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Multi-lane hybrid traffic flow model:
quantifying the impacts of lane-changing maneuvers on traffic flow

Jorge A. Laval [*] and Carlos F. Daganzo [†]

October 25, 2004

Abstract. A multi-lane traffic flow model realistically captures the disruptive effects of lane-changing vehicles by recognizing their limited ability to accelerate. While they accelerate, these vehicles create voids in the traffic stream that affect its character. Bounded acceleration explains two features of freeway traffic streams: the capacity drop of freeway bottlenecks, and the quantitative relation between the discharge rate of moving bottlenecks and bottleneck speed. The model combines a multilane kinematic wave module for the traffic stream, with a detailed constrained-motion model to describe the lane-changing maneuvers, and a behavioral demand model to trigger them. The behavioral demand model has only one parameter. It was held constant in all experiments.

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1 Introduction

Lane-changing activity on busy freeways has received increasing attention in the literature during the last decade. Research has produce both conjectures [1–5] and empirical evidence [6–9] suggesting how lane changes affect the traffic stream (eg, its capacity), but a quantitative understanding of the actual impacts has not yet been reached.

This paper attempts to fill this gap. It proposes that a lane-changing vehicle creates a temporary moving bottleneck on its target lane while it accelerates to the speed prevailing on that lane. Since the ensuing slow-down on the target lane can trigger other disruptive lane changes (DLC's), the multilane facility is modeled as a set of streams linked by the DLC's. The resulting model only needs one more parameter than the simplest kinematic wave model (which requires three), yet it explains several puzzling observations without re-calibration.

Existing traffic flow models do not address the DLC phenomenon. Extensions to kinematic wave (KW) theory [10, 11] with lane-changing [12–18] are inadequate because they treat DLC maneuvers as a fluid that can accelerate instantaneously and therefore does not sufficiently hinder vehicles behind. Some microscopic simulation models can consider realistic accelerations, but they have not yet been used successfully to model DLC's. Recent experience with these models [19] indicates that lane-changing greatly increases the complexity of the model specification and estimation process.

To overcome these problems, a hybrid approach was developed in [20] that combines the best features of microscopic and macroscopic models: the parsimony of the KW model (for the traffic stream) with the accuracy of microscopic models (for slow vehicles). Slow vehicles are treated in this reference as moving bottlenecks within a *single* KW stream, as in the kinematic wave theory of moving bottlenecks (KW-MB) [21–24]. Unfortunately, this requires as an input the maximum possible passing rates, which can only be guessed approximately.

This obstacle is removed in this paper by modeling each lane as a separate KW stream interrupted by the lane-changing particles, and then recognizing that the maximum passing rate on a target lane is zero. The incremental transfer (IT) principle for multilane KW problems [12], and a behavioral demand model for lane-changing with only one parameter are used to predict the flow transfers between neighboring lanes. A constrained motion model that captures a vehicle's ability to accelerate after changing lanes, recognizing both mechanical resistance and the restrictions imposed by downstream traffic, is used to generate DLC trajectories.

The remainder of this paper is organized as follows. §2 formulates the multilane KW model and shows how discrete moving bottlenecks can be incorporated in it. §3 presents a method for quantizing lane-changes. §4 contains examples showing the agreement of the model with known behavior of lane-drops and slow moving obstructions. Finally, a discussion is provided in §5.

2 Multilane KW model

A continuum multilane extension of the KW model for a highway with $n = 2$ lanes was first presented in [13]; see also [14, 15] . For $n > 2$ the conservation equation for a single lane, ℓ, is:

$$\frac{\partial k_\ell}{\partial t} + \frac{\partial q_\ell}{\partial x} = \Phi_\ell, \qquad \ell = 1 \ldots n \qquad (1)$$

where $k_\ell(t, x)$ and $q_\ell(t, x)$ give the density and flow on ℓ at the time-space point (t, x). The inhomogeneous term Φ_ℓ is the *net lane-changing rate onto* lane ℓ, in units of vehicles per unit time per unit distance. It was postulated in [15] that Φ_ℓ was a linear function of the set of k_ℓ's, but the idea was not applied successfully at the time because effective numerical methods [16, 25–27] had not yet been developed. The next two subsections refine the idea in [13] and show how it can be implemented in discrete time.

2.1 The continuum lane-changing model

Define the vector $k(t, x) \doteq [k_1(t, x), \ldots, k_n(t, x)]$ and let $\Phi_{\ell\ell'}(k, t, x), \ell \neq \ell'$ be a function giving the *lane-changing rate* from lane ℓ to lane ℓ'. Then, the net lane-changing rate onto lane ℓ is:

$$\Phi_\ell = \sum_{\ell' \neq \ell} \Phi_{\ell'\ell} - \Phi_{\ell\ell'}. \qquad (2)$$

To complete the formulation we specify functions $\Phi_{\ell\ell'}$ that realistically represent the competition between drivers' desires for changing lanes, and the availability of space in the target lane. We do not assume the $\Phi_{\ell\ell'}$'s are linear in k.

To strike a balance between people's desires for changing lanes and the available space capacity, we first specify functions of (k, t, x) defining: (a) a desired lane-changing rate from ℓ to ℓ' (ie, a demand for lane-changing in units of veh/time-space) $L_{\ell\ell'}, \ell \neq \ell'$, (b) a desired set of through flows on ℓ, T_ℓ, and (c) the available capacity (units of flow) on lane ℓ, μ_ℓ. Formally,

$$L_{\ell\ell'} = L_{\ell\ell'}(k, t, x), \tag{3a}$$
$$T_\ell = T_\ell(k, t, x), \tag{3b}$$
$$\mu_\ell = \mu_\ell(k_\ell, t, x). \tag{3c}$$

Demand functions L and T are obtained by disaggregating with a choice model the sending (or demand) function of KW theory; the capacity function μ_ℓ is the receiving (or supply) function of KW theory [25, 26, 28]. A competition mechanism, $\mathcal{F}$, then determines the *actual* lane-changing rates $\Phi_{\ell\ell'}$ and through flows q_ℓ from (3), ie:

$$(\Phi_{\ell-1,\ell}, q_\ell, \Phi_{\ell+1,\ell}) \doteq \mathcal{F}(L_{\ell-1,\ell}, T_\ell, L_{\ell+1,\ell}, \mu_\ell). \tag{4}$$

The function $\mathcal{F}$ should reflect sensible priority rules, which depend upon the nature of the lane-changing maneuvers (discretional or mandatory). For discretional lane changes we use the IT recipe, which allocates differentials of flow to their desired target on a first-come-first-served basis. Mandatory lane changes can be modeled similarly but are not studied in this paper.

2.2 The discrete formulation

It is assumed from now on that the fundamental diagram (FD) on each lane is triangular with free-flow speed u, wave speed $-w$ and jam density per lane κ. All freeway lanes are partitioned into small sections of length Δx and time is discretized into steps of duration Δt; see Fig. 1. The grid for time, distance and lane is $(t_j \doteq j\Delta t, x_i \doteq i\Delta x, \ell)$. For numerical stability we set

$$\Delta x \doteq u\Delta t. \tag{5}$$

We will use indices i and j to denote the calculated values at a discrete point (t_j, x_i); ie, $k_{i\ell}^j$ is the discrete approximation of $k_\ell(t_j, x_i)$. In the discrete world, the conservation equation becomes

$$\frac{k_{i\ell}^{j+1} - k_{i\ell}^j}{\Delta t} + \frac{q_{i\ell}^j - q_{i-1,\ell}^j}{\Delta x} = \sum_{\ell' \neq \ell} \Phi_{i-1,\ell'\ell}^j - \Phi_{i\ell\ell'}^j, \forall\ \ell. \tag{6}$$

This equation is ready for stepping through time, as there is only one term with time index $j+1$. At each iteration one calculates L, T and μ for every cell with (3) using the current densisites k^j as arguments. One then computes the lane-changing rates and through flows $q_{i\ell}^j, q_{i-1,\ell}^j, \Phi_{i-1,\ell'\ell}^j$ and $\Phi_{i\ell\ell'}^j$ with the IT principle (4) and then evaluates $k_{i\ell}^{j+1}$ with (6).

The desired aggregate number of advancing moves in Δt is given by the sending function for triangular FDs: $\Delta t \min\{uk_{i\ell}^j, Q\}$. The desired number of lane-changing moves is then

$$L_{i\ell\ell'}^j \Delta t \Delta x \doteq \pi_{i\ell\ell'}^j \Delta t \min\{uk_{i\ell}^j, Q\}, \qquad \forall\ \ell, \forall\ \ell' \neq \ell, \tag{7}$$

where $\pi_{i\ell\ell'}^j \Delta t$ is the fraction of choice-makers wishing to change from lane ℓ to lane ℓ' in Δt.

We will assume for maximum simplicity that the choice probability is proportional to the positive speed difference between lanes, $\Delta v_{i\ell\ell'}^j \doteq \max\{0, v_{i\ell'}^j - v_{i\ell}^j\}$, where $v_{i\ell}^j$ is the average speed on lane ℓ; ie [1]

$$\pi_{i\ell\ell'}^j \Delta t \doteq \frac{\Delta v_{i\ell\ell'}^j}{u} \frac{\Delta t}{\tau}, \qquad \forall\ \ell, \forall\ \ell' \neq \ell. \tag{8}$$

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[1] We propose using Eddie's generalized space-mean speed on a look-ahead section downstream of the current position during an evaluation period immediately preceding the current instant. Our tests used negligible evaluation periods comparable with the vehicle spacing.

The parameter τ has units of time. It follows now that the probability of staying in the same lane in the next Δt is $1 - \sum_{\ell' \neq \ell} \pi_{i\ell\ell'} \Delta t$, which will be in $(0,1]$ if we choose $\Delta t \ll \tau$. Thus, the demand for through flow is

$$T_{i\ell}^j \Delta t \doteq k_{i\ell} \Delta x (1 - \sum_{\ell' \neq \ell} \pi_{i\ell\ell'}^j \Delta t) \qquad \forall \; \ell, \tag{9}$$

The available capacity is given by the receiving function for a triangular FD:

$$\mu_{i\ell}^j \Delta t \doteq \Delta t \min\{w(\kappa - k_{i\ell}^j), Q\}. \tag{10}$$

This completes the formulation of the multilane KW model. A more detailed description of the algorithm is given in [29]. Note that equations (7)-(10) have only introduced one additional parameter, τ. Appendix A demonstrates that the discrete model is numerically stable and converges to a continuum solution as the lattice is refined. Thus, Δt and Δx are not parameters of the model; they should be simply chosen as small as possible.

This model, however, is not yet sufficiently realistic. It ignores that the disruption caused by a lane change depends on the initial speed of the lane-changing vehicle and on its ability to accelerate. As a result, it underestimates the effects of DLC 's. The following section shows how to remedy this problem by inserting discrete particles in the stream.

3   Discrete lane-changes: The hybrid model

The basic idea consists in quantizing the lane-changing rates from model (3)-(10) to generate discrete lane-changes, and then treating the temporary obstructions as blockages that move with bounded accelerations on the target lane with the method in [30]. This is possible because the blockages have a known (zero) passing rate and a trajectory that can be determined endogenously, eg with the constrained motion (CM) model of vehicles kinetics in [20]. In the CM model the particle has the initial speed of the sending lane but is constrained both by its ability to accelerate and by the downstream flow on its target lane.

To quantize the process we can simply evaluate the cumulative number of lane changes from (i, ℓ) to $(i+1, \ell')$ by time t_j, $\eta_{i\ell\ell'}^j$, and then use the "floor" function $\lfloor \eta_{i\ell\ell'}^j \rfloor$ to generate integer jumps. In our tests, we added a degree of randomness, by generating particles as outcomes of Poisson variables with mean $\eta_{i\ell\ell'}^j - \eta_{i\ell\ell'}^{j-1}$, but this is not necessary. The complete hybrid model is very parsimonious since it only requires the three usual KW parameters (free-flow speed, capacity and jam density), and the relaxation time for lane-changing, τ. It also requires the parameters of the CM model but these are readily available and do not pose an estimation problem.

4   Empirical tests

The following numerical experiments assume that $\tau = 3$ secs and the triangular FD on each lane has: free-flow speed $u = 60$ mph, congested wave speed $w = -15$ mph and jam density per lane $\kappa = 150$ vpmpl. Additionally, all lane-changing vehicles are assumed to have the acceleration capabilities of a typical car.

4.1   Lane-drops

The experiments in [31] show a consistent reduction in flow after the onset of congestion on bottlenecks caused by freeway lane-drops. The sample corresponds to 6 weekdays on two different three-lane freeways: the M4 motorway near London, UK; and the I-494 freeway in Minneapolis, Minnesota, USA. In both cases flow drops in the range of 6 to 12% were observed.

The proposed model reproduces these phenomena. Consider a 0.5 km (.3 mi) 3-lane freeway with a lane-drop at $x = 0.33$ km (.2 mi); see Fig. 2a. At $t = 0$ the freeway is empty and the input demand gradually increases up to the point that a queue forms and spills back to the freeway entrance. The numerical rescaled N-curves produced by our method (with $\Delta t = 0.3$ secs) for all lanes combined shown in part (b) were measured at the 4 evenly-spaced locations defined in part (a).

It can be seen from part (b) how the capacity drops from 2890 vph to 2656 vph at $t = 16$ min after the formation of the queue, which represents an 8.3% drop. Part (c) of the figure shows the vehicle accumulation

on the shoulder lane for $0.17 \leq x \leq 0.33$. Notice how the breakdown takes place at the same time when the accumulation exceeds six vehicles. This is in agreement with recent empirical observations at merge bottlenecks [8]. This may or may not be a coincidence. After all, a lane-drop is similar to a merge in that vehicles near the bottleneck are forced onto the adjacent lane.

4.2 Capacity flow downstream of a moving obstruction

Reference [24] describes the results of several experiments where constructions with a range of controlled speeds v were introduced in two freeways. It was found that there was a reproducible relation between the capacity of the moving bottleneck, Q_m (the queue discharge rate,) and v, ie

$$Q_m = Q_m(v), \qquad \text{with } \frac{dQ_m}{dv} > 0 \text{ for } v > 30 \text{ mph.} \qquad (11)$$

This behavior is still not well understood. Existing theories either assume that $\frac{dQ_m}{dv} = 0$ (see [22, 23, 32]), or introduce (11) as an exogenous boundary condition [20]. By doing this, however, all the factors that could affect capacity are ignored.

To test the hybrid model we simulated the experiments in [24] for $v \in \{0, 5, 10, 20, \ldots 50\}$ and $n \in \{2, 3, 4\}$. The curves in Fig. 3 plot the resulting (normalized) capacity, $\rho = Q_m(v)/Q_m(u)$, versus v. They match the empirical data of [24] (circles in the figure) reasonably well, considering that no parameters were re-calibrated.

The model predicts two regimes. Regime 1 ($v > 30$) where $\rho(v)$ increases, and regime 2 ($v < 30$) where $\rho(v)$ decreases. Although [24] the does not describe experiments in this latter regime, we note from the previous section that the hybrid model predicts $\rho(0)$ reasonably well. Thus, it is reasonable to assume that the trend reversal is real.

The two-regime phenomenon is qualitatively explained by the spatial location of lane changes. Fig. 4 shows the k-maps for one simulation for two bottleneck speeds, $v = 1$ and 20 mph. Lane-changing locations are depicted as bold dots in the figure. Note from the white regions how passing maneuvers reduce the flow downstream of the moving obstruction, as expected. The key observation here is that the closer a maneuver is to the bottleneck, the bigger the void in front of the lane-changer. This is because the lane-speed differences are greater close to the bottleneck. The reason for the decline in regime 1 is a direct consequence of this effect, because as v increases the lane-speed differences drop and lane-changing become less and less disruptive.

Note from Fig. 4a that there is a critical distance from the bottleneck where lane-changing no longer has an effect and that for very low speeds a significant number of lane changes occur beyond this critical distance. The trend of regime 2 is then explained because as $v \rightarrow 0$, more and more lane changes take place far upstream from the bottleneck where lane-speeds are similar. Of course, this result is a consequence of our simple [2] lane-choice model (8), but it clearly highlights the importance of lane-changing location relative to the bottleneck.

5 Discussion

A four-parameter multilane hybrid model for traffic flow that recognizes the bounded accelerations of lane-changing vehicles has been introduced. The model appears to explain the reduction in flow observed after the onset of congestion at freeway lane-drops and the relationship between the discharge rate of a moving bottleneck and its speed and The good match suggests that DLC theory may also describe the mechanisms governing other common bottleneck types.

Our preliminary research shows that DLC theory also explains the "capacity drop" phenomenon observed at merge bottlenecks. This suggests that bounded accelerations are at the heart of bottleneck behavior. This information should allow ITS practitioners to manage capacity more efficiently by controlling the actual variables that induce the drop in capacity. Since the spatial distribution of lane-changes appears to play a key role, one can enhance traffic flow by controlling the location of lane changes; eg, by posting lane-changing bans or advisories [33] at key locations.

The framework proposed in this paper is general and can be extended to model multiple traffic streams both on freeways (eg, merges and diverges) and on urban streets. This should not be a difficult task because the boundary conditions imposed by multiple stream problems (ie, the merging and diverging process at ramps,

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[2] very high speed differences should discourage lane changes.

roundabouts and signalized/unsignalized intersections) can be handled very naturally with the multilane modeling framework.
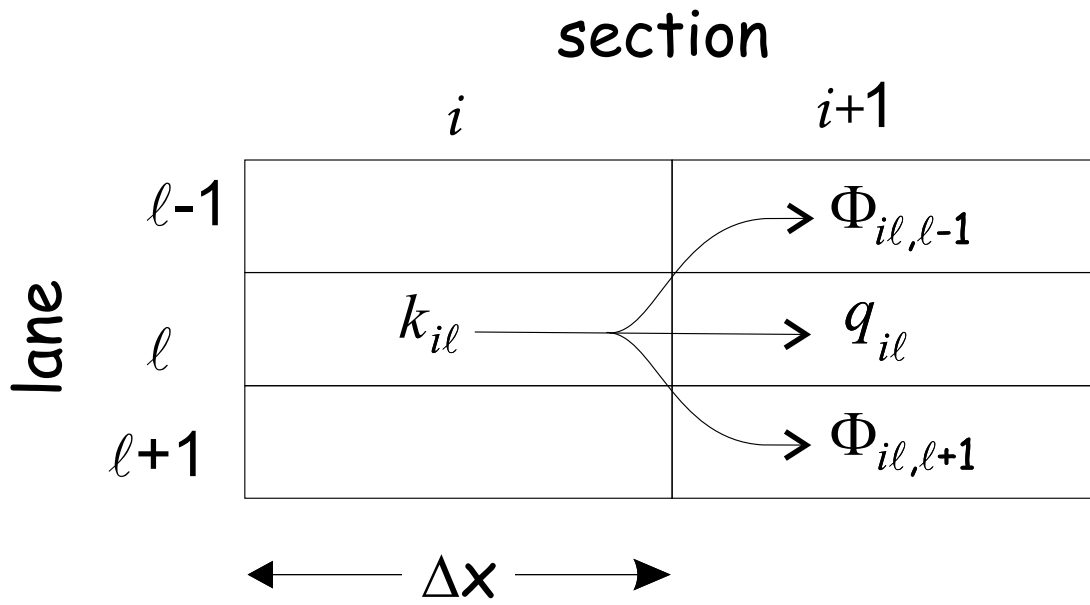
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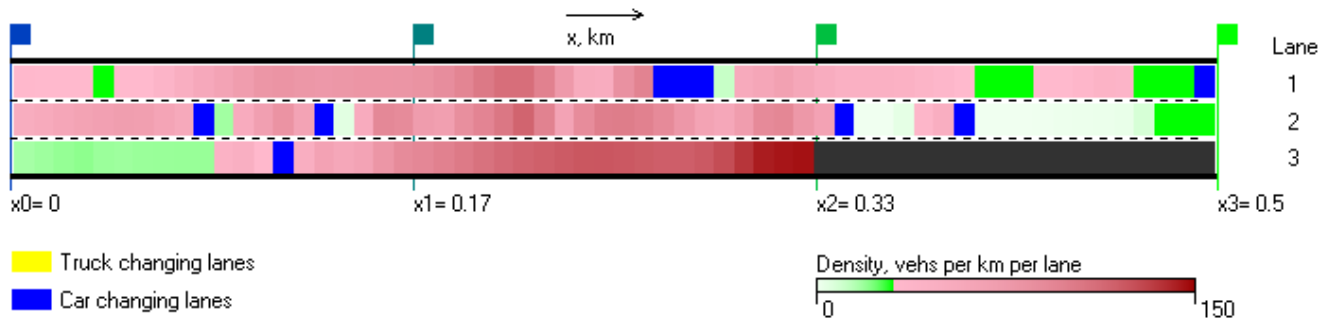
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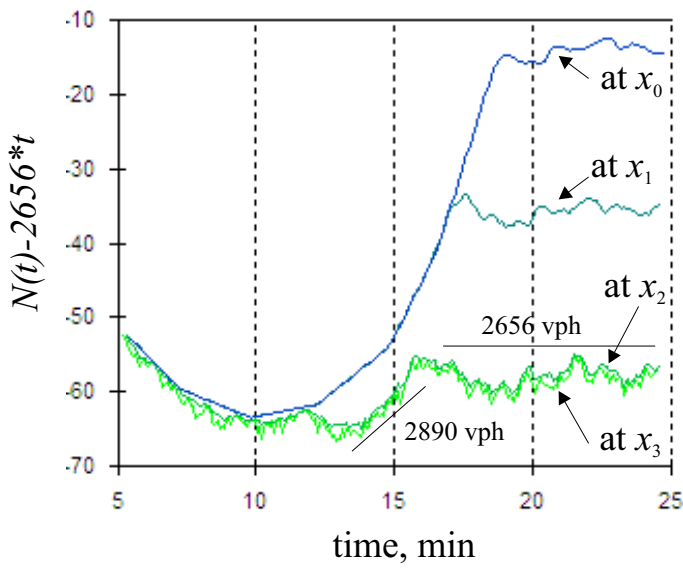
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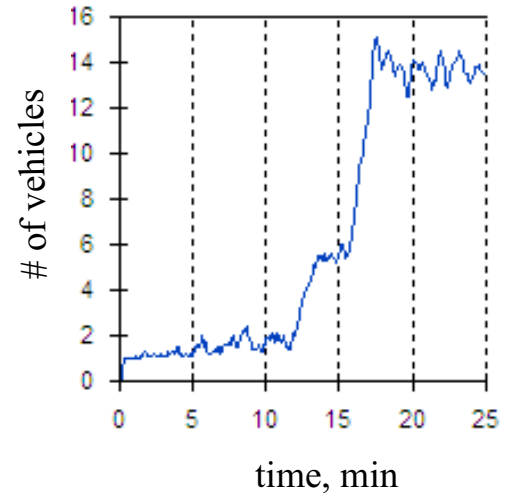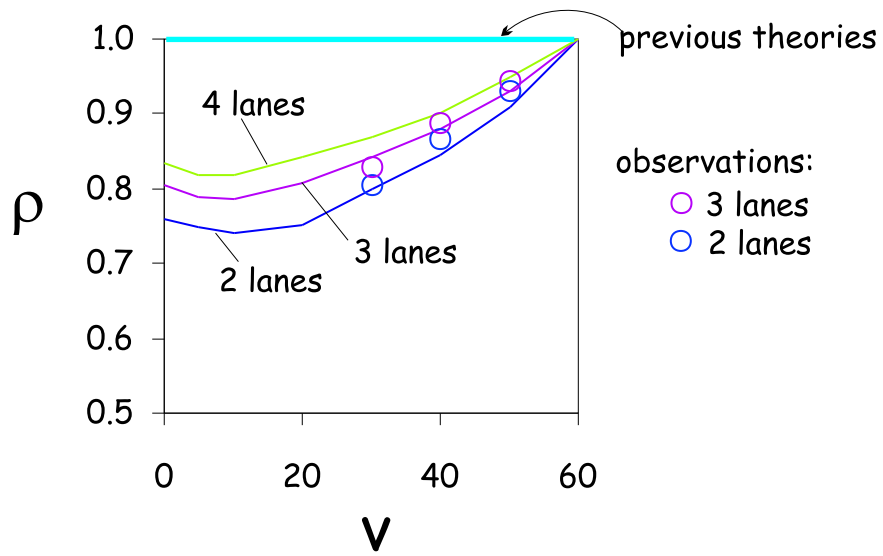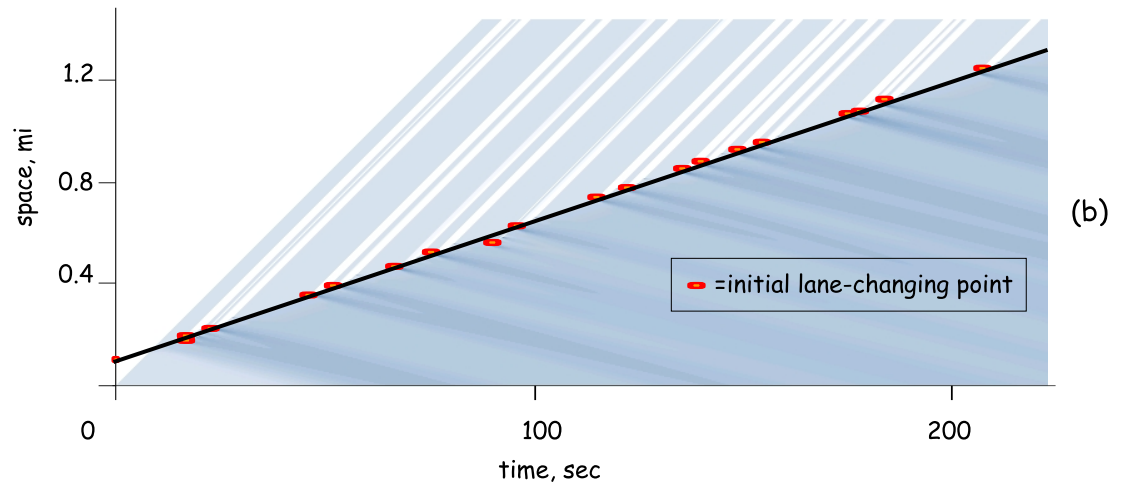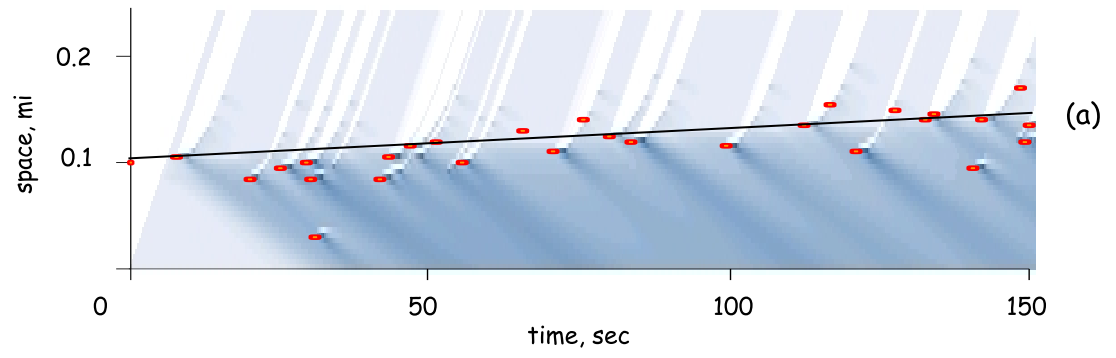
Figure 1: Discretized freeway representation.

Figure 2: Simulation of a 25-minute rush on a lane-drop: (a) configuration and snapshot at $t = 2$ min; (b) rescaled N-curves (c) vehicle accumulation on shoulder lane for $0.17 \leq x \leq 0.33$.

Figure 3: Dimensionless bottleneck discharge rate, ρ, as a function of the slow vehicle speed v for $n \in \{2, 3, 4\}$ lanes.

Figure 4: k-maps for a moving obstruction traveling at (a) $v = 1$ mph and (b) $v = 20$ mph.
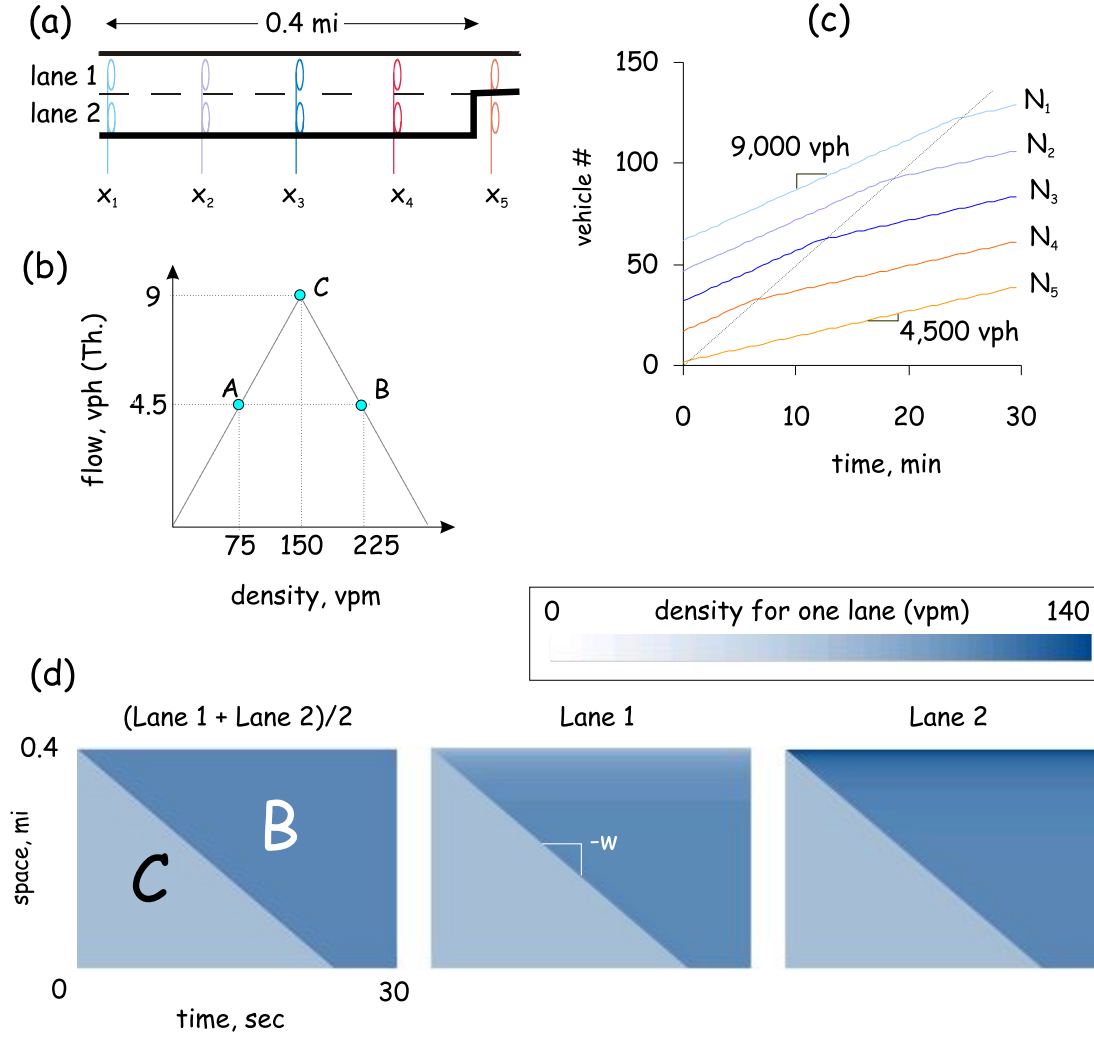
Figure A1: (a) A 2x1 lane-drop; (b) exact solution in the fundamental diagram; (c) numerical N-curves; (d) numerical k-map with $\Delta t = 0.2$ sec.

A   Appendix: Hybrid model convergence demonstration

In this section we examine the example of a lane-drop to show the convergence of model (3)-(10). The parameter values used in this section are: $u = -w = 60$ mph, $\kappa = 150$ vpmpl and $\tau = 3$ sec.

A.1   The numerical experiment

Consider a 0.4-mile, 2-lane freeway segment that has a lane-drop on lane 2 at $x = 0.4$ mi; see Fig. A1a. At $t = 0$ all lanes have optimal density $k_0 = 75$ vpmpl and the input demand is also k_0 for $t > 0$.

The numerical N-curves predicted by our method ($\Delta t = 0.2$ sec) for all lanes combined are shown in part (c), which are measured at the 5 evenly-spaced locations shown in part (a). This figure shows that this solution consists of an instantaneous transition between states C and B; see part (b) of the figure. This solution coincides with the (single-pipe) KW solution and indicates that the multi-lane model does not affect the discharge rate of the bottleneck. [3]

When we look at individual lanes separately, we observe reasonable patterns. Consider part (d) of the figure, which shows the k-maps of the numerical solution for each lane and for both lanes combined. Note how

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[3]This coincidence is a property of triangular fundamental diagrams and should not be true in general.

the density on lane 2 near the bottleneck is approx 140 vpm and decreases gradually upstream. The opposite happens on lane 1 with a density of k_0 at the bottleneck and increases gradually upstream. This behavior is reasonable, as common experience indicates that vehicles on the ending lane tend to drive slower the closer to the bottleneck they are.

To show the convergence of the model one must show that it is stable and consistent [34]. This is done next.
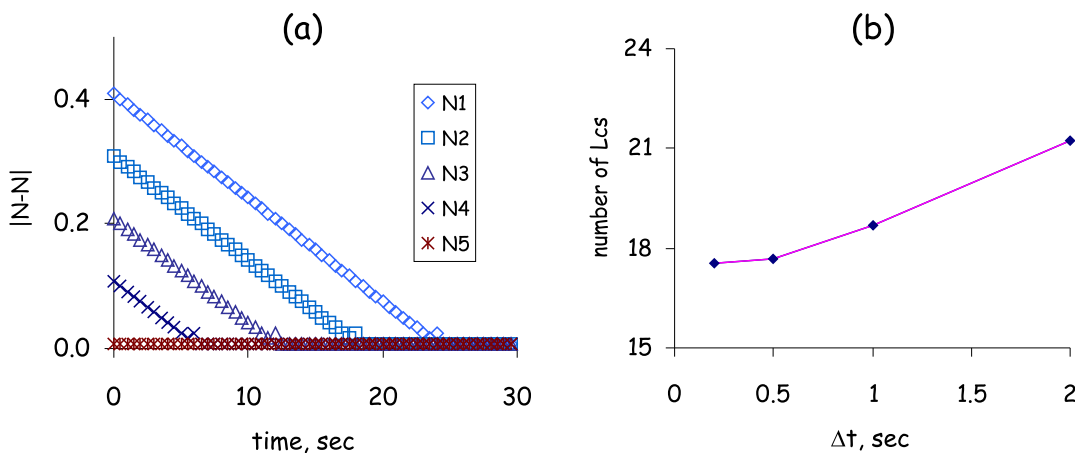
A.2 Stability

Let $N_0(x)$ be the initial N-profile used in the previous example; ie, the N-values along the road at $t = 0$, which in this case are given by $N_0(x) = 150x$. Also let a perturbed profile $\tilde{N}_0(x) = 152x$. If $\mathcal{M}_j$ is the operator obtained after j iterations of the numerical scheme (3)-(10), stability means that for all j:

$$\| N_0 - \tilde{N}_0 \| \leq \epsilon \qquad \Rightarrow \| \mathcal{M}_j N_0 - \mathcal{M}_j \tilde{N}_0 \| \leq \epsilon. \qquad (A1)$$

for some norm $\| \cdot \|$. Condition (A1) states that errors in the input data do not grow with time.

Fig. A2a shows the values of the errors $\| \mathcal{M}_j N_0 - \mathcal{M}_j \tilde{N}_0 \|$ for the L_∞-norm, $\Delta t = 0.5$ sec and $t_j \leq 30$ sec. It is clear that (A1) is satisfied since the errors decay as time passes.

Figure A2: (a) Error propagation showing stability of $\mathcal{M}$; (b) total number of lane-changes as a function of Δt.

A.3 Consistency

Let $\mathcal{M}_{\Delta t}$ be the operator obtained after one iteration of $\mathcal{M}$ with time-step Δt, and let $\mathcal{K}_{\Delta t}$ be the exact continuum operator up to time Δt. The model is consistent if

$$\| \mathcal{M}_{\Delta t} N_0 - \mathcal{K}_{\Delta t} N_0 \| \rightarrow 0 \qquad \text{as} \quad \Delta t \rightarrow 0. \qquad (A2)$$

Condition (A2) indicates that the solution of $\mathcal{M}$ tends to the continuum solution as the grid is refined.

Fig. A3 shows the k-maps of the lane-drop example for different mesh sizes. It becomes clear that the solution tends to Fig. A1d, obtained with $\Delta t = 0.2$ sec. Fig. A2b shows the total number of lane-changes (on the entire freeway during the whole simulation period) as a function of Δt, which also converges to approx. 17.5 as $\Delta t \rightarrow 0$.
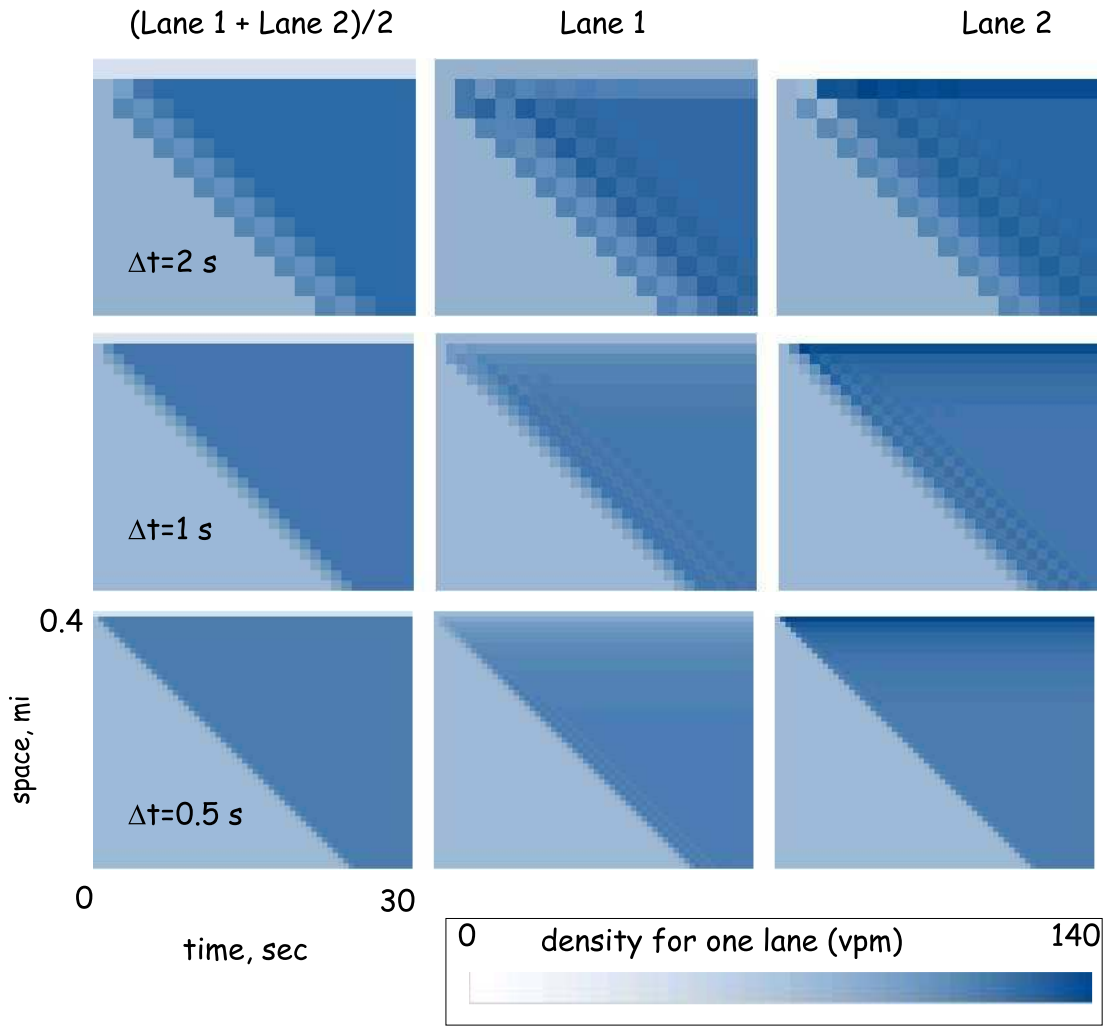
Figure A3: k-maps of the lane-drop example for $\Delta t = 2$ sec (top,) $\Delta t = 1$ sec (middle) and $\Delta t = 0.5$ sec (bottom.)