MULTI-LEVEL TRELLIS CODED MODULATION AND MULTI-STAGE DECODING*

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Outline of Paper

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- 90° rotationally invariant multi-level trellis codes
- Conclusions

Introduction

- The goal in designing a coded modulation system is to achieve a good trade-off between coding gain, decoding complexity, and decoding delay.

- Multi-level coding is a powerful technique for constructing bandwidth efficient coded modulation codes. Good multi-level coding schemes can be designed by using previously known codes as component codes.

- Multi-stage decoding provides a simple decoder implementation for multi-level codes with a small loss in coding gain.

- For coded QAM, the total power gain over uncoded QAM is composed of two parts: the coding gain ($\gamma(C)$) and the shaping gain ($\gamma(S)$).

- At a bit error rate of $10^{-5} \sim 10^{-6}$, the maximum coding gain is about 7.5 dB, and the maximum shaping gain is about 1.5 dB.

- Because these gains can be achieved independently, for coded QAM we focus on coding gain only and choose the signal set to be Z^N (N dimensional integer lattice).

- For coded MPSK, we also focus only on coding gain, since no shaping gain is possible.

- The phase invariant property (or phase symmetry) is useful in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after a temporary loss of synchronization. It is desirable for a coded modulation system to have as much phase symmetry as possible.

- We present necessary and sufficient conditions for a QAM code to be 90° rotationally invariant, and some 90° rotationally invariant multi-level codes are constructed.

Multi-level Trellis Coding Based on Set Partitioning

- Figure 1 shows a multi-level trellis coding scheme based on set partitioning. Λ_0 is a signal set, Λ_i is a subset of Λ_{i-1}, and Λ_m is the all-zero vector. $C_1, C_2, \ldots, C_m$ represent the different component codes, and the overall multi-level code is denoted by C.

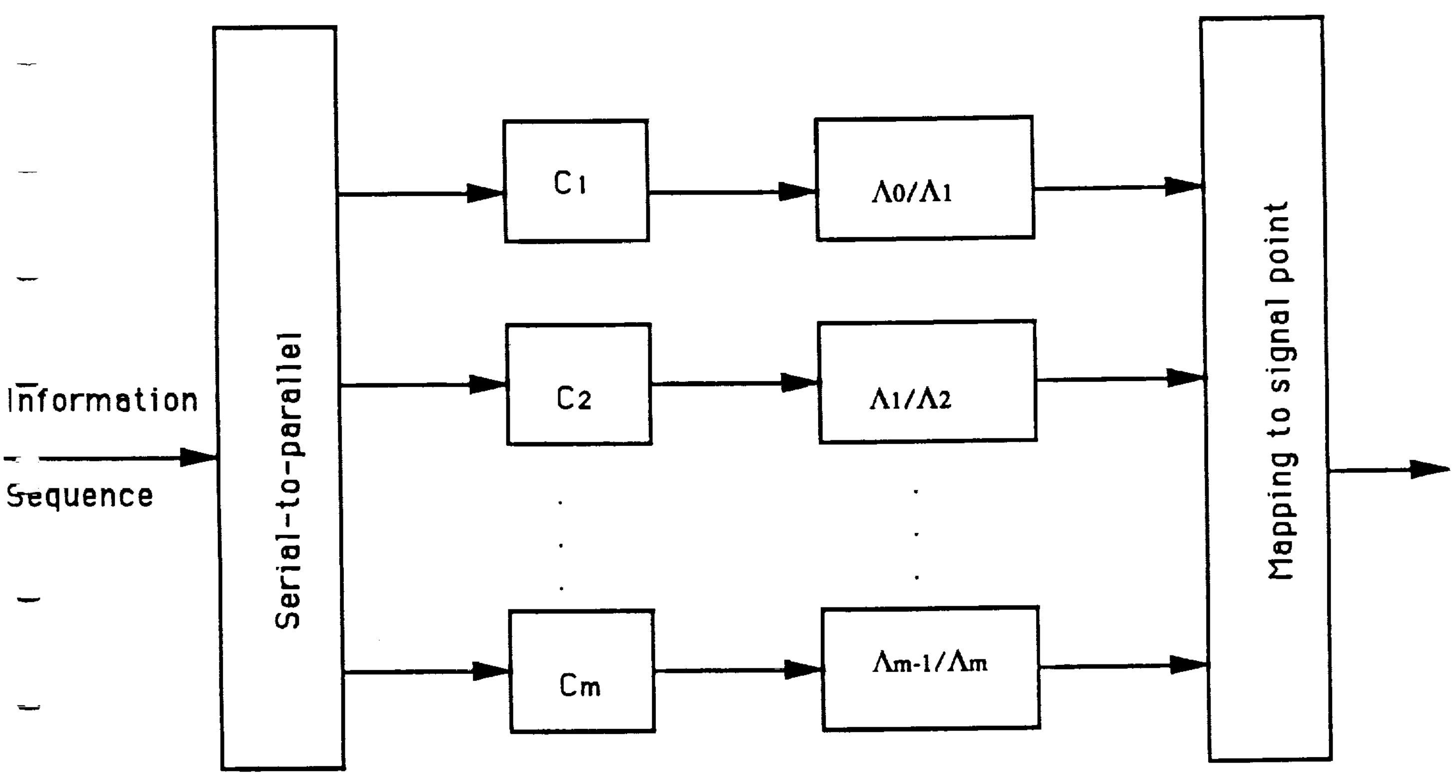


Fig.1 Multi-level trellis coding based on set partitioning

Related previous work

- Leech (1964) and Leech and Sloane (1971) used a multi-level structure to construct lattices.
- Multi-level codes using "proper indexing", which is the same as Ungerboeck's "set partitioning", of two dimensional signal sets was proposed by Imai and Hirakawa (1977). They also presented a multi-stage decoding method using a posteriori probabilities based on channel statistics.
- Ginzburg (1984) designed multi-level multi-phase codes for a continuous channel by using set partitioning and algebraic block codes.
- Sayegh (1986) showed how Imai and Hirakawa's method can be combined with set partitioning to create multi-level block coded modulation systems.
- Pottie and Taylor (1989) proposed a hierarchy of codes to match the partitioning of signal sets by generalizing Imai and Hirakawa's and Ginzburg's coding schemes.

- Calderbank (1989) investigated the path multiplicity for a variety of multi-level codes.
- Tanner (1990?)studied linking subspaces of vector spaces to guarantee a large minimum separation between signals in the resulting signal set so that good multi-level codes can be designed.

Basic multi-level trellis codes

- This construction is based on two-way partition chains, where all component codes are binary codes (block or convolutional).
- Let Δ_i be the minimum squared Euclidean distance (MSED) of Λ_i for $i = 0, 1, \ldots, m$.
- Let d_i be the minimum Hamming distance of binary code C_i for $i = 1, 2, \ldots, m$.
- Then the MSED of the multi-level code is (Leech & Sloane, Ginzburg, Sayegh, etc.)

$$D(C) = \min\{d_i \Delta_{i-1}, 1 \leq i \leq m\}$$

- The normalized redundancy $\rho(C)$ is defined as (Forney) the number of redundant bits per two dimensional signal (symbol).

- The spectral efficiency $\eta(C)$ is defined as (Ungerboeck) the number of information bits per two dimensional signal (symbol).

- Basic multi-level codes with normalized redunduncy $\rho(C) = 1$ bit/symbol were presented by Yamaguchi and Imai (1987).

- Basic multi-level codes with smaller normalized redundancies can be constructed by using two-way partition chains with multi-dimensional signal sets and binary convolutional or block codes. Some four and eight dimensional basic multi-level codes were constructed by Wu and Zhu (1990?).

- We present some new basic multi-level codes based on set partitioning of one and two dimensional signal sets. Some of these new codes have non-integer normalized redundancies $\rho(C)$.

- **Example 1.** A three-level trellis code using an 8-PSK signal set with mapping by set partitioning is shown in Figure 2.

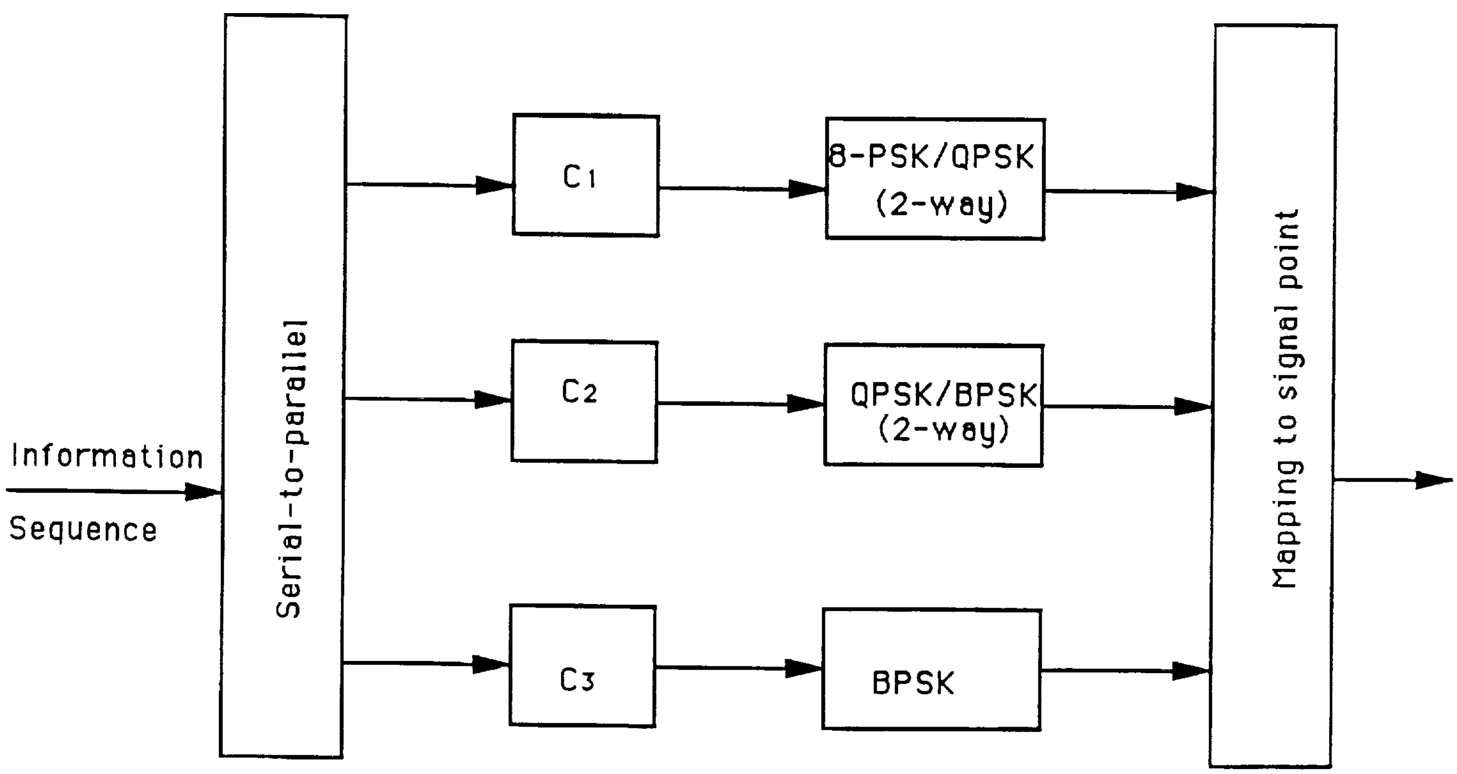


Fig.2 Multi-level code of Example 1

- **Let** C_1 **be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16,**
  C_2 **an 8-state rate-3/4 convolutional code with free distance 4,**

  $C_3 = P_n$, **the** $(n, n-1)$ **single parity check code.**
- **The spectral efficiency of this multi-level code is**

$$\eta(C) = 1 + (n-1)/n \text{ bits/symbol}$$

- **The minimum free squared Euclidean distance is**

$$\begin{aligned} D(C) &= \min\{0.586 \times 16, 2 \times 4, 4 \times 2\} \\ &= 8 \end{aligned}$$

- **The nominal coding gain (Ungerboeck) over uncoded QPSK is**

$$\gamma(C) = 10 \log_{10} \left(\frac{D(C)}{D(QPSK)} \right) = 6.02dB.$$

- **The 256-state, rate-2/3,** $\eta(C) = 2$ **bits/symbol Ungerboeck code has** $D(C) = 7.515$ **and** $\gamma(C) = 5.75$ **dB.**

Multi-Stage Decoding of Example 1

- A three-level multi-stage decoder for Example 1 is shown in Figure 3.

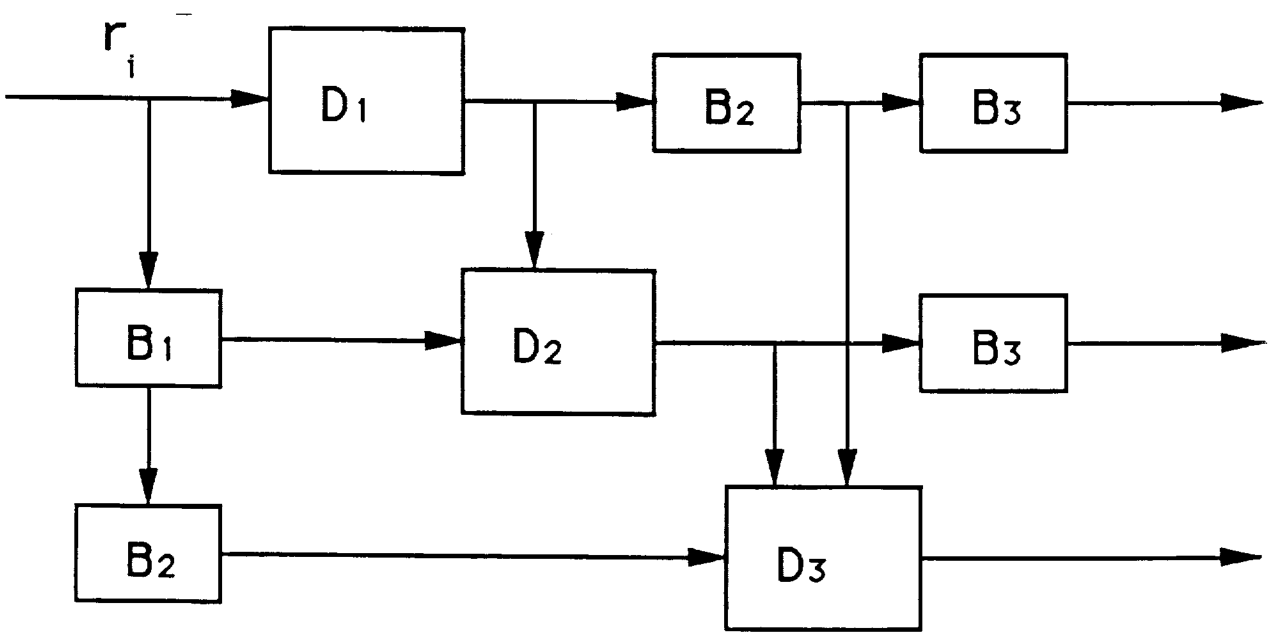


Fig. 3 Multi-stage decoding for Example 1

- **The normalized complexity N_D of multi-stage decoding is the number of required binary operations (additions and comparisons) per 2 dimensional symbol.**

- **For a 2^ν-state, k input bit, n output bit convolutional (trellis) code, the Add-Compare-Select (ACS) operation of the Viterbi algorithm requires 2^k additions and a comparison of 2^k numbers, or $2^k - 1$ binary comparisons, for each of the 2^ν states, so its complexity is $2^{k+\nu+1} - 2^\nu$. (This number should be normalized to the complexity per 2 dimensional symbol.)**

First-stage of Decoding

- For each state transition period, the symbol metrics of both QPSK subsets (see Figure 4) must be computed.
  Complexity = 2 binary operations/symbol

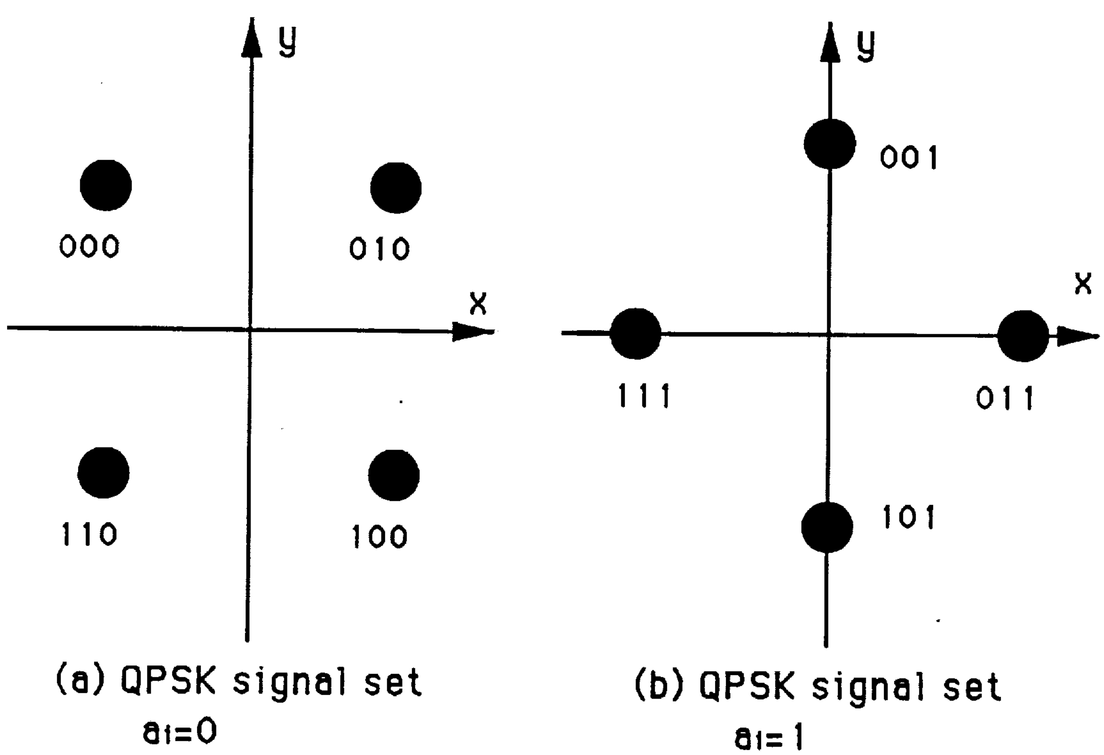


Fig.4

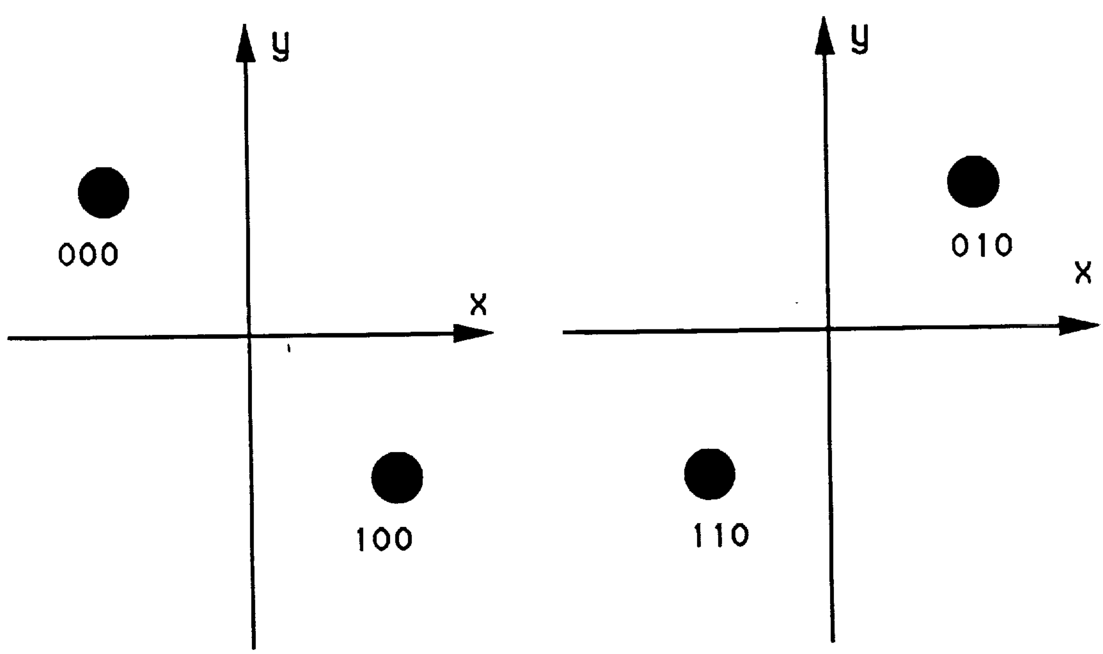
- Then the branch metrics within each state transition period must be computed by adding the four symbol metrics on each branch.
  Complexity = 6 binary operations/symbol

- The ACS operation of the Viterbi algorithm is then used to determine the surviving path at each state.
  Complexity = 12 binary operations/symbol

<u>Second-stage of Decoding</u>

- The decoded information from the first stage is passed on to the second-stage.

- For each state transition period, the symbol metrics of both BPSK subsets (see Figure 5) must be computed.
  Complexity = 2 binary operations/symbol



(a) BPSK signal set
a_1b_1=00

(b) BPSK signal set
a_1b_1=01

Fig.5

- Then the branch metrics within each state transition period must be computed by adding the four symbol metrics on each branch.
  Complexity = 6 binary operations/symbol

- The ACS operation of the Viterbi algorithm is then used to determine the surviving path at each state.
  Complexity = 30 binary operations/symbol

- If the parallel transitions in the trellis are resolved by table-look-up, the complexity reduces to 14 binary operations/symbol.

Third-stage of Decoding (assume $n = 32$)

- The decoded information from the first and second stages is made available to the third stage.
- For each state transition period, the metrics of both the 0 and 1 symbols must be computed.
  Complexity = 2 binary operations/symbol
- In this case, the branch metrics are the symbol metrics computed above (one symbol per trellis branch).
- After 8 branches (32 symbols) in the first and second trellis are decoded, the Viterbi algorithm is used to make a decoding decision for the block code $C_3 = P_{32}$.
  Complexity = 6 binary operations/symbol
- The total decoding complexity is $N_D = 66$ binary operations per 2 dimensional symbol, or $N_D = 50$ not counting the parallel transitions.

- If we take C_3 to be P_n, where $n \rightarrow \infty$ (i.e., a 2-state, non-redundant, catastrophic trellis code), the multi-level code has $1 + 3 + 4 = 8$ input bits and $4 + 4 + 4 = 12$ output bits for every four 8-PSK transmitted symbols. Overall, this can be viewed as a $16 \times 8 \times 2 = 256$-state 8-dimensional trellis code.

- Without considering the computation of the symbol and branch metrics, the ACS complexity of maximum likelihood decoding of the overall trellis code is

$$(2^{8+8+1} - 2^8)/4 = 2^{15} - 2^6 > 3 \times 10^4$$

binary operations/2 dimensional symbol.

- Note that the complexity of the multi-stage decoder in this example is only about 0.2% of the complexity of the overall maximum likelihood decoder.

- However, the performance of the multi-stage decoder is close to that of the maximum likelihood decoder.

- For the 256-state Ungerboeck code, the ACS complexity alone is 1792 binary operations/symbol.

- **<u>Example 2.</u> The one dimensional partition chain $Z/2Z/4Z/\ldots$ has MSED $1/4/16/\ldots$ (see Figure 6).**

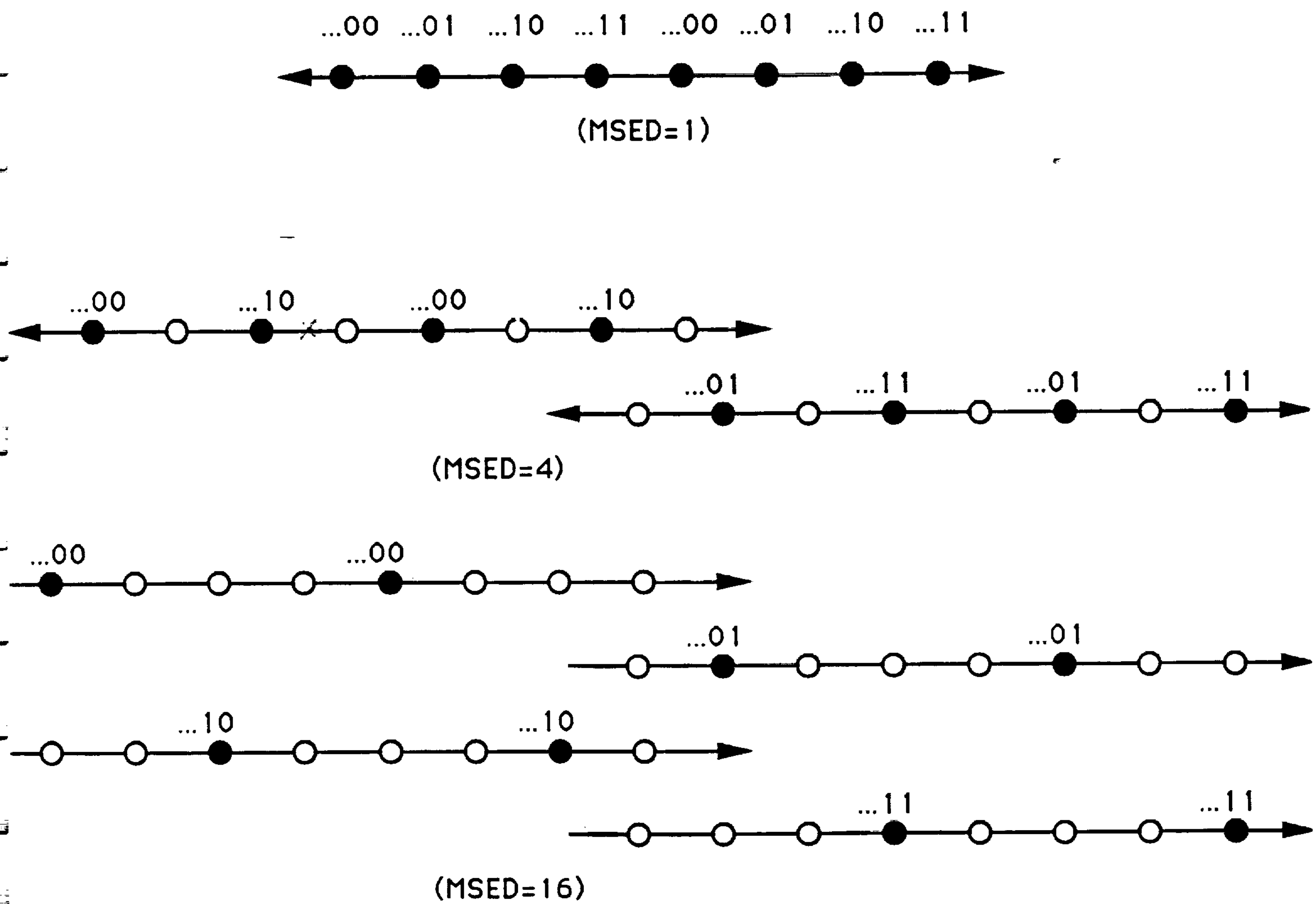


Fig.6 Set partitioning of Z

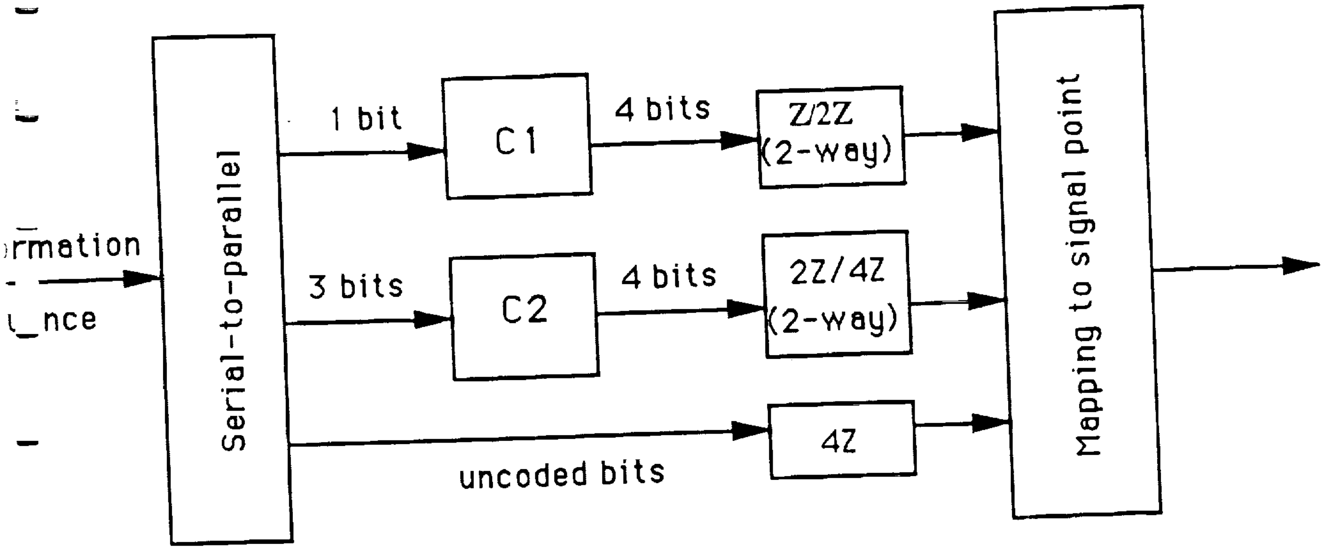


Fig.7 Multi-level code of Example 2.

- **Let C_1 be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16,**

  C_2 be an 8-state rate-3/4 convolutional code with free distance 4,

  and $C_3, C_4, \ldots$ be rate-1 codes (no coding).

- **Since there are two levels of coding, this is a two-level code (see Figure 7).**
- **The normalized redundancy $\rho(C)$ is 2 bits per symbol.**
- **The MSED is**

$$D(C) = \min\{1 \times 16, \;\; 4 \times 4, \;\; 16\} = 16$$

- The <u>nominal coding gain</u> (Forney) is

$$\gamma(C) = 10\log_{10}\frac{D(C)}{2^{\rho(C)}} = 6.02(\mathbf{dB})$$

- This code has the same nominal coding gain and normalized redundancy as the 24-dimensional Leech lattice Λ_{24} but much less decoding complexity.

- Due to a large path multiplicity, the effective coding gain of this two-level code is less than the nominal coding gain. To reduce the path multiplicity, we can choose longer convolutional codes (with larger constraint lengths and free distances).

- For example, if C_1 is a 32-state rate-1/4 convolutional code with free distance 18 and C_2 is a 32-state rate-3/4 convolutional code with free distance 5, the path multiplicity is reduced and the effective coding gain is closer to 6.02 dB.

sing multi-stage decoding, an additional ss of coding gain occurs, but the decoding omplexity is less than a 64-state ngerboeck code and much less than the eech lattice Λ_{24}.

ble 1. Comparison of multi-level trellis codes 1 other codes (spectral efficiency $\eta(C) = 4$ /symbol) using 8-PAM modulation

odes	#S	R_i	$\gamma(C)$	N_D	D
o-level	16 & 8	1/4 & 3/4	5.81	116	14
-level	32 & 8	1/4 & 3/4	5.81	130	16
o-level	32 & 32	1/4 & 3/4	5.81	350	20
	256		5.81	$\sim$ 1264	
gerboeck	32	2/3	4.77	232	5
ngerboeck	64	2/3	5.44	456	6

ne decoding delay of a multi-level trellis code proportional to

$$D \triangleq \frac{1}{2} \sum N_i K_i,$$

iere N_i is the dimensionality of the signals cosets) associated with a branch transition of e ith component code and K_i is the nstraint length of the ith component code.

Multi-level trellis codes based on a set partition chain with strictly increasing distances

- Multi-level trellis codes using multi-dimensional signal sets can achieve higher spectral efficiencies (lower normalized redundancies) than multi-level codes based on two dimensional signal sets.

- For two-way partitioning of multi-dimensional signal sets, the MSED at successive partition levels may be equal. For example, the partition chain $Z^4/D_4/RZ^4/RD_4/2Z^4/2D_4/\ldots$ of the four dimensional integer lattice Z^4 has distances $1/2/2/4/4/8/\ldots$, where R represents the rotation operation, $R^2 = 2$, and D_4 is the densest known four dimensional lattice.

- Reducing the number of component codes can reduce the decoding delay and the path multiplicity.

- Some partition levels can be joined to form a new multi-way partition chain with strictly increasing distances. For example, the partition chain $Z^4/D_4/RD_4/2D_4/\ldots$ has distances $1/2/4/8/\ldots$. Since $|Z^4/D_4| = 2$ and $|R^iD_4/R^{i+1}D_4| = 4$ for $i = 0, 1, 2, \ldots$, the first component code can be a binary code, and other component codes can be binary input, 4-ary output codes or codes over $GF(4)$.

- The lower bound on the MSED of these multi-level codes is given by

$$D(C) \geq \min\{d_i\Delta_{i-1},\ \ 1 \leq i \leq m\}$$

where d_i is now the minimum free Hamming distance of code C_i (binary or 4-ary).

Example 3. This code is based on the partition chain $Z^4/D_4/RD_4/\ldots$ and includes two component codes (see Figure 8):

C_1 is an 8 state rate-3/4 convolutional code with free distance 4 and C_2 is an (N, N-1) block code over GF(4) with minimum distance 2 (4 states).

- The normalized redundancy is $\rho(C) = \frac{1}{8} + \frac{1}{N}$, the MSED is 4, and the nominal coding gain is

$$\gamma(C) = 5.64 - \frac{3.01}{N} \text{ (dB)} = 5.48 \text{ dB } (N = 19)$$

- The decoding complexity is $N_D = 37$, and the decoding delay is $D = 24$ (excluding the decoding delay of the block code).

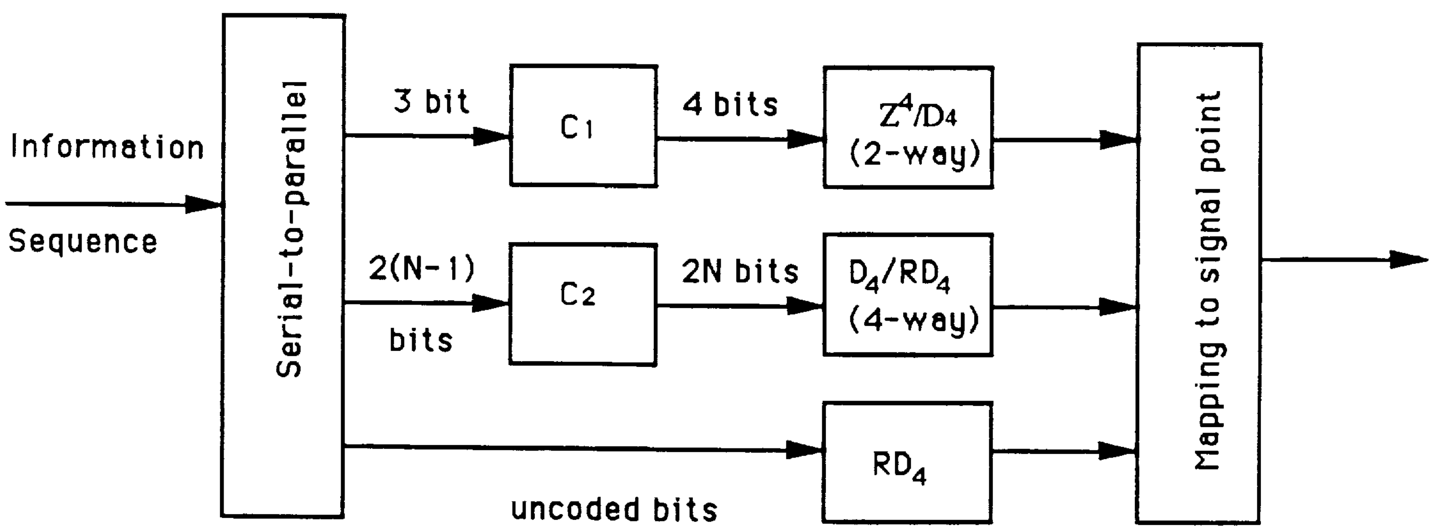


Fig.8 Multi-level code of Example 3.

- The 64-state, rate-4/5 Ungerboeck code for Z^4 has $\gamma(C) = 5.48$ dB, $N_D \approx 496$, and $D = 12$.

Combined Ungerboeck-type and multi-level trellis codes

- For the above two classes of multi-level codes, each output symbol of a component encoder corresponds to a single coset of a subset of a signal constellation (two dimensional or multi-dimensional). For Ungerboeck-type codes, all the encoder output symbols associated with a single trellis branch correspond to a single coset of a subset of a signal constellation. Ungerboeck-type codes can be used as component codes at some levels in conjunction with a multi-way partition chain.
- Instead of using several high rate codes at higher levels of partitioning, we use an Ungerboeck-type code to reduce the decoding delay and path multiplicity.
- The usual lower bound on the MSED cannot be applied to this construction. A more general lower bound on the MSED of these multi-level codes (Kasami & Lin) is given by

$$D(C) \geq \min\{D(C_i),\ \ 1 \leq i \leq m\}$$

where $D(C_i)$ is the MSED of code C_i.

Example 4. The encoding structure is shown in Figure 9.

- Let C_1 be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16,

  C_2 a 16-state rate-7/8 trellis code with MSED 8 (Pietrobon, Deng, et.al., 1990).

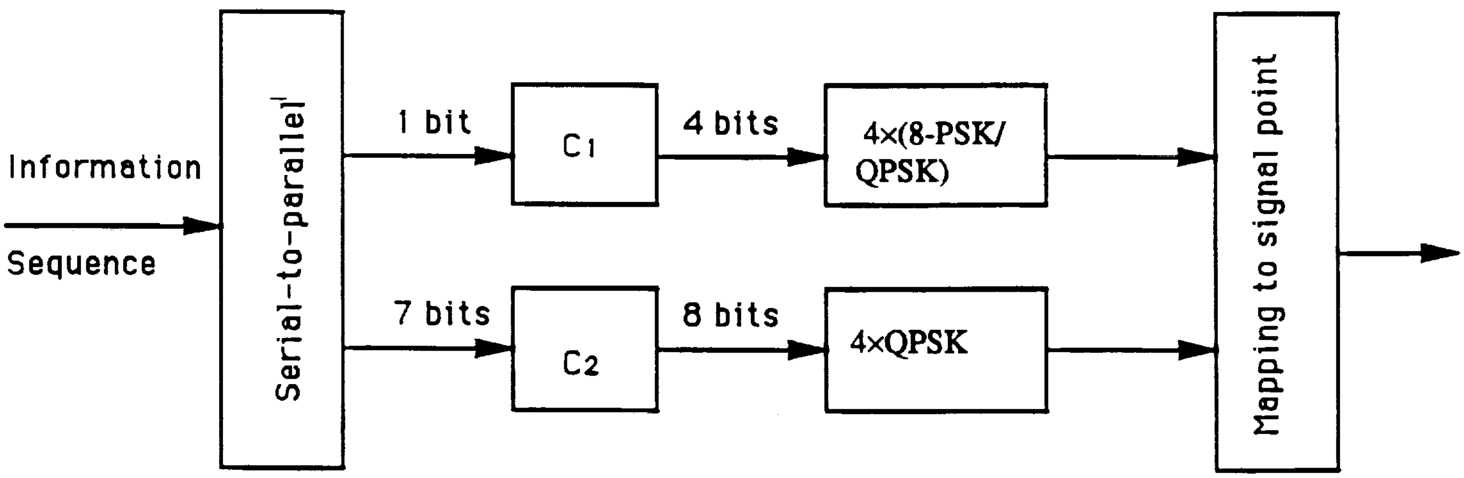


Fig.9 Multi-level code of example 4.

- The 64-state, rate-4/5, $\eta(C) = 2$, $4 \times$ 8-PSK code constructed by Pietrobon, Deng, et.al. (1990) has $D(C) = 7.029, \gamma(C) = 5.46$ dB , $N_D \approx 496$, and $D = 12$.

- Encoding procedure:

  The information sequence is divided into blocks of 8 bits each:
  the first bit in each block enters encoder C_1, and the 4 output bits specify 4 consecutive cosets of $4\times$(8-PSK/QPSK),

  the other 7 bits of the block enter encoder C_2 and the 8 output bits specify a $4\times$ QPSK signal.

- The spectral efficiency of this multi-level code is

$$\eta(C) = 8/4 = 2 \text{ bits/symbol}$$

- The MSED is

$$D(C) = \min\{16 \times 0.586, 8\} = 8$$

  where $D(C_1) = 16 \times 0.586$.

- The nominal coding gain over uncoded QPSK is $\gamma(C) = 6.02$ dB.

- The decoding complexity for multi-stage decoding is $N_D \approx 100$ binary operations per symbol.

- The decoding delay is only $D = 16$, which is less than a multi-level code with more stages.

Generalized multi-level trellis codes

- The previous examples were all based on Ungerboeck's set partitioning. A modified set partitioning method can be used to construct generalized multi-level trellis codes.

- Example 5. The four dimensional 8-state code $C(Z^4/RD_4)$ constructed by Wei (1987) has MSED 4. Mapping the same binary code to $RZ^4/2D_4$ rather than Z^4/RD_4, we obtain a trellis code, denoted by $C_2(RZ^4/2D_4)$, with MSED 8. Using an 8-state rate-1/3 convolutional code as the first component code $C_1(Z^2/RZ^2)$ and $C_2(RZ^4/2D_4)$ as the second component code gives the two-level trellis code shown in Figure 10.

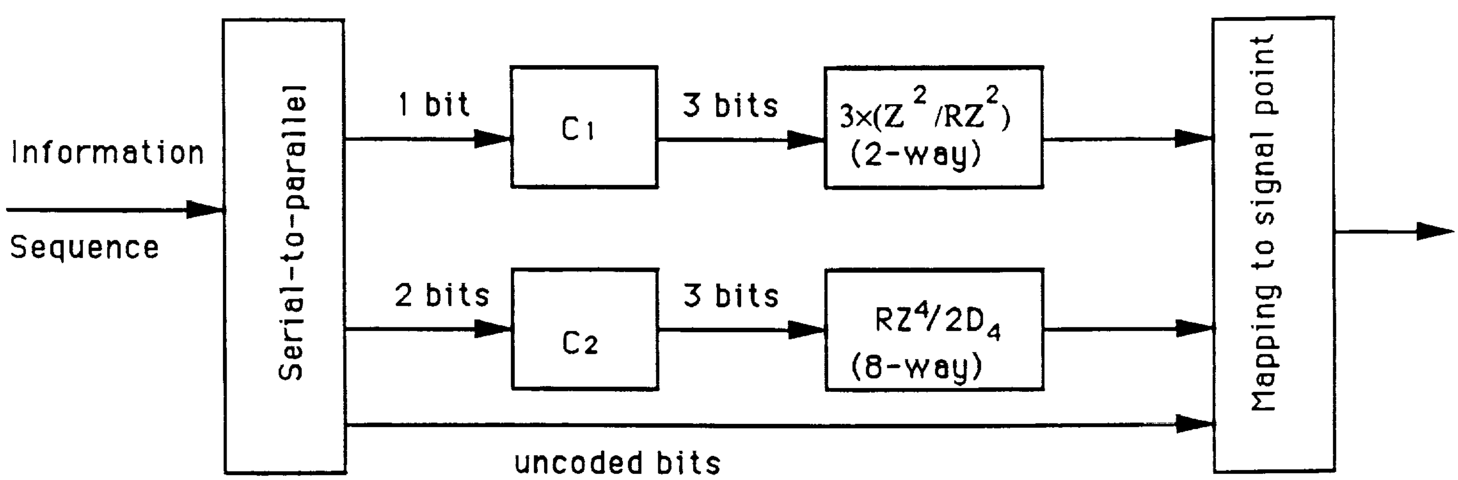


Fig.10 Multi-level code of Example 5.

- Encoding procedure:
  The information sequence is divided into blocks of a specified number of bits according to the desired spectral efficiency:
  the first 2 bits in each block enter encoder C_1, and the 6 output bits specify 6 consecutive cosets of Z^2/RZ^2, i.e., 3 consecutive cosets of Z^4/RZ^4,
  the next 6 bits of the block enter encoder C_2, and the 9 output bits specify 3 consecutive cosets of $RZ^4/2D_4$.
  Together with uncoded bits, each coded block determines 3 consecutive four dimensional signals, i.e., 6 two dimensional signals.
- Note that the first coding level partitions a six dimensional signal set whereas the second coding level partitions a four dimensional signal set.
- The nominal coding gain is $\gamma(C) = 5.52$ dB, which is 1.00 dB greater than Wei's code.
- The decoding delay of this two-level code is $D = 15$, whereas the delay of Wei's code is $D = 6$.
- The decoding complexity of this two-level code is $N_D = 56$, whereas the complexity of Wei's code is $N_D = 44$.

Example 6. Consider the generalized multi-level trellis code shown in Figure 11.

The first component code, associated with the partition $Z/2Z$, is a 32 state rate-1/2 convolutional code with free distance 8.

The second component code, associated with the partition $2Z^8/2D_8$, is a single parity check block code of length N.

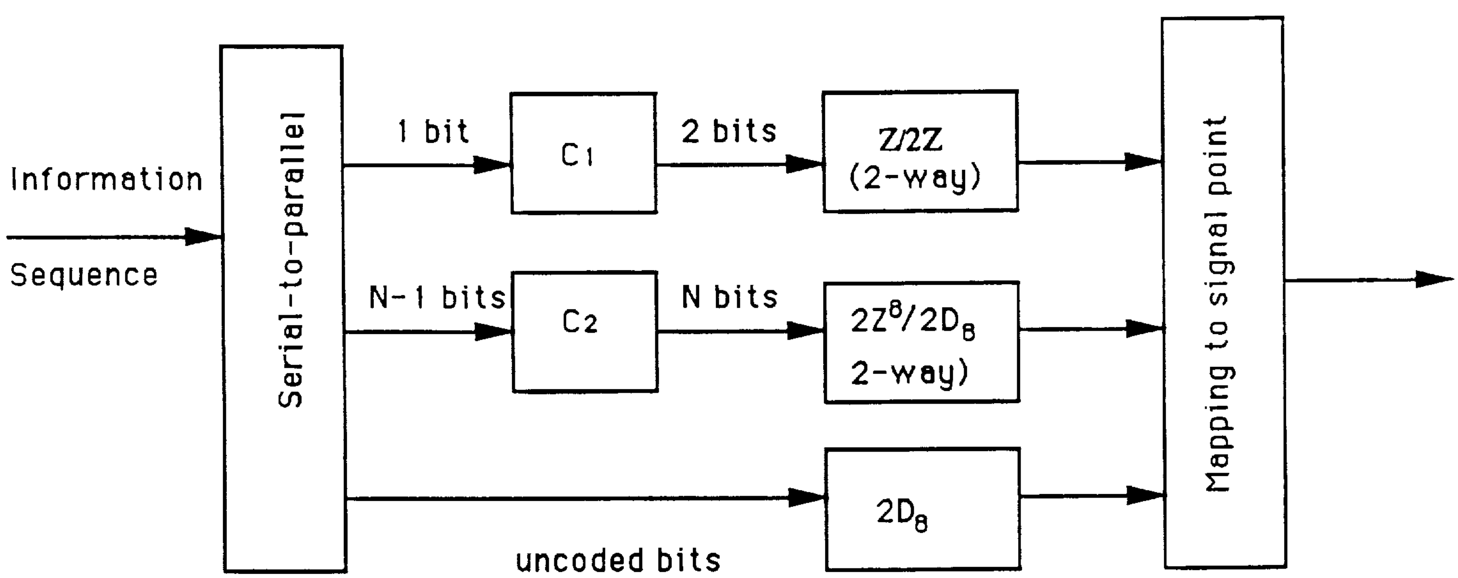


Fig.11 Multi-level code of Example 6.

- Encoding procedure (spectral efficiency = m bits/symbol):

- The information sequence is divided into blocks of $4Nm$ bits each, with three subsequences of length $4N, N-1$, and $4Nm-4N-(N-1)$ corresponding to C_1, C_2, and uncoded bits, respectively.

- Each output bit of encoder C_1 specifies a coset of partition $Z/2Z$, i.e., $8N$ output bits specify N cosets of $Z^8/2Z^8$.

- Each output bit of encoder C_2 specifies a coset of $2Z^8/2D_8$, i.e., N output bits specify N cosets of $2Z^8/2D_8$.

- Together with the $4Nm-4N-(N-1)$ uncoded bits, each coded block of N eight dimensional signals, i.e., $4N$ two dimensional signals contains $4Nm$ bits of information and the spectral efficiency in m bits/symbol.

- **The normalized redundancy is** $\rho(C) = 1 + \frac{1}{4N}$ **and the MSED is 8. Therefore the nominal coding gain is**

$$\gamma(C) = 6.02 - \frac{3.01}{4N} \text{ (dB)}.$$

- **The decoding complexity is** $N_D = 140$ **and the decoding delay (excluding the block code) is** $D = 5$.

- **The 128-state, rate-4/5 Ungerboeck code for** Z^8 **has** $\gamma(C) = 5.27$ **dB, number of nearest neighbors** $N_{free} = 112, N_D \approx 992$**, and** $D = 28$.

- **In general, let Λ_0 be a signal set and $\Lambda_0^{(1)}$ be a set such that $\underbrace{\Lambda_0 \times \cdots \times \Lambda_0}_{j_0} = \underbrace{\Lambda_0^{(1)} \times \cdots \times \Lambda_0^{(1)}}_{k_0}$, for some integers $j_0, k_0 \geq 1$.**

- **If there are $2m$ sets $\Lambda_{i-1}^{(i)}$ and $\Lambda_i^{(i)}$, for $i = 1, 2, \ldots, m$, where $\Lambda_m^{(m)}$ is the empty set, satisfying the following conditions:**

  (1) $\underbrace{\Lambda_{i-1}^{(i-1)} \times \cdots \times \Lambda_{i-1}^{(i-1)}}_{j_i} = \underbrace{\Lambda_{i-1}^{(i)} \times \cdots \times \Lambda_{i-1}^{(i)}}_{k_i}$, for $i = 2, 3, \ldots, m$, and for some $j_i, k_i \geq 1$;

  (2) $\Lambda_{i-1}^{(i)} \supseteq \Lambda_i^{(i)}$, for $i = 1, 2, \ldots, m$; then we can construct a multi-level code having the form shown in Figure 12:

$$C = C_1\left(\Lambda_0^{(1)}/\Lambda_1^{(1)}\right) + C_2\left(\Lambda_1^{(2)}/\Lambda_2^{(2)}\right) + \ldots + C_m\left(\Lambda_{m-1}^{(m)}/\Lambda_m^{(m)}\right)$$

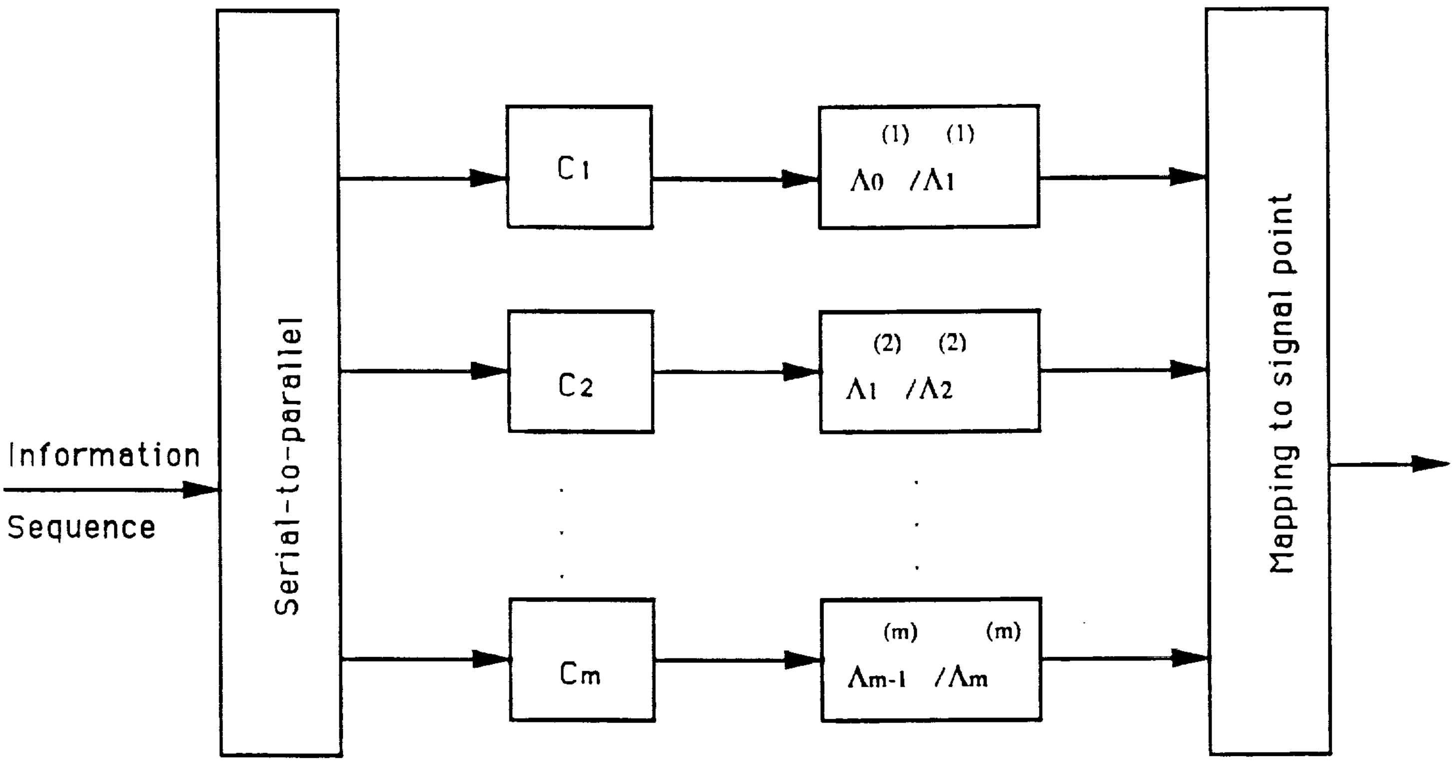


Fig.12 The generalized multi-level coding scheme

- **The MSED of this multi-level code is lower bounded by**

$$D(C) \geq \min\{D[C_i(\Lambda_{i-1}^{(i)}/\Lambda_i^{(i)})],\ 1 \leq i \leq m\}$$

where $D[C_i(\Lambda_{i-1}^{(i)}/\Lambda_i^{(i)})]$ **is MSED of the** i**th component code.**

Level spanning multi-level trellis codes

- Level spanning provides an approach to constructing rotationally invariant multi-level trellis codes.
- However, the lower bound on MSED of generalized multi-level codes may not hold for this class of codes.

Example 7. Consider the multi-level coding scheme shown in Figure 13.

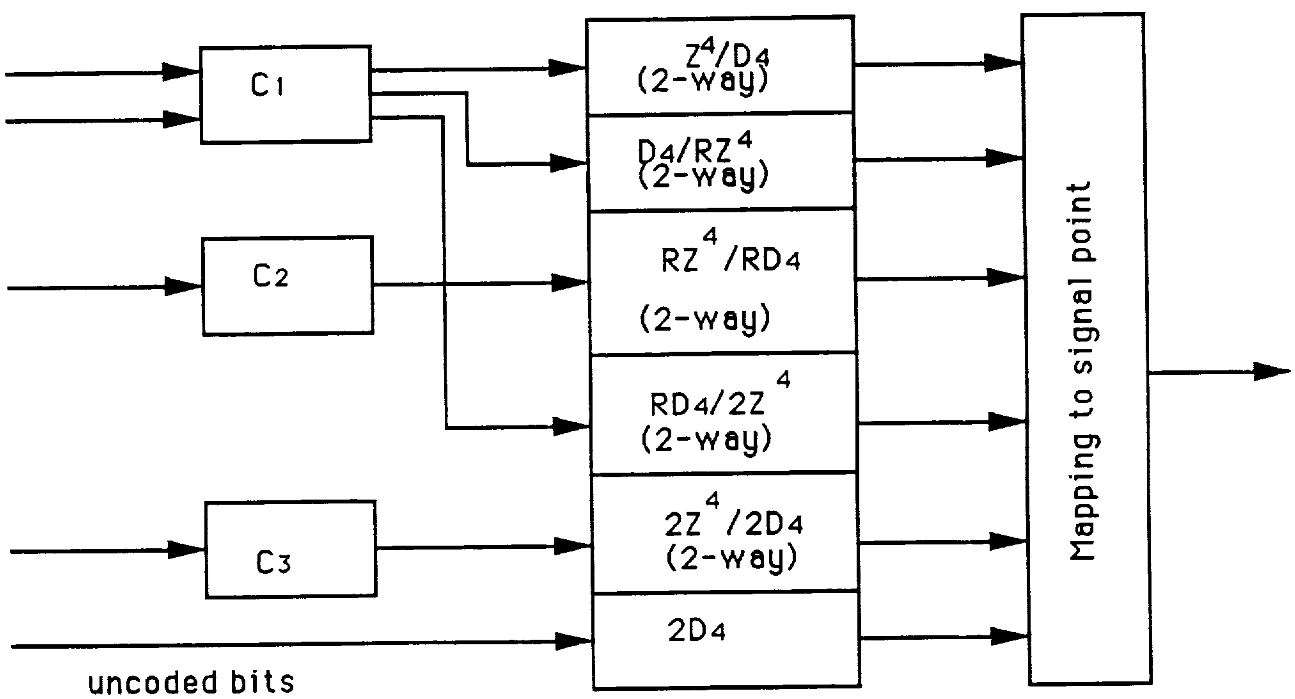


Fig. 13 Four dimensional multi-level trellis code with level spanning of Example 7

- C_1 is a 16-state rate-2/3 Ungerboeck code, which has MSED 6 when used with the partition $Z^2/RZ^2/2Z^2$.
- C_2 is an 8-state rate-7/8 binary convolutional code with free distance 3.
- C_3 is a 2-state (8, 7) block code with minimum distance 2.
- Encoding procedure (spectral efficiency $= m$ bits/symbol):
- The information sequence is divided into blocks of $16m$ bits each, with four subsequences of length 16, 7, 7, and $16m$ - 30 corresponding to C_1, C_2, C_3, and uncoded bits, respectively.
- Each state transition period of encoder C_1 outputs three bits which specify cosets of the partitions $Z^4/D_4, D_4/RZ^4$, and $RD_4/2Z^4$, respectively, i.e., 24 output bits specify 8 cosets of each partition.

- **Each output bit of encoder C_2 specifies a coset of RZ^4/RD_4, i.e., 8 output bits specify 8 cosets of RZ^4/RD_4.**

- **Each output bit of encoder C_3 specifies a coset of $2Z^4/2D_4$, i.e., 8 output bits specify 8 cosets of $2Z^4/2D_4$.**

- **Together with 16m - 30 uncoded bits, each coded block of 8 four dimensional signals, i.e., 16 two dimensional signals, contains 16m bits of information and the spectral efficiency is m bits/symbol.**

- **Since the MSED's of Z^4, D_4, and RD_4 are the same as Z^2, RZ^2, and $2Z^2$, respectively, the MSED of C_1 is the same as the corresponding Ungerboeck code, i.e., $D(C_1) = 6$. Therefore, assuming the lower bound on MSED holds in this case, $D(C) = \min\{6, 3 \times 2, 2 \times 4, 8\} = 6$.**

- **The normalized redundancy is $\rho(C) = 5/8$, the nominal coding gain is $\gamma(C) = 5.90$ dB, the decoding complexity of multi-stage decoding is $N_D = 108$, and the decoding delay is D = 56.**

- **However, due to the uncertainty regarding the bound, the actual values of $D(C)$ and $\gamma(C)$ may be less than stated above.**

- It can be shown that the two bits corresponding to the partition levels D_4/RZ^4 and $RD_4/2Z^4$ are the only ones affected by a 90° phase rotation. So this code can be combined with a differential encoder to achieve 90° rotational invariance as shown in Figure 14.

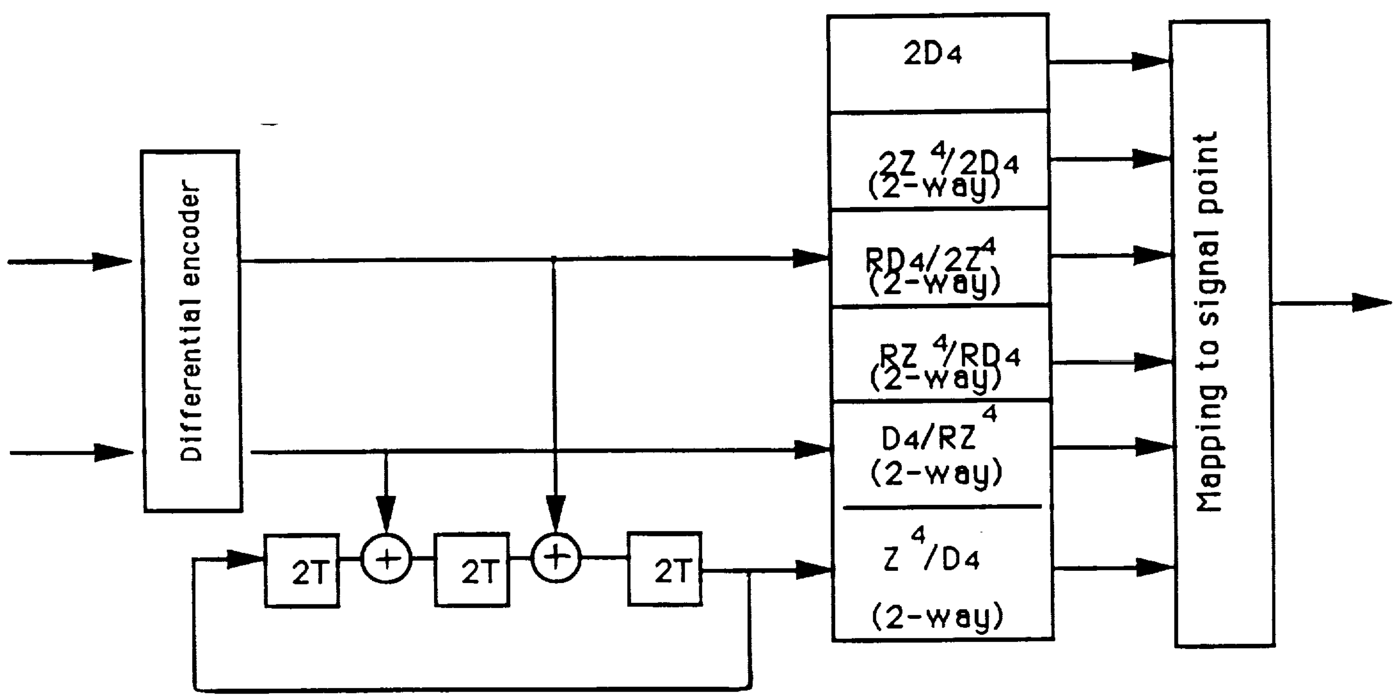


Fig. 14 Diagram of four dimensional 90 rotationally invariant encoder with a differential encoder

Conclusions

- Several constructions for multi-level trellis codes are presented and many codes with better performance than previously known codes are found. These codes provide a flexible trade-off between coding gain, decoding complexity, and decoding delay.

- New multi-level trellis coded modulation schemes using generalized set partitioning methods are developed for QAM and PSK signal sets.

- New rotationally invariant multi-level trellis codes which can be combined with differential encoding to resolve phase ambiguity are presented.

Appendix B

New Multi-Level Codes over $GF(q)$