambda' \neq \lambda$ and therefore the $e_\lambda \equiv u_\lambda e u^*_\lambda$ are mutually orthogonal projections in $\mathcal{M}$ with $\mathcal{E}_1(e_\lambda) = \mu_{\mathcal{A}}^{-1}$. Since their (strong) sum $p = \sum_\lambda e_\lambda$ is again an orthogonal projection we have $p \leq 1$ and thus $\mathcal{E}_1(p) \leq \mathcal{E}(1) = 1$. This implies the bound $N\mu_{\mathcal{A}}^{-1} \leq 1$ and thus our claim. □

We shall say that a sector $[\rho]$ is of *type I* if $\vee_{I\in\mathcal{I}}\rho(\mathcal{A}(I))$ is a type I von Neumann algebra, namely ρ is a type I representation of the quasi local C*-algebra $\overline{\cup_{s>o}\mathcal{A}(-s,s)}$.

Corollary 14. *If $\mathcal{A}$ is completely rational on a separable Hilbert space, then all factor representations of $\mathcal{A}$ on separable Hilbert spaces are of type I.*

Proof Assuming the contrary, by Corollary 59 we get an infinite family $[\rho_\lambda]$ of different irreducible sectors. This is in contradiction with the preceding proposition. □

We end this section by recalling the following (see [10]).

Proposition 15. *Let $\mathcal{A}$ be a completely rational net with modular PCT on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is separable.*

Proof We chose a pair $I \subset \tilde{I}$ of intervals and a type I factor $\mathcal{N}$ between $\mathcal{A}(I)$ and $\mathcal{A}(\tilde{I})$. The vacuum vector Ω is separating for $\mathcal{A}(\tilde{I})$, hence for $\mathcal{N}$. Thus $\mathcal{N}$ admits a faithful normal state, hence it is countably decomposable. Being of type I, $\mathcal{N}$ is countably generated. Being cyclic for $\mathcal{A}(I)$, Ω is also cyclic for $\mathcal{N}$ so $\mathcal{H} = \overline{\mathcal{N}\Omega}$ is separable. □

4 The structure of sectors for the (time $= 0$) LR net

This section contains a study of the sector strucure for the net obtained by the LR construction, by means of the braiding symmetry. It will be continued in the next section by a different approach.

Let $\mathcal{N}$ be an infinite factor and $\{[\rho_i]\}$ a rational system of sectors of $\mathcal{N}$, namely the $[\rho_i]$'s form a finite family of mutually different irreducible finite-dimensional sectors of $\mathcal{N}$ which is closed under conjugation and taking the irreducible components of compositions. The identity sector is usually labeled as ρ_0. We call

$$\mathcal{M} \supset \mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}}$$

the *LR inclusion*, the canonical inclusion constructed in [28] where $\mathcal{M}$ is a factor, $\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}} \subset \mathcal{M}$ is irreducible with finite index and

$$\lambda = \bigoplus_i \rho_i \otimes \rho_i^{\mathrm{opp}}$$

for $\lambda \in \mathrm{End}(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$ as the restriction of $\gamma : \mathcal{M} \to \mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}}$. We shall give an alternative characterization of this inclusion in Proposition 45.

The same construction works in slightly more generality, by replacing $\mathcal{N}^{\mathrm{opp}}$ with a factor $\mathcal{N}_1$ and $\{\rho_i^{\mathrm{opp}}\}_i$ by $\{\rho_i^j\}_i \subset \mathrm{End}(\mathcal{N}_1)$ where $\rho \to \rho^j$ is an anti-linear invertible tensor functor of the tensor category generated by $\{\rho_i\}_i$ to the tensor category generated by $\{\rho_i^j\}_i$. Extensions of our results to this case are obvious, but sometimes useful, and will be considered possibly implicitly.

The following is due to Izumi [22]. Since it is easy to give a proof in our context, we include a proof here.

Lemma 16. *For every ρ_i, the $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sector $[\rho_i \otimes \mathrm{id}][\gamma] = [\mathrm{id} \otimes \bar{\rho}_i^{\mathrm{opp}}][\gamma]$ is irreducible and each irreducible $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sector arising from $\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}} \subset \mathcal{M}$ is of this form, where γ is regarded as an $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sector. If $[\rho_i] \neq [\rho_j]$ as $\mathcal{A}$-$\mathcal{A}$ sectors, then $[\rho_i \otimes \mathrm{id}][\gamma] \neq [\rho_j \otimes \mathrm{id}][\gamma]$ as $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sectors. We have $[\rho_i \otimes \rho_j^{\mathrm{opp}}][\gamma] = \sum_k N_{i\bar{j}}^k [\rho_k \otimes \mathrm{id}][\gamma]$ as $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sectors, where $N_{i\bar{j}}^k$ is the structure constant for $\{\rho_i\}_i$.*

Proof Set $[\sigma] = [\rho_i \otimes \mathrm{id}][\gamma]$ and compute $[\sigma][\bar{\sigma}]$. Since $[\bar{\gamma}] = [\iota]$, where ι is the inclusion map of $\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}}$ into $\mathcal{M}$ regarded as a $\mathcal{M}$-$(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$ sector, and $[\gamma][\iota] = [\lambda] = \sum_k [\rho_k \otimes \rho_k^{\mathrm{opp}}]$, we have $[\sigma][\bar{\sigma}] = \sum_k [\rho_i \rho_k \bar{\rho}_i \otimes \rho_k^{\mathrm{opp}}]$, and this contains the identity only once. So $[\rho_i \otimes \mathrm{id}][\gamma]$ is an irreducible $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sector. We can similarly prove that if $[\rho_i] \neq [\rho_j]$, then $[\rho_i \otimes \mathrm{id}][\gamma] \neq [\rho_j \otimes \mathrm{id}][\gamma]$.

We next set $[\sigma'] = [\mathrm{id} \otimes \bar{\rho}_i^{\mathrm{opp}}][\gamma]$ as an $(\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}})$-$\mathcal{M}$ sector, which is also irreducible. We compute

$$[\sigma][\bar{\sigma}'] = [\rho_i \otimes \mathrm{id}][\lambda][\mathrm{id} \otimes \rho_i^{\mathrm{opp}}] = \sum_k [\rho_i \rho_k \otimes \rho_k^{\mathrm{opp}} \rho_i^{\mathrm{opp}}],$$

which contains the identity only once. So we have $[\rho_i \otimes \mathrm{id}][\gamma] = [\mathrm{id} \otimes \bar{\rho}_i^{\mathrm{opp}}][\gamma]$.

The rest is now easy. $\square$

Let us now assume we have a strongly additive, Haag dual, irreducible net of factors $\mathcal{A}(I)$ on $\mathbb{R}$ with a rational *system of irreducible sectors* $\{[\rho_i]\}_i$ (with $\rho_0 = \mathrm{id}$), namely $\{[\rho_i]\}_i$ is a family of finitely many different irreducible sectors of $\mathcal{A}$ with finite dimension stable under conjugation and irreducible components of compositions.

One may construct [42, 28] a net of subfactors $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}} \subset \mathcal{B}$ so that the corresponding canonical endomorphism restricted on $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ is given by $\bigoplus_i \rho_i \otimes \rho_i^{\mathrm{opp}}$. We call this $\mathcal{B}$ the *LR net.* For $\mathcal{A}^{\mathrm{opp}}$, we use $\varepsilon^{\mathrm{opp}}(\rho_k^{\mathrm{opp}}, \rho_l^{\mathrm{opp}}) = j(\varepsilon(\rho_k, \rho_l))^*$, where j is the anti-isomorphism from $\mathcal{A}$ to $\mathcal{A}^{\mathrm{opp}}$. In order to distinguish two braidings, we write ε^+ and ε^-.

In other words, the LR net here is obtained as the time zero fields from the canonical two-dimensional net constructed in [28]: it is a local net, but if $\mathcal{A}$ is translation covariant with positive energy, $\mathcal{B}$ is translation covariant without the spectrum condition (the translation on $\mathcal{B}$ are space translations).

Then the net of inclusion $\mathcal{A}\otimes\mathcal{A}^{\rm opp}(I)\subset\mathcal{B}(I)$ is a net of subfactors in the sense of [28, Section 3], that is, we have a vacuum vector with Reeh-Schlieder property and consistent conditional expectations. We denote by γ the canonical endomorphism of $\mathcal{B}$ into $\mathcal{A}\otimes\mathcal{A}^{\rm opp}$ and its restriction to $\mathcal{A}\otimes\mathcal{A}^{\rm opp}$ by λ. We may suppose that also λ is localized in I. We shorten our notation by setting $\mathcal{N}\equiv\mathcal{A}(I)$ and $\mathcal{M}=\mathcal{B}(I)$. We thus have $\lambda(x)=\sum_i V_i(\rho_i\otimes\rho_i^{\rm opp})(x)V_i^*$, where V_i's are isometries in $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ with $\sum_i V_iV_i^*=1$.

We follow [21] for the terminology of $(\mathcal{N}\otimes\mathcal{N}^{\rm opp})$-$\mathcal{M}$ sectors, and so on, and study the sector structure of the subfactor $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$ in this section. In other words we study the sector structure of a single subfactor, not the structure of superselection sectors of the net, though we will be interested in this structure for the net in the next section. So the terminology *sector* is used for a subfactor, not for a net, in this section. However the inclusion $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$ has extra structure inherited by the inclusion of nets $\mathcal{A}\otimes\mathcal{A}^{\rm opp}\subset\mathcal{B}$, that is there are the left and right unitary braid symmetries and the extension and restriction maps. We first note that $\{[\rho_i\otimes\rho_j^{\rm opp}]\}_{ij}$ gives a system of irreducible $\mathcal{A}\otimes\mathcal{A}^{\rm opp}$-$\mathcal{A}\otimes\mathcal{A}^{\rm opp}$ sectors.

This gives the description of the principal graph of $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$ as a corollary as follows, which was first found by Ocneanu in [35] for his asymptotic inclusion. Label even vertices with (i,j) for $[\rho_i\otimes\rho_{\bar j}^{\rm opp}]$ and odd vertices with k for $[\rho_k\otimes{\rm id}][\gamma]$ and draw an edge with multiplicity N_{ij}^k between the even vertex (i,j) and the odd vertex k. The connected component of this graph containing the vertex $(0,0)$ is the principal graph of the subfactor $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$.

Now we consider the α-induction introduced in [28] and further studied in [49, 2], namely if σ is a localized endomorphism of $\mathcal{A}\otimes\mathcal{A}^{\rm opp}$, we set

$$\alpha_\sigma^\pm=\gamma^{-1}\cdot{\rm Ad}(\varepsilon^\pm(\sigma,\lambda))\cdot\sigma\cdot\gamma. \tag{7}$$

(The notation in [28] is $\sigma^{\rm ext}$).

Recall that if σ is an endomorphism of $\mathcal{A}\otimes\mathcal{A}^{\rm opp}$ localized in the interval I, then $\alpha_\sigma^\pm$ is an endomorphism of $\mathcal{B}$ localized in a positive/negative half-line containing I, yet, as shown in [2, I], $\alpha_\sigma^\pm$ restricts to an endomorphism of $\mathcal{M}=\mathcal{B}(I)$. We will denote this restriction by the same symbol $\alpha_\sigma^\pm$.

Lemma 17. *The $\mathcal{M}$-$\mathcal{M}$ sectors $[\alpha^+_{\rho_i\otimes{\rm id}}]$ are irreducible and mutually different.*

Proof We compute $\langle\alpha^+_{\rho_i\otimes{\rm id}},\alpha^+_{\rho_j\otimes{\rm id}}\rangle$, the dimension of the intertwiner space between $\alpha^+_{\rho_i\otimes{\rm id}}$ and $\alpha^+_{\rho_j\otimes{\rm id}}$, by using [2, I, Theorem 3.9]. This number is then equal to

$$\langle\bigoplus_k\rho_k\rho_i\otimes\rho_k^{\rm opp},\rho_j\otimes{\rm id}\rangle=\delta_{ij}.$$

This gives the conclusion. □

Lemma 18. *As $\mathcal{M}$-$\mathcal{M}$ sectors, we have $[\alpha^+_{\rho_i\otimes{\rm id}}]=[\alpha^+_{{\rm id}\otimes\rho_i^{\rm opp}}]$.*

Proof By a similar argument to the proof of the above lemma, we know that $[\alpha^+_{\mathrm{id}\otimes\rho_i^{\mathrm{opp}}}]$ is also irreducible. [2, I, Theorem 3.9] gives

$$\langle[\alpha^+_{\rho_i\otimes\mathrm{id}}],[\alpha^+_{\mathrm{id}\otimes\rho_i^{\mathrm{opp}}}]\rangle = \langle\bigoplus_k \rho_k\rho_i\otimes\rho_k^{\mathrm{opp}},\mathrm{id}\otimes\rho_i^{\mathrm{opp}}\rangle = 1,$$

which gives the conclusion. □

We then have the following corollary.

Corollary 19. *The set of irreducible $\mathcal{M}$-$\mathcal{M}$ sectors appearing in the decomposition of $\alpha^+_{\rho_i\otimes\rho_j^{\mathrm{opp}}}$ for all i,j is $\{[\alpha^+_{\rho_i\otimes\mathrm{id}}]\}_i$.*

The next theorem is useful for studying the subfactors arising from disconnected intervals for a conformal net. For the rest of this section we shall assume the braiding to be non-degenerate.

Theorem 20. *Assume the braiding to be non-degenerate and suppose an irreducible $\mathcal{M}$-$\mathcal{M}$ sector $[\beta]$ appears in decompositions of both $\alpha^+_{\rho_i\otimes\rho_j^{\mathrm{opp}}}$ and $\alpha^-_{\rho_k\otimes\rho_l^{\mathrm{opp}}}$ for some i,j,k,l. Then $[\beta]$ is the identity of $\mathcal{M}$.*

Proof α^+ and α^- map sectors localized in bounded intervals to soliton sectors localized in right unbounded and left unbounded half-lines, respectively. Hence $[\beta]$ is localized in a bounded interval. By the above corollary, we may assume that $[\beta]=[\alpha^+_{\rho_i\otimes\mathrm{id}}]$ for some i, hence $\rho_i\otimes\mathrm{id}$ must have trivial monodromy with λ, i.e., $\varepsilon(\rho_i\otimes\mathrm{id},\lambda)\varepsilon(\lambda,\rho_i\otimes\mathrm{id})=1$, which in turn gives $\varepsilon(\rho_i,\rho_k)\varepsilon(\rho_k,\rho_i)=1$ for all k. The non-degeneracy assumption gives $[\rho_i]=[\mathrm{id}]$ as desired. □

We now define an endomorphism of $\mathcal{M}$ by $\beta_{ij}=\alpha^+_{\rho_i\otimes\mathrm{id}}\alpha^-_{\mathrm{id}\otimes\rho_j^{\mathrm{opp}}}$. More explicitly, we have $\beta_{ij}=\gamma^{-1}\cdot\mathrm{Ad}(U^{+-}_{ij})\cdot(\rho_i\otimes\rho_j^{\mathrm{opp}})\cdot\gamma$, where

$$U^{+-}_{ij}=\sum_k V_k(\varepsilon^+(\rho_i,\rho_k)\otimes\varepsilon^{-,\mathrm{opp}}(\rho_j^{\mathrm{opp}},\rho_k^{\mathrm{opp}}))(\rho_i\otimes\rho_j^{\mathrm{opp}})(V_k^*).$$

Note that if we define similarly

$$U^{++}_{ij}=\sum_k V_k(\varepsilon^+(\rho_i,\rho_k)\otimes\varepsilon^{+,\mathrm{opp}}(\rho_j^{\mathrm{opp}},\rho_k^{\mathrm{opp}}))(\rho_i\otimes\rho_j^{\mathrm{opp}})(V_k^*),$$

we then have $\alpha^+_{\rho_i\otimes\rho_j^{\mathrm{opp}}}=\gamma^{-1}\cdot\mathrm{Ad}(U^{++}_{ij})\cdot(\rho_i\otimes\rho_j^{\mathrm{opp}})\cdot\gamma$. By [2],1 Prop. 18, we have

$$[\beta_{ij}]=[\alpha^+_{\rho_i\otimes\mathrm{id}}][\alpha^-_{\mathrm{id}\otimes\rho_j^{\mathrm{opp}}}]=[\alpha^-_{\rho_j\otimes\mathrm{id}}][\alpha^+_{\rho_i\otimes\mathrm{id}}]=[\alpha^-_{\mathrm{id}\otimes\rho_j^{\mathrm{opp}}}][\alpha^+_{\rho_i\otimes\mathrm{id}}]$$

as $\mathcal{M}$-$\mathcal{M}$ sectors.

The following proposition is originally due to Izumi [22] (with a different proof) and first due to Ocneanu [37] in the setting of the asymptotic inclusion. (Also see [13].)

Proposition 21. *Each $[\beta_{ij}]$ is an irreducible $\mathcal{M}$-$\mathcal{M}$ sector and these are mutually different for different pairs of i,j. Each irreducible $\mathcal{M}$-$\mathcal{M}$ sector arising from $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}\subset\mathcal{M}$ is of this form.*

Proof We compute

$$\langle\beta_{ij},\beta_{kl}\rangle=\langle\alpha^+_{\rho_i\otimes\mathrm{id}}\alpha^-_{\mathrm{id}\otimes\rho_j^{\mathrm{opp}}},\alpha^+_{\rho_k\otimes\mathrm{id}}\alpha^-_{\mathrm{id}\otimes\rho_l^{\mathrm{opp}}}\rangle=\langle\alpha^+_{\bar\rho_k\rho_i\otimes\mathrm{id}},\alpha^-_{\mathrm{id}\otimes\rho_l^{\mathrm{opp}}\bar\rho_j^{\mathrm{opp}}}\rangle.$$

The only sector which can be contained in $[\alpha^+_{\bar\rho_k\rho_i\otimes\mathrm{id}}]$ and $[\alpha^-_{\mathrm{id}\otimes\rho_l^{\mathrm{opp}}\bar\rho_j^{\mathrm{opp}}}]$ is the identity by the above proposition. So the above number is $\delta_{ik}\delta_{jl}$. Since the square sums of the statistical dimensions for $\{\rho_i\otimes\rho_j^{\mathrm{opp}}\}_{ij}$ and $\{\beta_{ij}\}_{ij}$ are the same, it completes the proof. $\square$

Note that here we have used the definition in [28] for the map $\rho_i\otimes\rho_j^{\mathrm{opp}}\mapsto\beta_{ij}$, and a general theory of this map has been studied in [2] under the name α-induction. But in [2], they assumed a certain condition, called chiral locality in the terminology of [3], and some results in [2] depend on this assumption, while the definition itself makes sense without it. Our mixed use of braidings ε^+ and ε^- here violates this chiral locality condition, so we can use the results in [2] here only when they are independent of the chiral locality assumption. For example, it is easy to see that the analogue of [2, I, Theorem 3.9] does not hold for our map here.

With the above proposition, we have the following description of the dual principal graph of $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}\subset\mathcal{M}$ as a corollary, which is originally due to Ocneanu [37]. (Also see [13].) Label even vertices with (i,j) for $[\beta_{i\bar j}]$ and odd vertices with k for $[\rho_k\otimes\mathrm{id}][\gamma]$ and draw an edge with multiplicity N_{ij}^k between the even vertex (i,j) and the odd vertex k. The connected component of this graph containing the vertex $(0,0)$ is the dual principal graph of the subfactor $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}\subset\mathcal{M}$, which is the same as the principal graph.

We next study the tensor category of the $\mathcal{M}$-$\mathcal{M}$ sectors.

Lemma 22. *Let V,W be intertwiners from $\rho_i\rho_k$ to ρ_m and from $\rho_j\rho_l$ to ρ_n, respectively, in $\mathcal{N}$. Then $V\otimes W^{*\mathrm{opp}}\in\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}$ in an intertwiner from $\beta_{ij}\beta_{kl}$ to β_{mn}.*

Proof By a direct computation. $\square$

Then we easily get the following from the above lemma. (The quantum $6j$-symbols for subfactors have been introduced in [36] as a generalization for classical $6j$-symbols. See [12, Chapter 12] for details.)

Theorem 23. *In the above setting, the tensor categories of $(\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}})$-$(\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}})$ sectors and $\mathcal{M}$-$\mathcal{M}$ sectors with quantum $6j$-symbols are isomorphic.*

5 Relations with the quantum double

This section contains our main results.

Here below we will consider an inclusion $\mathcal{A} \subset \mathcal{B}$ of nets of factors. We shall say that $\mathcal{A} \subset \mathcal{B}$ has finite index if there is a consistent family of conditional expectations $\mathcal{E}_I : \mathcal{B}(I) \to \mathcal{A}(I)$, $I \in \mathcal{I}$ and $[\mathcal{B}(I) : \mathcal{A}(I)]_{\mathcal{E}_I} < \infty$ does not depend on $I \in \mathcal{I}$. The independence of the index of the interval I automatically holds if there is a vector (vacuum) with Reeh-Schlieder property and $\mathcal{E}_I$ preserves the vacuum state (standard nets, see [28]). The index will be simply denoted by $[\mathcal{B} : \mathcal{A}]$.

Proposition 24. *Let $\mathcal{A} \subset \mathcal{B}$ be a finite-index inclusion of nets of factors as above. If $\mathcal{A}$ and $\mathcal{B}$ are completely rational then*

$$\mu_{\mathcal{A}} = \mathbf{I}^2 \mu_{\mathcal{B}}$$

with $\mathbf{I} = [\mathcal{B} : \mathcal{A}]$.

Proof If $\mathcal{N}_1, \mathcal{N}_2$ are factors, we shall use the symbol

$$\mathcal{N}_1 \overset{\alpha}{\perp} \mathcal{N}_2$$

to indicate that $\mathcal{N}_1 \subset \mathcal{N}_2'$ and $[\mathcal{N}_2' : \mathcal{N}_1] = \alpha$.

Let $E = I_1 \cup I_2 \in \mathcal{I}_2$; we will show that

$$\begin{array}{ccc} \mathcal{B}(E) & \overset{\mu_{\mathcal{B}}}{\perp} & \mathcal{B}(E') \\ \mathbf{I}^2 \cup & & \cup\, \mathbf{I}^2 \\ \mathcal{A}(E) & \overset{\mathbf{I}^2\mu_{\mathcal{A}}}{\perp} & \mathcal{A}(E') \end{array}$$

where $\mathcal{A}(E) \subset \mathcal{B}(E)$ has index $\mathbf{I}^2$ because $\mathcal{A}(E) \cong \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$, $\mathcal{B}(E) \cong \mathcal{B}(I_1) \otimes \mathcal{B}(I_2)$ and $[\mathcal{B}(I_i) : \mathcal{A}(I_i)] = \mathbf{I}$.

In the diagram, the commutants are taken in the Hilbert space $\mathcal{H}_{\mathcal{B}}$ of $\mathcal{B}$, hence $\mathcal{B}(E) \overset{\mu_{\mathcal{B}}}{\perp} \mathcal{B}(E')$ is obvious.

We now show that on $\mathcal{H}_{\mathcal{B}}$

$$\mathcal{A}(E) \overset{\mathbf{I}^2\mu_{\mathcal{A}}}{\perp} \mathcal{A}(E').$$

Let $\gamma : \mathcal{B} \to \mathcal{A}$ be a canonical endomorphism with $\lambda = \gamma|_{\mathcal{A}}$ localized in an interval I_0; then the net $I \mapsto \mathcal{A}(I)$ on $\mathcal{H}_{\mathcal{B}}$ $(I \supset I_0)$ is unitarily equivalent to the net

$$I \mapsto \lambda(\mathcal{A}(I)) \quad \text{on } \mathcal{H}_{\mathcal{A}}$$

and we may assume $I_0 \subset I_1$.

Then the correspondence associated with

$$\mathcal{A}(E)\text{-}\mathcal{A}(E') \quad \text{on } \mathcal{H}_{\mathcal{B}} \,,$$

namely $\mathcal{H}_\mathcal{B}$ with the natural commuting actions of $\mathcal{A}(E)$ and $\mathcal{A}(E')$, is unitarily equivalent to the one associated with

$$\lambda(\mathcal{A}(E))\text{-}\lambda(\mathcal{A}(E')) \quad \text{on } \mathcal{H}_\mathcal{A} \ ,$$

namely $\mathcal{H}_\mathcal{A}$ with the commuting actions of $\mathcal{A}(E)$ and $\mathcal{A}(E')$ obtained by composing their defining actions with the map $X \to \lambda(X)$. But

$$\lambda(\mathcal{A}(E)) = \lambda(\mathcal{A}(I_1) \vee \mathcal{A}(I_2)) = \lambda(\mathcal{A}(I_1)) \vee \mathcal{A}(I_2)$$

and $\lambda(\mathcal{A}(E')) = \mathcal{A}(E')$ hence the $\mathcal{A}(E)$-$\mathcal{A}(E')$ correspondence on $\mathcal{H}_\mathcal{B}$ is unitarily equivalent to

$$(\lambda(\mathcal{A}(I_1)) \vee \mathcal{A}(I_2))\text{-}\mathcal{A}(E') \quad \text{on } \mathcal{H}_\mathcal{A}$$

and its index is

$$[\hat{\mathcal{A}}(E) : \lambda(\mathcal{A}(I_1)) \vee \mathcal{A}(I_2)] = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)][\mathcal{A}(E) : \lambda(\mathcal{A}(I_1)) \vee \mathcal{A}(I_2)] = \mu_\mathcal{A}\mathbf{I}^2.$$

It follows from the diagram that

$$\mathbf{I}^2\mu_\mathcal{A} = \mu_\mathcal{B}\mathbf{I}^2 \cdot \mathbf{I}^2,$$

thus, $\mathbf{I}^2\mu_\mathcal{B} = \mu_\mathcal{A}$. □

The following Proposition may be generalized to the case of a finite-index inclusion $\mathcal{A} \subset \mathcal{B}$ as above.

Proposition 25. *Let $\mathcal{A}$ be completely rational with modular PCT and $\mathcal{B} \supset \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ be the LR net. Then also $\mathcal{B}$ is completely rational with modular PCT.*

Proof Let $E = I_1 \cup I_2$ and I_3 the bounded connected component of E'. Set $\mathcal{C} \equiv \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$. Then the conditional expectation $\mathcal{E}_I : \mathcal{B}(I) \to \mathcal{C}(I)$ associated with the interval I, where I is the interior of $\bar{I}_1 \cup \bar{I}_2 \cup \bar{I}_3$, maps $\mathcal{B}(E)$ onto $\hat{\mathcal{C}}(E)$, because $\mathcal{E}_I(\mathcal{B}(E)) \subset \mathcal{C}(I_3)' \cap \mathcal{C}(I) = \hat{\mathcal{C}}(E)$, thus

$$\mathcal{E} \equiv \mathcal{E}_0 \cdot \mathcal{E}_I|_{\mathcal{B}(E)} \tag{8}$$

is a finite-index expectation of $\mathcal{B}(E)$ onto $\mathcal{C}(E)$, where $\mathcal{E}_0$ is the expectation of $\hat{\mathcal{C}}(E)$ onto $\mathcal{C}(E)$. Therefore $\mu_\mathcal{B} < \infty$ follows by a diagram similar to the one in (5) (with $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ instead of $\mathcal{A}$), as we know a priori that the vertical inclusions have a finite index, while the bottom horizontal inclusion has finite index by the argument given there.

Then the strong additivity of $\mathcal{B}$ follows easily, and so its modular PCT property, but we omit the arguments that are not essential here (if $\mathcal{A}$ is conformal case this follows directly because then also $\mathcal{B}$ is conformal).

We now show the split property of $\mathcal{B}$. For notational convenience we treat the case of two separated intervals, rather than that of an interval and the complement of a larger interval. It will be enough to show that the above expectation (8) satisfies

$$\mathcal{E}(b_1 b_2) = \mathcal{E}(b_1)\mathcal{E}(b_2)\ , \quad b_i \in \mathcal{C}(I_i)\ ,$$

and $\mathcal{E}(\mathcal{B}(I_i)) \subset \mathcal{C}(I_i)$, as we may then compose a normal product state $\varphi_1 \otimes \varphi_2$ of $\mathcal{C}(I_1) \vee \mathcal{C}(I_2) \simeq \mathcal{C}(I_1) \otimes \mathcal{C}(I_2)$ with $\mathcal{E}$ to get a normal product state of $\mathcal{B}(I_1) \vee \mathcal{B}(I_2)$.

Let $R_i^{(h)} \in \mathcal{B}(I_h)$, $h = 1, 2$, be elements satisfying the relations (15) so that $\mathcal{B}(I_h)$ is generated by $\mathcal{C}(I_h)$ and $\{R_i^{(h)}\}_i$. With $b_h \in \mathcal{B}(I_h)$ we then have

$$b^{(h)} = \sum_i a_i^{(h)} R_i^{(h)}\ , \quad a_i^{(h)} \in \mathcal{C}(I_h)\ ,$$

hence

$$b^{(1)} b^{(2)} = \sum_{i,j} a_i^{(1)} a_j^{(2)} R_i^{(1)} R_j^{(2)}\ ,$$

so we have to show that $\mathcal{E}(R_i^{(1)} R_j^{(2)}) = 0$ unless $i = j = 0$. Now $R_i^{(1)} = u_i R_i^{(2)}$ for some unitary $u_i \in \hat{\mathcal{C}}(E)$ and

$$\mathcal{E}_I(R_i^{(2)} R_j^{(2)}) = \mathcal{E}_I(\sum_k C_{ij}^{k\,(2)} R_k^{(2)}) = C_{ij}^{0\,(2)} = \delta_{\bar{i}j} C_{ij}^{0\,(2)}\ ,$$

(see Appendix A for the definition of the C_{ij}^k), hence

$$\mathcal{E}(R_i^{(1)} R_j^{(2)}) = \mathcal{E}(u_i R_i^{(2)} R_j^{(2)}) = \mathcal{E}_0(u_i \mathcal{E}_I(R_i^{(2)} R_j^{(2)})) = \mathcal{E}_0(u_i C_{i\bar{i}}^{0\,(2)}) = \mathcal{E}_0(u_i) C_{i\bar{i}}^{0\,(2)}\ ,$$

which is 0 if $i \neq 0$ because $\mathcal{E}_0(u_i) \in \mathcal{C}(E)$ is an intertwiner between irreducible endomorphisms localized in I_1 and I_2, while $\mathcal{E}_0(u_0) = \mathcal{E}_0(1) = 1$. $\square$

We get the following corollary, where the last part will follow from Proposition 36 later.

Corollary 26. *Let $\mathcal{A}$ be completely rational and*

$$\mathcal{A} \otimes \mathcal{A}^{\rm opp} \subset \mathcal{B}$$

be the LR inclusion. Then

$$\mu_{\mathcal{A}}^2 = \mathbf{I}_{\rm global}^2 \mu_{\mathcal{B}}$$

where $\mathbf{I}_{\rm global} = \sum d(\rho_i)^2$.

In particular, $\mu_{\mathcal{B}} = 1$ if and only if $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the LR inclusion.

Proof By Propositions 24, 25 and 36. $\square$

Lemma 27. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be irreducible, Haag dual nets on separable Hilbert spaces. Assume that each sector of $\mathcal{A}_1$ is of type I. If ρ is an irreducible localized endomorphism of $\mathcal{A}_1 \otimes \mathcal{A}_2$, then*

$$\rho \simeq \rho_1 \otimes \rho_2$$

with ρ_i irreducible localized endomorphisms of $\mathcal{A}_i$.

Proof Let π be a DHR representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ (see Appendix B) on a separable Hilbert space $\mathcal{H}$. Then $\pi(\mathfrak{A}_1)$ and $\pi(\mathfrak{A}_2)$ generate the von Neumann algebra $\mathcal{B}(\mathcal{H})$, where $\mathfrak{A}_i$ denotes the quasi-local C*-algebra associated by $\mathcal{A}_i$. Hence $\pi(\mathfrak{A}_1)''$ and $\pi(\mathfrak{A}_2)''$ are factors.

Let $\pi_i \equiv \pi|_{\mathcal{A}_i}$, where we identify $\mathcal{A}_1$ with $\mathcal{A}_1 \otimes \mathbb{C}$ and $\mathcal{A}_2$ with $\mathbb{C} \otimes \mathcal{A}_2$, then π_i is easily seen to be localizable in bounded intervals (namely if $I_1 \in \mathcal{I}$, the restriction of π_1 to the C^*-algebra generated by $\{\mathcal{A}_i(I) : I \in I_1', I \in \mathcal{I}\}$ extends to a normal representation of $\mathcal{A}_i(I_1')$). Therefore π_i is unitarily equivalent to a localized endomorphism of $\mathcal{A}_i$. As π_1 is a factor representation, by assumption $\pi(\mathfrak{A}_1)''$ is a type I factor and so is $\pi(\mathfrak{A}_2)''$. We then have a decomposition

$$\pi = \pi_1 \otimes \pi_2 \ .$$

This conclude s the proof. □

Corollary 28. *Let $\mathcal{A}$ be a completely rational net on a separable Hilbert space. The only irreducible finite dimensional sectors of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ are*

$$[\rho_i \otimes \rho_j^{\mathrm{opp}}]$$

with $[\rho_i]$, $[\rho_j]$ irreducible sectors of $\mathcal{A}$.

Proof Immediate by Lemma 14 and the above Lemma. □

Lemma 29. *Let $\mathcal{A}$ be completely rational and $\mathcal{B} \supset \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ the LR net. If σ is an irreducible localized endomorphism of $\mathcal{B}$ and $\sigma \prec \alpha^+_\rho$, $\sigma \prec \alpha^-_{\rho'}$ for some localized endomorphism ρ, ρ' of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$, then σ is localized in a bounded interval.*

Proof The thesis follows because $\sigma \prec \alpha^+_\rho$ is localized in a right half-line and $\sigma \prec \alpha^-_\rho$ in a left half-line. □

The following lemma extends Theorem 20.

Lemma 30. *Let $\mathcal{A}$ be a completely rational net, $\{[\rho_i]\}_i$ the system of all irreducible sectors with finite dimension, and $\mathcal{B} \supset \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$ the LR net. The following are equivalent:*

(i) *The braiding of the net $\mathcal{A}$ is non-degenerate.*

(ii) $\mathcal{B}$ *has no non-trivial localized endomorphism (localized in a bounded interval, finite index).*

Proof We use now an argument in [7]. Let σ be a non-trivial irreducible localized endomorphism of $\mathcal{B}$ localized in an interval, with $d(\sigma) < \infty$.

By Frobenius reciprocity

$$\begin{aligned} \sigma &\prec \alpha^+_{\sigma^{\text{rest}}}, \\ \sigma &\prec \alpha^-_{\sigma^{\text{rest}}}, \end{aligned}$$

where $\sigma^{\text{rest}} = \gamma \cdot \sigma|_{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}}$ and $\gamma : \mathcal{B} \to \mathcal{A} \otimes \mathcal{A}^{\text{opp}}$ is a canonical endomorphism. Hence if $\rho_k \otimes \text{id} \prec \sigma^{\text{rest}}$ is an irreducible sector with $[\alpha^+_{\rho_k\otimes\text{id}}] = [\sigma]$, then by [28], Prop. 3.9, the monodromy of $\rho_k \otimes \text{id}$ with $\gamma|_{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}} = \sum \rho_i \otimes \rho_i^{\text{opp}}$ must be trivial, namely ρ_k is a non-trivial sector with degenerate braiding.

The converse is true, namely if ρ_k is a non-trivial degenerate sector, then $\alpha^+_{\rho_k\otimes\text{id}}$ is a non-trivial sector of $\mathcal{B}$ localized in a bounded interval. □

Lemma 31. *Let $\mathcal{A}$ be a completely rational net with modular PCT and let $\{[\rho_i]\}_i$ be the system of all finite dimensional sectors of $\mathcal{A}$. If $E = I_1 \cup I_2 \in \mathcal{I}_2$, then*

$$\lambda_E = \bigoplus_i \rho_i \bar{\rho}_i|_{\mathcal{A}(E)},$$

where $\lambda_E = \gamma_E|_{\mathcal{A}(E)}$, the ρ_i's are localized in I_1 and the $\bar{\rho}_i$'s are localized in I_2

Proof Let $j = \text{Ad}J$, where J is the modular conjugation of $\mathcal{A}(0,\infty)$. Given $I \in \mathcal{I}$ we may identify $\mathcal{A}(I)^{\text{opp}}$ with $j(\mathcal{A}(I)) = \mathcal{A}(-I)$. We define a net $\tilde{\mathcal{A}}$ on $\mathbb{R}$ setting

$$\tilde{\mathcal{A}}(I) \equiv \mathcal{A}(I) \otimes \mathcal{A}(I)^{\text{opp}} = \mathcal{A}(I) \otimes \mathcal{A}(-I), \quad I \in \mathcal{I}.$$

With $I = (a,b)$ with $0 < a < b$ and $E = I \cup -I$, let $\gamma_E : \hat{\mathcal{A}}(E) \to \mathcal{A}(E)$ be the canonical endomorphism and $\lambda_E \equiv \gamma_E|_{\mathcal{A}(E)}$. We identify now λ_E with an endomorphism of η_I of $\tilde{\mathcal{A}}(I)$ and want to show that η_I extends to a localized endomorphism of $\tilde{\mathcal{A}}$.

The proof is similar to the one of Theorem 12. With $d > c > b$, by Lemma 11 there is an extension η of $\eta_{(a,b)}$ to $\tilde{\mathcal{A}}(a,d)$ with $\eta|_{\tilde{\mathcal{A}}(b,d)} = \text{id}$ and a canonical endomorphism $\eta_{(a,d)}$ acting trivially on $\mathcal{A}(a,c)$ with a unitary $u \in \tilde{\mathcal{A}}(a,d)$ such that

$$\eta = \text{Ad}u \cdot \eta_{(a,d)}.$$

Therefore $\text{Ad}u|_{\tilde{\mathcal{A}}(-\infty,c)}$ is an extension of $\eta_{(a,b)}$ to $\tilde{\mathcal{A}}(-\infty,c)$ which acts trivially on $\tilde{\mathcal{A}}(-\infty,a)$ and on $\tilde{\mathcal{A}}(b,c)$. Letting $c \to \infty$ we obtain the desired extension of $\eta_{(a,b)}$ to $\tilde{\mathcal{A}}$, that we still denote by η.

Now, by Lemma 27 for $\tilde{\mathcal{A}}$, every irreducible subsector of η will be equivalent to $\rho_h \otimes (j \cdot \rho_k \cdot j)$ for some h,k, hence each irreducible subsector of λ_E must be equivalent to $\rho_h \cdot \bar{\rho}_k|_{\mathcal{A}(E)}$ where ρ_h is localized in (a,b) and ρ_k is localized in $(-b,-a)$. By Theorem 9 this is possible if and only if $h = k$. □

Corollary 32. *Let $\mathcal{A}$ be completely rational with modular PCT. The following are equivalent.*

(i) *The net $\mathcal{A}$ has no non-trivial sector with finite dimension.*

(ii) *The net $\mathcal{A}$ has no non-trivial sector (with finite or infinite dimension).*

(iii) $\mu_{\mathcal{A}} = 1$*, namely $\mathcal{A}(E)' = \mathcal{A}(E')$ for all $E \in \mathcal{I}_2$.*

Proof (i) $\Rightarrow$ (ii): It will be enough to show that every sector (possibly with infinite dimension) ρ of $\mathcal{A}$ contains the identity sector. Given $E = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{I}$, we may suppose that ρ is localized in I_1 and choose a sector ρ' equivalent to ρ and localized in I_2. If u is a unitary with $\mathrm{Ad}u \cdot \rho = \rho'$, then $u \in \hat{\mathcal{A}}(E)$, hence $u \in \mathcal{A}(E)$ by assumptions. Now $\mathcal{A}(E) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ by the split property, hence there exists a conditional expectation $\mathcal{E} : \mathcal{A}(E) \to \mathcal{A}(I_1)$ with $\mathcal{E}(u) \neq 0$, thus $\mathcal{E}(u)$ is a non-zero intertwiner between ρ and the identity.

(ii) $\Rightarrow$ (iii) follows by Lemma 31.

(iii) $\Rightarrow$ (i) follows by Th. 9 (or by Lemma 31). $\square$

The condition $\mu_{\mathcal{A}} = 1$ is however compatible with the existence of soliton sectors.

Note also that the condition that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ has depth ≤ 2 (equivalently $\hat{\mathcal{A}}(E)$ is the crossed product of $\mathcal{A}(E)$ by a finite-dimensional Hopf algebra) is equivalent to the innerness of the sector λ extending λ_E (because λ_E is implemented by a Hilbert space of isometries in $\hat{\mathcal{A}}(E)$ [26]), hence it is equivalent to the the property that all irreducible sectors of $\mathcal{A}$ have dimension 1 by Lemma 31.

The following is the main result of this paper.

Theorem 33. *Let $\mathcal{A}$ be completely rational with modular PCT. Then*

$$\mu_{\mathcal{A}} = \mathbf{I}_{\text{global}} \equiv \sum d(\rho_i)^2$$

and $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the LR inclusion associated with $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ and all the finite-dimensional irreducible sectors $[\rho_i]$ of $\mathcal{A}$.

Proof $\hat{\mathcal{A}}(E) \supset \mathcal{A}(E)$ contains the LR inclusion by the following Proposition 36. Since $\mu_{\mathcal{A}} = \mathbf{I}_{\text{global}}$ by Lemma 31 it has to coincide with the LR inclusion. $\square$

Corollary 34. *Let $\mathcal{A}$ be completely rational and conformal. The inclusions $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ are all isomorphic for $E \in \mathcal{I}_2$.*

Proof If $I \in \mathcal{I}$ and the ρ_i's are localized in I, for any given $I_1 \in \mathcal{I}$ there is a Möbius transformation giving rise to an isomorphism of $\mathcal{A}(I)$ with $\mathcal{A}(I_1)$ carrying the ρ_i's to endomorphisms localized in I_1. Therefore the isomorphism class of $\{\mathcal{A}(E), \lambda_E\}$ is independent of $E \in \mathcal{I}_2$. Hence the LR inclusions based on that are isomorphic. $\square$

Indeed, by using the uniqueness of the III_1 injective factor [6, 19] and the classification of its finite depth subfactors [40] we have the following.

Corollary 35. *Let $\mathcal{A}$ be completely rational and conformal. The isomorphism class of the inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$, $E \in \mathcal{I}_2$, dependes only on the tensor category of the sectors of $\mathcal{A}$, not on its model realization.*

Proof If $\mathcal{A}$ is non-trivial and I is an interval, the $\mathcal{A}(I)$ is a III_1 factor and, as the split property hols, $\mathcal{A}(I)$ is injective (see e.g. [27]). Thus $\mathcal{A}(I)$ is the unique injective III_1 factor [19].

By Popa's theorem [40], if $\mathcal{N}$ is a III_1 injective factor and $\mathcal{T} \subset \mathrm{End}(\mathcal{N})$ a rational tensor category isomorphic to the tensor category of sectors of $\mathcal{A}$ (as abstract tensor categories), then there exists an isomorphism of $\mathcal{N}$ with $\mathcal{A}(I)$ implementing the equivalence between the two tensor categories.

Since the LR inclusion $\mathcal{N} \otimes \mathcal{N}^{\mathrm{opp}} \subset \mathcal{M}$ clearly depends, up to isomorphism, only on $\mathcal{N}$ and the tensor category $\mathcal{T} \subset \mathrm{End}(\mathcal{N})$, it is then isomorphic to $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. $\square$

We now show that, even in the infinite index case, the two-interval inclusion always contains the LR inclusion associated with any rational system of irreducible sectors.

Proposition 36. *Let $\mathcal{A}$ be completely rational with modular PCT j and $E = I \cup -I \in \mathcal{I}_2$ a symmetric 2-interval and $\{[\rho_i]\}$ a rational system of irreducible sectors of $\mathcal{A}$ with finite dimension, with the ρ_i's localized in I. Let $R_i \in (\mathrm{id}, \bar{\rho}_i\rho_i)$ be non-zero intertwiners, where $\bar{\rho}_i = j \cdot \rho_i \cdot j$.*

If $\mathcal{M}$ is the von Neumann subalgebra of $\hat{A}(E)$ generated by $\mathcal{A}(E)$ and $\{R_i\}_i$, then $\mathcal{M} \supset \mathcal{A}(E)$ is isomorphic to the LR inclusion associated with $\{[\rho_i]\}_i$, in particular

$$[\mathcal{M} : \mathcal{A}(E)] = \sum_i d(\rho_i)^2.$$

More generally this holds true if the assumption of complete rationality is relaxed with possibly $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \infty$.

Proof Denoting by $\mathcal{N}$ the factor $\mathcal{A}(0, \infty)$, we may assume $\bar{I} \subset (0, \infty)$ and consider the ρ_i as endomorphisms of $\mathcal{N}$. Let then V_i be the isometry standard implementation of ρ_i as in [17]. Since $JV_iJ = V_i$, we have

$$\rho_i\bar{\rho}_i(X)V_i = V_iX$$

for all $X \in \mathcal{N} \vee \mathcal{N}'$, hence for all local operators X by strong additivity.

Since ρ_i is irreducible, $(\mathrm{id}, \rho_i\bar{\rho}_i)$ is one-dimensional, thus R_i is a multiple of V_i and we may assume $R_i = \sqrt{d(\rho_i)}V_i$, thus

$$R_i^* R_i = d(\rho_i). \tag{9}$$

Now V_iV_j is the standard implementation of $\rho_i\rho_j$ on $\mathcal{N}$ hence by [17, Proposition A.4], we have

$$R_iR_j = \sum_k C_{ij}^k R_k, \tag{10}$$

where C_{ij}^k is the canonical intertwiner between $\rho_k\bar\rho_k$ and $\rho_i\rho_j\bar\rho_i\bar\rho_j$ given by

$$C_{ij}^k = \sum_h w_h j(w_h) \simeq \sum_h w_h \otimes j(w_h), \tag{11}$$

where the w_h's form an orthonormal basis of isometries in $(\rho_k, \rho_i\rho_j)$.

Setting $\rho_0 = \mathrm{id}$, we also have

$$R_i^* = d(\rho_i)C_{\bar{i}i}^{0*}R_{\bar{i}}. \tag{12}$$

Indeed the above equality holds up to sign by the j-invariance of both members [17, Lemma A.3], but the $-$ sign does not occur because both members have positive expectation values on the vacuum vector.

Now by the split property $\mathcal{A}(E) = \mathcal{A}(I) \vee \mathcal{A}(-I) \simeq \mathcal{A}(I) \otimes \mathcal{A}(-I)$ and $\mathcal{A}(-I) = j(\mathcal{A}(I))$ can be identified with $\mathcal{A}(I)^{\mathrm{opp}}$, therefore $\mathcal{M}$ is isomorphic to the algebra generated by $\mathcal{A}(I)\otimes\mathcal{A}(I)^{\mathrm{opp}}$ and multiple of isometries R_i satisfying the above relations. Moreover, there exists a conditional expectation from $\mathcal{M}$ to $\mathcal{A}(I) \otimes \mathcal{A}(I)^{\mathrm{opp}}$.

Corollary 46 then gives the desired isomorphism between $\mathcal{A}(E) \subset \mathcal{M}$ and the LR inclusion. (The Longo-Rehren inclusion in [31], as well as in [28], is dual to the one in this paper, but it does not matter here. Notice further that, in the conformal case, the 2-interval inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is manifestly self-dual.)

The above proof works also in the case case $\mu_{\mathcal{A}} = \infty$ thanks to Prop. 45. $\square$

Corollary 37. *Let $\mathcal{A}$ be completely rational with modular PCT. Then the braiding of the tensor category of all sectors of $\mathcal{A}$ is non-degenerate.*

Proof With the notations in Corollary 26 we have $\mu_{\mathcal{A}}^2 = \mathbf{I}_{\mathrm{global}}^2\mu_{\mathcal{B}}$. On the other hand $\mathbf{I}_{\mathrm{global}}^2 = \mathbf{I}_{\mathrm{global}}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}})$, hence

$$\mathbf{I}_{\mathrm{global}}(\mathcal{A}\otimes\mathcal{A}^{\mathrm{opp}}) = \mu_{\mathcal{A}}^2 = \mathbf{I}_{\mathrm{global}}^2\mu_{\mathcal{B}}$$

therefore $\mu_{\mathcal{B}} = 1$. By Corollary 32 we $\mathcal{B}$ has no non-trivial sector localized in a bounded interval and this is equivalent to the non-degeneracy of the braiding by Lemma 30. $\square$

That $\mu_{\mathcal{A}} = \mathbf{I}_{\mathrm{global}}$ implies the non-degeneracy of the braiding has been noticed in [32, Corollary 4.3].

An immediate consequence of Corollary 37 follows from the work [41], where a model independent construction of Verlinde's matrices S and T has been performed, provided the braiding symmetry is non-degenerate, thus providing a corresponding representation of the modular group $SL(2,\mathbb{Z})$. Hence we have:

Corollary 38. *The Verlinde's matrices T and S constructed in [41] are non-degenerate, hence there exists an associated representation of the modular group $SL(2, \mathbb{Z})$.*

Corollary 39. *Let $\mathcal{A}$ be completely rational with modular PCT. Every sector of $\mathcal{A}$ is a direct sum of finite dimensional sectors.*

Proof Assuming the contrary, by Proposition 59 we have an irreducible sector $[\rho]$ with infinite dimension. Let $E = I_1 \cup I_2 \in \mathcal{I}_2$ with ρ localized in I_1 and ρ' equivalent to ρ and localized in I_2. Let u be a unitary in (ρ, ρ'). Then $u \in \hat{\mathcal{A}}(E)$, hence it has a unique expansion

$$u = \sum_i x_i R_i, \quad x_i \in \mathcal{A}(E),$$

where R_i are as in Proposition 36. As $xu = u\rho(x)$, $x \in \mathcal{A}(I_1)$, we have

$$x \sum_i x_i R_i = \sum_i x_i R_i \rho(x) = \sum_i x_i (\rho_i \cdot \bar{\rho}_i)(\rho(x)) R_i = \sum_i x_i \rho_i(\rho(x)) R_i \quad \forall x \in \mathcal{A}(I_1),$$

thus $x x_i = x_i \rho_i(\rho(x))$ for all i. As there is a $x_i \neq 0$, by the split property there is a non-zero intertwiner between $\rho_i \cdot \rho$ and the identity. As ρ_i and ρ are irreducible, this implies that ρ is finite dimensional, contradicting our assumption. □

Corollary 40. *Let $\mathcal{A}$ be conformal and completely rational. Then every representation on a separable Hilbert space is Möbius covariant with positive energy.*

Proof By the preceding result every such representation is a direct sum of irreducible sectors with finite dimension. According to [16] every finite dimensional sector is covariant with positive energy, thus also a direct sum of such sectors. □

6 n-Interval Inclusions

In this section we extend the results on the 2-interval subfactors to arbitrary multi-interval subfactors. Let $\mathcal{A}$ be a local, irreducible net on S^1. We assume $\mathcal{A}$ to be completely rational with modular PCT, so that our previous analysis applies. Alternatively $\mathcal{A}$ may be assumed to be conformal with $\mu_{\mathcal{A}} = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ finite and independent of the 2-interval E; this setting will be needed to derive Cor. 7.

If $E \in \mathcal{I}_n$ we set

$$\mu_n = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] \ .$$

With this notation $\mu_{\mathcal{A}} = \mu_2$. We also consider the situation occurring in representations different from the vacuum representation: if ρ is a localizable representation of $\mathcal{A}$ (i. e. a DHR representation, that, on S^1, are just the locally normal representations), we set $\mu_n^\rho = [\rho(\mathcal{A}(E'))' : \rho(\mathcal{A}(E))]$.

Lemma 41. $\mu_n^\rho = \mu_1^\rho \, \mu_n$, $\quad \forall n \in \mathbb{N}$.

Proof Let $E = I_1 \cup I_2 \cup \cdots \cup I_n \in \mathcal{I}_n$. We may suppose that ρ is an endomorphism of $\mathcal{A}$ localized in I_1. Since ρ acts trivially on E', we have $\rho(\mathcal{A}(E'))' = \mathcal{A}(E')' = \hat{\mathcal{A}}(E)$, thus the inclusion $\rho(\mathcal{A}(E)) \subset \rho(\mathcal{A}(E'))'$ is a composition

$$\rho(\mathcal{A}(E)) \subset \mathcal{A}(E) \subset \rho(\mathcal{A}(E'))' = \hat{\mathcal{A}}(E) \ ;$$

by the split property $\rho(\mathcal{A}(E)) \subset \mathcal{A}(E)$ is isomorphic to $\rho(\mathcal{A}(I_1)) \otimes \mathcal{A}(I_2 \cup \cdots \cup I_n) \subset \mathcal{A}(I_1) \otimes \hat{\mathcal{A}}(I_2 \cup \cdots \cup I_n)$, therefore

$$\mu_n^\rho = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] \cdot [\mathcal{A}(I_1) : \rho(\mathcal{A}(I_1)] \ .$$

$\square$

Lemma 42. $\mu_n^\rho = d(\rho)^2 \, \mu_2^{n-1}$, $\quad \forall n \in \mathbb{N}$.

Proof By the index-statistics theorem [25] we have $\mu_1^\rho = d(\rho)^2$, hence, by Lemma 41, we only need to show that $\mu_n = \mu_2^{n-1}$. We proceed inductively. If $n = 1$ the claim is trivially true. Assume the claim for a given n and let $E_n = I_1 \cup \cdots \cup I_n \in \mathcal{I}_n$ and $E_{n+1} = I_1 \cup \cdots \cup I_n \cup I_{n+1} \in \mathcal{I}_{n+1}$. Then

$$\mathcal{A}(E_{n+1}) = \mathcal{A}(E_n) \vee \mathcal{A}(I_{n+1}) \subset \hat{\mathcal{A}}(E_n) \vee \mathcal{A}(I_{n+1}) \subset \hat{\mathcal{A}}(E_{n+1}) \ ,$$

thus, by the split property, $\mu_{n+1} = \mu_n \cdot [\hat{\mathcal{A}}(E_{n+1}) : \hat{\mathcal{A}}(E_n) \vee \mathcal{A}(I_{n+1})]$ and, by the inductive assumption, we have to show that $\hat{\mathcal{A}}(E_n) \vee \mathcal{A}(I_{n+1}) \subset \hat{\mathcal{A}}(E_{n+1})$ is equal to μ_2. But the commutant of this latter inclusion $\mathcal{A}(I'_{n+1}) \cap \mathcal{A}(E'_n) \subset \mathcal{A}(E'_{n+1})$ has index is μ_2 because, by the split property, turns out to be isomorphic to $\mathcal{A}(I_\ell \cup I_r) \otimes \mathcal{A}(L) \supset \hat{\mathcal{A}}(I_\ell \cup I_r) \otimes \mathcal{A}(L)$, namely to a 2-interval inclusion tensored by a common factor, where I_ℓ and I_r are the two intervals of E'_{n+1} contiguous to I_{n+1} and L is the remaining $(n-1)$-subinterval of E'_{n+1}. $\square$

Theorem 43. *Let $\mathcal{A}$ be a local, irreducible completely rational net with modular PCT. Let $E = \cup_{i=1}^n I_i \in \mathcal{I}_n$ and $\lambda^{(n)} = \gamma^{(n)}|\mathcal{A}(E)$ where $\gamma^{(n)}$ is a canonical endomorphism from $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$. Then*

$$\lambda^{(n)} \cong \bigoplus_{i_1, \dots, i_n} N^0_{i_1 \dots i_n} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_n} \ , \tag{13}$$

where $\{[\rho_i]\}_i$ are all the irreducible sectors with finite statistics, ρ_{i_k} being localized in I_k. $N^0_{i_1 \dots i_n}$ is the multiplicity of the identical endomorphism in the product $\rho_{i_1} \dots \rho_{i_n}$.

The same results hold true if complete rationality is replaced by conformal invariance and assuming $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \mathbf{I}_{\text{global}} < \infty$ independently of the 2-interval E.

Proof Let I be an interval which contains $\cup_i I_i$ and let ρ_{i_k}, $k = 1, \dots, n$, be irreducible endomorphisms localized in I_k, respectively. Then the intertwiner space between $\rho_{i_1}\rho_{i_2}\cdots\rho_{i_n}$, considered as an endomorphism of $\mathcal{A}(I)$, and the identity has dimension $N^0_{i_1\dots i_n}$. We are using here the equivalence between local and global intertwiners, that holds either by strong additivity of by conformal invariance [17]. These intertwiners are multiples of isometries in $\hat{\mathcal{A}}(E)$. Thus, by the argument leading to Th. 9, $\rho_{i_1}\rho_{i_2}\cdots\rho_{i_n}|_{\mathcal{A}(E)}$ is contained in $\lambda^{(n)}$ with multiplicity $N^0_{i_1\dots i_n}$. We have thus proved the inclusion $\succ$ in (13). Now the dimension of the endomorphism on the right hand side of (13) has been computed in [50]. For the sake of selfcontainedness we repeat the argument:

$$\begin{aligned}\sum_{i_1,\dots,i_n} N^0_{i_1\dots i_n} d(\rho_1)\cdots d(\rho_n) &= \sum_{i_1,\dots,i_{n-1}} \left(\sum_{i_n} N^{\overline{i_n}}_{i_1\dots i_{n-1}} d(\rho_{i_n})\right) d(\rho_{i_1})\cdots d(\rho_{i_{n-1}}) \\ &= \sum_{i_1,\dots,i_{n-1}} \big(d(\rho_1)\cdots d(\rho_{i_{n-1}})\big)^2 = \left(\sum_i d(\rho_i^2)\right)^{n-1}, \quad (14)\end{aligned}$$

where we have used Frobenius reciprocity $N^0_{i_1\dots i_n} = N^{\overline{i_n}}_{i_1\dots i_{n-1}}$, the fact $d(\rho) = d(\overline{\rho})$ and the identity $\sum_i \langle\rho_i, \rho\rangle d(\rho_i) = d(\rho)$. On the other hand, we have

$$d(\lambda^{(n)}) = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \mu_{\mathcal{A}}^{n-1} = \mathbf{I}^{n-1}_{\text{global}} = (\sum_i d(\rho_i)^2)^{n-1},$$

where the first equality is obvious, the second is given by Lemma 42 and the last one follows from the results of the preceding section. Thus the endomorphisms on both sides of (13) have the same dimension, hence they are equivalent.

The last claim in the statement follows by the same arguments and the equivalence between local and global intertwiners. $\square$

Corollary 44. *Let $\mathcal{A}$ be as in Th. 43. If $E \in \mathcal{I}_n$, then $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the n-th iterated LR inclusion associated with $\mathcal{N} \equiv \mathcal{A}(I)$, $I \in \mathcal{I}$, and the system of all sectors of $\mathcal{A}$ (considered as sectors of $\mathcal{N}$).*

In particular, for a fixed $n \in \mathbb{N}$, the isomorphism class of $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ depends only on the superselection structure of $\mathcal{A}$ and not on $E \in \mathcal{I}_n$.

Proof Let $E = I_1 \cup \cdots \cup I_n \in \mathcal{I}_n$ with $\bar{E} \subset (0, \infty)$ and $n = 2^k$. It follows by Lemma 42 and the split property that

$$[\hat{\mathcal{A}}(E \cup -E) : \hat{\mathcal{A}}(E) \vee \hat{\mathcal{A}}(-E)] = \mathbf{I}_{\text{global}} .$$

On the other hand, if the ρ_i's are localized in I_1, then the algebra generated by $\hat{\mathcal{A}}(E) \vee \hat{\mathcal{A}}(-E)$ and the standard implementation isometries V_i of $\rho_i|_{\hat{\mathcal{A}}(E)}$ is the associated LR inclusion, analogously as in Th. 33, and is contained in $\hat{\mathcal{A}}(E \cup -E)$, hence coincides with that by the equality of the indices.

The corollary then follows in the case $n = 2^k$ by induction, once we note that at each step the extension $\alpha^+_{\rho_i \otimes \text{id}}$ from $\hat{\mathcal{A}}(E) \vee \hat{\mathcal{A}}(-E)$ to $\hat{\mathcal{A}}(E \cup -E)$ is $\rho_i|_{\hat{\mathcal{A}}(E \cup -E)}$.

The same is then true for an arbitrary n by taking relative commutants. $\square$

7 Examples and further comments

Our results may be first illustrated by considering the case of an inclusion of completely rational, local conformal irreducible nets $\mathcal{A} \subset \mathcal{B}$, where $\mathcal{A} = \mathcal{B}^G$ is the fixed-point of $\mathcal{B}$ with respect to the action of a finite group G and $\mu_{\mathcal{B}} = 1$. Then $[\mathcal{B} : \mathcal{A}] = |G|$, thus by Prop. 24, $\mathbf{I}_{\text{global}}(\mathcal{A}) = \mu_{\mathcal{A}} = |G|^2$. Now $\mathcal{A}$ has the DHR [9] irreducible sectors $[\rho_\pi]$ associated with $\pi \in \hat{G}$ and

$$\sum_{\pi \in \hat{G}} d(\rho_\pi)^2 = |G|,$$

therefore $\mathcal{A}$ has extra irreducible sectors $[\sigma_i]$ with

$$\sum_i d(\sigma_i)^2 = |G|^2 - |G|.$$

For example, in the case of Ising model, we have $\mathcal{A} = \mathcal{B}^{\mathbb{Z}_2}$ as above (but with $\mathcal{B}$ twisted local, yet this does not alter our discussion), thus $\mu_{\mathcal{A}} = 4$ and thus $\sum d(\rho_i)^2 = 4$, so the standard three sectors are the only irreducible sectors.

On the other hand, in the situation studied in [34], the superselection category of $\mathcal{A}$ is equivalent to the representation category of a twisted quantum double $D^\omega(G)$ with $\omega \in H^3(G, \mathbb{T})$. Since $D^\omega(G)$ is semisimple we again have

$$\sum_{\sigma \in \widehat{D^\omega(G)}} d(\sigma)^2 = \dim D^\omega(G) = |G|^2 = \mu_{\mathcal{A}}.$$

One may compare this with the situation occurring on a higher dimensional spacetime. There the strong additivity property may be replaced by the requirement that $\mathcal{A}(\mathcal{O}' \cap \tilde{\mathcal{O}})' \cap \mathcal{A}(\tilde{\mathcal{O}}) = \mathcal{A}(\mathcal{O})$ if $\mathcal{O} \subset \tilde{\mathcal{O}}$ are double cones. If $E \equiv \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are double cones with space-like separated closure, the split property gives a natural isomorphism of $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ with $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ and

$$[\mathcal{A}(E')' : \mathcal{A}(E)] = \mathbf{I}_{\text{global}} = \sum_{\pi \in \hat{G}} d(\rho_\pi)^2 = |G|,$$

where G is the gauge group and the ρ_π's are the DHR sectors [9] (there is no extra sectors). The reason for this difference is that on S^1 the complement of a 2-interval is still a 2-interval, thus the inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is self-dual, while on the Minkowski spacetime the spacelike complement of $\mathcal{O}_1 \cup \mathcal{O}_2$ is a connected region producing no charge transfer inclusion.

The index $\mu_{\mathcal{A}}$ in the models given by the loop group construction for $SU(n)_k$ has been computed in [50]. Our results apply in particular to these nets and the 2-interval inclusion is the LR inclusion associated with the corresponding irreducible sectors $\{[\rho_i]\}_i$.

We note that in this case the 2-interval inclusion is *not* the asymptotic inclusion of the corresponding Jones-Wenzl subfactor [24, 48], even up to tensoring by a common injective III_1 factor. Consider $SU(2)_k$ as an example. The net has $k+1$ sectors and if we choose the standard generator, we get a corresponding subfactor of Jones with principal graph A_{k+1}, up to tensoring a common injective factor of type III_1, as in [47]. If we apply the construction of the asymptotic inclusion to this subfactor, we get a "quantum double" of only the sectors corresponding to the even vertices of A_{k+1}. We get the same result, if we apply the LR construction to the system of $\mathcal{N}$-$\mathcal{N}$ sectors (or $\mathcal{M}$-$\mathcal{M}$ sectors). But the construction of a subfactor from 4 intervals gives a "quantum double" of the system of *all* the sectors, both even and odd. If we want to get this system from the asymptotic inclusion or the Longo-Rehren inclusion, we have to use also bimodules/sectors corresponding to the odd vertices of the (dual) principal graph. In order to get this LR inclusion from the construction of the asymptotic inclusion, we need to proceed as follows. Let $\{[\rho_i]\}_i$ be the set of all the sectors for the net arising from the loop group construction for $SU(n)_k$ as above. Then for a fixed interval $I \subset S^1$, we consider $(\bigoplus_i \rho_i)(\mathcal{A}(I)) \subset \mathcal{A}(I)$ which has finite index and finite depth. Take a hyperfinite II_1 subfactor $P \subset Q$ with the same higher relative commutants as $(\bigoplus_i \rho_i)(\mathcal{A}(I)) \subset \mathcal{A}(I)$. Then the tensor categories of the sectors with quantum $6j$-symbols of $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ and $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ are isomorphic. For this reason, the index of the asymptotic inclusion of the Jones subfactor with principal graph A_{k+1} is half of that of the subfactor arising from 4 intervals and the net for $SU(2)_k$. For $SU(n)_k$, this ratio of the two indices is n.

Finally we notice that there are models like the $SO(2N)_1$ WZW models, see [1] or [34], where all irreducible sectors have dimension one, yet the superselection category $\mathcal{C}$ is modular in agreement with our results. In these cases the fusion graph is disconnected, therefore the equivalent categories of $\mathcal{M}-\mathcal{M}$ and of $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}} - \mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}$ sectors are proper subcategories of the categories $\mathcal{C}\times\mathcal{C}^{\mathrm{opp}} \simeq \mathcal{D}(\mathcal{C})$, where $\mathcal{D}(\mathcal{C})$ is the quantum double of $\mathcal{C}$.

We close this section with a few questions. Does there exist a net with only trivial sectors and non-trivial 2-interval inclusions (thus $\mu_{\mathcal{A}} = \infty$)? Does complete rationality imply strong additivity? Is the LR inclusion the only extension of $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}}$ with the given canonical endomorphism $\bigoplus_i \rho_i \otimes \rho_i^{\mathrm{opp}}$?

A The crossed product structure of the LR inclusion

Let $\mathcal{N}$ be an infinite factor and $\{[\rho_i]\}_i$ a rational system of irreducible sectors of $\mathcal{N}$. The LR inclusion [28] is a canonical inclusion $\mathcal{N}\otimes\mathcal{N}^{\mathrm{opp}} \subset \mathcal{M}$ associated with $\mathcal{N}$ and $\{[\rho_i]\}_i$ such that

$$\lambda \simeq \bigoplus_i \rho_i \otimes \rho_i^{\mathrm{opp}} ,$$

where λ is the restriction to $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ of the canonical endomorphism of $\mathcal{M}$ into $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$.

In [28] such an inclusion is obtained by a canonical choice of the intertwiners $T\in(\mathrm{id},\lambda)$ and $S\in(\lambda,\lambda^2)$ that characterize the canonical endomorphism [26] (Q-system). We now show the universality property of this inclusion and its crossed product structure, that will provide a different realization of it. By LR inclusion we will mean the upward LR inclusion.

We shall consider the free *-algebra $\mathcal{M}_0$ generated by $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ and elements R_i satisfying the relations

$$\begin{cases} R_i x = (\rho_i\otimes\rho_i^{\rm opp})(x)R_i, \qquad x\in\mathcal{N}\otimes\mathcal{N}^{\rm opp}\ , \\ R_i^*R_i = d(\rho_i)\ , \\ R_iR_j = \sum_k C_{ij}^k R_k\ , \\ R_i^* = d(\rho_i)C_{i\bar{i}}^{0*}R_{\bar{i}}\ , \end{cases} \tag{15}$$

where C_{ij}^k is the canonical intertwiner between $\rho_k\otimes\rho_k^{\rm opp}$ and $\rho_i\rho_j\otimes\rho_i^{\rm opp}\rho_j^{\rm opp}$ given by $C_{ij}^k=\sum_h w_h\otimes j(w_h)$, with j the antilinear isomorphism of $\mathcal{N}$ with $\mathcal{N}^{\rm opp}$, and the w_h's form an orthonormal basis of isometries in $(\rho_k,\rho_i\rho_j)$.

We equip $\mathcal{M}_0$ with the maximal C* semi-norm associated to the representations of $\mathcal{M}_0$ whose restriction to $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ are normal and denote by $\mathcal{M}$ the quotient of $\mathcal{M}_0$ modulo the ideal formed by the elements that are null with respect to this seminorm and refer to $\mathcal{M}$ as the free reduced pre-C^*-algebra generated by $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ and the R_i's.

Proposition 45. *Let $\mathcal{N}$ be an infinite factor with separable predual and $\{[\rho_i]\}_i$ a rational system of finite-dimensional irreducible sectors of $\mathcal{N}$.*

Let $\mathcal{M}$ be the free reduced pre-C^*-algebra generated by $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ and elements R_i satisfying the relations (15) as above.

Then $\mathcal{M}$ is a factor and $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$ is isomorphic to the LR inclusion associated with $\mathcal{N}$ and $\{[\rho_i]\}_i$.

In particular every element $X\in\mathcal{M}$ has a unique expansion

$$X=\sum_i x_iR_i, \quad x_i\in\mathcal{N}\otimes\mathcal{N}^{\rm opp}.$$

In other words: if $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ acts normally on a Hilbert space $\mathcal{H}$ and $R_i\in\mathcal{B}(\mathcal{H})$ are elements satisfying the relations (15), then the sub-algebra $\mathcal{M}$ of $B(\mathcal{H})$ generated by $\mathcal{N}\otimes\mathcal{N}^{\rm opp}$ and the R_i's is a factor and $\mathcal{N}\otimes\mathcal{N}^{\rm opp}\subset\mathcal{M}$ is isomorphic to the LR inclusion.

Proof Clearly all elements of $\mathcal{M}$ have the form

$$X=\sum_i x_iR_i, \quad x_i\in\mathcal{N}\otimes\mathcal{N}^{\rm opp}, \tag{16}$$

and we may suppose that $\mathcal{M}$ acts on a Hilbert space so that $\mathcal{N}$ and $\mathcal{N}^{\rm opp}$ are weakly closed.

We now construct an conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$. Setting $\rho_0 = \text{id}$, the expectation $\mathcal{E}$ may be defined by

$$\mathcal{E}(X) = x_0 \tag{17}$$

for X given by (16), once we show that this is well-defined. To this end we will apply the averaging argument in [23].

Let $\mathcal{J}$ be the set of all $x_0 \in \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ such that there exist $x_i \in \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$, $i > 0$, with $\sum_{i \geq 0} x_i R_i = 0$. Clearly $\mathcal{J}$ is a two-sided ideal of $\mathcal{N} \otimes \mathcal{N}^{\text{opp}}$, hence $\mathcal{J} = 0$ (as we want to show) or $\mathcal{J} = \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ (we may suppose $\mathcal{N}$ to be of type III). Suppose $\mathcal{J} \neq 0$ and let $X = 1 + \sum_{i>0} x_i R_i = 0$, thus

$$X = 1 + \sum_{i>0} u x_i R_i u^* = 1 + \sum_{i>0} u x_i \rho_i \otimes \rho_i^{\text{opp}}(u^*) R_i = 0$$

for all unitaries $u \in \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$. Letting u run in the unitary group of a simple injective subfactor $\mathcal{R}$ of $\mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ and taking a mean over this group, we have

$$X = 1 + \sum_{i>0} y_i R_i = 0,$$

where $y_i \in \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ intertwines id and $\rho_i \otimes \rho_i^{\text{opp}}$ on $\mathcal{R}$, thus on all $\mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ by the simplicity of $\mathcal{R}$. Since $\rho_i \otimes \rho_i^{\text{opp}}$ is irreducible, $y_i = 0$, $i > 0$, and we have $1 = 0$, a contradiction.

Notice now that

$$R_i R_i^* = d(\rho_i) R_i C_{\bar{i}i}^{0*} R_{\bar{i}} = d(\rho_i) \rho_i \otimes \rho_i^{\text{opp}}(C_{\bar{i}i}^{0*}) R_i R_{\bar{i}} = \sum_k d(\rho_i) \rho_i \otimes \rho_i^{\text{opp}}(C_{\bar{i}i}^{0*}) C_{i\bar{i}}^{k} R_k,$$

thus, by the conjugate equation in [25], we have

$$\mathcal{E}(R_i R_i^*) = d(\rho_i) \rho_i \otimes \rho_i^{\text{opp}}(C_{\bar{i}i}^{0*}) C_{i\bar{i}}^{0} = \frac{1}{d(\rho_i)} \ ,$$

so every $X \in \mathcal{M}$ has the unique expansion

$$X = \sum_i x_i R_i, \quad x_i = d(\rho_i) \mathcal{E}(X R_i^*) \ . \tag{18}$$

Denoting by $\mathcal{M}_1 \supset \mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ the LR inclusion associated with $\mathcal{N}$ and $\{[\rho_i]\}_i$, $\mathcal{M}_1$ is generated by $\mathcal{N} \otimes \mathcal{N}^{\text{opp}}$ and elements R_i', with an expectation $\mathcal{E}'$, satisfying the relations as in (15) and (18) [31, Section 5], hence the linear map

$$\Phi : X \equiv \sum_i x_i R_i \in \mathcal{M} \to \Phi(X) \equiv \sum_i x_i R_i' \in \mathcal{M}_1 \tag{19}$$

is clearly a homorphism of $\mathcal{M}$ onto $\mathcal{M}_1$, which is the identity on $\mathcal{N} \otimes \mathcal{N}^{\text{opp}}$. Φ is clearly one-to-one by the uniqueness of the expansion (18) both in $\mathcal{M}$ and in $\mathcal{M}_1$. $\square$

Note that the above Proposition gives an alternative construction of the LR inclusion, which is similar to Popa's construction of the symmetric enveloping algebra [39], as follows. Let $\mathcal{N}$ act standardly on $L^2(\mathcal{N})$ and V_i be the standard isometry implementing ρ_i. The *-algebra $\mathfrak{A}$ generated by $\mathcal{N}$ and $\mathcal{N}'$ is naturally isomorphic to the algebraic tensor product $\mathcal{N} \odot \mathcal{N}^{\rm opp}$ and the operators $R_i \equiv \sqrt{d(\rho_i)} V_i$ satisfy the relations (15) by [17, Appendix A]. By the above argument there exists a conditional expectation $\mathcal{E} : \mathfrak{B} \to \mathfrak{A}$, where $\mathfrak{B}$ is the *-algebra generated by $\mathfrak{A}$ and the V_i's. Taking a normal state φ of $\mathcal{N}$, the state $\tilde{\varphi} \equiv \varphi \odot \varphi^{\rm opp} \cdot \mathcal{E}$ of $\mathfrak{B}$ gives by the GNS representation the LR inclusion $\pi_{\tilde{\varphi}}(\mathfrak{A})'' \subset \pi_{\tilde{\varphi}}(\mathfrak{B})''$ (Prop. 45).

Corollary 46. *Let $\mathcal{N}$ be an infinite factor with separable predual and $\{[\rho_i]\}_i$ a rational system of finite-dimensional irreducible sectors of $\mathcal{N}$.*

Let $\mathcal{M}$ be a von Neumann algebra with $\mathcal{M} \supset \mathcal{N} \otimes \mathcal{N}^{\rm opp}$ and $R_i \in \mathcal{M}$ elements satisfying the relations (15). If $\mathcal{M}$ is generated by $\mathcal{N} \otimes \mathcal{N}^{\rm opp}$ and the R_i's, then $\mathcal{N} \otimes \mathcal{N}^{\rm opp} \subset \mathcal{M}$ is isomorphic to the LR inclusion associated with $\{[\rho_i]\}_i$.

In particular $(\mathcal{N} \otimes \mathcal{N}^{\rm opp})' \cap \mathcal{M} = \mathbb{C}$ and there exists a normal conditional expectation from $\mathcal{M}$ to $\mathcal{N} \otimes \mathcal{N}^{\rm opp}$.

Proof The proof is immediate, the isomorphism is obtained as in (19):

$$X \in \mathcal{M} \to \sum_i d(\rho_i) \mathcal{E}(X R_i^*) R_i' \ ,$$

(notations analogous to the ones in (19). □

In the following we shall iterate the LR construction, in order to describe the structure of multi-interval subfactors.

With $\mathcal{N}$ an infinite factor as above and $\{[\rho_i]\}_i$ a system of irreducible sectors with unitary braiding symmetry, let α^+ be the induction map from sectors $\rho_i \otimes \rho_j^{\rm opp}$ of $\mathcal{N} \otimes \mathcal{N}^{\rm opp}$ to sectors of the LR extension $\mathcal{M}_1 \equiv \mathcal{M}$ defined by formula (7). Then $\{\alpha^+_{\rho_i \otimes {\rm id}}\}_i$ is a system of irreducible sectors of $\mathcal{M}$ with braiding symmetry and we may construct the corresponding LR inclusion $\mathcal{M}_1 \otimes \mathcal{M}_1^{\rm opp} \subset \mathcal{M}_2$, where the opposite of $\alpha^+_{\rho_i \otimes {\rm id}}$ is $\alpha^+_{\bar{\rho}_i \otimes {\rm id}}$. We may then iterate the procedure to obtain a tower $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_{2^k} \subset \cdots$ and thus an inclusion

$$\mathcal{N}_n \subset \mathcal{M}_n \ , \quad n = 2^k \ ,$$

where $\mathcal{N}_n \equiv \mathcal{N} \otimes \mathcal{N}^{\rm opp} \otimes \mathcal{N} \otimes \cdots \mathcal{N} \otimes \mathcal{N}^{\rm opp}$ (2^k tensor factors). By construction this inclusion has index $\mathbf{I}^{n-1}_{\rm global}$ and we refer to it as the n-th iterated LR inclusion.

Proposition 47. *Let $n = 2^k$. The n-th iterated LR inclusion $\mathcal{N}_n \subset \mathcal{M}_n$ is irreducible. If $\gamma^{(n)} : \mathcal{M}_n \to \mathcal{N}_n$ is the canonical endomorphism, its restriction $\lambda^{(n)} = \gamma^{(n)}|_{\mathcal{N}_n}$ is given by*

$$\lambda^{(n)} \simeq \bigoplus_{i_1, i_2, \ldots, i_n} N^0_{i_1 i_2 \ldots i_n} \rho_{i_1} \otimes \rho_{i_2}^{\rm opp} \otimes \cdots \otimes \rho_{i_n}^{\rm opp} \ , \tag{20}$$

where $N^0_{i_1 i_2 \ldots i_n} \equiv \langle {\rm id}, \rho_{i_1} \bar{\rho}_{i_2} \cdots \bar{\rho}_{i_n} \rangle$.

Proof By a computation similar to the one in Sect. 6, $\lambda^{(n)}$ defined by formula (20) has dimension

$$d(\lambda^{(n)}) = \mathbf{I}^{n-1}_{\text{global}} ,$$

therefore the formula $\lambda^{(n)} = \gamma^{(n)}|_{\mathcal{N}_n}$ will follow by showing that $\rho_{i_1} \otimes \rho^{\text{opp}}_{i_2} \otimes \cdots \otimes \rho^{\text{opp}}_{i_n} \prec \gamma^{(n)}|_{\mathcal{N}_n}$ with multiplicity $N^0_{i_1 i_2 \dots i_n}$ and this will also imply the irreducibility of $\mathcal{N}_n \subset \mathcal{M}_n$ because then $\lambda^{(n)} \succ \text{id}$ with multiplicity one.

But $\rho_{i_1} \otimes \rho^{\text{opp}}_{i_2} \otimes \cdots \otimes \rho^{\text{opp}}_{i_n}$ is unitarily equivalent to $\rho_{i_1} \bar\rho_{i_2} \cdots \bar\rho_{i_n} \otimes \text{id} \otimes \cdots \otimes \text{id}$ in $\mathcal{M}_n$, by applying iteratively Lemma 18, hence we have the conclusion. $\square$

Let now $m < n = 2^k$ be an integer and set $\mathcal{N}_m$ be the alternate tensor product of k copies of $\mathcal{N}$ and $\mathcal{N}^{\text{opp}}$

$$\mathcal{N}_m \equiv \mathcal{N} \otimes \mathcal{N}^{\text{opp}} \otimes \mathcal{N} \otimes \cdots \mathcal{N} \otimes \mathcal{N}^{\text{opp}} , \quad m \text{ factors.}$$

We then define the m-th iterated LR inclusion

$$\mathcal{N}_m \subset \mathcal{M}_m ,$$

where $\mathcal{M}_m$ is defined as the relative commutant in $\mathcal{M}_n$ of the remaining $n-m$ copies of $\mathcal{N}$ and $\mathcal{N}^{\text{opp}}$, i.e. $\mathcal{M}_m = (\mathcal{N}'_m \cap \mathcal{N}_n)' \cap \mathcal{M}_n$. Note that $\mathcal{N}_m \subset \mathcal{M}_m$ is an irreducible inclusion of factors because $\mathcal{N}'_m \cap \mathcal{M}_m \subset \mathcal{N}'_n \cap \mathcal{M}_n = \mathbb{C}$.

Arguing similarly as above we then have:

Proposition 48. *Proposition 47 holds true for all positive integer n (in formula (20) $\rho^{\text{opp}}_{i_n}$ is ρ_{i_n} if n is odd).*

Proof Let $n = 2^k$. Let $\{V^\ell_{i_1 \dots i_n} : \ell = 1, 2, \dots N_{i_1 \dots i_n}\}$ be a basis of isometries in the space of elements in $\mathcal{M}_n$ that intertwine $\rho_{i_1} \otimes \rho^{\text{opp}}_{i_2} \cdots \otimes \rho^{\text{opp}}_{i_n}$ on $\mathcal{N}_n$. Arguing as in Prop. 45 we see that any element $X \in \mathcal{M}_n$ has a unique expansion

$$X = \sum_{i_1 \dots i_n} \sum_\ell x^\ell_{i_1 \dots i_n} V^\ell_{i_1 \dots i_n} , \qquad x^\ell_{i_1 \dots i_n} \in \mathcal{M}_n .$$

Using this expansion it is easy to check that for $m < n$ the factor $\mathcal{M}_m$ defined above is generate by $\mathcal{N}_m$ and the $V^\ell_{i_1 \dots i_n}$'s with $i_{m+1} = i_{m+2} = \cdots = i_n = 0$. The rest then follows easily. $\square$

B Nets on $\mathbb{R}$ and on S^1 and their representations.

In our paper we deal with nets on $\mathbb{R}$, rather than nets on S^1, for various reasons: because this is the natural language for our arguments, because our results are valid for nets that are not necessarily conformal and, finally, because even if our analysis were restricted to conformal nets on S^1, our proofs would require the analysis more general nets on $\mathbb{R}$ (the $t = 0$ LR net is not conformal).

In the next Section C we will however need to deal with nets on S^1 and their representations, and then conclude consequences for nets on $\mathbb{R}$. Although the relations between nets on $\mathbb{R}$ and on S^1 and their representations is straightforward, we will describe explicitely this point here for the convenience of the reader. However, for simplicity, we consider only the case of strongly additive, Haag dual nets.

Nets on S^1. Let $\mathcal{A}$ be a net of von Neumann algebras on S^1 on a separable Hilbert space satisfying Haag duality. We also assume the local von Neumann algebras $\mathcal{A}(I)$ to be properly infinite, which is automatically true if is the split property holds, or if $\mathcal{A}$ is conformal (except, of course, for the trivial net $\mathcal{A}(I) \equiv \mathbb{C}$).

A representation π of $\mathcal{A}$ is, by definition, a map $I \in \mathcal{I} \to \pi_I$ that associates to each interval $I \in \mathcal{I}$ of S^1 a representation, on a fixed Hilbert space, of the von Neumann algebra $\mathcal{A}(I)$ such that $\pi_{\tilde{I}}|_{\mathcal{A}(I)} = \pi_I$ if $I \subset \tilde{I}$. We shall say that π is *locally normal* if π_I is normal for all $I \in \mathcal{I}$ and that π is *localizable* if π_I is unitary equivalent to $\mathrm{id}|_{\mathcal{A}(I)}$ for all $I \in \mathcal{I}$. As the $\mathcal{A}(I)$'s are properly infinite the two notions coincide if π acts on a separable Hilbert space. Moreover every representation of $\mathcal{A}$ on a separable Hilbert space is automatically locally normal [45], thus localizable.

Denote by $C^*(\mathcal{A})$ the universal C^*-algebra [14] associated with $\mathcal{A}$ (see also [16]). For each $I \in \mathcal{I}$ there is a canonical embedding $\iota_I : \mathcal{A}(I) \to C^*(\mathcal{A})$ and $\iota_{\tilde{I}}|_{\mathcal{A}(I)} = \iota_I$ if $I \subset \tilde{I}$; we identify $\mathcal{A}(I)$ with $\iota_I(\mathcal{A}(I))$ if no confusion arises. There is a one-to-one correspondence between representations of the C^*-algebra $C^*(\mathcal{A})$ and representations of the net $\mathcal{A}$, given by $\pi \to \{I \to \pi_I \equiv \pi \cdot \iota_I\}$. Locally normal representations of the net $\mathcal{A}$ correspond, of course, to locally normal representations of $C^*(\mathcal{A})$. We shall always assume our representations to act on a separable Hilbert space, thus local normality is automatic.

As Haag duality holds, a localizable representation π of $C^*(\mathcal{A})$ is unitarily equivalent to a representation of the form $\sigma_0 \cdot \rho$, where σ_0 is the representation of $C^*(\mathcal{A})$ corresponding of the identity representation of $\mathcal{A}$ (we shall however not need this result).

Nets on $\mathbb{R}$. Given a net $\mathcal{A}$ of von Neumann algebras on S^1 satisfying Haag duality we may associate a net $\mathcal{A}_0$ of Neumann algebras on $\mathbb{R} = S^1 \smallsetminus \{\infty\}$ (identification by Cayley transform) by setting

$$\mathcal{A}_0(I) = \mathcal{A}(I) ,$$

for all bounded intervals I of $\mathbb{R}$. We call $\mathcal{A}_0$ the *restriction* of $\mathcal{A}$ to $\mathbb{R}$. Clearly, if $\mathcal{A}$ is strongly additive, then $\mathcal{A}_0$ is also strongly additive and satisfies Haag duality on $\mathbb{R}$ in the form

$$\mathcal{A}(I)' = \mathcal{A}(\mathbb{R} \smallsetminus I) , \tag{21}$$

where $I \subset \mathbb{R}$ is either an interval or an half-line (a, ∞) or $(-\infty, a)$, $a \in \mathbb{R}$.

Here, if $E \subset \mathbb{R}$ has non-empty interior, we denote by $\mathfrak{A}_0(E)$ the C*-algebra generated by the von Neumann algebras $\mathcal{A}_0(I)$'s as I runs in the intervals contained in the region E and set $\mathcal{A}_0(E) = \mathfrak{A}_0(E)''$.

Conversely, let now $\mathcal{A}_0$ be a strongly additive net of properly infinite von Neumann algebras $\mathcal{A}_0(I)$ on the (bounded, non-trivial) intervals of $\mathbb{R}$ satisfying Haag duality (21).

We may compactify $\mathbb{R}$ to $S^1 = \mathbb{R} \cup \{\infty\}$ and extend $\mathcal{A}_0$ to a net $\mathcal{A}$ on the intervals of S^1 by defining

$$\mathcal{A}(I) \equiv \mathcal{A}_0(S^1 \smallsetminus I)' \tag{22}$$

if I is an interval whose closure contains the point ∞. Clearly, $\mathcal{A}$ is the unique Haag dual net on S^1 whose restriction to $\mathbb{R}$ is $\mathcal{A}_0$; we thus call $\mathcal{A}$ the *extension* of $\mathcal{A}_0$ to S^1.

We state explicitely this one-to-one in the following.

Lemma 49. *Let $\mathcal{A}$ be a net on S^1 satisfying Haag duality and strong additivity. Then its restriction $\mathcal{A}_0$ to $\mathbb{R}$ satifies strong additivity and Haag duality on $\mathbb{R}$.*

Conversely if $\mathcal{A}_0$ is a Haag dual (21), strongly additive net on $\mathbb{R}$, then its extension $\mathcal{A}$ to S^1 is strongly additive and Haag dual.

Moreover $\mathcal{A}_0$ satisfies the split property on $\mathbb{R}$ if and only if $\mathcal{A}$ satisfies the split property on S^1.

Proof The proof is immediate. The statement concerning the split property follows because an inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ is split iff the commutant inclusion $\mathcal{M}' \subset \mathcal{N}'$ is split. $\square$

We now consider the relation between representations of a net $\mathcal{A}$, satisfying Haag duality and strong additivity on S^1 as in Lemma 49 and its restriction $\mathcal{A}_0$ on $\mathbb{R}$.

A *DHR representation* π_0 of $\mathcal{A}_0$ is, by definition, a representation π_0 of $\mathfrak{A}_0(\mathbb{R})$ such that $\pi_0|_{\mathfrak{A}_0(\mathbb{R} \smallsetminus I)}$ is unitarily equivalent to $\mathrm{id}|_{\mathfrak{A}_0(\mathbb{R} \smallsetminus I)}$ for every bounded non-trivial interval I of $\mathbb{R}$, cf. [9].

Clearly a localizable representation π of $\mathcal{A}$ determines a DHR representation π_0 of $\mathcal{A}_0$; indeed π_0 is consistently defined on $\cup_{a>0}\mathcal{A}(-a,a)$ by

$$\pi_0(X) = \pi_I(X),\ X \in \mathcal{A}(I)\ ,$$

where $I \equiv (-a,a)$, hence on all $\mathfrak{A}(\mathbb{R})$ by continuity. We call π_0 the *restriction* of π to $\mathcal{A}_0$.

Conversely, as we shall see, every DHR representation π_0 of $\mathfrak{A}_0(\mathbb{R})$ determines uniquely a localizable representation π of $\mathcal{A}$.

A *localized endomorphism* ρ of $\mathcal{A}_0$ is, by definition, an endomorphism of $\mathfrak{A}_0(\mathbb{R})$ such that $\rho|_{\mathfrak{A}_0(I')} = \mathrm{id}|_{\mathfrak{A}_0(I')}$ for some interval $I \subset \mathbb{R}$; one then says that ρ is localized in I. ρ is transportable if for each interval I_1 there is an endomorphism ρ_1 localized in I_1 and (unitarily) equivalent to ρ (as representations of $\mathfrak{A}_0(\mathbb{R})$). By Haag duality then $\rho_1 = \mathrm{Ad}u \cdot \rho$, where the unitary u belongs to $\mathcal{A}_0(\tilde{I})$, if $\tilde{I}$ is any interval containing both I and I_1. In this paper (as is often the case) *transportability is assumed* in the definition of localized endomorphism.

By a classical simple argument [9], a DHR representation π_0 of $\mathfrak{A}_0(\mathbb{R})$ is unitarily equivalent to a (transportable) endomorphism ρ of $\mathfrak{A}_0(\mathbb{R})$ localized in each given interval I; it is enough to put

$$\rho(X) \equiv U\pi_0(X)U^*,\ X \in \mathfrak{A}_0(\mathbb{R}) ,$$

where U is a unitary intertwiner between $\pi_0|_{\mathfrak{A}_0(\mathbb{R}\smallsetminus I)}$ and $\mathrm{id}|_{\mathfrak{A}_0(\mathbb{R}\smallsetminus I)}$.

Proposition 50. *Let $\mathcal{A}$ be a strongly additive, Haag dual net on S^1 and $\mathcal{A}_0$ be its restriction to $\mathbb{R}$, as in Lemma 49.*

If π is a localizable representation of $\mathcal{A}$, its restriction π_0 to $\mathcal{A}_0$ is a DHR representation of $\mathcal{A}_0$.

Conversely, if π_0 is a DHR representation of $\mathcal{A}_0$, there exists a (obviously unique) localizable representation π of $\mathcal{A}$ whose restriction to $\mathcal{A}_0$ is π_0.

Proof By the above discussion, we only show that if π_0 is a DHR representation of $\mathcal{A}_0$, there exists a localizable representation π of $\mathcal{A}$ such that $\pi_I = \pi_0|_{\mathcal{A}(I)}$ if I is a bounded interval of $\mathbb{R}$.

Indeed, if the closure of I contains the point ∞, we can define π_I as the normal extension of $\pi_0|_{\mathfrak{A}_0(I\smallsetminus\{\infty\})}$, once we show the necessary normality property. Now the normality of $\pi_0|_{\mathfrak{A}_0(I\smallsetminus\{\infty\})}$ does not depend on the unitary equivalence class of π_0, thus we may replace π_0 by a DHR endomorphism ρ of $\mathcal{A}_0$ localized in interval $I_1 \subset \mathbb{R}$ with $I_1 \cap I = \varnothing$. But then $\rho|_{\mathfrak{A}_0(I\smallsetminus\{\infty\})}$ is the identity, hence normal. □

By definition, the sectors of $\mathcal{A}$ (resp. of $\mathcal{A}_0$) are the unitary equivalence classes of localizable representations of $\mathcal{A}$ (resp. of DHR representations of $\mathcal{A}_0$). By the above discussions, the two classes are in one-to-one correspondence.

On the other hand localizable representations of $\mathcal{A}$ corresponds to localizable representations of $C^*(\mathcal{A})$ and DHR representations of $\mathcal{A}_0$ are equivalent to DHR localized endomorphisms of $\mathcal{A}_0$, hence we have the following.

Corollary 51. *Let $\mathcal{A}_0$ be a strongly additive, Haag dual as in (21), net on $\mathbb{R}$ and $\mathcal{A}$ be its extension to S^1. The restriction map $\pi \to \pi_0$ gives rise to a natural one-to-one correspondence between unitary equivalence classes of localizable representations of $C^*(\mathcal{A})$ and unitary equivalence classes of DHR localized endomorphisms of $\mathcal{A}_0$.*

In particular $\pi(C^*(\mathcal{A}))'' = \pi_0(\mathfrak{A}_0(\mathbb{R}))''$, so π is of type I iff π_0 is of type I.

Proof It remains to check the last part of the statement. As $C^*(\mathcal{A})$ is generated (as C*-algebra) by the von Neumann algebras $\mathcal{A}(I)$ as I runs in the intervals of S^1, one has $\pi(C^*(\mathcal{A}))'' = \vee_I \pi_I(\mathcal{A}(I))$, thus clearly $\pi(C^*(\mathcal{A}))'' \supset \pi_0(\mathfrak{A}_0(\mathbb{R}))''$.

On the other hand if I is an interval of S^1, by local normality and strong additivity we have $\pi_I(\mathcal{A}(I)) = \pi_I(\mathcal{A}(I \smallsetminus \{\infty\})) \subset \pi_0(\mathfrak{A}_0(\mathbb{R}))''$, hence $\pi(C^*(\mathcal{A}))'' \subset \pi_0(\mathfrak{A}_0(\mathbb{R}))''$. □

The naturality in the above corollary means that the tensor categories of localizable representations of $C^*(\mathcal{A})$ and of DHR localized endomorphisms of $\mathcal{A}_0$ are equivalent, but we do not need this form of the above statement.

C Disintegration of locally normal representations and of sectors.

Takesaki and Winnink [44] have shown that a locally normal state decomposes into locally normal states, if the split property holds. We shall show here analogous results for localizable representations (sectors). Our arguments work, however, along the same lines to show that locally normal representations decompose into locally normal representations, also on higher dimensional manifolds.

We begin with a simple Lemma.

Lemma 52. *Let $\mathcal{M}$ be a von Neumann algebra, $\mathfrak{L} \subset \mathcal{M}$ a σ-weakly dense C^*-subalgebra and $J \subset \mathfrak{L}$ a right ideal of $\mathfrak{L}$.*

If π is a representation of $\mathfrak{L}$ on a Hilbert space $\mathcal{H}$ such that $\pi|_J$ is σ-weakly continuous and $\overline{\pi(J)\mathcal{H}} = \mathcal{H}$, then π is σ-weakly continuous, thus it extends uniquely to a normal representation of $\mathcal{M}$.

Proof It is sufficient to show that π is σ-weakly continuous on the unit ball of $\mathfrak{L}$, see e.g. [45]. Let then $\{a_i\}_i$ be a bounded net of elements $a_i \in \mathfrak{L}$ such that $a_i \to 0$ σ-weakly. If $t \in B(\mathcal{H})$ is a σ-weak limit point of $\{\pi(a_i)\}_i$, we have to show that $t = 0$. By considering a subnet, if necessary, we may assume $\pi(a_i) \to t$. Given $h \in J$, we have $a_i h \in J$ and $a_i h \to 0$, thus $\pi(a_i h) \to 0$ because $\pi|_J$ is σ-weakly continuous, therefore

$$t\pi(h) = \lim_i \pi(a_i)\pi(h) = \lim_i \pi(a_i h) = 0,$$

and this entails $t = 0$ because h is arbitrary and $\pi(J)\mathcal{H}$ is dense in $\mathcal{H}$. □

We shall use the well-known fact that the C*-algebra of compact operators on a separable Hilbert space $\mathcal{H}$ has only one non-degenerate (i.e. not containing the zero representation) representation, up to multiplicity, hence a unique normal extension to $B(\mathcal{H})$.

Corollary 53. *Let $\mathcal{N}$ be a type I factor with separable predual, $K \subset \mathcal{N}$ the ideal of compact operator relative to $\mathcal{N}$ and $\mathfrak{L}$ a C^*-algebra with $K \subset \mathfrak{L} \subset \mathcal{M}$.*

If π is a representation of $\mathfrak{L}$ such that $\pi|_K$ is non-degenerate, then π is σ-weakly continuous, thus it extends uniquely to a normal representation of $\mathcal{N}$.

Proof Immediate because any non-degenerate representation of K is σ-weakly continuous and K is σ-weakly dense in $\mathcal{N}$. □

Let $\mathcal{A}$ be a net of von Neumann algebras on S^1 over a separable Hilbert space satisfying the split property and Haag duality.

If $I, \tilde{I}$ are intervals, we write $I \subset\subset \tilde{I}$ if the closure of I is contained in the interior of $\tilde{I}$. For each pair of intervals $I \subset\subset \tilde{I}$ we choose an intermediate type I factor

$\mathcal{N}(I,\tilde{I})$ between $\mathcal{A}(I)$ and $\mathcal{A}(\tilde{I})$ and let $K(I,\tilde{I})$ be the compact operators of $\mathcal{N}(I,\tilde{I})$ (there is a canonical choice for $\mathcal{N}(I,\tilde{I})$ [10], but this does not play a role here). We denote by $\mathcal{I}_{\mathbb{Q}}$ the set of intervals with rational endpoints and by $\mathfrak{A}$ the C^*-subalgebra of $C^*(\mathcal{A})$ generated by all $K(I,\tilde{I})$ as $I \subset\subset \tilde{I}$ run in $\mathcal{I}_{\mathbb{Q}}$. Clearly $\mathfrak{A}$ is norm separable.

If $I_1 \subset\subset \tilde{I}_1 \subset I_2 \subset\subset \tilde{I}_2$ then clearly $\mathcal{N}(I_1,\tilde{I}_1) \subset \mathcal{N}(I_2,\tilde{I}_2)$, but $K(I_1,\tilde{I}_1)$ is not included in $K(I_2,\tilde{I}_2)$. For this reason we define the C*-algebras associated to pairs of intervals $I \subset\subset \tilde{I}$

$$\mathfrak{L}(I,\tilde{I}) \equiv \mathcal{N}(I,\tilde{I}) \cap \mathfrak{A} \ .$$

As $\mathcal{N}(I,\tilde{I})$ is the multiplier algebra of $K(I,\tilde{I})$, $\mathfrak{L}(I,\tilde{I})$ consists of elements of $\mathfrak{A}$ that are multipliers of $K(I,\tilde{I})$.

By definition $K(I,\tilde{I}) \subset \mathfrak{L}(I,\tilde{I}) \subset \mathcal{N}(I,\tilde{I})$ and $\mathfrak{A}$ is the C^*-subalgebra of $C^*(\mathcal{A})$ generated by all $\mathfrak{L}(I,\tilde{I})$ as $I \subset\subset \tilde{I}$ run in $\mathcal{I}_{\mathbb{Q}}$.

Lemma 54. *If $I_1 \subset\subset \tilde{I}_1 \subset I_2 \subset\subset \tilde{I}_2$ are intervals then*

$$\mathfrak{L}(I_1,\tilde{I}_1) \subset \mathfrak{L}(I_2,\tilde{I}_2) \ .$$

Proof $\mathfrak{L}(I_1,\tilde{I}_1) \subset \mathcal{N}(I_1,\tilde{I}_1) \subset \mathcal{N}(I_2,\tilde{I}_2)$, thus

$$\mathfrak{L}(I_1,\tilde{I}_1) \subset \mathcal{N}(I_2,\tilde{I}_2) \cap \mathfrak{A} = \mathfrak{L}(I_2,\tilde{I}_2) \ .$$

$\square$

Proposition 55. *Let π be a locally normal representation of $C^*(\mathcal{A})$. Then $\pi|_{\mathfrak{A}}$ is a representation of $\mathfrak{A}$ and $\pi|_{K(I,\tilde{I})}$ is non-degenerate for every of pair of intervals $I \subset\subset \tilde{I}$.*

Conversely, if σ is a representation of $\mathfrak{A}$ such that $\sigma|_{K(I,\tilde{I})}$ is non-degenerate for all intervals $I, \tilde{I} \in \mathcal{I}_{\mathbb{Q}}$, $I \subset\subset \tilde{I}$, there exists a unique locally normal representation $\tilde{\sigma}$ of $C^*(\mathcal{A})$ that extends σ.

Moreover equivalent representations $C^*(\mathcal{A})$ correspond to equivalent representations of $\mathfrak{A}$.

Proof The only non-trivial part is that σ extends to a locally normal representation $\tilde{\sigma}$ of $C^*(\mathcal{A})$. If $I \subset\subset \tilde{I}$ are intervals in $\mathcal{I}_{\mathbb{Q}}$, we denote by $\tilde{\sigma}_{I,\tilde{I}}$ the unique normal extension of $\sigma|_{\mathfrak{L}(I,\tilde{I})}$ to $\mathcal{N}(I,\tilde{I})$ given by Corollary 53.

Given an interval I, we choose $I_1, \tilde{I}_1 \in \mathcal{I}_{\mathbb{Q}}$, $I_1 \subset\subset \tilde{I}_1$ such that $I \subset\subset I_1$ and set

$$\tilde{\sigma}_I \equiv \tilde{\sigma}_{I_1,\tilde{I}_1}|_{\mathcal{A}(I)} \ ,$$

We have to show that $\tilde{\sigma}_I$ is well-defined, then $I \to \tilde{\sigma}_I$ is clearly a representation of $\mathcal{A}$.

Indeed, let $I_2, \tilde{I}_2 \in \mathcal{I}_\mathbb{Q}$ with $I_2 \subset\subset \tilde{I}_2$ be another pair such that $I \subset\subset I_2$. We can choose $I_3, \tilde{I}_3 \in \mathcal{I}_\mathbb{Q}$ such that $I \subset\subset I_3 \subset\subset \tilde{I}_3 \subset\subset I_1 \cap I_2$. Then by Lemma 54 $\mathfrak{L}(I_3, \tilde{I}_3) \subset \mathfrak{L}(I_i, \tilde{I}_i)$, $i = 1, 2$, and therefore

$$\tilde{\sigma}_{I_3,\tilde{I}_3} = \tilde{\sigma}_{I_1,\tilde{I}_1}|_{\mathcal{N}(I_3,\tilde{I}_3)} = \tilde{\sigma}_{I_2,\tilde{I}_2}|_{\mathcal{N}(I_3,\tilde{I}_3)} \ .$$

This concludes the proof. □

Proposition 56. *Let π be a locally normal representation of $C^*(\mathcal{A})$ on a separable Hilbert space and denote by $\pi_\mathfrak{A}$ be the restriction of π to $\mathfrak{A}$. If*

$$\pi_\mathfrak{A} = \int_X^\oplus \pi_\lambda d\mu(\lambda)$$

is a decomposition into irreducible representations π_λ (which always exists), then π_λ extends to a locally normal representation $\tilde{\pi}_\lambda$ of $C^*(\mathcal{A})$ for almost all λ.

Proof By Proposition 55, it is sufficient to show that there exists a null set $E \subset X$ such that $\pi_\lambda|_{K(I,\tilde{I})}$ is non-degenerate for $\lambda \notin E$ and all $I, \tilde{I} \in \mathcal{I}_\mathbb{Q}$ with $I \subset\subset \tilde{I}$. This is clear for a fixed pair $I, \tilde{I}$ of the family, because $\pi_{K(I,\tilde{I})}$ is non-degenerate. Then the statement follows since the considered family of $K(I, \tilde{I})$'s is countable. □

Proposition 57. *With the notations in the Proposition 56, if $\pi(C^*(\mathcal{A}))''$ is a factor not of type I, then for each $\lambda \in X$ the set $X_\lambda \equiv \{\lambda' \in X, \pi_{\lambda'} \simeq \pi_\lambda\}$ has measure zero.*

Proof The set X_λ is measurable by Lemma 60 below. We have $\mu(X \smallsetminus X_\lambda) > 0$, as otherwise π would be quasi-equivalent to π_λ, hence $\pi(\mathfrak{A})''$ would be a type I factor. If $\mu(X_\lambda) > 0$, then $\pi_\mathfrak{A}$ would be the direct sum of two inequivalent representations

$$\pi_\mathfrak{A} = \int_{X_\lambda}^\oplus \pi_\lambda d\mu(\lambda) \oplus \int_{X \smallsetminus X_\lambda}^\oplus \pi_\lambda d\mu(\lambda)$$

which is not possible since $\pi(\mathfrak{A})''$ is a factor. □

Corollary 58. *If there exists a localizable representation π of $C^*(\mathcal{A})$ with $\pi(C^*(\mathcal{A}))''$ a factor not of type I, then there exist uncountably many inequivalent irreducible localizable representations of $C^*(\mathcal{A})$.*

Proof If the representation π is factorial not of type I, then the family of the π_λ's in the above proposition contains an uncountable set of mutually inequivalent irreducible localizable representations as desired. □

Corollary 59. *Let $\mathcal{A}_0$ be a strongly additive, split net of von Neumann algebras on the intervals of $\mathbb{R}$ which is Haag dual as in (21). If there exists a DHR localized endomorphism ρ of $\mathcal{A}_0$ with $\rho(\mathfrak{A}_0(\mathbb{R}))''$ a factor not of type I, then there exist uncountably many inequivalent irreducible DHR localized endomorphisms of $\mathcal{A}_0$.*

Proof Immediate by Corollary 58 and Corollary 51. $\square$

Before concluding this appendix we have to prove a Lemma that has been used. Let $\mathfrak{A}$ be any separable C^*-algebra and σ a representation of $\mathfrak{A}$. Choose a sequence of elements $a_\ell \in \mathfrak{A}$ dense in the unit ball $\mathfrak{A}_1$, a sequence $\varphi_i \in \mathfrak{A}^*$ dense in the Banach space of normal linear functionals $(\sigma(\mathfrak{A})'')_*$ associated with σ. A linear functional $\varphi \in \mathfrak{A}^*$ is then normal with respect to σ if and only if

$$\forall k \in \mathbb{N},\ \exists i \in \mathbb{N} : |\varphi(a_\ell) - \varphi_i(a_\ell)| \leq \frac{1}{k},\ \forall \ell \in \mathbb{N}. \tag{23}$$

We thus have the following.

Lemma 60. *Let $\mathfrak{A}$ be a separable C^*-algebra, π a representation of $\mathfrak{A}$ on a separable Hilbert space and $\pi = \int_X^\oplus \pi_\lambda d\mu(\lambda)$ a direct integral decomposition into a.e. irreducible representations π_λ of $\mathfrak{A}$. For any irreducible representation σ of $\mathfrak{A}$, the set $X_\sigma \equiv \{\lambda, \pi_\lambda \simeq \sigma\}$ is measurable.*

Proof Let $\xi = \int_X^\oplus \xi(\lambda) d\mu(\lambda)$ be a vector with $\xi(\lambda) \neq 0$, for all $\lambda \in X$, and consider the functional of $\mathfrak{A}$ given by $\varphi_\lambda = (\pi_\lambda(\cdot)\xi(\lambda), \xi(\lambda))$.

As both σ and π_λ are irreducible, we have $\sigma \simeq \pi_\lambda$ if and only if φ_λ is normal with respect to σ. With the previous notations, we then have by eq. (23)

$$X_\sigma = \bigcap_k \bigcup_i \bigcap_\ell X_{ik\ell}$$

where

$$X_{ik\ell} = \{\lambda \in X : |\varphi_\lambda(a_\ell) - \varphi_i(a_\ell)| \leq \frac{1}{k}\}.$$

As $X_{ik\ell}$ is measurable, also X_σ is measurable. $\square$

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