Multi Objective Optimization

Handout November 4, 2011

(A good reference for this material is the book “multi-objective optimization by K. Deb)

Multiple Objective Optimization

- So far we have dealt with single objective optimization, e.g. Objective (S) is a scalar.
- For many problems there are competing objectives. For example,
  - A. expected investment return versus risk with decisions about stock mixtures in a portfolio or
  - B. expected speed of autonomous vehicle through a course versus risk of having an accident
- Competing objectives means that
  - the optimal solutions for each objective are different
  - changing the values of the decision vector to improve one objective might result in a decrease in the other objective.
- What are other examples of multi-objective optimization problems?

Learning Objectives

- Motivation for Multi Objective Optimization
- Understanding and Visualizing Trade-offs
- Domination, Non-Domination and Pareto Optimality
- Key Features of Good Multi Objective Optimization Algorithms
- Challenges in Developing Effective Multi Objective Optimization Algorithms
- Advantages of Using Evolutionary Heuristics
- Calculating Fitness for Multi-Objective Genetic Algorithms

Multi Objective Optimization: Problem Formulation

- Minimize (or Maximize) $F_1(x)$
- Minimize (or Maximize) $F_2(x)$

$\ddots$

- Minimize (or Maximize) $F_m(x)$

- Subject to $g_j(x) \geq 0$, $j=1,\ldots, J$
- $h_k(x) = 0$, $k=1,\ldots,K$
- $x \in D$ So the "decision space" is D
- ,If $A_i \leq x_i \leq B_{i,,,}$,$i=1,\ldots,n$, then D is a hypercube defined by the "box constraints" (A_i,B_i)

Dominated Solutions

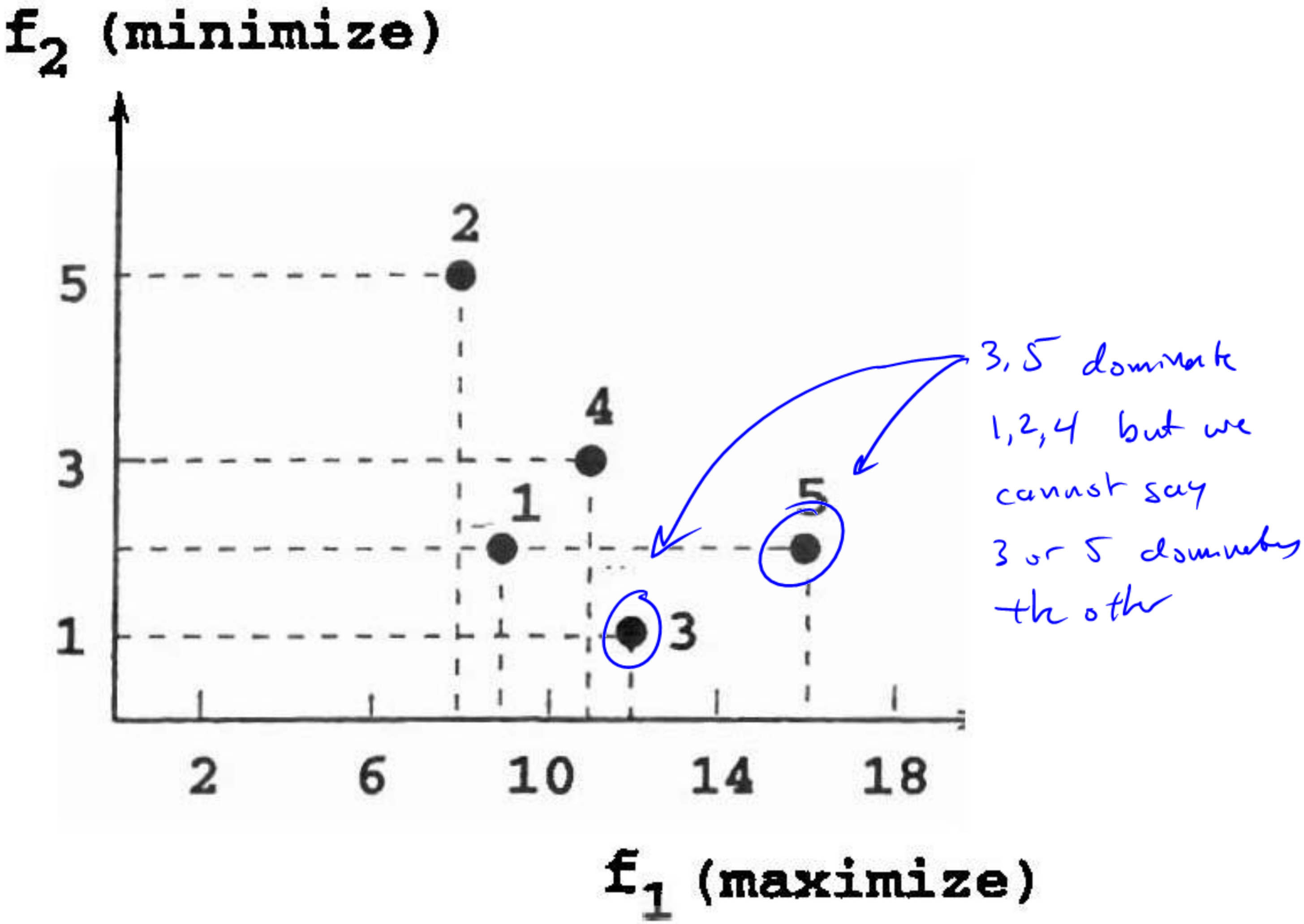
- Assume we want to minimize both $F_1(x)$ and $F_2(x)$

- A solution x_1 is said to **dominate** a solution x_2 if both of the following are true:
  - A. $F_1(x_1) \leq F_1(x_2)$ and $F_2(x_1) \leq F_2(x_2)$
  - B. $F_1(x_1) < F_1(x_2)$ or $F_2(x_1) < F_2(x_2)$
- In other words, x_1 dominates x_2 if x_1 is not worse for any of the functions (condition A) and is better in at least one of the functions (condition B)
- Note if you are maximizing one or both of the functions, the direction of the inequalities will change.

Pareto Optimality

- **Non-Domination:** A solution x* is **non-dominated** in set *S* if there does **not** exist a solution x^ ε *S* which dominates x*
- Let *D* be the feasible set of solutions for a Multi Objective Optmization Problem
- **Pareto-Optimality:** A solution x* is **pareto optimal** if there does **not** exist a solution x^ ε *D* which dominates x*.
- **Pareto Front:** The set of all possible pareto-optimal solutions is called the **pareto front**
- The aim of a multi objective optimization algorithm is to deduce the pareto front or a near optimal front

Figure 14



f2 (minimize)
2
5
4
3
1
5
1
3
2
6
10
14
18
f1 (maximize)
3, 5 dominate
1,2,4 but we
cannot say
3 or 5 dominates
the other

Figure 14: Identifying Non Dominating Solutions

- In Figure 14 we are trying to maximize F and minimize F_2 we see that solutions 1 and 5 have the same value of F_2, but 5 has a larger value of F_1.

- Which is the better solution 1 or 5? Why? Is one of them dominated?

- What about the comparison between solution 1 and 2? Is one of them dominated by the others.

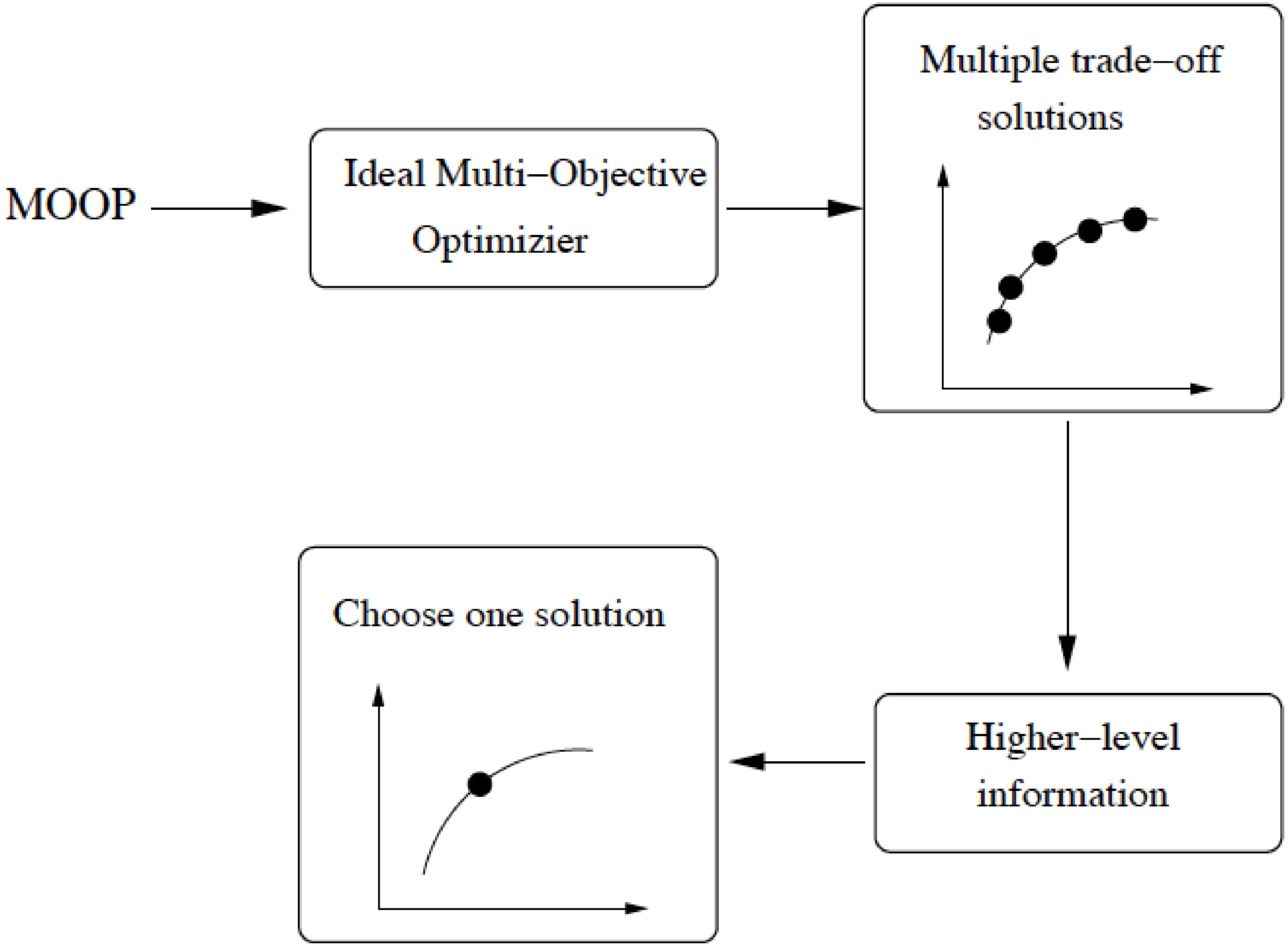
- Which are the points x* that are non dominated, i.e. there is no other solution that dominates x*/

Board comments on Figure 14

- Clearly 5 dominates 1 since they have the same F_2 value and 5's F_1 value is higher than 1's and goal for f1 is maximize.
- 1 clearly dominates 2 since f2 is lower and f1 is higher.

- The non dominated points are 3 and 5 since they have eq to or less values of f2 in comparison to the other points and eq to or greater values of f1. Between each other, neither dominates the other.

Multi Objective Approach



Equations for two objectives for Cantilever Problem

d = diameter   l = length

$$\text{Min } f_1(d, l) = \frac{\rho \pi}{4} d^2 l$$

weight

constant

$$\text{Min } f_2(d, l) = \delta = \frac{64 P}{3 E \pi} \frac{l^3}{d^4}$$

end deflection

we'd like to minimize both of these variables

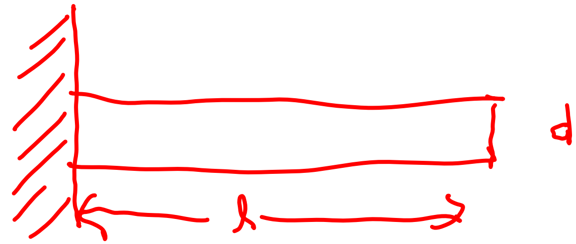
$$\ni \frac{32 P}{\pi} \frac{l}{d^3} = \sigma_{max} \leq S_y$$

constant

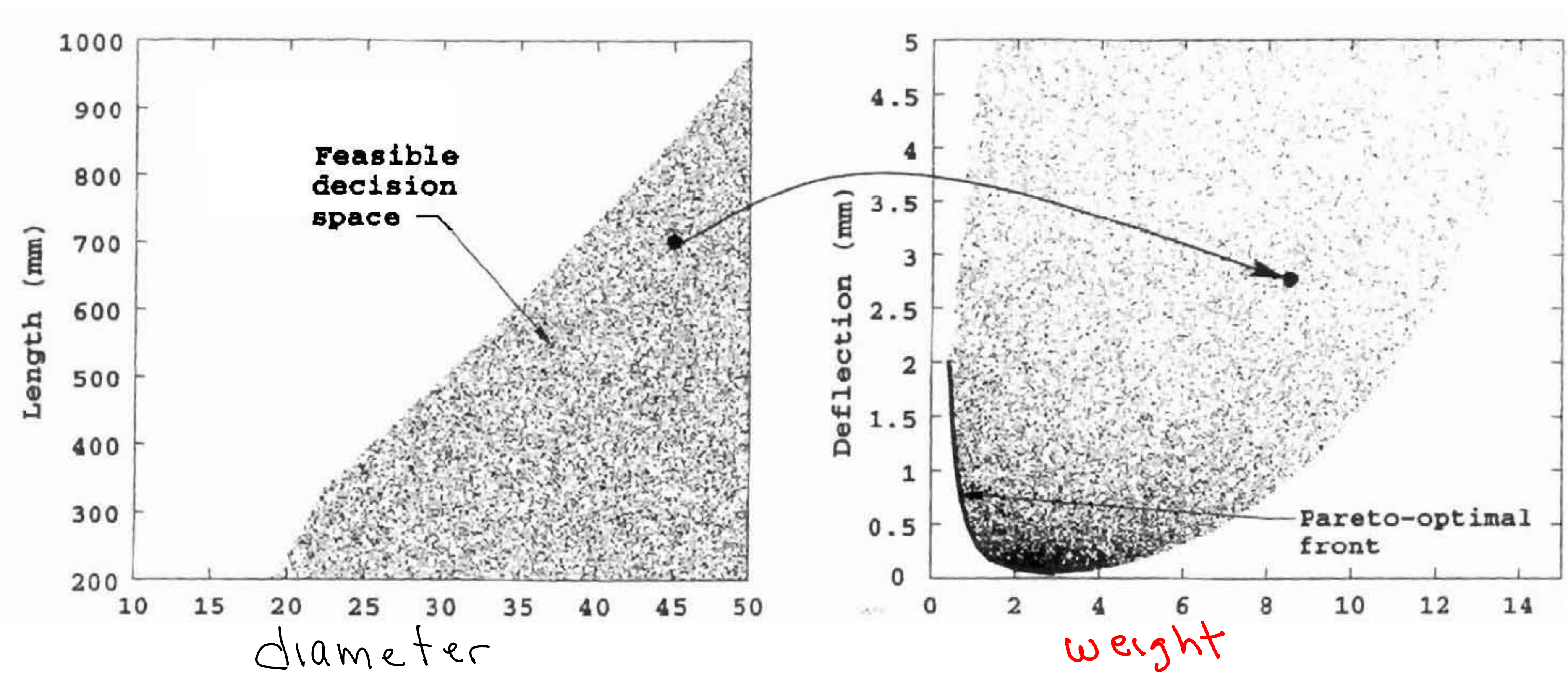
$$\delta \leq \delta_{max}$$

(stress less than max   & end deflection δ is smaller

Figure 10



d
l



Length (mm)
1000
900
800
700
600
500
400
300
200
10
15
20
25
30
35
40
45
50
Feasible decision space
diameter
Deflection (mm)
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
2
4
6
8
10
12
14
Pareto-optimal front
weight

Table 1

Which solution is best?

Table 1 Five solutions for the cantilever design problem.

Solution	d (mm)	l (mm)	Weight (kg) f_1	Deflection (mm) f_2
A	18.94	200.00	0.44	2.04
B	21.24	200.00	0.58	1.18
C	34.19	200.00	1.43	0.19
D	50.00	200.00	3.06	0.04
E	33.02	362.49	2.42	1.31

Figure 11

A-E are solutions from table 1

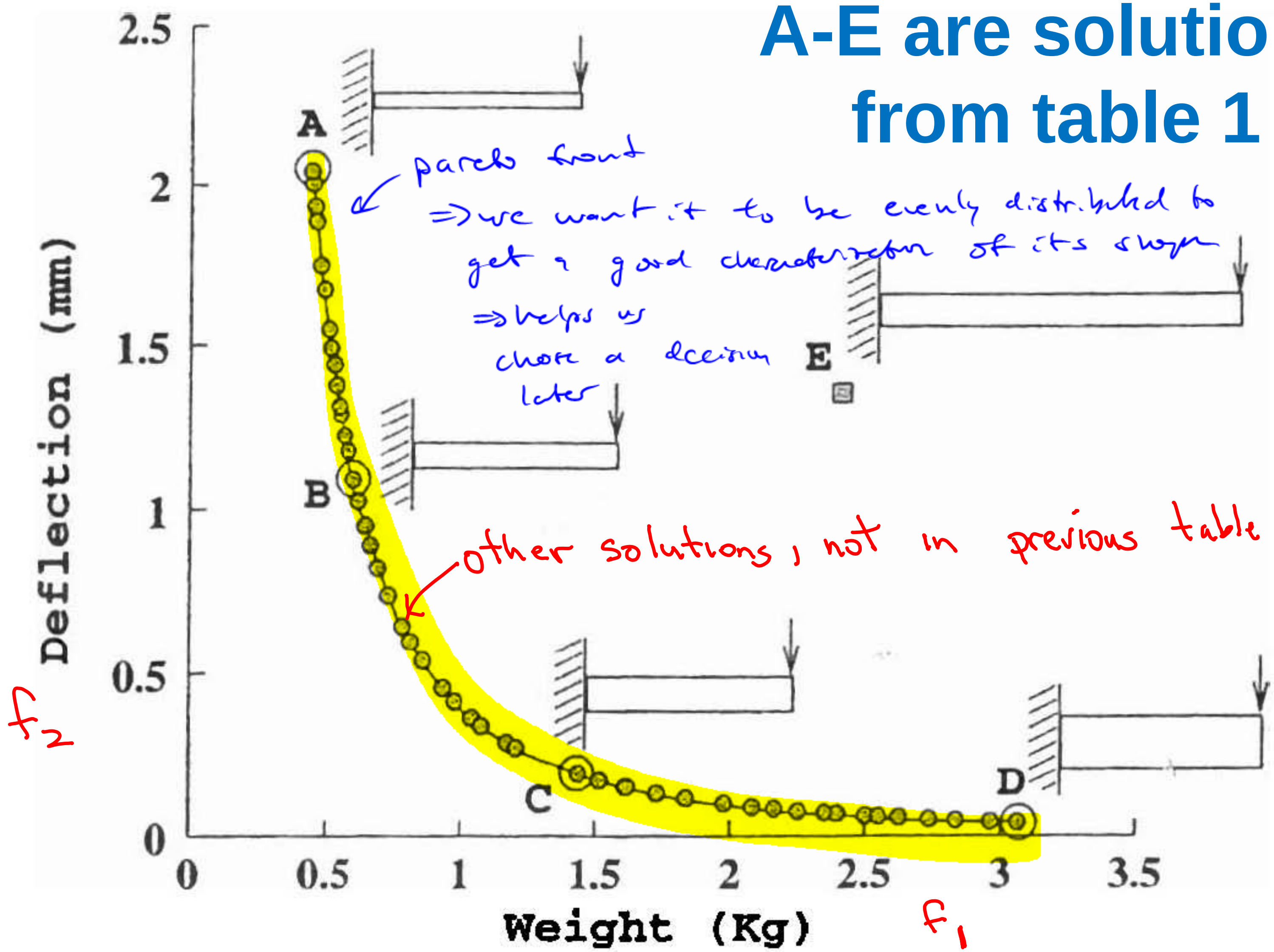
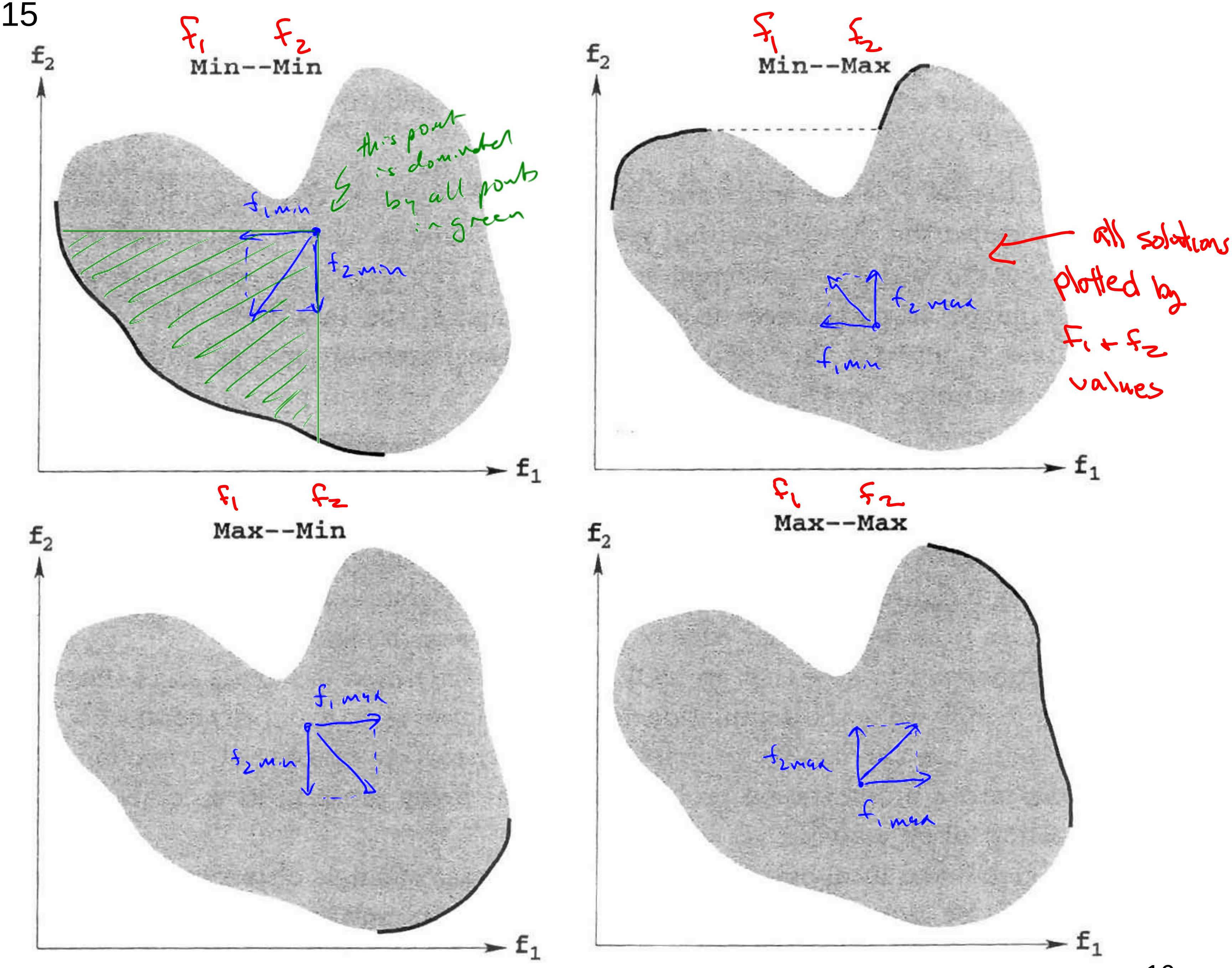


Figure 1 of Pareto fronts

- In the following slide we are assuming the two cost functions have been evaluated for thousands of points and the points are plotted in terms of their f_1 (horizotal) and f_2 (vertical) values.
- All these points are in a gray cloud in Figure 15.
- The difference between the 4 plots is related to whether you are trying to minimize or ~~maximzie~~ maximize f_1 and f_2

we can think of it like a gravity vector pushing us towards a pareto front. See next slide

Figure 15



f1 f2
Min--Min
this point is dominated by all points in green
f_{1min}
f_{2min}
f1 f2
Min--Max
all solutions plotted by f1 + f2 values
f_{2max}
f_{1min}
f1 f2
Max--Min
f_{1max}
f_{2min}
f1 f2
Max--Max
f_{2max}
f_{1max}
f_1
f_2

Two Goals in Ideal Multi-Objective Optimization

- Converge on the Pareto Optimal Front

  ⇒ these points dominate

- Maintain as diverse a distribution as possible.

  ⇒ want to have an evenly distributed Pareto Optimal front so as to allow us to characterize the shape and choose the best solution for our problem.

  i.e. the Pareto Optimal front is the best set of all the solutions across our multi-objective vector. Now we have to make an engineering decision, and we'd like a set of diverse choices.

Multi Objective Problems: Optimization Methods

- Classical Methods
  - Convert Multi Objective Problem into multiple Single Objective Problems
  - Each Single Objective Problem can be solved via conventional or heuristic methods
- Evolutionary Methods
  - Population based approach with retention of good trade-off solutions is employed
  - No need to solve multiple Single Objective Problems

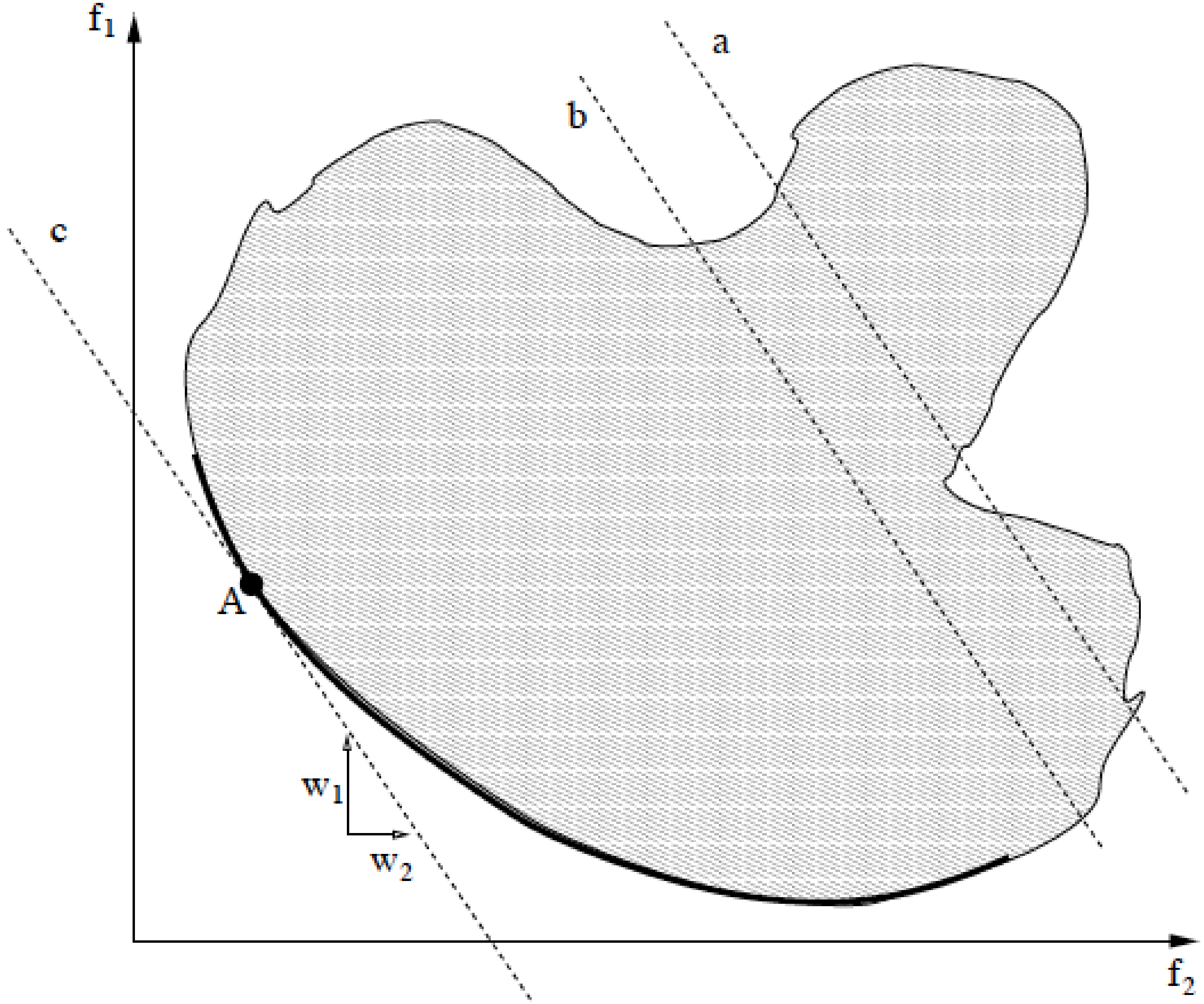
COMMON APPROACHES FOR APPROXIMATING THE TRADEOFF CURVE

- Weighting Method ⇒ converts multi-objective function into a single-objective problem
  - assign weights to each objective and then optimize the weighted sum of the objectives
- Constraint Method
  - optimize one objective, convert other objectives into constraints

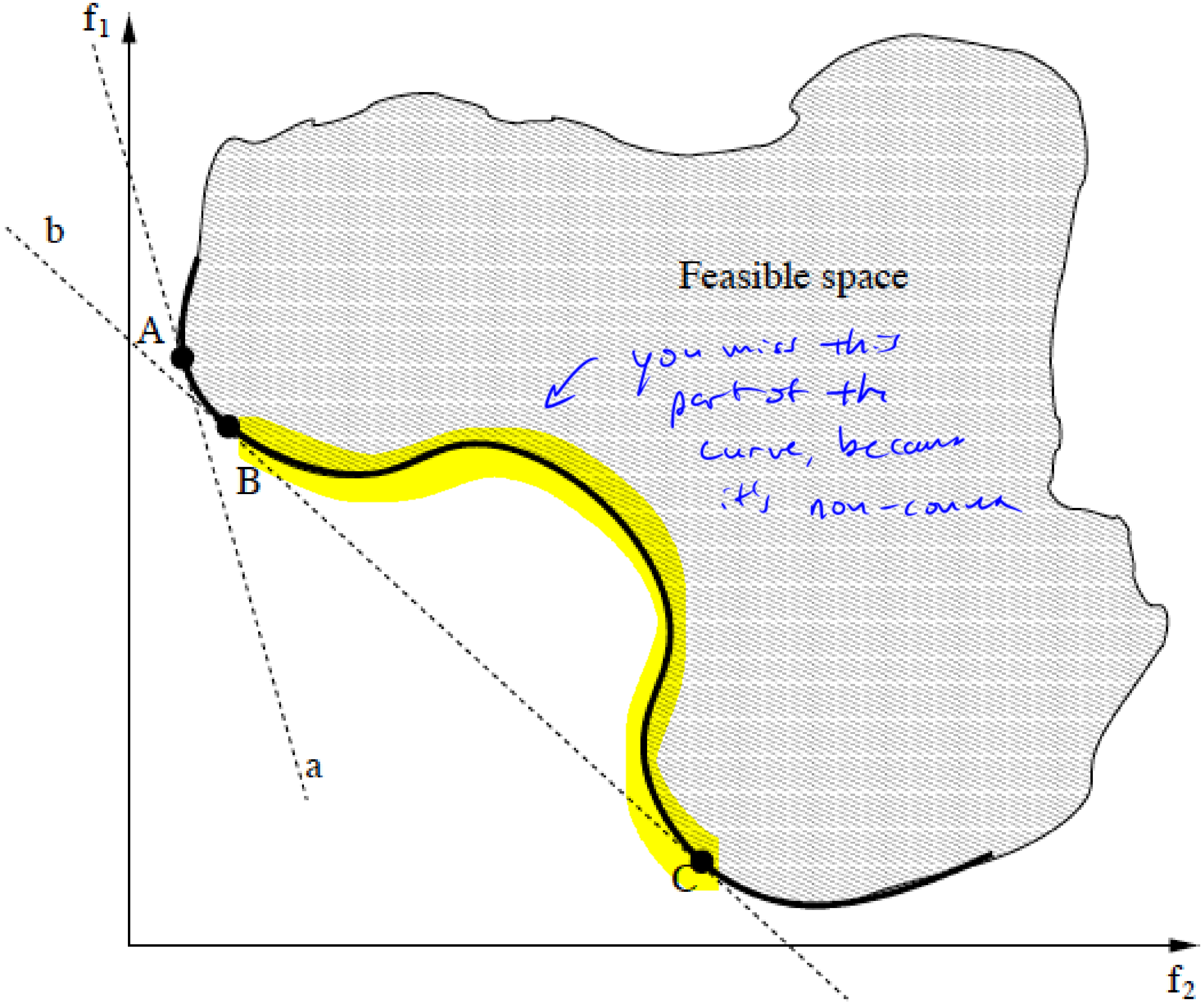
Weighted Method to Solve Multi Objective Problems with Single Objective Optimization

- **Replace**
- Minimize ($F_1(x)$
- Minimize $F_2(x)$
- subject to $g_j(x) \geq 0$, j=1,…, J
- x ε D
- **With**
- Minimize $r_k \; F_1(x) + F_2(x)$
- subject to $g_j(x) \geq 0$, j=1,…, J
- x ε D
- So r_k is a ratio of weights on $F_1(x)$ and $F_2(x)$
- Solve this for many values of r_k, k=1,…M_k to attempt to get different points on the Pareto Front

Weighted Sum Method



Weighted Sum Method: Non-Convexity



Problems with Weighted Sum Method for Multiple Objectives

- You must solve a Single Objective many times for each ratio of $w_1/w_{2.}$
- No control over area of objective space searched.
- Approach will not work on non-convex parts of tradeoff curve. (The solution is always where the tangent is w1 /w2. If there are two such points, then the optimal is the lower of those two points.)

Figure Another Classical Method: Constraint Method for Multi Objective Optimization

- Optimize one objective and constrain all the others
- **Constraint Method:**
- **Minimize F_2**
- **Subject to $F_1 \delta R_i$** constraint on one objective

- **Solve the problem for many values of R_i.**
- **Again has the problem that must solve a single optimization problem many times**
- **Solution highly depends on the values of R_i chosen**