# Multi-order asymptotic expansion of blow-up solutions for autonomous ODEs. I - Method and Justification 

Taisei Asai ${ }^{1}$, Hisatoshi Kodani ${ }^{2}$, Kaname Matsue*2,3, Hiroyuki Ochiai ${ }^{2}$, and Takiko Sasaki ${ }^{4,5}$<br>1 Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan<br>${ }^{2}$ Institute of Mathematics for Industry, Kyushu University, Fukuoka 819-0395, Japan<br>${ }^{3}$ International Institute for Carbon-Neutral Energy Research (WPI-I ${ }^{2}$ CNER), Kyushu University, Fukuoka 819-0395, Japan<br>${ }^{4}$ Department of Mathematical Engineering, Faculty of Engineering, Musashino University, 3-3-3<br>Ariake, Koto-ku, Tokyo 135-8181, Japan<br>${ }^{5}$ Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

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#### Abstract

In this paper, we provide a systematic methodology for calculating multi-order asymptotic expansion of blow-up solutions near blow-up for autonomous ordinary differential equations (ODEs). Under the specific form of the principal term of blow-up solutions for a class of vector fields, we extract algebraic objects determining all possible orders in the asymptotic expansions. Examples for calculating concrete multi-order asymptotic expansions of blow-up solutions are finally collected.


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## 1 Introduction

Asymptotic expansion is an important and useful tool for analyzing concrete behavior of functions or solutions of differential equations at infinity and/or near given points (e.g. [4). Its importance is widely understood in physical and mathematical problems in analyzing functions. A famous investigation involving calculations of asymptotic expansion around given points, including singularities, in the field of differential equations is the Painlevé test (e.g. [14, 25, 26]), which roughly aims at verifying whether all solutions of differential equations of the form

$$
\begin{equation*}
\frac{d u}{d z}=f(z, u), \quad z \in \mathbb{C} \tag{1.1}
\end{equation*}
$$

are single valued and all possible (movable) singularities $z$ are poles. Note that the differential equation whose solutions satisfy the above properties is often said to possess the Painlevé property. In wide investigations for the criterion of the Painlevé property, a concept of resonance (e.g. [26]) has successfully played a key role in determining asymptotic expansion of solutions with their dependence on initial points. The presence of resonance is algebraically verified and is applicable even when solutions possess movable singularities. Conceptually, multi-order asymptotic expansion of solutions is systematically determined once all resonances (around a given singularity $z_{0}$, possibly at infinity $z_{0}=\infty$ ) are determined.

Our main interest in the present paper is blow-up solutions of (autonomous) ordinary differential equations (ODEs for short) depending on the real-time variable $t$ :

$$
\begin{equation*}
\frac{d \mathbf{y}}{d t}=f(\mathbf{y}), \quad \mathbf{y}\left(t_{0}\right)=\mathbf{y}_{0} \in \mathbb{R}^{n} \tag{1.2}
\end{equation*}
$$

where $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a smooth function. A solution $\mathbf{y}(t)$ is said to blow up at $t_{\max }<\infty$ if its modulus diverges as $t \rightarrow t_{\max }-0$. The finite value $t_{\max }$ is known as the blow-up time. Properties of blow-up solutions such as the existence and asymptotic behavior are of great importance in the field of (ordinary, partial, delay, etc.) differential equations, and are widely investigated in decades from many aspects (e.g. [5, 6, 8, 13, 15]). Asymptotic behavior of blow-up solutions is often referred to as determination of blow-up rates, which is characterized as the following form:

$$
\mathbf{y}(t) \sim \mathbf{u}(\theta(t)) \quad \text { as } \quad t \rightarrow t_{\max }
$$

for a function $\mathbf{u}$, where

$$
\theta(t)=t_{\max }-t
$$

and $t_{\max }$ is assumed to be a finite value and to be known a priori. In many studies, however, the function $\mathbf{u}$ is assumed to consist of a single term, and investigations of the concrete form of $\mathbf{u}$ as a function with multiple terms are limited. One of the reasons could be the difficulty even for deriving the principal term, namely the first term, of $\mathbf{u}$ except special cases (e.g. [9]). One would expect that the similar approach to the Painlevé test can be applied to determining multi-order asymptotic expansion of blow-ups. It should be mentioned here that the Painlevé test typically begins with the anzatz of solutions being the Frobenius serie 11 , namely

$$
\begin{equation*}
u(z)=\left(z-z_{0}\right)^{-\rho} \sum_{m=0}^{\infty} a_{m}\left(z-z_{0}\right)^{m}, \quad a_{m} \in \mathbb{C} \tag{1.3}
\end{equation*}
$$

when $z_{0}$ is a regular singular point (cf. [4]). On the other hand, there is no guarantee that blow-up solutions always admit the similar form to (1.3), namely

$$
\begin{equation*}
y(t)=\theta(t)^{-\rho} \sum_{m=0}^{\infty} a_{m} \theta(t)^{m}, \quad a_{m} \in \mathbb{R} \tag{1.4}
\end{equation*}
$$

In particular, no information about the successive terms of blow-up solutions is provided a priori in general. Nevertheless, the blow-up time $t_{\text {max }}$ is interpreted as a movable singularity in $\mathbb{C}$ because,

[^1]as far as the (complex) analytic continuation is admitted, the system (1.2) is also considered as the system depending on complex time variable $z$ restricted to $\{t \equiv \operatorname{Re} z>0\}$, and $t_{\text {max }}$ depends on initial points $\mathbf{y}_{0}$, and the multi-order asymptotic expansion of blow-up solutions contributes to understanding the blow-up behavior itself in detail, and the classification of $t_{\max }$ as singularities, as investigated in the Painlevé test.

In recent years, the third author and his collaborators have developed a framework to characterize blow-up solutions from the viewpoint of dynamics at infinity (e.g. 28, 29]) based on preceding works (e.g. 12) and have derived machineries of computer-assisted proofs for the existence of blow-up solutions as well as their qualitative and quantitative features (e.g. [27, 30, 31, 33]). As in the present paper, finite-dimensional vector fields with scale invariance in an asymptotic sense, asymptotic quasi-homogeneity defined precisely in Definition 2.2, are mainly concerned. In this framework, dynamics at infinity is appropriately characterized, and blow-up solutions are shown to be characterized by dynamical properties of invariant sets, such as equilibria, "at infinity". In particular, hyperbolicity of such invariant sets induces the leading asymptotic approximation of blow-up solutions of the form $a_{0}\left(t_{\max }-t\right)^{-\rho}$ as in (1.4) uniquely determined by the original vector field. By means of the terminology in the field of (partial) differential equations, such blow-ups are said to be type-I.

One would then expect that multi-order asymptotics of blow-up solutions can be clarified by the dynamical property of invariant sets "at infinity". In particular, some characteristics determining multi-order asymptotics, like the resonance in the Painlevé test, can be extracted from the dynamical property of blow-up solutions.

The present subject, consisting of two papers, aims at providing a systematic methodology to calculate multi-order asymptotic expansions of blow-up solutions for autonomous ODEs with their justifications (Part I), and investigating the correspondence of the characterization of these asymptotic expansions to dynamical properties of blow-up solutions (Part II). We pay attention to multi-order asymptotics of type-I blow-ups and, with general asymptotic series of blow-up solutions under an appropriate assumption, derive the associated system of differential equations which all asymptotic terms are inductively solved.

The main result in the present paper is, under the above setting, to observe that the following algebraic objects essentially determine all terms appeared in asymptotic expansions of blow-up solutions:

- The balance law determining coefficients of the leading terms of blow-up solutions (Definition 3.2).
- The blow-up power-determining matrix determined by the quasi-homogeneous part of vector fields and blow-up solutions (Definition 3.3).
- The blow-up power eigenvalues defined by eigenvalues of the blow-up power-determining matrix (Definition 3.3).

The combination of standard integration of ODEs with the above algebraic objects, lower order terms if exist, and typical assumptions in asymptotic expansions provide a systematic algorithm to calculate asymptotic expansions of blow-up solutions of an arbitrary order. In particular, all possible terms and parameter dependence of blow-up solutions are essentially determined by blowup power eigenvalues, and hence these eigenvalues play a role like resonance in the Painlevé test.

The rest of this paper is organized as follows. In Section 2 the precise definition of the class of vector fields we mainly treat is presented. We then introduce a function measuring the asymptotic degree of functions with respect to $\theta(t)$, which plays a role in estimating asymptotic relations among terms appeared in our asymptotic expansions of blow-up solutions.

In Section 3 we derive a systematic methodology to calculate multi-order asymptotic expansion of (type-I) blow-up solutions under a mild assumption. Quasi-homogeneity of vector fields in an asymptotic sense extracts algebraic objects essential to determine asymptotic expansions, which shall be called blow-up power eigenvalues associated with blow-up solutions. More precisely, we show that the vector fields along type-I blow-up solutions induce the matrices determined by the quasi-homogeneous part of vector fields and coefficients of principal terms of blow-up solutions, which are called blow-up power-determining matrices. Eigenvalues of such matrices, which are exactly blow-up power eigenvalues in our terminology, and lower order terms of vector fields are then shown to determine all possible terms in asymptotic expansion of blow-up solutions. Once these objects are determined, standard integrator of linear ODEs yields asymptotic expansions of blow-up solutions of an arbitrary order with the extraction of their parameter dependence in a formal sense. Justification of such formal expansions as the asymptotic expansion of the original blow-up solutions is also discussed under mild assumptions.

In Section 4 examples of multi-order asymptotic expansions of blow-up solutions are presented for showing the applicability of our present methodology. Examples below aim at demonstrating the following significance, respectively:

- Section 4.1 collects the simplest cases: the one-dimensional ODEs. We shall provide the basic strategy based on the presented approach to determine multi-order asymptotic expansions of blow-up solutions.
- Section 4.2 demonstrates the asymptotic expansion of blow-up solutions for a rational vector field containing a component independent of scaling. In particular, an asymptotically quasihomogeneous vector field is treated.
- Sections 4.3 and 4.4 demonstrate blow-up solutions such that parameters in coefficients of vector fields can affect the order of $\theta(t)$ in asymptotic expansions. In Section 4.3 a rational vector field is treated. In Section 4.4, the third order asymptotic expansion is also calculated.
- Section 4.5 treats asymptotically quasi-homogeneous vector fields possessing two different blow-up solutions whose orders of $\theta(t)$ in the first terms are identical.
- Section 4.6 treats asymptotically quasi-homogeneous vector fields possessing the similar type of blow-ups to Section 4.5 while blow-up power-determining matrix admits a nontrivial Jordan normal form. This section also exhibits an example of the presence of qualitatively different asymptotic behavior of solutions among components. In the field of partial differential equations, this phenomenon is recognized as nonsimultaneous blow-up (e.g. [17, 18]).

Note that the first terms of all blow-up solutions listed here are already calculated in [28, 29] or their references, while a criterion for determining the first terms of blow-ups is briefly reviewed in Part II 23].

In Appendix, proofs of several statements in Section 3 are provided, where advanced machineries in the theory of dynamical systems, such as time-variable compactifications for asymptotically autonomous systems (e.g. [34), are applied.

Finally, several examples showing multi-order asymptotic expansions of blow-up solutions are revisited in Part II [23], where the correspondence to dynamical properties obtained through desingularized vector fields are also described.

## 2 Setting and the asymptotic degree of functions

Our main concern is the Cauchy problem of an autonomous system of ODEs

$$
\begin{equation*}
\mathbf{y}^{\prime}=\frac{d \mathbf{y}(t)}{d t}=f(\mathbf{y}(t)), \quad \mathbf{y}\left(t_{0}\right)=\mathbf{y}_{0} \tag{2.1}
\end{equation*}
$$

where $t \in\left[t_{0}, T\right)$ with $t_{0}<T \leq \infty, f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a $C^{r}$ function with $r \geq 2$ and $\mathbf{y}_{0} \in \mathbb{R}^{n}$. In this section, we introduce a class of ODEs which we mainly treat, and a function measuring the asymptotic degree of functions as $t \rightarrow t_{\max }$ with respect to $\theta(t)$. We also provide several estimates of such degrees we mainly use later to obtain multi-order asymptotic expansions of blow-up solutions.

Remark 2.1. In what follows, we shall use the following notations for describing asymptotic relations based on [11]. Consider two real continuous functions $f(\epsilon), g(\epsilon)$ defined on a small interval $\left(0, \epsilon_{0}\right]$ and asymptotic behavior as $\epsilon \rightarrow+0$.

- $f=O(g)$ iff there are constant $\bar{\epsilon}_{0} \in\left(0, \epsilon_{0}\right]$ and $C>0$ such that $|f(\epsilon)| \leq C|g(\epsilon)|$ for all $\epsilon \in\left(0, \bar{\epsilon}_{0}\right]$.
- $f=o(g)$, or $f \ll g$ iff $\lim _{\epsilon \rightarrow+0}(f(\epsilon) / g(\epsilon))$ exists and is 0 .
- $f \sim g$ iff $\lim _{\epsilon \rightarrow+0}(f(\epsilon) / g(\epsilon))$ exists and is 1 .
- $f=O_{s}(g)$ iff $f=O(g)$ and $f \neq o(g)$.


### 2.1 Asymptotically quasi-homogeneous vector fields

First of all, we review a class of vector fields in our present discussions.
Definition 2.2 (Asymptotically quasi-homogeneous vector fields, cf. [10, 28). Let $f_{0}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a function. Let $\alpha_{1}, \ldots, \alpha_{n}$ be nonnegative integers with $\left(\alpha_{1}, \ldots, \alpha_{n}\right) \neq(0, \ldots, 0)$ and $k>0$. We say that $f_{0}$ is a quasi-homogeneous function of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k$ if

$$
f_{0}\left(s^{\Lambda_{\alpha}} \mathbf{x}\right)=s^{k} f_{0}(\mathbf{x}) \quad \text { for all } \mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)^{T} \in \mathbb{R}^{n} \text { and } s>0
$$

whert ${ }^{3}$

$$
\Lambda_{\alpha}=\operatorname{diag}\left(\alpha_{1}, \ldots, \alpha_{n}\right), \quad s^{\Lambda_{\alpha}} \mathbf{x}=\left(s^{\alpha_{1}} x_{1}, \ldots, s^{\alpha_{n}} x_{n}\right)^{T}
$$

Next, let $X=\sum_{i=1}^{n} f_{i}(\mathbf{x}) \frac{\partial}{\partial x_{i}}$ be a continuous vector field on $\mathbb{R}^{n}$. We say that $X$, or simply $f=\left(f_{1}, \ldots, f_{n}\right)^{T}$ is a quasi-homogeneous vector field of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k+1$ if each component $f_{i}$ is a quasi-homogeneous function of type $\alpha$ and order $k+\alpha_{i}$.

[^2]Finally, we say that $X=\sum_{i=1}^{n} f_{i}(\mathbf{x}) \frac{\partial}{\partial x_{i}}$, or simply $f$ is an asymptotically quasi-homogeneous vector field of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k+1$ at infinity if there is a quasi-homogeneous vector field $f_{\alpha, k}=\left(f_{i ; \alpha, k}\right)_{i=1}^{n}$ of type $\alpha$ and order $k+1$ such that

$$
f_{i}\left(s^{\Lambda_{\alpha}} \mathbf{x}\right)-s^{k+\alpha_{i}} f_{i ; \alpha, k}(\mathbf{x})=o\left(s^{k+\alpha_{i}}\right)
$$

as $s \rightarrow+\infty$ uniformly for $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in S^{n-1} \equiv\left\{\mathbf{x} \in \mathbb{R}^{n} \mid \sum_{i=1}^{n} x_{i}^{2}=1\right\}$.
A fundamental property of quasi-homogeneous functions and vector fields is reviewed here.
Lemma 2.3. A $C^{1}$ quasi-homogenous function $f_{0}$ of type $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k$ satisfies the following differential equation:

$$
\begin{equation*}
\sum_{l=1}^{n} \alpha_{l} y_{l} \frac{\partial f_{0}}{\partial y_{l}}(\mathbf{y})=k f_{0}(\mathbf{y}) \tag{2.2}
\end{equation*}
$$

This equation is rephrased as

$$
\left(\nabla_{\mathbf{y}} f_{0}(\mathbf{y})\right)^{T} \Lambda_{\alpha} \mathbf{y}=k f_{0}(\mathbf{y})
$$

Proof. Differentiating the identity

$$
f_{0}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)=s^{k} f_{0}(\mathbf{y})
$$

in $s$ and put $s=1$, we obtain the desired equation (2.2).
The same argument yields that, for any quasi-homogenenous function $f_{0}$ of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k$,

$$
\sum_{l=1}^{n} \alpha_{l} s^{\alpha_{l}} y_{l} \frac{\partial f_{0}}{\partial y_{l}}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)=k s^{k} f_{0}(\mathbf{y})
$$

and

$$
\sum_{l=1}^{n} \alpha_{l}\left(\alpha_{l}-1\right) s^{\alpha_{l}} y_{l} \frac{\partial f_{0}}{\partial y_{l}}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)+\sum_{j, l=1}^{n} \alpha_{j} \alpha_{l}\left(s^{\alpha_{j}} y_{j}\right)\left(s^{\alpha_{l}} y_{l}\right) \frac{\partial^{2} f_{0}}{\partial y_{j} \partial y_{l}}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)=k(k-1) s^{k} f_{0}(\mathbf{y})
$$

for any $\mathbf{y} \in \mathbb{R}^{n}$. In particular, each partial derivative satisfies

$$
\begin{equation*}
\frac{\partial f_{0}}{\partial y_{l}}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)=O\left(s^{k-\alpha_{l}}\right), \quad \frac{\partial^{2} f_{0}}{\partial y_{j} \partial y_{l}}\left(s^{\Lambda_{\alpha}} \mathbf{y}\right)=O\left(s^{k-\alpha_{j}-\alpha_{l}}\right) \tag{2.3}
\end{equation*}
$$

as $s \rightarrow 0, \infty$ for any $\mathbf{y} \in \mathbb{R}^{n}$, as long as $f_{0}$ is $C^{2}$ in the latter case. In particular, for any fixed $\mathbf{y} \in \mathbb{R}^{n}$, both derivatives are $O(1)$ as $s \rightarrow 1$. In other words, we have quasi-homogeneous relations for partial derivatives in the above sense.
Lemma 2.4. A quasi-homogeneous vector field $f=\left(f_{1}, \ldots, f_{n}\right)$ of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k+1$ satisfies the following differential equation:

$$
\begin{equation*}
\sum_{l=1}^{n} \alpha_{l} y_{l} \frac{\partial f_{i}}{\partial y_{l}}(\mathbf{y})=\left(k+\alpha_{i}\right) f_{i}(\mathbf{y}) \quad(i=1, \ldots, n) \tag{2.4}
\end{equation*}
$$

This equation can be rephrased as

$$
\begin{equation*}
(D f)(\mathbf{y}) \Lambda_{\alpha} \mathbf{y}=\left(k I+\Lambda_{\alpha}\right) f(\mathbf{y}) \tag{2.5}
\end{equation*}
$$

Proof. By Lemma 2.3, we obtain (2.4). For (2.5), we recall that $D f=\left(\frac{\partial f_{i}}{\partial y_{l}}\right)$ is Jacobian matrix, while $k I+\Lambda_{\alpha}$ is the diagonal matrix with diagonal entries $k+\alpha_{i}$. Finally $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right)^{T}$ is the column vector, so that the left-hand side of (2.5) is the product of two matrices and one column vector.

Throughout successive sections, consider an (autonomous) $C^{r}$ vector field (2.1) with $r \geq 2$, where $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is asymptotically quasi-homogeneous of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k+1$ at infinity.

### 2.2 The asymptotic degree of functions as $t \rightarrow t_{\max }$

Our concern here is to introduce a quantity to measure the asymptotic behavior of functions as $t \rightarrow t_{\max }-0$ so that a given series of functions, in particular describing blow-up solutions, is verified to be an asymptotic series as $t \rightarrow t_{\max }-0$. To this end, we shall introduce the asymptotic degree of continuous scalar- and vector-valued functions as $t \rightarrow t_{\max }-0$. It should be a generalization of degree of polynomials and rational functions. The following notion indeed generalizes such well-known degree only in a neighborhood of $t_{\max }$ with $t<t_{\text {max }}$.

Definition 2.5 (Asymptotic degree $\operatorname{deg}_{\theta}$ ). For a given scalar-valued function $h=h(t)$, let

$$
\begin{equation*}
I_{h}:=\left\{\gamma \in \mathbb{R} \mid h(t)=o\left(\theta(t)^{\gamma}\right) \text { as } t \rightarrow t_{\max }-0\right\} \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{deg}_{\theta}(h):=\sup I_{h} \tag{2.7}
\end{equation*}
$$

For a given $n$-vector-valued function $\mathbf{h}=\mathbf{h}(t) \equiv\left(h_{1}(t), \ldots, h_{n}(t)\right)^{T}$, define

$$
\operatorname{deg}_{\theta}(\mathbf{h}):=\min _{i=1, \ldots, n} \operatorname{deg}_{\theta}\left(h_{i}\right)
$$

The value $\operatorname{deg}_{\theta}$ measures the behavior of functions with respect to the power function $\theta(t)^{\gamma}$ as $t \rightarrow t_{\max }-0$, and generalizes the degree of polynomial functions of $\theta(t)$. Indeed, $\operatorname{deg}_{\theta}$ is defined as suprema of $\gamma$ satisfying the corresponding asymptotic estimates because various elementary functions such as logarithmic ones are also taken into account. For example, as $s \rightarrow+0, f_{0}(s)=\ln s$ is $o\left(s^{a}\right)$ for any $a<0$, whereas $f_{0}(s) \neq O(1)$. On the other hand, the above definition is consistent with the asymptotic relation of the identically zero function ${ }^{4}$. Indeed, if $h(t) \equiv 0$, the definition implies that $\operatorname{deg}_{\theta}(h)=\infty$ and is consistent with the asymptotic relation $\lim _{t \rightarrow t_{\max }-0} h(t) \theta(t)^{-\gamma}=0$ for any $\gamma \in \mathbb{R}$.

### 2.2.1 Fundamental properties

Fundamental properties of $\operatorname{deg}_{\theta}$ used in the successive arguments are summarized here. First, we have the following properties by definition.

Lemma 2.6. Let $I_{h}$ be the set given in (2.6) for a scalar function $h(t)$.

1. If $I_{h} \neq \mathbb{R}$, it is the interval of the form $\left(-\infty, \gamma_{h}\right)$ or $\left(-\infty, \gamma_{h}\right)$ for some $\gamma_{h} \in \mathbb{R}$.

[^3]2. If $f(t) \leq g(t)$ holds for all $t$ near and less than $t_{\max }$, then $\operatorname{deg}_{\theta}(g) \leq \operatorname{deg}_{\theta}(f)$.

Proof. 1. $\gamma \in I_{h}$ and $\gamma^{\prime}<\gamma$ imply $\gamma^{\prime} \in I_{h}$.
2. The assumption implies $I_{g} \subset I_{f}$, and hence the statement holds.

The degree of several elementary functions are collected below, which are fully applied to estimating the orders of functions in asymptotic expansions of blow-up solutions.

Example 2.7. The results below follow from the definition.

- $\operatorname{deg}_{\theta}(c)=0$ for any constant $c \neq 0$.
- $\operatorname{deg}_{\theta}(0)=+\infty$.
- $\operatorname{deg}_{\theta}\left(\theta(t)^{\gamma}\right)=\gamma$.
- $\ln \theta(t)=o\left(\theta(t)^{\gamma}\right)$ for any $\gamma<0$, while $\lim _{t \rightarrow t_{\max }-0} \ln \theta(t)=-\infty$. Therefore $\operatorname{deg}_{\theta}(\ln \theta(t))=0$.
- $\cos (\lambda \ln \theta(t))=O_{s}(1)$ and hence $\operatorname{deg}_{\theta}(\cos (\lambda \ln \theta(t)))=0$.
- $\sin (\lambda \ln \theta(t))=O_{s}(1)$ and hence $\operatorname{deg}_{\theta}(\sin (\lambda \ln \theta(t)))=0$.

Next, we summarize basic several asymptotic relations based on the l'Hôpital's theorem so that readers who are not familiar with asymptotic relations can easily understand the subsequent arguments.

Lemma 2.8 (cf. [4]). For a given $x_{0} \in \mathbb{R}$ and a real-valued function $f=f(x)$ continuous in $\left(x_{0}-\delta, x_{0}\right)$ for some $\delta>0$, let $\mathbf{F}_{f}(x)$ be a primitive function of $f(x)$. Then the following asymptotic relations hold as $x \rightarrow x_{0}-0$; if $f(x) \sim a\left(x_{0}-x\right)^{-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ with $M \in \mathbb{Z}_{\geq 0}$,

1. $\mathbf{F}_{f}(x) \sim[-a /(1-b)]\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ if $b>1$.
2. assuming $b<1$ and if $\mathbf{F}_{f}(x)$ of $f(x)$ is chosen to vanish as $x \rightarrow x_{0}-0$, then $\mathbf{F}_{f}(x) \sim$ $[-a /(1-b)]\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$, otherwise $\mathbf{F}_{f}(x) \sim C$, where $C$ is a non-zero constant.
In the former case, we further obtain $\mathbf{F}_{f}(x)=o\left(\left(x_{0}-x\right)^{1-b-\epsilon}\right)$ for any $\epsilon>0$ as $x \rightarrow x_{0}-0$.
3. $\int_{x}^{x_{0}} f(\eta) d \eta \equiv\left(\lim _{x \rightarrow x_{0}-0} \mathbf{F}_{f}(x)\right)-\mathbf{F}_{f}(x) \sim[a /(1-b)]\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ if $b<1$.
4. $\mathbf{F}_{f}(x) \sim \frac{a}{M+1}\left(\ln \left(x_{0}-x\right)\right)^{M+1}$ if $b=1$.

Proof. The l'Hôpital's theorem and the assumption yield

$$
\mathbf{F}_{f}(x) \sim a \int^{x}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \quad \text { as } \quad x \rightarrow x_{0}-0
$$

provided that both sides diverge, or converge to 0 . The right-hand side is given as

$$
a\left\{\sum_{l=0}^{M} \frac{M!}{(M-l)!}\left(\frac{-1}{1-b}\right)^{l+1}\left(\ln \left(x_{0}-x\right)\right)^{M-l}\right\}\left(x_{0}-x\right)^{1-b}+C
$$

with the integral constant $C$, provided $b \neq 1$. In particular, we have

$$
a \int^{x}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \sim \frac{-a}{1-b}\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}+C .
$$

Note that the right-hand side converges to $C$ as $x \rightarrow x_{0}-0$ if $b<1$.
Now we prove our statements. If $b>1$, the function $\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ diverges as $x \rightarrow x_{0}-0$ and hence

$$
a \int^{x}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \sim \frac{-a}{1-b}\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M},
$$

which shows the first statement. On the other hand, if $b<1$, then

$$
a \int^{x}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \sim C
$$

provided $C \neq 0$, because $\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ converges to 0 as $x \rightarrow x_{0}-0$ and hence the constant $C$ is dominant as $x \rightarrow x_{0}-0$, while $\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}$ becomes dominant when $\mathbf{F}_{f}(x) \rightarrow 0$ as $x \rightarrow x_{0}-0$. Moreover, we have

$$
\lim _{x \rightarrow x_{0}-0} \frac{\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M}}{\left(x-x_{0}\right)^{1-b-\epsilon}}=\lim _{x \rightarrow x_{0}-0} \frac{\left(\ln \left(x_{0}-x\right)\right)^{M}}{\left(x_{0}-x\right)^{-\epsilon}}=0
$$

for any $\epsilon>0$. As a summary, we have the second statement.
Consider the integral $\int_{x}^{x_{0}} f(\eta) d \eta$ instead, assuming $b<1$. By the l'Hôpital's theorem, we have

$$
\int_{x}^{x_{0}} f(\eta) d \eta \sim a \int_{x}^{x_{0}}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \quad \text { as } \quad x \rightarrow x_{0}-0
$$

The integral of the right-hand side is

$$
a \int_{x}^{x_{0}}\left(x_{0}-\eta\right)^{-b}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta=-a\left\{\sum_{l=0}^{M} \frac{M!}{(M-l)!}\left(\frac{-1}{1-b}\right)^{l+1}\left(\ln \left(x_{0}-x\right)\right)^{M-l}\right\}\left(x_{0}-x\right)^{1-b},
$$

because $b<1$. Substituting this identity into the asymptotic relation, we have

$$
\int_{x}^{x_{0}} f(\eta) d \eta \sim \frac{a}{1-b}\left(x_{0}-x\right)^{1-b}\left(\ln \left(x_{0}-x\right)\right)^{M} \quad \text { as } \quad x \rightarrow x_{0}-0,
$$

namely, we have the third statement.
We go back to the primitive function $\mathbf{F}_{f}(x)$. When $b=1$, direct calculation yields

$$
a \int^{x}\left(x_{0}-\eta\right)^{-1}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta=\frac{-a}{M+1}\left(\ln \left(x_{0}-x\right)\right)^{M+1}+C
$$

with a constant $C$. Therefore we have

$$
a \int^{x}\left(x_{0}-\eta\right)^{-1}\left(\ln \left(x_{0}-\eta\right)\right)^{M} d \eta \sim \frac{-a}{M+1}\left(\ln \left(x_{0}-x\right)\right)^{M+1} \quad \text { as } \quad x \rightarrow x_{0}-0
$$

and hence the last statement is proved.

Based on the above arguments, we have the following basic estimates of $\mathrm{deg}_{\theta}$ involving integrals, depending on powers.

Lemma 2.9. For any $t_{0}<t_{\text {max }}$, the following results hold.

1. If $\gamma \leq-1$,

$$
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta\right)=\gamma+1
$$

2. If $\gamma>-1$,

$$
\operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }} \theta(\eta)^{\gamma} d \eta\right)=\gamma+1
$$

3. If $\gamma>-1$, there is a constant $C \in \mathbb{R}$ such that

$$
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta+C\right) \geq \gamma+1
$$

Proof. 1. If $\gamma<-1$,

$$
\begin{aligned}
\frac{1}{\theta(t)^{\gamma+1}} \int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta & =\frac{1}{\theta(t)^{\gamma+1}} \frac{-1}{\gamma+1}\left(\theta(t)^{\gamma+1}-\theta\left(t_{0}\right)^{\gamma+1}\right) \\
& =\frac{-1}{\gamma+1}\left(1-\left(\theta(t) / \theta\left(t_{0}\right)\right)^{-(\gamma+1)}\right) \\
& \rightarrow \frac{-1}{\gamma+1} \quad \text { as } \quad t \rightarrow t_{\max }-0
\end{aligned}
$$

If $\gamma=-1$,

$$
\int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta=\ln \frac{\theta\left(t_{0}\right)}{\theta(t)}
$$

which implies, for any $\tilde{\gamma}>0$,

$$
\theta(t)^{\tilde{\gamma}} \int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta=\theta(t)^{\tilde{\gamma}} \ln \frac{\theta\left(t_{0}\right)}{\theta(t)} \rightarrow 0
$$

holds as $t \rightarrow t_{\text {max }}-0$.
2. If $\gamma>-1$,

$$
\frac{1}{\theta(t)^{\gamma+1}} \int_{t}^{t_{\max }} \theta(\eta)^{\gamma} d \eta=\frac{1}{\theta(t)^{\gamma+1}} \frac{1}{\gamma+1}\left(\theta(t)^{\gamma+1}-\theta\left(t_{\max }\right)^{\gamma+1}\right)=\frac{1}{\gamma+1}
$$

3. Similar to the previous arguments, we have

$$
\int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta=\frac{1}{\gamma+1}\left(\theta\left(t_{0}\right)^{\gamma+1}-\theta(t)^{\gamma+1}\right) \rightarrow \frac{\theta\left(t_{0}\right)^{\gamma+1}}{\gamma+1}
$$

as $t \rightarrow t_{\max }-0$, which yields

$$
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} \theta(\eta)^{\gamma} d \eta\right)= \begin{cases}\gamma+1 & \text { if } \theta\left(t_{0}\right)=0 \\ 0 & \text { otherwise }\end{cases}
$$

Let $\Theta_{\gamma}(t)$ be a primitive function of $\theta(t)^{\gamma}$. Note that, for any constant $C \in \mathbb{R}, \Theta_{\gamma}(t)+C$ is also a primitive function of $\theta(t)^{\gamma}$, and that $C$ can be chosen so that the corresponding primitive function $\Theta_{\gamma}^{0}(t)$ converges to 0 as $t \rightarrow t_{\max }-0$. In the present case, we see that

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\Theta_{\gamma}^{0}(t)\right)=\gamma+1 \quad \text { letting } \quad \Theta_{\gamma}^{0}(t)=-(\gamma+1)^{-1} \theta(t)^{\gamma+1} \tag{2.8}
\end{equation*}
$$

The fundamental properties of $\operatorname{deg}_{\theta}$ are stated below.
Proposition 2.10. Consider real-valued continuous functions $f_{0}(t), g_{0}(t)$ defined in $\left[t_{0}, t_{\max }\right)$ with $t_{\max }<\infty$. Let $\gamma_{f_{0}}=\operatorname{deg}_{\theta}\left(f_{0}(t)\right)$ and $\gamma_{g_{0}}=\operatorname{deg}_{\theta}\left(g_{0}(t)\right)$.

1. If $f_{0}(t) \sim a \theta(t)^{-b}(\ln \theta(t))^{M}$ with $a \neq 0$ as $t \rightarrow t_{\max }-0$, we have

$$
\operatorname{deg}_{\theta}\left(\int^{t} f_{0}(\eta) d \eta\right)= \begin{cases}0 & \text { if } b<1 \text { and the integral constant } C \text { is not } 0 \\ 1-b & \text { otherwise }\end{cases}
$$

2. $\operatorname{deg}_{\theta}\left(f_{0}(t)+g_{0}(t)\right) \geq \min \left\{\gamma_{f_{0}}, \gamma_{g_{0}}\right\}$.
3. $\operatorname{deg}_{\theta}\left(f_{0}(t) g_{0}(t)\right) \geq \gamma_{f_{0}}+\gamma_{g_{0}}$.
4. If $\gamma_{f_{0}} \leq-1$, then $\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} f_{0}(\eta) d \eta\right) \geq \gamma_{f_{0}}+1$.
5. If $\gamma_{f_{0}}>-1$, then $\operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }} f_{0}(\eta) d \eta\right) \geq \gamma_{f_{0}}+1$.
6. If $\gamma_{f_{0}}>-1$, for the primitive function $\mathbf{F}_{f_{0}}(t)$ of $f_{0}(t)$ satisfying ${ }^{5} \lim _{t \rightarrow t_{\max }-0} \mathbf{F}_{f_{0}}(t)=0$, we have $\operatorname{deg}_{\theta}\left(\mathbf{F}_{f_{0}}(t)\right) \geq \gamma_{f_{0}}+1$.

Remark 2.11. When $\gamma_{f_{0}}>-1$ we have

$$
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} f_{0}(\eta) d \eta\right)=0
$$

unless integral constants are chosen appropriately (cf. Lemmas 2.8-2 and 2.9-3).

[^4]Proof. Statement 1 is just the paraphrase of Lemma 2.8.
Next we prove Statement 2. If $\gamma \in I_{f_{0}} \cap I_{g_{0}}$, then

$$
\lim _{t \rightarrow t_{\max }} \frac{f_{0}(t)+g_{0}(t)}{\theta(t)^{\gamma}}=\lim _{t \rightarrow t_{\max }} \frac{f_{0}(t)}{\theta(t)^{\gamma}}+\lim _{t \rightarrow t_{\max }} \frac{g_{0}(t)}{\theta(t)^{\gamma}}=0+0=0
$$

which implies that $\gamma \in I_{f_{0}+g_{0}}$. In other words, $I_{f_{0}} \cap I_{g_{0}} \subset I_{f_{0}+g_{0}}$. The statement then follows by taking the supremum on both sides.

Next we prove Statement 3. Let $\gamma_{f_{0}} \in I_{f_{0}}$ and $\gamma_{g_{0}} \in I_{g_{0}}$. Our claim is to prove $\gamma_{f_{0}}+\gamma_{g_{0}} \in I_{f_{0} g_{0}}$, which follows from the relation

$$
\lim _{t \rightarrow t_{\max }} \frac{f_{0}(t) g_{0}(t)}{\theta(t)^{\gamma_{f_{0}}+\gamma_{g_{0}}}}=\lim _{t \rightarrow t_{\max }} \frac{f_{0}(t)}{\theta(t)^{\gamma_{f_{0}}}} \lim _{t \rightarrow t_{\max }} \frac{g_{0}(t)}{\theta(t)^{\gamma_{g_{0}}}}=0 \cdot 0=0
$$

which implies that

$$
I_{f_{0}}+I_{g_{0}} \equiv\left\{\gamma_{1}+\gamma_{2} \mid \gamma_{1} \in I_{f_{0}}, \gamma_{2} \in I_{g_{0}}\right\} \subset I_{f_{0} g_{0}}
$$

The statement then follows by taking the supremum on both sides.
Next, we prove Statement 4. For all $\gamma<\gamma_{f_{0}}$, the relation

$$
\lim _{t \rightarrow t_{\max }-0} \frac{f_{0}(t)}{\theta(t)^{\gamma}}=0
$$

implies that

$$
\left|f_{0}(t)\right| \leq \theta(t)^{\gamma} \quad \text { near } \quad t=t_{\max }
$$

Then it follows from Lemma 2.6 that

$$
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} f_{0}(\eta) d \eta\right) \geq \operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t}\left|f_{0}(\eta)\right| d \eta\right) \geq \operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} \theta(t)^{\gamma} d \eta\right)=\gamma+1
$$

which proves the statement. Note that the rightmost equality follows from Lemma 2.9
Next, we prove Statement 5. For all $\gamma \in\left(-1, \gamma_{f_{0}}\right)$, the relation

$$
\lim _{t \rightarrow t_{\max }-0} \frac{f_{0}(t)}{\theta(t)^{\gamma}}=0
$$

implies that

$$
\operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }} f_{0}(\eta) d \eta\right) \geq \operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }}\left|f_{0}(\eta)\right| d \eta\right) \geq \operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }} \theta(t)^{\gamma} d \eta\right)=\gamma+1
$$

which proves the statement.
Finally, we prove the statement 6. Consider $\gamma<\gamma_{f_{0}}$. Let $\Theta_{\gamma}^{0}(t)$ be the primitive function of $\theta(t)^{\gamma}$ given in (2.8). By assumption, the l'Hôpital's theorem can be applied to $\mathbf{F}_{f_{0}}$ and $\Theta_{\gamma}^{0}(t)$, indicating

$$
\lim _{t \rightarrow t_{\max }-0} \frac{\mathbf{F}_{f_{0}}(t)}{\Theta_{\gamma}^{0}(t)}=\lim _{t \rightarrow t_{\max }-0} \frac{f_{0}(t)}{\theta(t)^{\gamma}}=0
$$

Because $\gamma<\gamma_{f_{0}}$ can be chosen arbitrarily, we have $\operatorname{deg}_{\theta}\left(\mathbf{F}_{f_{0}}(t)\right) \geq \gamma_{f_{0}}+1$.

### 2.2.2 Degree and the fundamental matrices of Euler-type homogeneous ODEs

As seen in Section 3, the fundamental matrix of the following homogeneous system is fully applied to asymptotic expansions of blow-up solutions:

$$
\frac{d \mathbf{v}}{d t}=\theta(t)^{-1} A \mathbf{v}, \quad A \in M_{n}(\mathbb{R})
$$

The fundamental matrix is given by $\theta(t)^{-A}=\exp (-A \ln \theta(t))=\left(\theta(t)^{A}\right)^{-1}$. Any matrix $A \in$ $M_{n}(\mathbb{R})$ and the associated fundamental matrix $\theta(t)^{A}$ has the following real Jordan normal forms, respectively:

$$
\begin{align*}
P^{-1} A P & =J_{1}\left(\lambda_{1}\right) \oplus \cdots \oplus J_{d_{r}}\left(\lambda_{d_{r}}\right) \oplus K_{1}\left(\lambda_{r e, 1}, \lambda_{i m, 1}\right) \oplus \cdots \oplus K_{d_{c}}\left(\lambda_{r e, d_{c}}, \lambda_{i m, d_{c}}\right), \\
P^{-1} \theta(t)^{A} P & =\theta(t)^{J_{1}\left(\lambda_{1}\right)} \oplus \cdots \oplus \theta(t)^{J_{d_{r}}\left(\lambda_{d_{r}}\right)} \oplus \theta(t)^{K_{1}\left(\lambda_{r e, 1}, \lambda_{i m, 1}\right)} \oplus \cdots \oplus \theta(t)^{K_{d_{c}}\left(\lambda_{r e, d_{c}}, \lambda_{i m, d_{c}}\right)}, \tag{2.9}
\end{align*}
$$

where

$$
B_{1} \oplus B_{2} \oplus \cdots \oplus B_{m} \equiv\left(\begin{array}{cccc}
B_{1} & & & O \\
& B_{2} & & \\
& & \ddots & \\
O & & & B_{m}
\end{array}\right), \quad B^{\oplus s}=\underbrace{B \oplus \cdots \oplus B}_{s}
$$

for any real squared matrices $B, B_{1}, \ldots, B_{m}$, and the above real Jordan blocks are expressed by

$$
\begin{gather*}
J_{l}(\lambda)=\lambda I_{m_{l}}+N_{m_{l}}, \quad N_{m_{l}}=\left(\begin{array}{ccccc}
0 & 1 & & & O \\
& 0 & 1 & & \\
& & 0 & \ddots & \\
& & & \ddots & 1 \\
O & & & & 0
\end{array}\right) \in M_{m_{l}}(\mathbb{R}),  \tag{2.10}\\
K_{l}\left(\lambda_{r e}, \lambda_{i m}\right)=R\left(\lambda_{r e}, \lambda_{i m}\right)^{\oplus s_{l}}+N_{2 s_{l}}^{2}, \quad R\left(\lambda_{r e}, \lambda_{i m}\right)=\left(\begin{array}{cc}
\lambda_{r e} & \lambda_{i m} \\
-\lambda_{i m} & \lambda_{r e}
\end{array}\right) . \tag{2.11}
\end{gather*}
$$

Here $I_{m}$ denotes the $m$-dimensional identity matrix. The scalar function $\theta(t)$ to the above matrices are given as follows:

$$
\begin{aligned}
\theta(t)^{J_{l}(\lambda)}= & \theta(t)^{\lambda}\left(I_{m_{l}}+\ell(t) N_{m_{l}}+\frac{\ell(t)^{2}}{2!} N_{m_{l}}^{2}+\cdots+\frac{\ell(t)^{m_{l}-1}}{\left(m_{l}-1\right)!} N_{m_{l}}^{m_{l}-1}\right), \\
\theta(t)^{K_{l}\left(\lambda_{r e}, \lambda_{i m}\right)}= & \theta(t)^{\lambda_{r e}} R\left(\cos \left(\lambda_{i m} \ell(t)\right), \sin \left(\lambda_{i m} \ell(t)\right)\right)^{\oplus s_{l}} \\
& \cdot\left(I_{2 s_{l}}+\ell(t) N_{2 s_{l}}^{2}+\frac{\ell(t)^{2}}{2!} N_{2 s_{l}}^{4}+\cdots+\frac{\ell(t)^{s_{l}-1}}{\left(s_{l}-1\right)!} N_{2 s_{l}}^{2\left(s_{l}-1\right)}\right), \\
\ell(t)= & \ln (\theta(t)) .
\end{aligned}
$$

This Jordan normal form shows that the matrix $\theta(t)^{A}$, as well as $\theta(t)^{-A}$, consists of functions $\theta(t)^{\gamma}, \ln \theta(t), \cos (\lambda \ln \theta(t))$ and $\sin (\lambda \ln \theta(t))$. The degree $\operatorname{deg}_{\theta}$ of functions appeared in the above fundamental matrix are collected in Example 2.7 Combining with Proposition 2.10 we obtain the following inequalities.

Proposition 2.12. For any real m-dimensional and $2 m$-dimensional vector-valued continuous functions $\mathbf{a}_{m}(t)$ and $\mathbf{b}_{2 m}(t)$ defined in $\left(t_{0}, t_{\max }\right)$ with $t_{\max }<\infty$ and any constant matrix $P \in$ $M_{m}(\mathbb{R})$, we have

$$
\begin{align*}
\operatorname{deg}_{\theta}\left(P \mathbf{a}_{m}(t)\right) & \geq \operatorname{deg}_{\theta}\left(\mathbf{a}_{m}(t)\right)  \tag{2.12}\\
\operatorname{deg}_{\theta}\left(\theta(t)^{J(\lambda)} \mathbf{a}_{m}(t)\right) & \geq \lambda+\operatorname{deg}_{\theta}\left(\mathbf{a}_{m}(t)\right),  \tag{2.13}\\
\operatorname{deg}_{\theta}\left(\theta(t)^{K\left(\lambda_{r e}, \lambda_{i m}\right)} \mathbf{b}_{2 m}(t)\right) & \geq \lambda_{r e}+\operatorname{deg}_{\theta}\left(\mathbf{b}_{2 m}(t)\right), \tag{2.14}
\end{align*}
$$

where $J(\lambda)$ and $K\left(\lambda_{r e}, \lambda_{i m}\right)$ are m-dimensional and $2 m$-dimensional Jordan matrices of the form (2.10) and (2.11), respectively.

Proof. By Example 2.7 the values of $\operatorname{deg}_{\theta}$ for each entry of the constant matrix $P$ is either $+\infty$ or 0. The first statement then follows from the combination of statements in Proposition 2.10, Similarly, the values of $\operatorname{deg}_{\theta}$ for each entry in $\theta(t)^{J(\lambda)}$ are identically either $+\infty$ or $\lambda$, while those for each entry in $\theta(t)^{K\left(\lambda_{r e}, \lambda_{i m}\right)}$ are identically either $+\infty$ or $\lambda_{r e}$. The conclusion then follows from the combination of statements in Proposition 2.10.

We further consider the asymptotic degree of the fundamental matrix on invariant subspaces of $A$. In general, eigenvalues of $A$ with positive (resp. negative) real parts associate the invariant projector $P_{+}$(resp. $P_{-}$) such that $A P_{-}=P_{-} A$ (resp. $A P_{+}=P_{+} A$ ) and that Image $\left(P_{+}\right)$(resp. Image $\left(P_{-}\right)$) is generated by (generalized) eigenvectors $A$ associated with eigenvalues with (resp. negative) real part: Now $\mathbb{R}^{n}$ possesses the following decomposition into the invariant subspaces associated with $\{\lambda \in \operatorname{Spec}(A) \mid \operatorname{Re} \lambda>0\},\{\lambda \in \operatorname{Spec}(A) \mid \operatorname{Re} \lambda=0\}$ and $\{\lambda \in \operatorname{Spec}(A) \mid \operatorname{Re} \lambda<0\}$, respectively (e.g. [7, 32]):

$$
\begin{equation*}
\mathbb{R}^{n}=\mathbb{E}_{A}^{u} \oplus \mathbb{E}_{A}^{c} \oplus \mathbb{E}_{A}^{s} \tag{2.15}
\end{equation*}
$$

Let $P_{+}$be the invariant projector onto $\mathbb{E}_{A}^{u}$ and $P_{-}$be the invariant projector onto $\mathbb{E}_{A}^{s}$. In particular, we have

$$
A P_{ \pm}=P_{ \pm} A, \quad \theta(t)^{A} P_{ \pm}=P_{ \pm} \theta(t)^{A}
$$

for any $t<t_{\text {max }}$ under our considerations.
Our interest here is the asymptotic degree of vector-valued functions of the form $\theta(t)^{-A} P_{ \pm} \mathbf{h}(t)$. The following estimates are mainly used to construct asymptotic expansions of blow-ups.

Proposition 2.13. Let $\mathbf{h}(t)$ be a continuous n-vector-valued function with $\operatorname{deg}_{\theta}(\mathbf{h}(t))=\gamma>-1$. Then the following inequalities hold:

$$
\begin{align*}
\operatorname{deg}_{\theta}\left(P_{-}\left(\mathbf{C}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} \mathbf{h}(\eta) d \eta\right)\right) & \geq \min \left\{\min _{\substack{\lambda \in \operatorname{Spec}(A), \operatorname{Re} \lambda<0}}(-\operatorname{Re} \lambda), \gamma+1\right\},  \tag{2.16}\\
\operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{+} \mathbf{h}(\eta) d \eta\right) & \geq \gamma+1 \tag{2.17}
\end{align*}
$$

[^5]where $\mathbf{C} \in \mathbb{R}^{n}$ is a constant vector. Finally, let $P^{-1} A P=\Lambda$ be the Jordan normal form of the matrix $A$ and $\mathbf{H}_{\mathbf{h}}(t)$ be the primitive function of $\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{h}(t)$ such that
\[

$$
\begin{equation*}
\lim _{t \rightarrow t_{\max }-0}\left(\mathbf{H}_{\mathbf{h}}(t)\right)_{i}=0 \quad \text { whenever } \quad \operatorname{deg}_{\theta}\left(\left\{\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{h}(t)\right\}_{i}\right)>-1 \tag{2.18}
\end{equation*}
$$

\]

Then we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(P\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\Lambda} P_{-} \mathbf{H}_{\mathbf{h}}(t)\right) \geq \gamma+1 \tag{2.19}
\end{equation*}
$$

Proof. First, for fixed $\eta \in\left[t_{0}, t\right]$, apply the normalization to $(\theta(\eta) / \theta(t))^{A}$ following (2.9):

$$
\begin{aligned}
P^{-1}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P & =\left(\frac{\theta(\eta)}{\theta(t)}\right)^{J_{1}\left(\lambda_{1}\right)} \oplus \cdots \oplus\left(\frac{\theta(\eta)}{\theta(t)}\right)^{J_{d_{r}}\left(\lambda_{d_{r}}\right)} \\
& \oplus\left(\frac{\theta(\eta)}{\theta(t)}\right)^{K_{1}\left(\lambda_{r e, 1}, \lambda_{i m, 1}\right)} \oplus \cdots \oplus\left(\frac{\theta(\eta)}{\theta(t)}\right)^{K_{d_{c}}\left(\lambda_{r e, d_{c}}, \lambda_{i m, d_{c}}\right)}
\end{aligned}
$$

Letting $P_{l}^{(r)}$ be the invariant projector onto the invariant subspace associated with the real eigenvalue $\lambda_{l}$, we have

$$
\begin{align*}
\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{l}^{(r)} \mathbf{h}(\eta) & =P\left(0 \oplus \cdots \oplus 0 \oplus\left(\frac{\theta(\eta)}{\theta(t)}\right)^{J_{l}\left(\lambda_{l}\right)} \oplus 0 \oplus \cdots \oplus 0\right) P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) \\
& \equiv P\left(\frac{\theta(\eta)}{\theta(t)}\right)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) \tag{2.20}
\end{align*}
$$

Similarly, letting $P_{l}^{(c)}$ be the invariant projector onto the invariant subspace associated with the complex conjugate eigenvalues $\left\{\lambda_{r e, l} \pm \sqrt{-1} \lambda_{i m, l}\right\}$, we have

$$
\begin{aligned}
\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{l}^{(c)} \mathbf{h}(\eta) & =P\left(0 \oplus \cdots \oplus 0 \oplus\left(\frac{\theta(\eta)}{\theta(t)}\right)^{K_{l}\left(\lambda_{r e, l}, \lambda_{i m, l}\right)} \oplus 0 \oplus \cdots \oplus 0\right) P^{-1} P_{l}^{(c)} \mathbf{h}(\eta) \\
& \equiv P\left(\frac{\theta(\eta)}{\theta(t)}\right)^{\overline{K_{l}\left(\lambda_{r e, l}, \lambda_{i m, l}\right)}} P^{-1} P_{l}^{(c)} \mathbf{h}(\eta)
\end{aligned}
$$

Now consider the integral

$$
\begin{equation*}
\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{l}^{(r)} \mathbf{h}(\eta) d \eta \tag{2.21}
\end{equation*}
$$

The integral is also written as

$$
\begin{align*}
\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{l}^{(r)} \mathbf{h}(\eta) d \eta & =\int_{t_{0}}^{t} P\left(\frac{\theta(\eta)}{\theta(t)}\right)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta \\
& =P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \int_{t_{0}}^{t} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta \tag{2.22}
\end{align*}
$$

Moreover, if the improper integral

$$
\mathbb{I}_{1}=\int_{t_{0}}^{t_{\max }} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta
$$

converges, then the integral is also written as

$$
\begin{align*}
& \int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{l}^{(r)} \mathbf{h}(\eta) d \eta \\
& \quad=P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \int_{t_{0}}^{t_{\max }} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta-P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \int_{t}^{t_{\max }} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta \\
& \quad \equiv P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}}\left(\mathbb{I}_{1}+\mathbb{I}_{2}(t)\right) \tag{2.23}
\end{align*}
$$

Now we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\theta(t)^{\Lambda} P^{-1} P_{l}^{(r)} \mathbf{h}(t)\right) \geq \lambda_{l}+\gamma, \quad \operatorname{deg}_{\theta}\left(\theta(t)^{\Lambda} P^{-1} P_{l}^{(c)} \mathbf{h}(t)\right) \geq \operatorname{Re} \lambda_{r e, l}+\gamma \tag{2.24}
\end{equation*}
$$

by using Proposition 2.12 There are two cases depending on the value

$$
\begin{equation*}
\nu_{l}:=\operatorname{deg}_{\theta}\left(\theta(t)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(t)\right) \tag{2.25}
\end{equation*}
$$

From (2.24), we have $\nu_{l} \geq \lambda_{l}+\gamma$.
Case 1. $\nu_{l} \leq-1$.
In this case, the integral (2.21) with the equivalent form (2.22) is directly estimated through Proposition $2.10-4$ to obtain

$$
\begin{aligned}
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta\right) & \geq \nu_{l}+1 \\
& \geq \lambda_{l}+\gamma+1
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}}\left\{\int_{t_{0}}^{t} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta\right\}\right) \geq \gamma+1 \tag{2.26}
\end{equation*}
$$

by Proposition 2.12. In particular, for any constant vector $\mathbf{C}$, we have

$$
\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}}\left\{\mathbf{C}+\int_{t_{0}}^{t} \theta(\eta)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta\right\}\right) \geq \min \left\{-\lambda_{l}, \gamma+1\right\}
$$

from Proposition 2.10 2.
Case 2. $\nu_{l}>-1$.

In this case, the improper integral $\mathbb{I}_{1}$ converges and the integral (2.21) with the equivalent form (2.23) is considered, instead of (2.22). Proposition 2.10 yields

$$
\begin{align*}
\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}}\left(\mathbb{I}_{1}+\mathbb{I}_{2}(t)\right)\right) & \geq \min \left\{\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \mathbb{I}_{1}\right), \operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \mathbb{I}_{2}(t)\right)\right\} \\
\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \mathbb{I}_{1}\right) & \geq-\lambda_{l} \\
\operatorname{deg}_{\theta}\left(\mathbb{I}_{2}(t)\right) & \geq \nu_{l}+1 \quad(\text { from Proposition 2.10-5) } \\
& \geq \lambda_{l}+\gamma+1, \\
\operatorname{deg}_{\theta}\left(P \theta(t)^{-\overline{J_{l}\left(\lambda_{l}\right)}} \mathbb{I}_{2}(t)\right) & \geq \gamma+1 \tag{2.27}
\end{align*}
$$

Combining two cases, we have

$$
\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} P_{l}^{(r)}\left(\mathbf{C}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{h}(\eta) d \eta\right) \geq \min \left\{-\lambda_{l}, \gamma+1\right\}
$$

for any constant vector $\mathbf{C} \in \mathbb{R}^{n}$, where we have used the commutativity

$$
P_{l}^{(r)}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A}=\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} P_{l}^{(r)}
$$

for any $\eta \in\left[t_{0}, t\right]$, thank to the fact that $P_{l}^{(r)}$ is an invariant projector. The similar estimates follow replacing $\lambda_{l}$ and $P_{l}^{(r)}$ by $\lambda_{r e, l}$ and $P_{l}^{(c)}$, respectively, for the case of complex conjugate eigenvalues. The inequality (2.16) then follows from the property of projections

$$
P_{-}=\sum_{l ; \lambda_{l}<0} P_{l}^{(r)}+\sum_{l ; \operatorname{Re} \lambda_{r e}, l<0} P_{l}^{(c)}
$$

As for $P_{+}$, the inequality (2.17) follows from (2.27) and

$$
P_{+}=\sum_{l ; \lambda_{l}>0} P_{l}^{(r)}+\sum_{l ; \operatorname{Re} \lambda_{r e, l}>0} P_{l}^{(c)}
$$

Finally consider the last inequality (2.19). Our main interest here is (2.20) replacing $\theta(t)$ by $\theta(t) / \theta\left(t_{0}\right)$. Let $\nu_{l}$ be the asymptotic degree defined in (2.25). If $\nu_{l}<-1$, our statement is exactly (2.26). Note that we have used the fact that $\operatorname{Re}\left(-\lambda_{l}\right)>0 \geq \nu_{l}+1$ in the present setting.

Now consider the case when $\nu_{l}>-1$. In this case, the improper integral

$$
\lim _{t \rightarrow t_{\max }-0} \int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(\eta) d \eta
$$

converges. Let $\overline{\mathbf{H}_{\mathbf{h} ; l}}(t)$ be the primitive function of $\left(\theta(t) / \theta\left(t_{0}\right)\right)^{\overline{J_{l}\left(\lambda_{l}\right)}} P^{-1} P_{l}^{(r)} \mathbf{h}(t)$ such that (2.18) holds, which is $n$-vector-valued. Proposition $2.10-6$ yields that

$$
\operatorname{deg}_{\theta}\left(\overline{\mathbf{H}_{\mathbf{h} ; l}}(t)\right) \geq \nu_{l}+1
$$

and hence

$$
\operatorname{deg}_{\theta}\left(P\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\overline{J_{l}\left(\lambda_{l}\right)}} P_{-} \overline{\mathbf{H}_{\mathbf{h} ; l}}(t)\right) \geq \nu_{l}-\lambda_{l}+1 \geq \gamma+1
$$

which completes our proof through the composition of the projection $P_{-}$.

## 3 Asymptotic expansion of type-I blow-ups

Now we move to provide a systematic methodology for deriving asymptotic expansions of blow-up solutions. When blow-up solutions are supposed to be given a priori, one of the main issues in blow-up studies is to characterize rough asymptotic behavior of blow-up solutions, well-known as the blow-up rates. Our aim here is to obtain successive terms of such blow-up solutions towards the precise description of blow-up behavior.

### 3.1 The fundamental setting

In the present study, we only pay attention to blow-up solutions $\mathbf{y}(t)$ of (2.1) possessing the following asymptotic behavior.

Assumption 3.1. The asymptotically quasi-homogeneous system (2.1) of type $\alpha$ and the order $k+1$ admits a solution

$$
\mathbf{y}(t)=\left(y_{1}(t), \ldots, y_{n}(t)\right)^{T}
$$

which blows up at $t=t_{\max }<\infty$ with the asymptotic behavior

$$
\begin{equation*}
y_{i}(t) \sim c_{i} \theta(t)^{-\alpha_{i} / k}, \quad t \rightarrow t_{\max }-0, \quad i=1, \ldots, n \tag{3.1}
\end{equation*}
$$

for some constants $c_{i} \in \mathbb{R}$.
Under Assumption 3.1, we shall write the blow-up solution $\mathbf{y}(t)$ as $\mathbf{y}(t)=\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}(t)$, equivalently

$$
\begin{equation*}
y_{i}(t)=\theta(t)^{-\alpha_{i} / k} Y_{i}(t), \quad \mathbf{Y}(t)=\left(Y_{1}(t), \ldots, Y_{n}(t)\right)^{T} \tag{3.2}
\end{equation*}
$$

and determine the concrete form of the factor $\mathbf{Y}(t)$.
As the first step, decompose the vector field $f$ into two terms as follows:

$$
f(\mathbf{y})=f_{\alpha, k}(\mathbf{y})+f_{\mathrm{res}}(\mathbf{y})
$$

where $f_{\alpha, k}$ is the quasi-homogeneous component of $f$ and $f_{\text {res }}$ is the residual (i.e., lower-order) terms. The componentwise expressions are

$$
f_{\alpha, k}(\mathbf{y})=\left(f_{1 ; \alpha, k}(\mathbf{y}), \ldots, f_{n ; \alpha, k}(\mathbf{y})\right)^{T}, \quad f_{\mathrm{res}}(\mathbf{y})=\left(f_{1 ; \mathrm{res}}(\mathbf{y}), \ldots, f_{n ; \mathrm{res}}(\mathbf{y})\right)^{T}
$$

respectively. Substituting (3.2) into (2.1), we have

$$
\begin{aligned}
\frac{\alpha_{i}}{k} \theta(t)^{-\alpha_{i} / k-1} Y_{i}+\theta(t)^{-\alpha_{i} / k} Y_{i}^{\prime} & =f_{i ; \alpha, k}(\mathbf{y})+f_{i ; \mathrm{res}}(\mathbf{y}) \\
& =\theta(t)^{-\left(k+\alpha_{i}\right) / k} f_{i ; \alpha, k}(\mathbf{Y})+f_{i ; \mathrm{res}}(\mathbf{y})
\end{aligned}
$$

where we have used the quasi-homogeneity of $f_{\alpha, k}(\mathbf{y})$ in the last equality. Multiplying $\left(t_{\max }-t\right)^{\frac{\alpha_{i}}{k}+1}$ for each $i$, we finally have

$$
\theta(t) \frac{d}{d t} Y_{i}=-\frac{\alpha_{i}}{k} Y_{i}+f_{i ; \alpha, k}(\mathbf{Y})+\theta(t)^{\frac{\alpha_{i}}{k}+1} f_{i ; \mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)
$$

with the vector form

$$
\begin{align*}
\theta(t) \frac{d}{d t} \mathbf{Y} & =-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y})+\theta(t)^{\frac{1}{k} \Lambda_{\alpha}+I} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)  \tag{3.3}\\
\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} & \equiv \operatorname{diag}\left(\theta(t)^{\alpha_{1} / k}, \ldots, \theta(t)^{\alpha_{n} / k}\right)
\end{align*}
$$

Another form equivalent to (3.3) is

$$
\begin{equation*}
\frac{d}{d t} \mathbf{Y}=\theta(t)^{-1}\left\{-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y})\right\}+\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right) \tag{3.4}
\end{equation*}
$$

While the residual term $f_{i ; \text { res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)$ can diverge as $t \rightarrow t_{\text {max }}-0$, the product $\theta(t)^{\frac{1}{k} \Lambda_{\alpha}+I} f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)$ goes to zero as $t \rightarrow t_{\text {max }}-0$ from the asymptotic quasi-homogeneity of $f$.

The system (3.4) is actually nonautonomous, but the most singular term $\theta(t)^{-1}\left\{-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y})\right\}$ as $t \rightarrow t_{\max }-0$ has the common $t$-dependence $\theta(t)^{-1}$. We thus focus on this singular term and consider the expansion of vector fields so that the linearization at an appropriate point and normalization of the system at the point can be applied to deriving the function $\mathbf{Y}(t)$ step by step.

### 3.2 Balance law

First note that the asymptotic behavior (3.1) indicates the fact that the function $\mathbf{Y}(t)$ is as smooth as $f$ in $t<t_{\text {max }}$ and there is a non-zero vector $\mathbf{Y}_{0} \in \mathbb{R}^{n}$ satisfying

$$
\begin{equation*}
\mathbf{Y}(t)=\mathbf{Y}_{0}+\tilde{\mathbf{Y}}(t), \quad \mathbf{Y}_{0}=\left(Y_{0,1}, \ldots, Y_{0, n}\right)^{T}, \quad \lim _{t \rightarrow t_{\max }-0} \tilde{\mathbf{Y}}(t)=\mathbf{0} \in \mathbb{R}^{n} \tag{3.5}
\end{equation*}
$$

The constant term $\mathbf{Y}_{0}$ determines the complete form of the lowest order term of $\mathbf{Y}(t)$. This term can be determined as a root of the time-independent system associated with (3.3). Taking the limit $t \rightarrow t_{\text {max }}-0$ in (3.3), we observe with continuity of the vector field and $\mathbf{Y}=\mathbf{Y}(t)$ that the system must satisfy

$$
\begin{equation*}
\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}_{0}=f_{\alpha, k}\left(\mathbf{Y}_{0}\right) \tag{3.6}
\end{equation*}
$$

provided that $\theta(t) \frac{d \mathbf{Y}}{d t}=o(\theta(t))$ holds as $t \rightarrow t_{\text {max }}$, which have to be justified later ${ }^{7}$.
Definition 3.2. We call the equation (3.6) a balance law for the blow-up solution $\mathbf{y}(t)$.
Note that (nonzero) roots of the balance law are not always uniquely determined. In other words, the equation (3.6) can have multiple different solutions.

### 3.3 Multi-order asymptotic expansions

We shall determine the non-constant term $\tilde{\mathbf{Y}}(t)$. Assume that a nonzero root $\mathbf{Y}_{0}$ of the balance law (3.6) is obtained. We do not know the concrete form of $\tilde{\mathbf{Y}}(t)$ at present 8 . We then expand

[^6]$\tilde{\mathbf{Y}}(t)$ as the general asymptotic series
\[

$$
\begin{align*}
\mathbf{Y}(t) & =\mathbf{Y}_{0}+\tilde{\mathbf{Y}}(t) \\
\tilde{\mathbf{Y}}(t) & =\sum_{j=1}^{\infty} \mathbf{Y}_{j}(t), \quad \mathbf{Y}_{j}(t) \ll \mathbf{Y}_{j-1}(t) \quad\left(t \rightarrow t_{\max }-0\right), \quad j=1,2, \ldots  \tag{3.7}\\
\mathbf{Y}_{j}(t) & =\left(Y_{j, 1}(t), \ldots, Y_{j, n}(t)\right)^{T}, \quad j=1,2, \ldots
\end{align*}
$$
\]

and solve $\mathbf{Y}_{j}(t)$ through the balance law for each $j$ inductively as the general framework of local asymptotic analysis for ordinary differential equations (cf. [4]), where the asymptotic relation $\mathbf{Y}_{j}(t) \ll \mathbf{Y}_{j-1}(t)$ for vector-valued functions are considered componentwise. Note that all the arguments below are valid only near $t=t_{\text {max }}$ with $t<t_{\text {max }}$.

### 3.3.1 The blow-up power-determining matrix and the blow-up power eigenvalues

Apply the Taylor's theorem to the right-hand side of (3.4) at $\mathbf{Y}=\mathbf{Y}_{0}$ :

$$
\begin{align*}
& \frac{d}{d t} \tilde{\mathbf{Y}}=\theta(t)^{-1}\left[\left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \tilde{\mathbf{Y}}+R_{\alpha, k}(\mathbf{Y})\right]+\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)  \tag{3.8}\\
& R_{\alpha, k}(\mathbf{Y})=f_{\alpha, k}(\mathbf{Y})-\left\{f_{\alpha, k}\left(\mathbf{Y}_{0}\right)+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right) \tilde{\mathbf{Y}}\right\}
\end{align*}
$$

where all terms involving the balance law (3.6) are cancelled. The remainder of the Taylor expansion of $f_{\alpha, k}$ is expressed as $R_{\alpha, k}$ and note that $R_{\alpha, k}\left(\mathbf{Y}_{0}\right)=0$.

Now we derive a governing system for the second term $\mathbf{Y}_{1}$ eliminating all possible negligible terms. From (3.7), the principal term of the left-hand side will be $d \mathbf{Y}_{1} / d t$, which have to be verified later (see the sentence before Remark (3.6). On the other hand, the principal terms in the right-hand side will involve the linear part $\left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \tilde{\mathbf{Y}}$ and $f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{0}\right)$. In other words, the other parts are negligible as $t \rightarrow t_{\max }$, compared with the above terms. As a summary, the asymptotically governing system for $\mathbf{Y}_{1}$ as $t \rightarrow t_{\max }-0$ we consider is

$$
\begin{equation*}
\frac{d}{d t} \mathbf{Y}_{1}=\theta(t)^{-1}\left\{-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right\} \mathbf{Y}_{1}+\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{0}\right) \tag{3.9}
\end{equation*}
$$

which is inhomogeneous and linear. Moreover, the coefficient of the homogeneous part is the product of $\theta(t)^{-1}$ and a constant matrix. We can therefore apply the fundamental matrix of linear homogeneous systems with constant coefficient matrix to obtaining the general form of solutions. In concrete problems, we can transform this constant matrix to a canonical form through a constant matrix and reduce the problem to $n$ inhomogeneous differential equations of the first order. The constant matrix $-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)$ plays a key role in our asymptotic analysis.

Definition 3.3 (Blow-up power eigenvalues). Suppose that a nonzero root $\mathbf{Y}_{0}$ of the balance law (3.6) is given. We call the constant matrix

$$
\begin{equation*}
A=-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right) \tag{3.10}
\end{equation*}
$$

[^7]the blow-up power-determining matrix for the blow-up solution $\mathbf{y}(t)$, and call the eigenvalues $\left\{\lambda_{i}\right\}_{i=1}^{n} \equiv \operatorname{Spec}(A)$ the blow-up power eigenvalues, where eigenvalues with nontrivial multiplicity are distinguished in this expression, except specifically noted.

Using the matrix $A$, (3.8) is simply rewritten as follows:

$$
\frac{d}{d t} \tilde{\mathbf{Y}}=\theta(t)^{-1}\left[A \tilde{\mathbf{Y}}+R_{\alpha, k}(\mathbf{Y})\right]+\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}\right)
$$

Similarly, the linear system (3.9) is rewritten as follows:

$$
\begin{equation*}
\frac{d}{d t} \mathbf{Y}_{1}=\theta(t)^{-1} A \mathbf{Y}_{1}+\mathbf{g}_{1}(t), \quad \mathbf{g}_{1}(t)=\theta(t)^{\frac{1}{k} \Lambda_{\alpha}} f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{0}\right) \tag{3.11}
\end{equation*}
$$

### 3.3.2 Multi-order asymptotic expansions

Let $\Psi(t) \equiv \theta(t)^{-A}$ be the fundamental matrix of the homogeneous system

$$
\frac{d}{d t} \mathbf{Y}_{1}=\theta(t)^{-1} A \mathbf{Y}_{1}
$$

Also, let $\mathbb{R}^{n}=\mathbb{E}_{A}^{u} \oplus \mathbb{E}_{A}^{c} \oplus \mathbb{E}_{A}^{s}$ be the decomposition of $\mathbb{R}^{n}$ into the invariant subspaces associated with $\operatorname{Spec}(A)$ mentioned in (2.15). The following proposition provides a simple form of $\mathbf{Y}_{1}(t)$ satisfying the asymptotic relation (3.7) with $j=1$ which is convergent under a mild assumption to $A$ and a suitable choice of initial points.

Proposition 3.4. Fix $t_{\max }>t_{0}$, assuming finite, and let $\mathbf{Y}_{0}$ be a nonzero root of the balance law (3.6) for (3.4). Assume that the associated blow-up power-determining matrix $A$ is hyperbolic:

$$
\operatorname{Spec}(A) \cap \sqrt{-1} \mathbb{R}=\emptyset
$$

Then the inhomogeneous system (3.11) admits a global-in-time solution (3.13) with

$$
\begin{equation*}
\lim _{t \rightarrow t_{\max }-0} \mathbf{Y}_{1}(t)=\mathbf{0} \in \mathbb{R}^{n} \tag{3.12}
\end{equation*}
$$

This solution can be written as
$\mathbf{Y}_{1}(t)=\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A}\left\{P_{-}\left(\mathbf{Y}_{1}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right\}, \quad \mathbf{Y}_{1}^{0} \in \mathbb{R}^{n}$
for $t \in\left(t_{0}, t_{\max }\right)$ with the initial time $t=t_{0}$, whenever $\left\|P_{-} \mathbf{Y}_{1}^{0}\right\|$ is sufficiently small. In particular, (3.11) locally generates a $C^{r} m_{A}$-parameter family of solutions $\mathbf{Y}_{1}(t)$ satisfying (3.12), where

$$
\begin{equation*}
m_{A}=\sharp\{\lambda \in \operatorname{Spec}(A) \mid \operatorname{Re} \lambda<0\} \equiv \operatorname{dim} \mathbb{E}_{A}^{s} \tag{3.14}
\end{equation*}
$$

The proof is left to Appendix B.1. The key point of the expression (3.13) is that free parameters describing the solution $\mathbf{Y}_{1}(t)$ are chosen only from $\mathbb{E}_{A}^{s}$. We see in examples in Part II [23] that integral constants stemming from $\mathbb{E}_{A}^{u}$ vanish in all cases, from asymptotic quasi-homogeneity of $f$ and hyperbolicity of $A$.

Multi-order asymptotic expansion of $\mathbf{Y}(t)$ can be derived inductively. In other words, the higher-order terms $\mathbf{Y}_{j}(t), j \geq 2$, can be also determined in the similar way to $\mathbf{Y}_{1}(t)$, while one attention mentioned below is needed. Now assume that, for given $j \geq 2$, the preceding terms $\left\{\mathbf{Y}_{l}(t)\right\}_{l=0}^{j-1}$ are calculated. Neglecting possible higher-order terms, the governing system for $\mathbf{Y}_{j}(t)$ is given as follows:

$$
\begin{align*}
\frac{d}{d t} \mathbf{Y}_{j}= & \theta(t)^{-1} A \mathbf{Y}_{j}+\mathbf{g}_{j}(t)  \tag{3.15}\\
\mathbf{g}_{j}(t)= & \theta(t)^{-1}\left\{R_{\alpha, k}\left(\mathbf{S}_{j-1}(\mathbf{Y})(t)\right)-R_{\alpha, k}\left(\mathbf{S}_{j-2}(\mathbf{Y})(t)\right)\right\} \\
& +\theta(t)^{\frac{1}{k} \Lambda_{\alpha}}\left\{f_{\mathrm{res}}\left(\mathbf{S}_{j-1}^{\theta}(\mathbf{Y})(t)\right)-f_{\mathrm{res}}\left(\mathbf{S}_{j-2}^{\theta}(\mathbf{Y})(t)\right)\right\} \tag{3.16}
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{S}_{j}(\mathbf{Y})(t) \equiv \sum_{\tilde{j}=0}^{j} \mathbf{Y}_{\tilde{j}}(t), \quad \mathbf{S}_{j}^{\theta}(\mathbf{Y})(t) \equiv \theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{j}(\mathbf{Y})(t), \quad j=0,1, \ldots  \tag{3.17}\\
& \mathbf{S}_{j}(\mathbf{Y})(t) \equiv \mathbf{S}_{j}^{\theta}(\mathbf{Y})(t) \equiv 0, \quad j<0
\end{align*}
$$

The system (3.15) is inhomogeneous and linear, like (3.11). In the similar way to solving (3.11), we obtain a general form of solutions $\mathbf{Y}_{j}(t)$. However, we have to pay attention to asymptotic relations (3.7). Although the validity of (3.7) depends on $f$ and $\left\{\mathbf{Y}_{l}(t)\right\}_{l=0}^{j-1}$, one can construct a solution of (3.15) such that free parameters describing $\mathbf{Y}_{j}(t)$ are not newly generated except those included in $\mathbf{Y}_{1}(t)$, which is uniquely determined. As shown in the proof of the following proposition, such a choice of solutions indicates the elimination of exponential terms generated by the homogeneous part of (3.15), namely $d \mathbf{Y}_{j} / d t=\theta(t)^{-1} A \mathbf{Y}_{j}$, in solutions $\mathbf{Y}_{j}(t)$, which would yield a necessary condition of the asymptotic relation (3.7).
Proposition 3.5. Fix $t_{\max }>t_{0}$, assuming finite, and let $\mathbf{Y}_{0}$ be a nonzero root of the balance law (3.6) for (3.4). Assume that the associated blow-up power-determining matrix $A$ is hyperbolic and that, for given $j \geq 2$, the preceding terms $\left\{\mathbf{Y}_{l}(t)\right\}_{l=0}^{j-1}$ are calculated. Then the integral

$$
\begin{equation*}
\mathbf{Y}_{j}(t)=\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A}\left\{P_{-}\left(\mathbf{Y}_{j}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right\} \tag{3.18}
\end{equation*}
$$

for $t \in\left(t_{0}, t_{\max }\right)$ with the initial time $t=t_{0}$ solves the inhomogeneous system (3.15). The constant vector $\mathbf{Y}_{j}^{0} \in \mathbb{R}^{n}$ can be chosen to satisfy

$$
\begin{equation*}
P_{-} \mathbf{Y}_{j}^{0}=P_{-} P \mathbf{H}_{j}\left(t_{0}\right) \tag{3.19}
\end{equation*}
$$

where the matrix $A$ has the Jordan structure through a nonsingular matrix $P: P^{-1} A P=\Lambda$, and $\mathbf{H}_{j}=\mathbf{H}_{j}(t)$ denotes the primitive function of $\left(\theta(t) / \theta\left(t_{0}\right)\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(t)$ satisfying

$$
\begin{equation*}
\lim _{t \rightarrow t_{\max }-0}\left(\mathbf{H}_{j}(t)\right)_{i}=0 \quad \text { whenever } \quad \operatorname{deg}_{\theta}\left(\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(t)\right)_{i}\right)>-1 \tag{3.20}
\end{equation*}
$$

The initial point

$$
\mathbf{Y}_{j}\left(t_{0}\right)=P_{-} \mathbf{Y}_{j}^{0}-P_{+} \int_{t_{0}}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta
$$

at $t=t_{0}$ is then uniquely determined once free parameters in $\mathbf{Y}_{1}(t)$ constructing $\mathbf{g}_{j}(t)$ are fixed.

The proof is left to Appendix B.2. The choice of $\mathbf{Y}_{j}^{0}$ in the proposition plays a key role in justification of multi-order asymptotic expansions: Proposition 3.13 and Theorem 3.15 stated below. Under this arrangement, we see in (3.18) that general solutions of the homogeneous part make no contributions to $\mathbf{Y}_{j}$ with $j \geq 2$. In such a case, it turns out that our choice in (3.11) that the governing term in the left-hand side of (3.8) being $d \mathbf{Y}_{1} / d t$ is valid.

Remark 3.6. For $j \geq 2$, the leading terms of the function $\mathbf{g}_{j}(t)$ are essentially $\mathbf{Y}_{j-1}(t)$ and $D f_{\mathrm{res}}\left(\mathbf{S}_{j-2}^{\theta}(\mathbf{Y})(t)\right) \mathbf{Y}_{j-1}(t)$ by the Taylor's theorem. The integral in (3.18) would therefore increase the order of $\theta(t)$ in $\mathbf{Y}_{j}(t)$, which would provide the asymptotic relation 3.7). As seen in Section 3.5. this expectation is justified under mild assumptions.

The above arguments are summarized as the following theorem. Obviously, the formal sum of systems (3.6), (3.11) and (3.15) for $j \geq 2$ recovers (3.4).
Theorem 3.7 (Formal asymptotic expansion of blow-up solutions). Let $\mathbf{y}(t)$ be a blow-up solution of (2.1) satisfying Assumption 3.1. Assume that a nonzero root $\mathbf{Y}_{0}$ of the balance law (3.6) is given and that the associated blow-up power-determining matrix $A$ is hyperbolic. Then $\mathbf{y}(t)$ with the initial time $t=t_{0}$ is formally expanded as follow 10 :

$$
\begin{align*}
\mathbf{y}(t) & =\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{0} \\
& +\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\left(\frac{1}{k} \Lambda_{\alpha}+A\right)} \sum_{j=1}^{\infty}\left\{P_{-}\left(\mathbf{Y}_{j}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right\} \tag{3.21}
\end{align*}
$$

where $\mathbf{g}_{1}(t)$ is given in (3.11), while $\mathbf{g}_{j}(t)$ for $j \geq 2$ are given in (3.16). The constant vector $\mathbf{Y}_{1}^{0} \in \mathbb{R}^{n}$ is arbitrarily chosen so that $\left\|P_{-} \mathbf{Y}_{1}^{0}\right\|$ is sufficiently small, while $\mathbf{Y}_{j}^{0} \in \mathbb{R}^{n}$ with $j \geq 2$ are chosen so that (3.19) holds. In particular, (3.11) and (3.15) for $j \geq 2$ generate an $m_{A}$-parameter family of formal expansions of $\mathbf{y}(t)$ with fixed $t_{\text {max }}$.

This is a systematic procedure of a candidate of asymptotic expansion of $\mathbf{y}(t)$ in an arbitrary order. On the other hand, the asymptotic relation

$$
\mathbf{y}(t)-\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \sum_{j=0}^{N} \mathbf{Y}_{j}(t) \ll \theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{N}(t) \quad\left(t \rightarrow t_{\max }\right)
$$

has to be independently investigated to claim that (3.21) is indeed an asymptotic expansion of $\mathbf{y}(t)$. In fact, Proposition 3.5 does not refer to the decay order of $\mathbf{Y}_{j}(t)$ so that asymptotic relation $\mathbf{Y}_{j}(t) \ll \mathbf{Y}_{j-1}(t)$ as $t \rightarrow t_{\max }-0$ is satisfied for all $j \in \mathbb{Z}_{\geq 1}$. Nevertheless, we see later that, under mild assumptions, the above solution satisfies (3.7) in an appropriate sense.

On the other hand, there is a nontrivial problem whether the (formal) series $\mathbf{Y}(t)$ is analytically meaningful, that is, convergent. The following theorem shows that the full system (3.4) possesses a global-in-time solution converging to a root $\mathbf{Y}_{0}$ of the balance law (3.6) and locally generates an $m_{A}$-parameter family of such a solution, indicating that the consideration of asymptotic expansion of the form (3.21) as an $m_{A}$-parameter family is consistent.

[^8]Theorem 3.8 (Convergence and parameter dependence of $\mathbf{Y}(t)$ ). For a blow-up solution associating $\mathbf{Y}(t)$, fix $t_{\max }$, assuming finite, and let $\mathbf{Y}_{0}$ be a nonzero root of the balance law (3.6) for (3.4). Assume that the associated blow-up power-determining matrix $A$ is hyperbolic. Then the system (3.4) has a solution $\mathbf{Y}(t)$ converging to $\mathbf{Y}_{0}$ as $t \rightarrow t_{\max }-0$ and generates an $m_{A}$-parameter family of solutions

$$
\mathbf{Y}(t)=\mathbf{Y}\left(t ; t_{\max }, C_{1}, \ldots, C_{m_{A}}\right)
$$

satisfying 3.7), where $m_{A}$ is given in 3.14), and free parameters $C_{1}, \ldots, C_{m_{A}}$ are chosen in a small neighborhood of $\mathbf{0} \in \mathbb{R}^{m_{A}}$. Moreover, the parameter family $\mathbf{Y}\left(t ; t_{\max }, C_{1}, \ldots, C_{m_{A}}\right)$ is $C^{r}$ with respect to $t$ with $t<t_{\max }$ and $C_{1}, \ldots, C_{m_{A}}$.

The right-sided compactification of time variable introduced in [34] is applied to the proof, which is left to Appendix B.3.

Remark 3.9 (Validity of the balance law (3.6)). Using the convergence stated in Theorem 3.8, our choice of terms in (3.3) for deriving the balance law (3.6) is confirmed to be valid. That is, all possible dominant terms in (3.3) as $t \rightarrow t_{\max }$ appears in (3.6). Indeed, the proof of Theorem 3.8, shown in Appendix B.3, implies that $\theta(t) \frac{d \mathbf{Y}}{d t}=O\left(\theta(t)^{\mu}\right)$ for some $\mu>0$, whenever $A$ is hyperbolic.

We can also prove that, for each $N \in \mathbb{N}$, the remainder $\mathbf{Y}_{N}^{c}(t) \equiv \mathbf{Y}(t)-S_{N}(\mathbf{Y})(t)$ also converges as $t \rightarrow t_{\max }$ whenever free parameters are small. The precise statement is below, which is proved in Appendix B. 4

Proposition $\mathbf{3 . 1 0}$ (Convergence of the remainder of $\mathbf{Y}(t)$ ). Under the same assumptions as Theorem [3.8, for each $N \in \mathbb{N}$, assume that vector-valued functions $\left\{\mathbf{Y}_{j}(t)\right\}_{j=0}^{N}$ are constructed as $\mathbf{Y}_{0}$ when $j=0$, (3.13) when $j=1$ and (3.18) for $j \geq 2$.

Then, for each $N \in \mathbb{N}$, the remainder $\mathbf{Y}_{N}^{c}(t) \equiv \mathbf{Y}(t)-S_{N}(\mathbf{Y})(t)$ of the formal asymptotic expansion of $\mathbf{Y}(t)$ converges to $\mathbf{0} \in \mathbb{R}^{n}$ as $t \rightarrow t_{\max }-0$ whenever free parameters $C_{1}, \ldots, C_{m_{A}}$ are small.

### 3.4 Complete determination of asymptotic expansion: a case study

In the above setting, the function $\mathbf{g}_{j}$ contained in the integrand consists of nonlinearities depending on $\mathbf{S}_{j-1}(\mathbf{Y})(t)$. Powers of asymptotic series are then determined by blow-up power eigenvalues $\left\{\lambda_{i}\right\}$, the quasi-homogeneous part $f_{\alpha, k}$ and the lower term $f_{\text {res }}$. In case that $f$ has a special form, the complete form of asymptotic expansions can be determined. For example, assume that the vector field $f$ is quasi-homogeneous polynomial. Then we know that $f_{\text {res }} \equiv 0$ and the equations for all $\mathbf{Y}_{j}$ become much simpler. When we further assume that $A$ is diagonalizable and that $\operatorname{Spec}(A) \subset \mathbb{R}$, 3.11) becomes linear and homogeneous, and hence the solution $\mathbf{Y}_{1}$ is written as the linear combination of $\left\{\theta(t)^{-\lambda_{i}}\right\}_{i=1}^{n}$. In the next step for solving $\mathbf{Y}_{2}$, the powers of $\theta(t)$ appeared in (3.15) with $j=2$ are determined as the following form:

$$
\left(\sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^{n} \\|\beta|=l}} \sum_{i=1}^{n} \beta_{i} \lambda_{i}\right)-1, \quad l=1,2, \ldots
$$

because $f_{\text {res }} \equiv 0$. Through the integration, we see that the powers of $\theta(t)$ in $\mathbf{Y}_{2}$ have the following form:

$$
\begin{equation*}
\sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^{n} \\|\beta|=l}} \sum_{i=1}^{n} \beta_{i} \lambda_{i}, \quad l=1,2, \ldots \tag{3.22}
\end{equation*}
$$

The same conclusion holds for $\mathbf{Y}_{j}$ with $j \geq 3$. We therefore expect that all possible powers of $\theta(t)$ appeared in $\mathbf{Y}(t)$ are restricted to the form (3.22), and hence the true powers of $\theta(t)$ in $y_{i}(t)$ are the sum of the above exponents and the principal exponent $-\alpha_{i} / k$. The similar argument yields the complete form of (formal) asymptotic expansions, even though $A$ admits nontrivial Jordan blocks.

Theorem 3.11. Consider (2.1) with a polynomial vector field $f$ which is quasi-homogeneous. Let $\mathbf{y}(t)=\left(y_{1}(t), \ldots, y_{n}(t)\right)^{T}$ be a blow-up solution for $f$ satisfying Assumption [3.1. Suppose that a nonzero root $\mathbf{Y}_{0}$ of the balance law is associated and that the induced blow-up power-determining matrix $A$ satisfies $\operatorname{Spec}(A) \subset \mathbb{R}$. Then $\mathbf{y}(t)$ has the form

$$
\begin{equation*}
y_{i}(t)=\theta(t)^{-\alpha_{i} / k} \sum_{j=0}^{\infty}\left\{\sum_{M=0}^{M_{j, k, \alpha, A}} \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^{n} \\|\beta|=j}} Y_{\beta, M, i} \theta(t)^{-\beta \cdot A \lambda}(\ln \theta(t))^{M}\right\} \tag{3.23}
\end{equation*}
$$

where $Y_{\beta, M}=\left(Y_{\beta, M, 1}, \ldots, Y_{\beta, M, n}\right)^{T} \in \mathbb{R}^{n}$ and

$$
\beta \cdot{ }_{A} \lambda \equiv \sum_{l=1}^{n} \beta_{l}\left(\lambda_{l}\right)_{-}, \quad(\mu)_{-} \equiv \min \{0, \mu\}
$$

for $\beta \in \mathbb{Z}_{\geq 0}^{n}$ and $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)^{T}$ with $\left\{\lambda_{i}\right\}_{i=1}^{n}=\operatorname{Spec}(A)$ counting the multiplicity. The integer $M_{j, k, \alpha, A}$ determined by $j, k, \alpha$, geometric multiplicity of elements in $\operatorname{Spec}(A)$ and size of associated Jordan blocks.

The proof is left in Appendix B The similar conclusion will be obtained when $f$ is polynomial which is not necessarily quasi-homogeneous. In this case, exponents of $\theta(t)$ and $\ln \theta(t)$ (3.23) depend on $f_{\text {res }}$.

Remark 3.12. When a pair of complex conjugate eigenvalues is contained in $\operatorname{Spec}(A)$, the function $\mathbf{Y}(t)$ includes terms of the following form:

$$
\theta(t)^{\eta} \cos ^{m_{1}}\left(\lambda_{i m} \ln \theta(t)\right) \sin ^{m_{2}}\left(\lambda_{i m} \ln \theta(t)\right)(\ln \theta(t))^{m_{3}}
$$

for some $\eta \in \mathbb{R}$, $\lambda_{\text {im }} \in \mathbb{R} \backslash\{0\}$ and $m_{1}, m_{2}, m_{3} \in \mathbb{Z}_{\geq 0}$. If such a term is included in $\mathbf{Y}_{N}(t)$ for some $N \geq 1$, the next term $\mathbf{Y}_{N+1}(t)$ needs lengthy calculations because the integral of the above term is required.

### 3.5 Justification of asymptotic expansions

We have observed that, in the construction of multi-order asymptotic expansions of blow-ups, free parameters corresponding to blow-up power eigenvalues with negative real parts appear. Next we
consider if the formal asymptotic expansion (3.21) of $\mathbf{y}(t)$ is certainly an asymptotic expansion in the original sense. It should be noted that, in our construction (3.21), the asymptotic relation $\mathbf{Y}_{l}(t) \ll \mathbf{Y}_{l-1}(t)$ in (3.7) is applied as the necessary condition to asymptotic expansions, which is not always sufficient to prove the requirement of asymptotic expansions

$$
\begin{equation*}
\frac{\theta(t)^{\alpha_{i} / k} y_{i}(t)-\sum_{j=0}^{N} Y_{j, i}(t)}{Y_{N, i}(t)}=\frac{\sum_{j=N+1}^{\infty} Y_{j, i}(t)}{Y_{N, i}(t)}=o(1) \quad \text { as } t \rightarrow t_{\max } \tag{3.24}
\end{equation*}
$$

for all $N \in \mathbb{N}$ and each $i=1, \ldots, n$, while the residual series $\sum_{j=N+1}^{\infty} Y_{j, i}(t)$ converges to 0 as $t \rightarrow t_{\text {max }}$ from Proposition 3.10 and the uniqueness of solutions. To verify the asymptotic relation (3.24) for (3.21), it requires concrete forms of $f$ as well as the blow-up power-determining matrix $A$ and a nonsingular matrix $P$ inducing the Jordan canonical form of $A$. Indeed, there can be a case that $Y_{N, i}(t)$ goes to zero much faster than $Y_{N+1, i}(t)$ as $t \rightarrow t_{\max }$ for some $N$, due to the situation that coefficients of low powers of $\theta(t)$ accidentally become zer 11 . Nevertheless, the lowest orders of $\theta(t)$ in the sequence $\left\{\mathbf{Y}_{j}(t)\right\}_{j=0}^{\infty}$ are expected to increase monotonously under a mild assumption, and hence the asymptotic series in the sense of (3.24) can be constructed by arranging several terms so that an asymptotic series in the original sense is obtained. In what follows, we prove that this presumption is true, at least, under several restrictions of $f_{\text {res }}$ and its derivatives. The degree function $\operatorname{deg}_{\theta}$ introduced in Section 2.2 plays a key role in estimating the degree of $\mathbf{Y}_{j}(t)$ for each $j \in \mathbb{Z}_{\geq 0}$ a priori, which yields the justification of our asymptotic expansions.

Proposition 3.13. Let (3.21) be the formal asymptotic expansion of $\mathbf{y}(t)$ obtained in Theorem 3.8 for an asymptotically quasi-homogeneous vector field $f$ of type $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and order $k+1$, and $\mathbf{Y}_{0}$ be the corresponding non-zero root of the balance law. For $i=1, \ldots, n$, define $\gamma_{i} \in \mathbb{R}$ by

$$
\begin{equation*}
\gamma_{i}:=\inf _{\tilde{\mathbf{x}} \in \mathbb{R}^{n}} \operatorname{deg}_{\theta}\left(f_{i ; \text { res }}\left(\theta(\cdot)^{-\frac{1}{k} \Lambda_{\alpha}} \tilde{\mathbf{x}}\right)\right) \tag{3.25}
\end{equation*}
$$

Assume that

- for $i=1, \ldots, n$,

$$
\begin{equation*}
\gamma_{i} \leq \inf _{\tilde{\mathbf{x}} \in \mathbb{R}^{n}} \inf _{\tilde{\mathbf{v}} \in \mathbb{R}^{n}} \operatorname{deg}_{\theta}\left(\left[D f_{\text {res }}\left(\theta(\cdot)^{-\frac{1}{k} \Lambda_{\alpha}} \tilde{\mathbf{x}}\right) \theta(\cdot)^{-\frac{1}{k} \Lambda_{\alpha}} \tilde{\mathbf{v}}\right]_{i}\right) \tag{3.26}
\end{equation*}
$$

- the following inequality holds:

$$
\frac{\alpha_{i}}{k}+\gamma_{i}>-1 \quad \text { for all } \quad i=1, \ldots, n
$$

Then we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\mathbf{Y}_{N}\right) \geq N \delta \quad \text { for all } \quad N \in \mathbb{Z}_{\geq 0} \tag{3.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta:=\min \left\{\min _{l=1, \ldots, n}\left\{\frac{\alpha_{l}}{k}+\gamma_{l}\right\}+1, \min _{\substack{i=1, \ldots, n \\ \operatorname{Re} \lambda_{i}<0}}\left(-\operatorname{Re} \lambda_{i}\right)\right\}>0 \tag{3.28}
\end{equation*}
$$

[^9]The proof is left to Appendix B
Remark 3.14. The asymptotic quasi-homogeneity of $f$ implies

$$
\gamma_{i} \geq-\frac{k+\alpha_{i}}{k}
$$

and hence the quantity $\delta$ in (3.28) is always nonnegative. In particular, $\delta$ becomes positive under our assumption. When $f$ is polynomial, we have the following sharper estimate:

$$
\gamma_{i} \geq-\frac{k+\alpha_{i}-1}{k}
$$

indicating $\delta \geq 1 / k$ if $\operatorname{Spec}(A) \subset\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq 0\}$. Note that the statement itself also holds for the case that each $\alpha_{i}$ is nonnegative real numbers (namely, not always integers) except $\alpha_{i} \equiv 0$ for all $i$, by generalizing the quasi-homogeneity of $f$ in Definition 2.2 to the $n$-tuple $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in$ $\left(\mathbb{R}_{\geq 0}\right)^{n} \backslash\{\mathbf{0}\}$ as the type of $f$.

The proposition shows that the magnitude of $\theta(t)$ in the series $\left\{\mathbf{Y}_{j}(t)\right\}_{j=0}^{\infty}$ eventually increases in the sense of (3.27). Therefore, arranging $\left\{\mathbf{Y}_{j}(t)\right\}_{j=0}^{\infty}$ componentwise, we can show that, for any finite sequence $\left\{\mathbf{Y}_{j}(t)\right\}_{j=0}^{N}$, the arranged sequence includes the finite order asymptotic expansion of $\mathbf{Y}(t)$. The precise statement is shown below.

Theorem 3.15 (Justification of asymptotic expansion). Consider an asymptotically quasi-homogeneous vector field $f$ of type $\alpha$ and order $k+1$. Let (3.21) be the formal asymptotic expansion of $\mathbf{y}(t)$ obtained in Theorem 3.8. Suppose that all assumptions in Proposition 3.13 are satisfied. Then, for any $N \in \mathbb{N}$, there exist a natural number $N_{1}=N_{1}(N)$ satisfying

$$
\lim _{N \rightarrow \infty} N_{1}(N)=\infty
$$

and an arranged sequence $\left\{\overline{\mathbf{Y}}_{j}(t)\right\}_{j=1}^{N}$ of the finite sequence $\left\{\mathbf{Y}_{j}\right\}_{j=1}^{N}$ obtained by the permutation of indices $\{1, \ldots, N\}$ such that the finite sum

$$
\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \sum_{j=1}^{N_{1}} \overline{\mathbf{Y}}_{j}(t)
$$

is the $N_{1}$-th order asymptotic expansion of $\mathbf{y}(t)$ as $t \rightarrow t_{\max }$ in the sense that 3.24) holds with $N=N_{1}$.

Proof. Let $N_{1}:=\min \{\lceil N \delta\rceil, N\}$, where $\delta>0$ is the number given in (3.28). Obviously $N_{1} \rightarrow \infty$ holds as $N \rightarrow \infty$. Proposition 3.13 indicates that all terms $\mathbf{Y}_{j}$ with $\operatorname{deg}_{\theta}\left(\mathbf{Y}_{j}\right) \leq N_{1}$ must be included in the sequence $\left\{\mathbf{Y}_{j}\right\}_{j=1}^{N}$. The sequence $\left\{\mathbf{Y}_{j}\right\}_{j=1}^{N}$ can be arranged componentwise to $\left\{\overline{\mathbf{Y}}_{j}\right\}_{j=1}^{N}$ so that $\operatorname{deg}_{\theta}\left(\overline{\mathbf{Y}}_{j}\right)$ is monotonously increasing in $j$. It follows by construction that

$$
\lim _{t \rightarrow t_{\max }} \frac{\bar{Y}_{j+1, i}(t)}{\bar{Y}_{j, i}(t)}=0, \quad j=0, \cdots, N_{1}-1
$$

where the case $j=0$ is excluded when $Y_{j, i}(t)=\bar{Y}_{j, i}(t)=0$. Proposition 3.13 also indicates that $\operatorname{deg}_{\theta}\left(\mathbf{Y}_{j}\right) \geq N_{1}$ for $j \geq N+1$. Moreover, Proposition 3.10 shows $\sum_{j=N+1}^{\infty} \mathbf{Y}_{j}(t) \rightarrow 0$ as $t \rightarrow t_{\max }$. In particular,

$$
\lim _{t \rightarrow t_{\max }} \frac{\sum_{j=N+1}^{\infty} Y_{j, i}(t)}{\bar{Y}_{N_{1}, i}(t)}=0
$$

holds and consequently we have

$$
\lim _{t \rightarrow t_{\max }} \frac{\sum_{j=N_{1}+1}^{N} \bar{Y}_{j, i}(t)+\sum_{j=N+1}^{\infty} Y_{j, i}(t)}{\bar{Y}_{N_{1}, i}(t)}=0
$$

holds for each $i=1, \cdots, n$.
Consequently, our formal asymptotic expansion (3.21) provides the true asymptotic expansion of the blow-up solution $\mathbf{y}(t)$ in the sense of Theorem 3.15 under mild assumptions to the asymptotic degree of the residual term $f_{\text {res }}$.

## 4 Examples

Examples of asymptotic expansions of blow-up solutions are collected. In some examples, correspondence of algebraic information for describing asymptotic expansions to dynamics at infinity is also revealed.

### 4.1 One-dimensional ODEs

First we demonstrate our methodology to one-dimensional ODEs to see effectiveness of the methodology and interpretations of results.

### 4.1.1 A simple example

The first example is

$$
\begin{equation*}
y^{\prime}=-y+y^{3}, \quad,=\frac{d}{d t} \tag{4.1}
\end{equation*}
$$

If the initial point $y(0)>0$ is sufficiently large, the corresponding solution would blow up in a finite time. To describe a blow-up solution precisely, we apply the (homogeneous) parabolic compactification and the time-scale desingularization to (4.1). First note that the ODE (4.1) is asymptotically homogeneous (namely $\alpha=(1)$ ) of order $k+1=3$, in particular $k=2$.

Our concern here is the asymptotic behavior of blow-up solutions of the following form:

$$
y(t)=\theta(t)^{-1 / 2} Y(t)
$$

which yields the following equation solving $Y(t)$ :

$$
\begin{equation*}
Y^{\prime}=-Y+\theta(t)^{-1}\left\{-\frac{1}{2} Y+Y^{3}\right\} \tag{4.2}
\end{equation*}
$$

Note that the existence of such blow-up solutions is discussed in [23]. Under the asymptotic expansion of the positive blow-up solution:

$$
Y(t)=\sum_{n=0}^{\infty} Y_{n}(t) \quad \text { with } \quad \lim _{t \rightarrow t_{\max }} Y(t)=Y_{0}>0
$$

the balance law requires $Y_{0}=1 / \sqrt{2}$, which is the coefficient of the principal term of $y(t)$. The blow-up power-determining matrix at $Y_{0}$, coinciding with the blow-up power eigenvalue, is

$$
\left\{-\frac{1}{2}+3 Y^{2}\right\}_{Y=Y_{0}}=1
$$

This eigenvalue has no contributions to $Y_{n}(t)$ with $n \geq 1$. Next we shall calculate the second term $Y_{1}(t)$. Using the Taylor expansion at $Y_{0}$, the system (4.2) is written by

$$
\sum_{n=1}^{\infty} Y_{n}^{\prime}=-\sum_{n=0}^{\infty} Y_{n}+\theta(t)^{-1} \sum_{m=1}^{\infty} \frac{1}{m!}\left\{\frac{d^{m}}{d Y^{m}}\left(-\frac{1}{2} Y+Y^{3}\right)\right\}_{Y=Y_{0}}\left(\sum_{n=1}^{\infty} Y_{n}\right)^{m}
$$

where the balance law is applied to cancel the principal terms. The governing equation for $Y_{1}$ near $t=t_{\text {max }}$ is then

$$
Y_{1}^{\prime}=-Y_{0}+\theta(t)^{-1} Y_{1}=-\frac{1}{\sqrt{2}}+\theta(t)^{-1} Y_{1}
$$

which is solved to obtain a bounded solution by (3.13):

$$
Y_{1}(t)=\theta(t)^{-1}\left[-\int_{t}^{t_{\max }} \theta(s)\left(-\frac{1}{\sqrt{2}}\right) d s\right]=\frac{1}{2 \sqrt{2}} \theta(t)
$$

This is consistent with the asymptotic relation $Y_{1}(t) \ll Y_{0}$ as $t \rightarrow t_{\max }$. The third term $Y_{2}(t)$ is also calculated. The governing equation for $Y_{2}$ obtained by the similar way to $Y_{1}$ is

$$
\begin{align*}
Y_{2}^{\prime} & =-Y_{1}+\theta(t)^{-1}\left\{Y_{2}+\frac{3 \sqrt{2}}{2} Y_{1}^{2}+Y_{1}^{3}\right\} \\
& =-\frac{1}{2 \sqrt{2}} \theta(t)+\theta(t)^{-1} Y_{2}+\frac{3 \sqrt{2}}{16} \theta(t)+\frac{1}{16 \sqrt{2}} \theta(t)^{2} \tag{4.3}
\end{align*}
$$

where we have used the fact that

$$
\frac{d^{m}}{d Y^{m}}\left(-\frac{1}{2} Y+Y^{3}\right) \equiv 0 \quad \text { for } \quad m \geq 4
$$

and all terms involving $Y_{1}^{\prime}$ are cancelled. Standard theory of ODEs yields that the general solution of the homogeneous equation $Y_{2}^{\prime}=\theta(t)^{-1} Y_{2}$ is $c_{2} \theta(t)^{-1}$ with a constant $c_{2}$, which is already appeared in calculations of $Y_{1}$. The bounded solution of (4.3) is then obtained by (3.18) with $j=2$ as follows:

$$
\begin{equation*}
Y_{2}(t)=\theta(t)^{-1}\left[-\int_{t}^{t_{\max }} \theta(s)\left\{-\frac{\sqrt{2}}{16} \theta(s)+\frac{1}{16 \sqrt{2}} \theta(s)^{2}\right\} d s\right]=\frac{\sqrt{2}}{48} \theta(t)^{2}-\frac{1}{64 \sqrt{2}} \theta(t)^{3} \tag{4.4}
\end{equation*}
$$

which is consistent with the asymptotic assumption $Y_{2}(t) \ll Y_{1}(t)$ as $t \rightarrow t_{\text {max }}$. Remark that the above form does not directly determine the corresponding term in $Y(t)$. In particular, coefficients of $\theta(t)^{m}$ with $m \geq 3$ can change depending on $Y_{n}(t), n \geq 3$. On the other hand, the similar calculations yield that there is no terms of $\theta(t)^{2}$ in $Y_{n}(t), n \geq 3$, by the asymptotic assumption.

Indeed, the constant $\delta$ defined in (3.28) is estimated as follows. First, $\alpha=1$ and $k=2$, and hence $\gamma_{1}=-1 / 2$ in (3.25). Because there are no negative blow-up power eigenvalues, $\delta$ is evaluated as

$$
\delta=\frac{1}{2}+1-\frac{1}{2}=1
$$

Proposition 3.13 indicates that $\operatorname{ord}_{\theta}\left(Y_{3}\right) \geq 3$ and hence there is no term $\theta(t)^{m}$ with $m<3$ in $Y_{n}(t)$, $n \geq 3$. In other words, the coefficient of $\theta(t)^{2}$ is determined as $\sqrt{2} / 48$. Summarizing the above arguments, we have the following result.
Theorem 4.1. The system 4.1) admits a blow-up solution with the following third order asymptotic expansion as $t \rightarrow t_{\max }$ :

$$
y(t) \sim \frac{1}{\sqrt{2}} \theta(t)^{-1 / 2}+\frac{1}{2 \sqrt{2}} \theta(t)^{1 / 2}+\frac{\sqrt{2}}{48} \theta(t)^{3 / 2} \quad \text { as } \quad t \rightarrow t_{\max }
$$

The higher-order asymptotic expansion of $y(t)$ is also obtained by calculating $Y_{n}(t), n \geq 3$, in the similar way to the above arguments. In the above expression, the parameter dependence of solutions appears in $t_{\max }$, which admits a various choice of initial points inducing blow-up solutions.

Remark 4.2 (Analyticity of $Y(t)$ at $t=t_{\max }$ ). The solution $Y(t)$ of (4.2) is actually real-analytic at $t=t_{\text {max }}$. Indeed, (4.1) is formally transformed into

$$
2 y^{-3} y^{\prime}=2\left(-y^{-2}+1\right)
$$

Using the identity

$$
\left(-1+y^{-2}\right)^{\prime}=\left(y^{-2}\right)^{\prime}=-2 y^{-3} y^{\prime}
$$

the above equation becomes

$$
\left(1-y^{-2}\right)^{\prime}=2\left(1-y^{-2}\right)
$$

The standard method of variable separation is thus applied to obtain

$$
y(t)=\left(1-\left(1-\frac{1}{y_{0}^{2}}\right) e^{2 t}\right)^{-1 / 2}
$$

for blow-up solutions with $y(0)=y_{0}>1$. Letting

$$
1-\frac{1}{y_{0}^{2}}=e^{-2 t_{\max }}
$$

the above solution is rewritten by

$$
\begin{equation*}
y(t)=\left(1-e^{-2 \theta(t)}\right)^{-1 / 2} \tag{4.5}
\end{equation*}
$$

The above function is also written by $y(t)=(2 \theta(t) h(t))^{-1 / 2}$, where

$$
h(t) \equiv \frac{1-e^{-2 \theta(t)}}{2 \theta(t)}
$$

is real-analytic at $t=t_{\max }$ with $\lim _{t \rightarrow t_{\max }} h(t)=1$. It is concluded that $y(t)=\theta(t)^{-1 / 2} Y(t)$, where $Y(t)=\frac{1}{\sqrt{2}} h(t)^{-1 / 2}$ is real analytic at $t=t_{\text {max }}$.

Remark 4.3. The function $h(t)$ appeared in Remark 4.2 is expressed as the generating function of the Bernoulli numbers (e.g. [3]). Indeed, the generating function is

$$
B(t)=\frac{t}{1-e^{-t}}=\sum_{l \geq 0} \frac{B_{l}}{l!} t^{l}
$$

where $\left\{B_{l}\right\}_{l \geq 0}$ are the Bernoulli numbers. Using the function $B(t)$, the exact solution $y(t)$ in 4.5) is rewritten as

$$
\begin{aligned}
y(t) & =\frac{1}{\sqrt{2}} \theta(t)^{-1 / 2} B(2 \theta(t))^{1 / 2}=\frac{1}{\sqrt{2}} \theta(t)^{-1 / 2}\{1+(B(2 \theta(t))-1)\}^{1 / 2} \\
& =\frac{1}{\sqrt{2}} \theta(t)^{-1 / 2} \sum_{l \geq 0}\binom{1 / 2}{l}(B(2 \theta(t))-1)^{l}
\end{aligned}
$$

From the binomial expansion and

$$
(B(2 \theta(t))-1)^{m}=\left(\sum_{l \geq 1} \frac{B_{l}}{l!}(2 \theta(t))^{l}\right)^{m}
$$

we have the expression

$$
y(t)=\frac{1}{\sqrt{2}} \sum_{n \geq 0} a_{n} \theta(t)^{-\frac{1}{2}+n}
$$

where $a_{0}=1$ and

$$
\begin{aligned}
a_{n} & =\sum_{l=1}^{n} \sum_{\substack{I=\left(i_{1}, \ldots, i_{n}\right) \\
|I|=l, w(I)=n}}\binom{1 / 2}{l} \frac{1}{l!} \frac{2^{i_{1}+2 i_{2}+\cdots+n i_{n}}}{(1!)^{i_{1}}(2!)^{i_{2}} \cdots(n!)^{i_{n}}} \frac{l!}{i_{1}!\cdots i_{n}!} B_{1}^{i_{1}} \cdots B_{n}^{i_{n}} \\
& =\sum_{l=1}^{n} \sum_{\substack{I=\left(i_{1}, \ldots, i_{n}\right) \\
|I|=l, w(I)=n}}\binom{1 / 2}{l} \frac{1}{l!} \frac{2^{n}}{(1!)^{i_{1}}(2!)^{i_{2}} \cdots(n!)^{i_{n}}} \frac{l!}{i_{1}!\cdots i_{n}!} B_{1}^{i_{1}} \cdots B_{n}^{i_{n}}
\end{aligned}
$$

where the multi-index $I=\left(i_{1}, \ldots, i_{n}\right)$ in the above sum runs over $\mathbb{Z}_{\geq 0}^{n}$ satisfying

$$
|I| \equiv i_{1}+\cdots+i_{n}=l, \quad w(I) \equiv i_{1}+2 i_{2}+\cdots+n i_{n}=n
$$

The first three coefficients are

$$
\begin{aligned}
& a_{0}=1, \quad a_{1}=B_{1}=\frac{1}{2} \\
& a_{2}=\frac{1}{2} \frac{2^{2}}{2!} B_{2}+\frac{1}{2}\left(-\frac{1}{2}\right) \frac{2^{2}}{2!} B_{1}^{2}=B_{2}-\frac{1}{2} B_{1}^{2}=\frac{1}{24}
\end{aligned}
$$

which yield the coincidence of coefficients with $y(t)$ which we have obtained in Theorem 4.1. The above observation shows the validity of our expansion methodology. On the other hand, the coefficient $a_{3}$ is calculated as

$$
\begin{aligned}
a_{3} & =\binom{1 / 2}{1} \frac{2^{3}}{(3!)^{3}} \frac{1!}{1!} B_{3}^{3}+\binom{1 / 2}{2} \frac{2^{3}}{(1!)^{1}(2!)^{1}} \frac{2!}{1!1!} B_{1}^{1} B_{2}^{1}+\binom{1 / 2}{3} \frac{2^{3}}{(1!)^{3}} \frac{3!}{3!} B_{1}^{3} \\
& =0-B_{1} B_{2}+\frac{1}{2} B_{1}^{3}=-\frac{1}{48}
\end{aligned}
$$

which indicates that we require calculating $Y_{3}(t)$ to obtain the correct coefficient of $\theta(t)^{3}$, which is different from that obtained in 4.4.

### 4.1.2 Ishiwata-Yazaki's example

The next example concerns with blow-up solutions of the following system:

$$
\begin{equation*}
u^{\prime}=a u^{\frac{a+1}{a}} v, \quad v^{\prime}=a v^{\frac{a+1}{a}} u \tag{4.6}
\end{equation*}
$$

where $a \in(0,1)$ is a parameter. In (4.6), the following results are obtained in preceding works, which are originally obtained in [20] and revisited in [29].

Remark 4.4 (cf. [20, 29]). Consider initial points $u(0), v(0)>0$. If $u(0) \neq v(0)$, then the solution $(u(t), v(t))$ blows up at $t=t_{\max }<\infty$ with the blow-up rate $O\left(\theta(t)^{-a}\right)$. On the other hand, if $u(0)=v(0)$, the solution $(u(t), v(t))$ blows up at $t=t_{\max }<\infty$ with the blow-up rate $O\left(\theta(t)^{-a /(a+1)}\right)$.

The above blow-up mechanism based on dynamics at infinity is discussed in 29. The main interest here is the multi-order asymptotic expansion of blow-up solutions for (4.6) mentioned in Remark 4.4.

First note that the system (4.6) has the first integral

$$
I=I(u, v):=v^{1-\frac{1}{a}}-u^{1-\frac{1}{a}}
$$

In other words, the time-differential of the functional $I$ along solutions of (4.6) is always zero, equivalently the level set $\{I(t)=C\}$ is invariant for (4.6). Indeed,

$$
\begin{aligned}
\frac{d}{d t} I(t) & =\left(1-\frac{1}{a}\right)\left\{v^{-1 / a} v^{\prime}-u^{-1 / a} u^{\prime}\right\} \\
& =(a-1)\left\{v^{-1 / a} v^{\frac{a+1}{a}} u-u^{-1 / a} u^{\frac{a+1}{a}} v\right\} \\
& =(a-1)\{v u-u v\}=0
\end{aligned}
$$

Using this functional,

$$
v=\left(I+u^{\frac{a-1}{a}}\right)^{\frac{a}{a-1}}
$$

and the system (4.6) is then reduced to a one-dimensional ODE

$$
\begin{equation*}
u^{\prime}=a u^{\frac{a+1}{a}}\left(u^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}} \tag{4.7}
\end{equation*}
$$

Blow-up solutions of the rate $O\left(\theta(t)^{-a}\right)$ correspond to $I \neq 0$, while those of the rate $O\left(\theta(t)^{-a /(a+1)}\right)$ correspond to $I=0$. We pay attention to the case $u(0)>v(0)$ when $I \neq 0$, in which case $I>0$ holds 12 .

[^10]First consider the case $I>0$, where the vector field (4.7) is asymptotically homogeneous of the order $1+a^{-1}$. Using the asymptotic expansion

$$
u(t)=\theta(t)^{-a} U(t)=\theta(t)^{-a}\left(\sum_{n=0}^{\infty} U_{n}(t)\right), \quad \lim _{t \rightarrow t_{\max }} U(t)=U_{0}
$$

the system becomes

$$
\begin{align*}
U^{\prime} & =a \theta(t)^{-1}\left\{-U+U^{\frac{a+1}{a}}\left(\theta(t)^{-(a-1)} U^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}}\right\} \\
& =a \theta(t)^{-1}\left\{-U+I^{\frac{a}{a-1}} U^{\frac{a+1}{a}}\left(I^{-1} \theta(t)^{-(a-1)} U^{1-\frac{1}{a}}+1\right)^{\frac{a}{a-1}}\right\} \tag{4.8}
\end{align*}
$$

The principal component in the above vector field is extracted as follows. Let

$$
f(t ; U):=\left(I^{-1} \theta(t)^{1-a} U^{1-\frac{1}{a}}+1\right)^{\frac{a}{a-1}}
$$

following (4.7). Because $\theta(t)^{1-a} U^{1-\frac{1}{a}} \ll I$ as $t \rightarrow t_{\max }$ holds by $a<1, I>0$ fixed and $U \rightarrow U_{0}$ as $t \rightarrow t_{\max }$, then the nonlinear term $f(t ; U)$ itself converges to 1 as $t \rightarrow t_{\max }$. We therefore know that the principal component of the vector field (4.8) is $a \theta(t)^{-1}\left\{-U+I^{\frac{a}{a-1}} U^{\frac{a+1}{a}}\right\}$, provided $I>0$. Using the binomial series,

$$
\begin{equation*}
\left(I^{-1} \theta(t)^{1-a} U^{1-\frac{1}{a}}+1\right)^{\frac{a}{a-1}}=\sum_{k=0}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} \theta(t)^{1-a} U^{\frac{a-1}{a}}\right)^{k} \tag{4.9}
\end{equation*}
$$

where

$$
\binom{\frac{a}{a-1}}{k}=\frac{\left(\frac{a}{a-1}\right)_{k}}{k!}, \quad\left(\frac{a}{a-1}\right)_{k}=\frac{a}{a-1}\left(\frac{a}{a-1}-1\right)\left(\frac{a}{a-1}-2\right) \cdots\left(\frac{a}{a-1}-k+1\right) .
$$

The latter is the well-known Pochhammer symbol. The balance law then yields

$$
-U_{0}+I^{\frac{a}{a-1}} U_{0}^{\frac{a+1}{a}}=0 \quad \Rightarrow \quad U_{0}=I^{-a^{2} /(a-1)}
$$

The corresponding blow-up power-determining matrix is

$$
\frac{d}{d U}\left(a\left\{-U+U^{\frac{a+1}{a}} I^{\frac{a}{a-1}}\right\}\right)_{U=U_{0}}=a\left\{-1+\frac{a+1}{a} U_{0}^{\frac{1}{a}} I^{\frac{a}{a-1}}\right\}=1
$$

With the binomial series (4.9), the governing equation for the second term $U_{1}(t)$ is

$$
\begin{aligned}
U_{1}^{\prime} & =\theta(t)^{-1} U_{1}+a \theta(t)^{-1} U_{0}^{\frac{a+1}{a}} I^{\frac{a}{a-1}} \sum_{k=1}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} \theta(t)^{1-a} U_{0}^{\frac{a-1}{a}}\right)^{k} \\
& =\theta(t)^{-1} U_{1}+a \theta(t)^{-1} I^{-\frac{a^{2}}{a-1}} \sum_{k=1}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} \theta(t)^{1-a} U_{0}^{\frac{a-1}{a}}\right)^{k} .
\end{aligned}
$$

The bounded solution is

$$
\begin{aligned}
U_{1}(t) & =\theta(t)^{-1}\left[-a \int_{t}^{t_{\max }} I^{-\frac{a^{2}}{a-1}} \sum_{k=1}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} \theta(s)^{1-a} U_{0}^{1-\frac{1}{a}}\right)^{k} d s\right] \\
& =\theta(t)^{-1}\left[-a I^{-\frac{a^{2}}{a-1}} \sum_{k=1}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} U_{0}^{1-\frac{1}{a}}\right)^{k} \frac{\theta(t)^{k(1-a)+1}}{k(1-a)+1}\right]
\end{aligned}
$$

Note that all series appeared in the above equalities have positive convergence radii. Therefore

$$
U_{1}(t)=I^{\frac{-2 a^{2}+1}{a-1}} \frac{a^{2}}{(1-a)(2-a)} \theta(t)^{1-a}-a I^{-\frac{a^{2}}{a-1}} \sum_{k=2}^{\infty}\binom{\frac{a}{a-1}}{k}\left(I^{-1} U_{0}^{1-\frac{1}{a}}\right)^{k} \frac{\theta(t)^{k(1-a)}}{k(1-a)+1} .
$$

As in the case of (4.1), coefficients of $\theta(t)^{k(1-a)}$ also depend on $U_{n}(t)$ with $n \geq 2$, while the coefficient of $\theta(t)^{1-a}$ is determined here from the asymptotic assumption. Indeed, the constant $\delta$ defined in (3.28) is estimated as follows. First, $\alpha=1$ and $k=1 / a$. Next, we shall extract the residual term $f_{\text {res }}$ so that $\gamma_{1}$ in (3.25) is evaluated. Note that $k$ is not always an integer, but the concept of quasi-homogeneity is generalized to any positive numbers. In (4.7), the vector field is written as

$$
f(u) \equiv a u^{\frac{a+1}{a}}\left(u^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}}=a I^{\frac{a}{a-1}} u^{\frac{a+1}{a}}+a\left(u^{\frac{a+1}{a}}\left(u^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}}-I^{\frac{a}{a-1}} u^{\frac{a+1}{a}}\right)
$$

and hence the residual term $f_{\text {res }}(u)$ is

$$
f_{\mathrm{res}}(u)=a\left(u^{\frac{a+1}{a}}\left(u^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}}-I^{\frac{a}{a-1}} u^{\frac{a+1}{a}}\right)
$$

Therefore, using $\alpha / k=a$, we have

$$
f_{\mathrm{res}}\left(\theta(t)^{-a} u\right)=a\left(\theta(t)^{-(a+1)} u^{\frac{a+1}{a}}\left(\theta(t)^{-a+1} u^{1-\frac{1}{a}}+I\right)^{\frac{a}{a-1}}-I^{\frac{a}{a-1}} \theta(t)^{-(a+1)} u^{\frac{a+1}{a}}\right)=O\left(\theta(t)^{-2 a}\right)
$$

as $t \rightarrow t_{\max }$, and hence $\gamma_{1}=-2 a$. Because there are no negative blow-up power eigenvalues, $\delta$ is evaluated as

$$
\delta=\frac{\alpha}{k}+1+\gamma_{1}=a+1-2 a=1-a
$$

Proposition 3.13 (with the assertion right after the proposition) indicates that $\operatorname{ord}_{\theta}\left(U_{m}\right) \geq m(1-a)$ for $m \geq 2$ and hence there is no term $\theta(t)^{1-a}$ in $U_{n}(t), n \geq 2$. As a summary, we have the second order asymptotic expansion

$$
U(t) \sim I^{-a^{2} /(a-1)}+I^{\frac{-2 a^{2}+1}{a-1}} \frac{a^{2}}{(1-a)(2-a)} \theta(t)^{1-a}
$$

as $t \rightarrow t_{\max }$. Back to the original coordinate, we have the second order asymptotic expansion of blow-up solution

$$
\begin{aligned}
u(t) & \sim I^{-a^{2} /(a-1)} \theta(t)^{-a}+I^{\frac{-2 a^{2}+1}{a-1}} \frac{a^{2}}{(1-a)(2-a)} \theta(t)^{1-2 a} \\
v(t) & \equiv\left(u(t)^{(a-1) / a}+I\right)^{a /(a-1)}=I^{a /(a-1)}\left(I^{-1} u(t)^{(a-1) / a}+1\right)^{a /(a-1)} \\
& \sim I^{\frac{a}{a-1}}-I^{\frac{1}{a-1}-a} \frac{a}{1-a} \theta(t)^{1-a}
\end{aligned}
$$

as $t \rightarrow t_{\text {max }}$. The higher-order asymptotic expansion is derived in the similar way.
Our asymptotic expansion has two parameters: $t_{\max }$ and $I \equiv v(0)^{1-\frac{1}{a}}-u(0)^{1-\frac{1}{a}}$, whose dynamical interpretation is mentioned in Part II [23].

Next consider the case $I=0$ :

$$
\begin{equation*}
u^{\prime}=a u^{2+\frac{1}{a}}, \tag{4.10}
\end{equation*}
$$

which is homogeneous of the order $2+a^{-1}$. Note that the order is different from that for $I>0$. Using the asymptotic expansion

$$
u(t)=\theta(t)^{-a /(a+1)} U(t)=\theta(t)^{-a /(a+1)}\left(\sum_{n=0}^{\infty} U_{n}(t)\right), \quad \lim _{t \rightarrow t_{\max }} U(t)=U_{0}
$$

the system becomes

$$
\begin{equation*}
U^{\prime}=a \theta(t)^{-1}\left(-\frac{1}{a+1} U+U^{2+\frac{1}{a}}\right) \tag{4.11}
\end{equation*}
$$

The balance law yields

$$
U_{0}=\left(\frac{1}{a+1}\right)^{\frac{a}{a+1}}
$$

Letting

$$
f_{0}(U):=a\left(-\frac{1}{a+1} U+U^{2+\frac{1}{a}}\right)
$$

the equation (4.11) is rewritten by

$$
U^{\prime}=\theta(t)^{-1}\left\{\left(U-U_{0}\right)+\sum_{m=2}^{\infty} \frac{1}{m!}\left(\frac{d^{m}}{d U^{m}} U^{2+\frac{1}{a}}\right)_{U=U_{0}}\left(U-U_{0}\right)^{m}\right\}
$$

by using the Taylor expansion of $f_{0}$ at $U_{0}$. The governing equation of the second term $U_{1}$ is thus

$$
U_{1}^{\prime}=\theta(t)^{-1} U_{1}
$$

whose general solution is $U_{1}(t)=c_{1} \theta(t)^{-1}$. In the above equation we have used the fact that $\frac{d f_{0}}{d U}\left(U_{0}\right)=1$. The asymptotic assumption requires $c_{1}=0$, and hence $U_{1}(t) \equiv 0$. The same arguments yield $U_{n}(t) \equiv 0$ for all $n \geq 1$. Consequently, the asymptotic expansion of the blow-up solution is

$$
\begin{equation*}
u(t) \sim\left(\frac{1}{a+1}\right)^{\frac{a}{a+1}} \theta(t)^{-a /(a+1)} \tag{4.12}
\end{equation*}
$$

which is in fact the exact solution of (4.10). As a summary, we obtain the following result for asymptotic expansions of blow-up solutions.
Theorem 4.5. The system 4.6) with fixed $a \in(0,1)$ admits blow-up solutions with the following second-order asymptotic expansions as $t \rightarrow t_{\max }$ :

$$
u(t) \sim I^{-a^{2} /(a-1)} \theta(t)^{-a}+I^{\frac{-2 a^{2}+1}{a-1}} \frac{a^{2}}{(1-a)(2-a)} \theta(t)^{1-2 a}, \quad v(t) \sim I^{\frac{a}{a-1}}-I^{\frac{1}{a-1}-a} \frac{a}{1-a} \theta(t)^{1-a}
$$

with $I>0$, where $I$ is a free parameter. When $I=0$, 4.6) admits a blow-up solution

$$
u(t)=v(t)=\left(\frac{1}{a+1}\right)^{\frac{a}{a+1}} \theta(t)^{-a /(a+1)}
$$

Note that the solution through the above argument coincides with that obtained by the method of separation of variables in (4.10).

### 4.2 Two-phase flow model

We move to multi-dimensional examples. In this example, the following system is concerned (see e.g. 21, 28] for the details of the system):

$$
\left\{\begin{array}{l}
\beta^{\prime}=v B_{1}(\beta)-c \beta-c_{1},  \tag{4.13}\\
v^{\prime}=v^{2} B_{2}(\beta)-c v-c_{2},
\end{array} \quad,=\frac{d}{d t},\right.
$$

where

$$
B_{1}(\beta)=\frac{\left(\beta-\rho_{1}\right)\left(\beta-\rho_{2}\right)}{\beta}, \quad B_{2}(\beta)=\frac{\beta^{2}-\rho_{1} \rho_{2}}{2 \beta^{2}}
$$

with $\rho_{2}>\rho_{1}>0$,

$$
c=\frac{v_{R} B_{1}\left(\beta_{R}\right)-v_{L} B_{1}\left(\beta_{L}\right)}{\beta_{R}-\beta_{L}}
$$

and $\left(c_{1}, c_{2}\right)=\left(c_{1 L}, c_{2 L}\right)$ or $\left(c_{1 R}, c_{2 R}\right)$, where

$$
\left\{\begin{array} { l } 
{ c _ { 1 L } = v _ { L } B _ { 1 } ( \beta _ { L } ) - c \beta _ { L } , } \\
{ c _ { 2 L } = v _ { L } ^ { 2 } B _ { 2 } ( \beta _ { L } ) - c v _ { L } , }
\end{array} \quad \left\{\begin{array}{l}
c_{1 R}=v_{R} B_{1}\left(\beta_{R}\right)-c \beta_{R} \\
c_{2 R}=v_{R}^{2} B_{2}\left(\beta_{R}\right)-c v_{R}
\end{array}\right.\right.
$$

Points $\left(\beta_{L}, v_{L}\right)$ and $\left(\beta_{R}, v_{R}\right)$ are given in advance. The following property immediately holds by simple calculations.

Proposition 4.6 ([28]). The system 4.13) is asymptotically quasi-homogeneous of type $(0,1)$ and order 2.

Following arguments in [28], we observe that there is a blow-up solution with the asymptotic behavior

$$
\begin{equation*}
\beta(t) \sim \rho_{2}, \quad v(t) \sim V_{0} \theta(t)^{-1} \quad \text { as } \quad t \rightarrow t_{\max }-0 \tag{4.14}
\end{equation*}
$$

which is consistent with arguments in [21. In particular, type-I blow-up solutions are observed.
Our main concern here is to derive multi-order asymptotic expansion of the blow-up solution (4.14) for (4.13). To this end, write the blow-up solution $(\beta(t), v(t))$ as follows:

$$
\begin{align*}
\beta(t) & =b(t), \quad v(t)=\theta(t)^{-1} V(t) \\
b(t) & =\sum_{n=0}^{\infty} b_{n}(t) \equiv b_{0}+\tilde{b}(t), \quad b_{0}=\rho_{2}, \quad b_{n}(t) \ll b_{n-1}(t) \quad\left(t \rightarrow t_{\max }-0\right), \quad n \geq 1  \tag{4.15}\\
V(t) & =\sum_{n=0}^{\infty} V_{n}(t) \equiv V_{0}+\tilde{V}(t), \quad V_{n}(t) \ll V_{n-1}(t) \quad\left(t \rightarrow t_{\max }-0\right), \quad n \geq 1
\end{align*}
$$

The balance law which $\left(b_{0}, V_{0}\right)$ satisfies can be easily derived. Substituting the form (4.15) into (4.13), we have

$$
\begin{aligned}
\beta^{\prime} & =b^{\prime} \\
& =\theta(t)^{-1} V B_{1}(b)-c b-c_{1}, \\
v^{\prime} & =\theta(t)^{-2} V+\theta(t)^{-1} V^{\prime} \\
& =\theta(t)^{-2} V^{2} B_{2}(b)-c \theta(t)^{-1} V-c_{2} .
\end{aligned}
$$

Dividing the first equation by $\theta(t)^{0} \equiv 1$ and the second equation by $\theta(t)^{-1}$, we have

$$
\begin{equation*}
\frac{d}{d t}\binom{b}{V}=\theta(t)^{-1}\binom{V B_{1}(b)}{-V+V^{2} B_{2}(b)}-\binom{c b+c_{1}}{c V+\theta(t) c_{2}} \tag{4.16}
\end{equation*}
$$

The balance law is then

$$
\binom{V_{0} B_{1}\left(b_{0}\right)}{-V_{0}+V_{0}^{2} B_{2}\left(b_{0}\right)}=\binom{0}{0}
$$

that is,

$$
V_{0} \frac{\left(b_{0}-\rho_{1}\right)\left(b_{0}-\rho_{2}\right)}{\beta}=0, \quad-V_{0}+V_{0}^{2} \frac{b_{0}^{2}-\rho_{1} \rho_{2}}{2 b_{0}^{2}}=0
$$

In the present case, $b_{0}=\rho_{2}$ is already determined as the principal term of $b(t)$, which satisfies the first equation. Substituting $b_{0}=\rho_{2}$ into the second equation, we have $V_{0}=2 \rho_{2} /\left(\rho_{2}-\rho_{1}\right)$, provided $V_{0} \neq 0$. As a summary, the root of the balance law (under (4.14)) is uniquely determined by

$$
\begin{equation*}
\left(b_{0}, V_{0}\right)=\left(\rho_{2}, \frac{2 \rho_{2}}{\rho_{2}-\rho_{1}}\right) . \tag{4.17}
\end{equation*}
$$

Then the evolutionary system for the residual terms $(\tilde{b}, \tilde{V})$ is given below:

$$
\frac{d}{d t}\binom{\tilde{b}}{\tilde{V}}=\theta(t)^{-1}\binom{V_{0}\left\{B_{1}\left(b_{0}+\tilde{b}\right)-B_{1}\left(b_{0}\right)\right\}+\tilde{V} B_{1}\left(b_{0}+\tilde{b}\right)}{-\tilde{V}+V_{0}^{2}\left\{B_{2}\left(b_{0}+\tilde{b}\right)-B_{2}\left(b_{0}\right)\right\}+\left(2 V_{0} \tilde{V}+\tilde{V}^{2}\right) B_{2}\left(b_{0}+\tilde{b}\right)}-\binom{c\left(b_{0}+\tilde{b}\right)+c_{1}}{c\left(V_{0}+\tilde{V}\right)+\theta(t) c_{2}}
$$

Next consider the second term $\left(b_{1}(t), V_{1}(t)\right)$. The strategy is essentially the same as the previous example, while the vector field is rational. Direct calculations yield

$$
\frac{d}{d \beta} B_{1}(\beta)=1-\rho_{1} \rho_{2} \beta^{-2}, \quad \frac{d}{d \beta} B_{2}(\beta)=\rho_{1} \rho_{2} \beta^{-3}
$$

Letting

$$
\begin{aligned}
\bar{f}\left(b_{0}+\tilde{b}, V_{0}+\tilde{V}\right) & :=\binom{V_{0}\left\{B_{1}\left(b_{0}+\tilde{b}\right)-B_{1}\left(b_{0}\right)\right\}+\tilde{V} B_{1}\left(b_{0}+\tilde{b}\right)}{-\tilde{V}+V_{0}^{2}\left\{B_{2}\left(b_{0}+\tilde{b}\right)-B_{2}\left(b_{0}\right)\right\}+\left(2 V_{0} \tilde{V}+\tilde{V}^{2}\right) B_{2}\left(b_{0}+\tilde{b}\right)} \\
& \equiv-\Lambda_{\alpha}\binom{b}{V}+f_{\alpha, k}(b, V) \quad(\text { with } k=1)
\end{aligned}
$$

with the aid of the balance law, we have

$$
D \bar{f}\left(b_{0}, V_{0}\right)=\left(\begin{array}{cc}
\left.V_{0} \frac{d}{d \tilde{b}} B_{1}(\beta)\right|_{\beta=b_{0}} & B_{1}\left(b_{0}\right) \\
\left.V_{0}^{2} \frac{d}{d \beta} B_{2}(\beta)\right|_{\beta=b_{0}} & -1+2 V_{0} B_{2}\left(b_{0}\right)
\end{array}\right)=\left(\begin{array}{cc}
V_{0}\left(1-\rho_{1} \rho_{2} b_{0}^{-2}\right) & \frac{\left(b_{0}-\rho_{1}\right)\left(b_{0}-\rho_{2}\right)}{b_{0}} \\
V_{0}^{2} \rho_{1} \rho_{2} b_{0}^{-3} & -1+2 V_{0} \frac{b_{0}^{2}-\rho_{1} \rho_{2}}{2 b_{0}^{2}}
\end{array}\right)
$$

which is the blow-up power determining matrix associated with the blow-up solution (4.14). Using (4.17), we have

$$
\begin{aligned}
A \equiv D \bar{f}\left(b_{0}, V_{0}\right) & =\left(\begin{array}{cc}
\frac{2 \rho_{2}}{\rho_{2}-\rho_{1}}\left(1-\rho_{1} \rho_{2}^{-1}\right) & 0 \\
\left(\frac{2 \rho_{2}}{\rho_{2}-\rho_{1}}\right)^{2} \rho_{1} \rho_{2} b_{0}^{-3} & -1+2 \frac{2 \rho_{2}}{\rho_{2}-\rho_{1}} \frac{\rho_{2}-\rho_{1}}{2 \rho_{2}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\frac{2}{} & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right) .
\end{aligned}
$$

We immediately know that eigenvalues are 2,1 , which are independent of $\rho_{1}, \rho_{2}$ and make no contributions to asymptotic behavior of the blow-up solution. Associated eigenvectors are

$$
\begin{aligned}
&\left(\begin{array}{cc}
2 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\binom{x_{1}}{x_{2}}=2\binom{x_{1}}{x_{2}} \Rightarrow \quad\binom{x_{1}}{x_{2}}=\binom{1}{\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}}} \\
&\left(\begin{array}{cc}
2 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\binom{x_{1}}{x_{2}}=\binom{x_{1}}{x_{2}} \quad \Rightarrow \quad\binom{x_{1}}{x_{2}}=\binom{0}{1} .
\end{aligned}
$$

Define

$$
P:=\left(\begin{array}{cc}
1 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right) \quad \Leftrightarrow \quad P^{-1}=\left(\begin{array}{cc}
1 & 0 \\
-\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)
$$

and multiply $P^{-1}$ to (4.16) from the left:

$$
\frac{d}{d t} P^{-1}\binom{b}{V}=\theta(t)^{-1}\left[\left(\begin{array}{cc}
2 & 0  \tag{4.18}\\
0 & 1
\end{array}\right) P^{-1}\binom{b}{V}+P^{-1} \bar{f}_{R}(b, V)\right]-c P^{-1}\binom{b}{V}-P^{-1}\binom{c_{1}}{\theta(t) c_{2}}
$$

where

$$
\begin{equation*}
\bar{f}_{\mathrm{R}}(b, V)=\bar{f}(b, V)-D \bar{f}\left(b_{0}, V_{0}\right)\binom{\tilde{b}}{\tilde{V}} \tag{4.19}
\end{equation*}
$$

is the second order residual term of $\bar{f}$ whose details are shown in Remark 4.7 below, and we have used

$$
\left(\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right) P^{-1}\binom{b_{0}}{V_{0}}+P^{-1} \bar{f}_{\mathrm{R}}\left(b_{0}, V_{0}\right)=\binom{0}{0}
$$

Remark 4.7. The detailed expression of $\bar{f}_{R}(b, V)$ around the root $(b, V)=\left(b_{0}, V_{0}\right)$ is derived by the Taylor's expansion at $(b, V)=\left(b_{0}, V_{0}\right)$ :

$$
\begin{aligned}
V B_{1}(b) & =\left(V_{0} \tilde{b}+\tilde{V} \tilde{b}^{2}\right) \frac{\left(2 b_{0} V_{0}^{-1}+\tilde{b}\right)}{b_{0}+\tilde{b}} \quad(\text { from } \\
& =V_{0} \tilde{b}\left[2 V_{0}^{-1}+\frac{\left.1-2 V_{0}^{-1}\right)}{b_{0}} \tilde{b}-\frac{\left(1-2 V_{0}^{-1}\right)}{b_{0}\left(b_{0}+\tilde{b}\right)} \tilde{b}^{2}\right]+\tilde{V} \tilde{b}\left[2 V_{0}^{-1}+\frac{1-2 V_{0}^{-1}}{b_{0}\left(b_{0}+\tilde{b}\right)} \tilde{b}\right] \\
& \equiv 2 \tilde{b}+\bar{f}_{R, 1}(b, V)
\end{aligned}
$$

$$
\begin{aligned}
-V+V^{2} B_{2}(b)= & -\left(V_{0}+\tilde{V}\right)+\frac{1}{2} V_{0}^{2}\left[1-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}+2 \frac{\rho_{1} \rho_{2}}{b_{0}^{3}} \tilde{b}+\rho_{1} \rho_{2}\left\{\frac{-3 b_{0} \tilde{b}^{2}-2 \tilde{b}^{3}}{b_{0}^{3}\left(b_{0}+\tilde{b}\right)^{2}}\right\}\right] \\
& +V_{0} \tilde{V}\left[1-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}+\rho_{1} \rho_{2}\left\{\frac{2 b_{0} \tilde{b}+\tilde{b}^{2}}{b_{0}^{2}\left(b_{0}+\tilde{b}\right)^{2}}\right\}\right]+\frac{1}{2} \tilde{V}^{2}\left\{1-\frac{\rho_{1} \rho_{2}}{\left(b_{0}+\tilde{b}\right)^{2}}\right\} \\
\equiv & -\left(V_{0}+\tilde{V}\right)+\frac{1}{2} V_{0}^{2}\left(1-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}+2 \frac{\rho_{1} \rho_{2}}{b_{0}^{3}} \tilde{b}\right)+V_{0} \tilde{V}\left(1-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}\right)+\bar{f}_{R, 2}(b, V) \\
= & \frac{\rho_{1} \rho_{2}}{b_{0}^{3}} V_{0}^{2} \tilde{b}+\left[-1+V_{0}\left\{1-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}\right\}\right] \tilde{V}+\bar{f}_{R, 2}(b, V) \quad(\text { from 4.17) }) \\
= & \frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} \tilde{b}+\tilde{V}+\bar{f}_{R, 2}(b, V),
\end{aligned}
$$

where we have used the following identities:

$$
\begin{aligned}
-\frac{\rho_{1} \rho_{2}}{\left(b_{0}+\tilde{b}\right)^{2}} & =-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}+\rho_{1} \rho_{2}\left\{\frac{2 b_{0} \tilde{b}+\tilde{b}^{2}}{b_{0}^{2}\left(b_{0}+\tilde{b}\right)^{2}}\right\} \\
& =-\frac{\rho_{1} \rho_{2}}{b_{0}^{2}}+2 \frac{\rho_{1} \rho_{2}}{b_{0}^{3}} \tilde{b}+\rho_{1} \rho_{2}\left\{\frac{-3 b_{0} \tilde{b}^{2}-2 \tilde{b}^{3}}{b_{0}^{3}\left(b_{0}+\tilde{b}\right)^{2}}\right\} .
\end{aligned}
$$

Therefore we obtain the expansion

$$
\binom{V B_{1}(b)}{-V+V^{2} B_{2}(b)}=\left(\begin{array}{cc}
2 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\binom{\tilde{b}}{\tilde{V}}+\bar{f}_{R}(b, V)
$$

around $\left(b_{0}, V_{0}\right)^{T}$, where

$$
\begin{aligned}
\bar{f}_{R}(b, V) & =\left(\bar{f}_{R, 1}(b, V), \bar{f}_{R, 2}(b, V)\right)^{T} \\
\bar{f}_{R, 1}(b, V) & =\frac{V_{0}-2}{b_{0}} \tilde{b}^{2}-\frac{V_{0}-2}{b_{0}\left(b_{0}+\tilde{b}\right)} \tilde{b}^{3}+\frac{2}{V_{0}} \tilde{V} \tilde{b}+\frac{\left(1-2 V_{0}^{-1}\right)}{b_{0}\left(b_{0}+\tilde{b}\right)} \tilde{V} \tilde{b}^{2} \\
\bar{f}_{R, 2}(b, V) & =\frac{1}{2} V_{0}^{2} \rho_{1} \rho_{2}\left\{\frac{-3 b_{0}-2 \tilde{b}}{b_{0}^{3}\left(b_{0}+\tilde{b}\right)^{2}}\right\} \tilde{b}^{2}+V_{0} \rho_{1} \rho_{2}\left\{\frac{2 b_{0}+\tilde{b}}{b_{0}^{2}\left(b_{0}+\tilde{b}\right)^{2}}\right\} \tilde{b} \tilde{V}+\frac{1}{2} \tilde{V}^{2}\left\{1-\frac{\rho_{1} \rho_{2}}{\left(b_{0}+\tilde{b}\right)^{2}}\right\} .
\end{aligned}
$$

Let

$$
\binom{e_{1}}{W_{1}}:=P^{-1}\binom{\tilde{b}}{\tilde{V}}
$$

and the governing terms are collected to form the equation for $\left(e_{1}, W_{1}\right)^{T}$ :

$$
\frac{d}{d t}\binom{e_{1}}{W_{1}}=\theta(t)^{-1}\left(\begin{array}{ll}
2 & 0 \\
0 & 1
\end{array}\right)\binom{e_{1}}{W_{1}}-\left(\begin{array}{cc}
1 & 0 \\
-\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\binom{c b_{0}+c_{1}}{c V_{0}}
$$

where the remaining term $\theta(t) c_{2}$ in the second component is omitted because, in the inhomogeneous term, it is obviously negligible compared with the constant term $c V_{0}$ as $t \rightarrow t_{\max }$. This system is
easily solved to obtain the bounded solution as follows:

$$
\begin{aligned}
e_{1}^{\prime} & =2 \theta(t)^{-1} e_{1}-\left(c \rho_{2}+c_{1}\right) \\
& \Leftrightarrow \quad e_{1}=\theta(t)^{-2}\left\{\left(c \rho_{2}+c_{1}\right) \int_{t}^{t_{\max }} \theta(s)^{2} d s\right\} \\
W_{1}^{\prime} & =\theta(t)^{-1} W_{1}+\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}}\left(c \rho_{2}+c_{1}\right)-\left(\frac{2 c \rho_{2}}{\rho_{2}-\rho_{1}}\right) \\
& \Leftrightarrow \quad W_{1}=\theta(t)^{-1}\left[-\left\{\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}}\left(c \rho_{2}+c_{1}\right)-\frac{2 c \rho_{2}}{\rho_{2}-\rho_{1}}\right\} \int_{t}^{t_{\max }} \theta(s) d s\right] .
\end{aligned}
$$

In particular, the bounded solution $\left(e_{1}(t), W_{1}(t)\right)$ is

$$
e_{1}(t)=\frac{1}{3}\left(c \rho_{2}+c_{1}\right) \theta(t), \quad W_{1}(t)=-\frac{1}{2}\left\{\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}}\left(c \rho_{2}+c_{1}\right)-\frac{2 c \rho_{2}}{\rho_{2}-\rho_{1}}\right\} \theta(t)
$$

In the original coordinate, we have

$$
\left.\begin{array}{rl}
\binom{b_{1}}{V_{1}} & =\left(\begin{array}{cc}
1 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\binom{e_{1}}{W_{1}} \\
& =\theta(t)\left(\begin{array}{cc}
1 & 0 \\
\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}} & 1
\end{array}\right)\left(-\frac{1}{2}\left\{\frac{4 \rho_{1}}{\left(\rho_{2}-\rho_{1}\right)^{2}}\left(c \rho_{2}+c_{1}\right)\right.\right. \\
b_{1} & \left.=\frac{1}{3}\left(c \rho_{2}+c_{1}\right) \theta(t)-\frac{2 c \rho_{2}}{\rho_{2}-\rho_{1}}\right\}
\end{array}\right),
$$

Now the constant $\delta$ defined in (3.28) is estimated as follows. As already observed, we know that $\alpha=(0,1)$ and $k=1$. We know that $\left(\gamma_{1}, \gamma_{2}\right)=(0,-1)$. Because there are no negative blow-up power eigenvalues, $\delta$ is evaluated as

$$
\delta=\min \left\{\frac{0}{1}+0, \frac{1}{1}-1\right\}+1=1
$$

Proposition 3.13 indicates that $\operatorname{ord}_{\theta}\left(\mathbf{Y}_{m}\right) \geq m$ for $m \geq 2$ and hence there is no term $\theta(t)^{m}$ with $m \leq 1$ in $\mathbf{Y}_{n}(t)=\left(b_{n}(t), V_{n}(t)\right), n \geq 2$. As a summary, we obtain the following result.
Theorem 4.8. The system 4.13) with $\left(c_{1}, c_{2}\right)=\left(c_{1 L}, c_{2 L}\right)$ admits a blow-up solution with the following second order asymptotic expansion as $t \rightarrow t_{\max }$ :

$$
\beta(t) \sim \rho_{2}+\frac{1}{3}\left(c \rho_{2}+c_{1}\right) \theta(t), \quad v(t) \sim \frac{2 \rho_{2}}{\rho_{2}-\rho_{1}} \theta(t)^{-1}+\left[-\frac{2 \rho_{1}}{3\left(\rho_{2}-\rho_{1}\right)^{2}}\left(c \rho_{2}+c_{1}\right)+\frac{c \rho_{2}}{\rho_{2}-\rho_{1}}\right] .
$$

We omit further calculations to obtain the third term $\left(b_{2}(t), V_{2}(t)\right)$, but it should be mentioned that the neglected term $\theta(t) c_{2}$ in the above calculations should be added because it is nontrivial whether $\theta(t) c_{2}$ is negligible compared with inhomogeneous terms in the system solving $\left(b_{2}, V_{2}\right)$. In the present problem, $\theta(t) c_{2}$ must be included in the $\left(b_{2}, V_{2}\right)$-system because $c V_{1}$ has the same order as $\theta(t) c_{2}$. The inhomogeneous term $\mathbf{g}_{2}$ in the equation for $\left(b_{2}(t), V_{2}(t)\right)$ can be the calculated from $\bar{f}_{R}$ extracting appropriate terms (cf. Remark 4.7) and remaining terms mentioned above.

Remark 4.9. Because $b_{0}=\rho_{2}>0$, the function $\bar{f}_{R}(b, V)$ in 4.19) is real analytic in a neighborhood of $(\tilde{b}, \tilde{V})=(0,0)$.

### 4.3 Andrews' system I

Consider the following system (cf. [2, 29) ${ }^{13}$ :

$$
\left\{\begin{array}{l}
\frac{d u}{d t}=\frac{1}{\sin \theta} v u^{2}-\frac{2 a \cos \theta}{\sin \theta} u^{3}  \tag{4.20}\\
\frac{d v}{d t}=\frac{a}{\sin \theta} u v^{2}+\frac{1-a}{\sin \theta} \frac{u v^{3}}{v+2 \cos \theta u}
\end{array}\right.
$$

where $a \in(0,1)$ and $\theta \in(0, \pi / 2)$ are constant parameters.
Remark 4.10. The following results are known for 4.20) with initial points $(u(0), v(0))$ such that both components are positive (cf. [2, 29]).

1. When $a<1 / 2$, all solutions with sufficiently large initial points blow up at $t=t_{\max }<\infty$ with the blow-up rate $O\left(\theta(t)^{-1 / 2}\right)$ as $t \rightarrow t_{\max }$.
2. When $a \in(1 / 2,1)$, there is a blow-up solution admitting the asymptotic behavior $O\left(\left\{\theta(t)^{-1} \log \theta(t)^{-1}\right\}^{1 / 2}\right)$ as $t \rightarrow t_{\text {max }}$.
3. When $a=1 / 2$, there is a blow-up solution admitting the asymptotic behavior $O\left(\left\{\theta(t)^{-1}\left(\log \theta(t)^{-1}\right)^{1 / 2}\right\}^{1 / 2}\right)$ as $t \rightarrow t_{\text {max }}$.

Our interest here is the blow-up behavior for $a \in(0,1 / 2)$. We easily see that the system (4.20) is homogeneous of the order 3, and arguments in [29] show that the first term of stationary blow-up solutions has the form

$$
u(t)=O\left(\theta(t)^{-1 / 2}\right), \quad v(t)=O\left(\theta(t)^{-1 / 2}\right)
$$

Before expanding $(u(t), v(t))$ directly, introduce

$$
\begin{equation*}
w=u \cos \theta, \quad s=\frac{t}{\sin \theta \cos \theta} \tag{4.21}
\end{equation*}
$$

in which case we have

$$
\frac{d u}{d s}=\frac{d u}{d t} \frac{d t}{d s}=\sin \theta \cos \theta \frac{d u}{d t}, \quad \frac{d w}{d s}=\cos \theta \frac{d u}{d s}=\sin \theta \cos ^{2} \theta \frac{d u}{d t}
$$

[^11]The vector field (4.20) is then transformed into

$$
\begin{aligned}
\frac{d w}{d s} & =\sin \theta \cos ^{2} \theta \frac{d u}{d t} \\
& =\sin \theta \cos ^{2} \theta\left(\frac{1}{\sin \theta} v u^{2}-\frac{2 a \cos \theta}{\sin \theta} u^{3}\right) \\
& =\sin \theta \cos ^{2} \theta\left(\frac{1}{\sin \theta} v\left(\frac{w}{\cos \theta}\right)^{2}-\frac{2 a \cos \theta}{\sin \theta}\left(\frac{w}{\cos \theta}\right)^{3}\right) \\
& =v w^{2}-2 a w^{3} \\
\frac{d v}{d s} & =\sin \theta \cos \theta \frac{d v}{d t} \\
& =\sin \theta \cos \theta\left(\frac{a}{\sin \theta}\left(\frac{w}{\cos \theta}\right) v^{2}+\frac{1-a}{\sin \theta} \frac{\left(\frac{w}{\cos \theta}\right) v^{3}}{v+2 w}\right) \\
& =a w v^{2}+(1-a) \frac{w v^{3}}{v+2 w} .
\end{aligned}
$$

As a summary, the system (4.20) is transformed into the following rational vector field independent of $\theta$ :

$$
\begin{equation*}
\frac{d w}{d s}=w^{2}(v-2 a w), \quad \frac{d v}{d s}=w v^{2} \frac{v+2 a w}{v+2 w} \tag{4.22}
\end{equation*}
$$

Because $\theta \in(0, \pi / 2)$, then $\sin \theta>0$ and $\cos \theta>0$, and hence dynamics of $(u, v)$ in $t$-timescale and $(w, v)$ in $s$-variable are mutually smoothly equivalent.

In what follows we shall consider the blow-up solution with the following blow-up rate

$$
w(t)=O\left(\tilde{\theta}(s)^{-1 / 2}\right), \quad v(t)=O\left(\tilde{\theta}(s)^{-1 / 2}\right), \quad \tilde{\theta}(s)=s_{\max }-s=\frac{2}{\sin (2 \theta)} \theta(t)
$$

Expand the solution $(w(s), v(s))$ as the asymptotic series

$$
\begin{align*}
& w(s)=\tilde{\theta}(s)^{-1 / 2} W(s) \equiv \tilde{\theta}(s)^{-1 / 2} \sum_{n=0}^{\infty} W_{n}(s), \quad W_{n}(s) \ll W_{n-1}(s), \quad \lim _{t \rightarrow t_{\max }} W_{n}(s)=W_{0}, \\
& v(s)=\tilde{\theta}(s)^{-1 / 2} V(s) \equiv \tilde{\theta}(s)^{-1 / 2} \sum_{n=0}^{\infty} V_{n}(s), \quad V_{n}(s) \ll V_{n-1}(s), \quad \lim _{t \rightarrow t_{\max }} V_{n}(s)=V_{0} \tag{4.23}
\end{align*}
$$

The balance law for the solution under the assumption $W_{0}, V_{0} \neq 0$ is

$$
\begin{equation*}
\frac{1}{2}=W_{0} V_{0}-2 a W_{0}^{2}, \quad \frac{1}{2}=W_{0} V_{0} \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}} \tag{4.24}
\end{equation*}
$$

We further assume $V_{0}+2 W_{0} \neq 0$ at a moment. Then we have

$$
\begin{aligned}
& V_{0}-2 a W_{0}=V_{0} \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}} \\
& \Leftrightarrow \quad\left(V_{0}+2 W_{0}\right)\left(V_{0}-2 a W_{0}\right)=V_{0}\left(V_{0}+2 a W_{0}\right) \\
& \Leftrightarrow \quad V_{0}^{2}-2 a V_{0} W_{0}+2 V_{0} W_{0}-4 a W_{0}^{2}=V_{0}^{2}+2 a V_{0} W_{0} \\
& \Leftrightarrow \quad 2(1-2 a) V_{0}=4 a W_{0} .
\end{aligned}
$$

From this identity, we have the constraint among $U_{0}$ and $V_{0}$ so that the balance law must be satisfied:

$$
\begin{equation*}
V_{0}=\frac{2 a}{1-2 a} W_{0} \tag{4.25}
\end{equation*}
$$

Substituting (4.25) into, say the first component of the balance law, we have

$$
\frac{1}{2}=\frac{2 a-2 a(1-2 a)}{1-2 a} W_{0}^{2}=\frac{4 a^{2}}{1-2 a} W_{0}^{2}
$$

We therefore have

$$
W_{0}= \pm \sqrt{\frac{1-2 a}{8 a^{2}}}, \quad V_{0}=\frac{2 a}{1-2 a} W_{0}= \pm \sqrt{\frac{1}{2(1-2 a)}}
$$

We shall choose the positive root according to the setting mentioned in Remark 4.10. Moreover, the root $V_{0}$ can be achieved as a real value by the assumption of $a$.

Next we derive the equation for the residual terms $W(s)-W_{0}$ and $V(s)-V_{0}$. First, substituting (4.23) into (4.20), we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{d W_{n}}{d s}=\tilde{\theta}(s)^{-1}\left[-\frac{1}{2} \sum_{n=0}^{\infty} W_{n}+\left(\sum_{n=0}^{\infty} W_{n}\right)^{2}\left\{\sum_{n=0}^{\infty} V_{n}-2 a \sum_{n=0}^{\infty} W_{n}\right\}\right] \\
& \sum_{n=1}^{\infty} \frac{d V_{n}}{d s}=\tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} \sum_{n=0}^{\infty} V_{n}+\left(\sum_{n=0}^{\infty} W_{n}\right)\left(\sum_{n=0}^{\infty} V_{n}\right)^{2} \frac{\sum_{n=0}^{\infty} V_{n}+2 a \sum_{n=0}^{\infty} W_{n}}{\sum_{n=0}^{\infty} V_{n}+2 \sum_{n=0}^{\infty} W_{n}}\right\} .
\end{aligned}
$$

In the next step, we derive the governing equation for $\left(W_{1}(t), V_{1}(t)\right)$. Because the original system is homogeneous, we have the following governing system for $\left(W_{1}(t), V_{1}(t)\right)$, which is exactly the linearized system of (4.20) at $\left(W_{0}, V_{0}\right)$ :

$$
\begin{align*}
\frac{d}{d s}\binom{W_{1}}{V_{1}}= & \tilde{\theta}(s)^{-1}\left\{-\frac{1}{2}\binom{W_{1}}{V_{1}}\right. \\
& +\left(\begin{array}{cc}
2 V_{0} W_{0}-6 a W_{0}^{2} & W_{0}^{2} \\
V_{0}^{2} \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}+W_{0} V_{0}^{2} \frac{2 a\left(V_{0}+2 W_{0}\right)-2\left(V_{0}+2 a W_{0}\right)}{\left(V_{0}+2 W_{0}\right)^{2}} & \left.2 V_{0} W_{0} \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}+W_{0} V_{0}^{2} \frac{\left(V_{0}+2 W_{0}\right)-\left(V_{0}+2 a W_{0}\right)}{\left(V_{0}+2 W_{0}\right)^{2}}\right)\binom{W_{1}}{V_{1}}
\end{array}\right\} \\
\equiv & \tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} I_{2}+C\left(W_{0}, V_{0} ; a\right)\right\}\binom{W_{1}}{V_{1}} \tag{4.26}
\end{align*}
$$

where

$$
C\left(W_{0}, V_{0} ; a\right)=\left(\begin{array}{c}
2 V_{0} W_{0}-6 a W_{0}^{2} \\
V_{0}^{2}\left\{\frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}-W_{0} \frac{2(1-a) V_{0}}{\left(V_{0}+2 W_{0}\right)^{2}}\right\}
\end{array} \quad V_{0} W_{0}\left\{2 \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}+V_{0} \frac{2(1-a) W_{0}}{\left(V_{0}+2 W_{0}\right)^{2}}\right\}\right)
$$

Calculation of blow-up power eigenvalues is reduced to that of the matrix $C\left(W_{0}, V_{0} ; a\right)$, which still looks complicating. Using the balance law, we derive the simpler form.

Remark 4.11. We derive several identities among $V_{0}$ and $W_{0}$ so that eigenpairs of $C\left(W_{0}, V_{0} ; a\right)$ are easily calculated. First, using the identity 4.25), we have

$$
\begin{equation*}
V_{0}+2 W_{0}=\left(1+\frac{1-2 a}{a}\right) V_{0}=\frac{1-a}{a} V_{0} \tag{4.27}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{0}+2 a W_{0}=(1+1-2 a) V_{0}=2(1-a) V_{0} \tag{4.28}
\end{equation*}
$$

Using these identities, we have

$$
\begin{equation*}
\frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}=\frac{2(1-a) V_{0}}{\frac{1-a}{a} V_{0}}=2 a \tag{4.29}
\end{equation*}
$$

and

$$
\begin{align*}
& \frac{2(1-a) V_{0} W_{0}}{\left(V_{0}+2 W_{0}\right)^{2}}=\frac{a}{1-a} \frac{2(1-a) W_{0}}{V_{0}+2 W_{0}} \quad(b y \text { (4.27) }) \\
& \quad=\frac{2 a W_{0}}{V_{0}+2 W_{0}}=\frac{(1-2 a) V_{0}}{V_{0}+2 W_{0}} \quad(b y(4.25)) \\
& \quad=\frac{(1-2 a) V_{0}}{\frac{1-a}{a} V_{0}} \quad(b y(4.27)) \\
& \quad=\frac{a(1-2 a)}{1-a} \tag{4.30}
\end{align*}
$$

The balance law (4.25) and identities obtained in Remark 4.11 yield

$$
\begin{aligned}
2 V_{0} W_{0}-6 a W_{0}^{2} & =(6 a-1) V_{0} W_{0}, \\
V_{0}^{2}\left\{\frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}-\frac{2(1-a) V_{0} W_{0}}{\left(V_{0}+2 W_{0}\right)^{2}}\right\} & =V_{0}^{2}\left\{2 a-\frac{a(1-2 a)}{1-a}\right\} \quad(\text { by (4.29) and (4.30) }) \\
& =\frac{a}{1-a} V_{0}^{2}, \\
V_{0} W_{0}\left\{2 \frac{V_{0}+2 a W_{0}}{V_{0}+2 W_{0}}+\frac{2(1-a) V_{0} W_{0}}{\left(V_{0}+2 W_{0}\right)^{2}}\right\} & =V_{0} W_{0}\left\{4 a+\frac{a(1-2 a)}{1-a}\right\} \quad(\text { by (4.29) and (4.30) }) \\
& =\frac{a(5-6 a)}{1-a} V_{0} W_{0} .
\end{aligned}
$$

Next, from the first identity of (4.24), we have

$$
\begin{aligned}
\frac{1}{2} & =V_{0} W_{0}-2 a W_{0}^{2} \\
& =V_{0} W_{0}-(1-2 a) V_{0} W_{0} \quad(\text { by (4.25) }) \\
& =2 a V_{0} W_{0}
\end{aligned}
$$

which yields

$$
\begin{equation*}
V_{0} W_{0}=\frac{1}{4 a} \tag{4.31}
\end{equation*}
$$

We use this identity to obtain

$$
\begin{aligned}
(6 a-1) V_{0} W_{0} & =\frac{3}{2}-\frac{1}{4 a} \quad(\text { by (4.31) }) \\
\frac{a(5-6 a)}{1-a} V_{0} W_{0} & =\frac{3}{2}-\frac{1}{4(1-a)}
\end{aligned}
$$

Therefore the matrix $C\left(W_{0}, V_{0} ; a\right)$ is simplified as follows:

$$
C\left(W_{0}, V_{0} ; a\right)=\left(\begin{array}{cc}
\frac{3}{2}-\frac{1}{4 a} & W_{0}^{2} \\
\frac{a}{1-a} V_{0}^{2} & \frac{3}{2}-\frac{1}{4(1-a)}
\end{array}\right) .
$$

In particular, we obtain

$$
A-I_{2} \equiv C\left(W_{0}, V_{0} ; a\right)-\frac{3}{2} I_{2}=\left(\begin{array}{cc}
-\frac{1}{4 a} & W_{0}^{2} \\
\frac{a}{1-a} V_{0}^{2} & -\frac{1}{4(1-a)}
\end{array}\right)
$$

Using the identity (4.31) again, we know that the matrix $A-I_{2}$ has the eigenvalue 0 associating the eigenvector $\left(W_{0}, V_{0}\right)$. In this case, another eigenvalue is easily calculated through the trace:

$$
\operatorname{trace}\left(A-I_{2}\right)=-\frac{1}{4 a(1-a)}
$$

From

$$
-\frac{1}{4 a(1-a)} I_{2}-\left(\begin{array}{cc}
-\frac{1}{4 a} & W_{0}^{2} \\
\frac{a}{1-a} V_{0}^{2} & -\frac{1}{4(1-a)}
\end{array}\right)=\left(\begin{array}{cc}
\frac{a}{1-a} V_{0} W_{0} & -W_{0}^{2} \\
-\frac{a}{1-a} V_{0}^{2} & V_{0} W_{0}
\end{array}\right)
$$

the associated eigenvector is

$$
\left(W_{0}, \frac{a}{1-a} V_{0}\right)^{T}
$$

As a consequence, eigenpairs of $A$ are calculated as follows:

$$
\left\{1,\binom{\sqrt{\frac{1-2 a}{8 a^{2}}}}{\sqrt{\frac{1}{2(1-2 a)}}}\right\}, \quad\left\{1-\frac{1}{4 a(1-a)},\binom{\sqrt{\frac{1-2 a}{8 a^{2}}}}{\frac{a}{1-a} \sqrt{\frac{1}{2(1-2 a)}}}\right\}
$$

Introducing a matrix

$$
P=\left(\begin{array}{cc}
\sqrt{\frac{1-2 a}{8 a^{2}}} & \sqrt{\frac{1-2 a}{8 a^{2}}} \\
\sqrt{\frac{1}{2(1-2 a)}} & \frac{a}{1-a} \sqrt{\frac{1}{2(1-2 a)}}
\end{array}\right)
$$

the blow-up power-determining matrix $A$ is diagonalized. Note that only the eigenvalue $\lambda \equiv$ $1-\frac{1}{4 a(1-a)}$ contributes the second term of asymptotic expansion, because it is always negative whenever $a \in(0,1 / 2)$. Therefore we know that the second term $\left(W_{1}(s), V_{1}(s)\right)$ consists only of a multiple of $\tilde{\theta}(s)^{-\lambda}$, namely

$$
W_{1}(s)=C_{1 w} \tilde{\theta}(s)^{-1+\frac{1}{4 a(1-a)}}, \quad V_{1}(s)=C_{1 v} \tilde{\theta}(s)^{-1+\frac{1}{4 a(1-a)}}
$$

with constants $C_{1 w}, C_{1 v}$.

Remark 4.12. The concrete form of the matrix $P$ is not necessary to obtain $\left(W_{1}(s), V_{1}(s)\right)$ itself as the solution of linear homogeneous system of ODEs. The concrete calculation of $P$ is necessary for solving $\left(W_{n}, V_{n}\right)$ with $n \geq 2$ because $\left(W_{1}, V_{1}\right)$ and other lower-order terms are included as inhomogeneous terms.

The remaining issue is to determine coefficients $C_{1 w}, C_{1 v}$. Rewriting the system (4.26),

$$
\frac{d}{d s}\binom{W_{1}}{V_{1}}=\tilde{\theta}(s)^{-1}\left[-\left(\begin{array}{cc}
\frac{1}{2} & 0  \tag{4.32}\\
0 & \frac{1}{2}
\end{array}\right)+\left(\begin{array}{cc}
\frac{3}{2}-\frac{1}{4 a} & \frac{1-2 a}{8 a^{2}} \\
\frac{a}{1-a} \frac{1}{2(1-2 a)} & \frac{3}{2}-\frac{1}{4(1-a)}
\end{array}\right)\right]\binom{W_{1}}{V_{1}}
$$

is our interest. In the previous arguments we have shown that

$$
W_{1}(s)=C_{1 w} \tilde{\theta}(s)^{-\lambda}, \quad V_{1}(s)=C_{1 v} \tilde{\theta}(s)^{-\lambda}
$$

Substituting this form into (4.32), we have

$$
\left\{\begin{array}{l}
\lambda C_{1 w}=\left\{1-\frac{1}{4 a}\right\} C_{1 w}+\frac{1-2 a}{8 a^{2}} C_{1 v}  \tag{4.33}\\
\lambda C_{1 v}=\frac{a}{2(1-a)(1-2 a)} C_{1 w}+\left\{1-\frac{1}{4(1-a)}\right\} C_{1 v}
\end{array}\right.
$$

Using $\lambda=1-\frac{1}{4 a(1-a)}$, we have the constraint

$$
C_{1 v}=\frac{2 a^{2}}{(1-a)(2 a-1)} C_{1 w}
$$

Letting $C_{1 w}$ be an arbitrary parameter, we have the complete form of $\left(W_{1}(s), V_{1}(s)\right)$ :

$$
W_{1}(s)=C_{1 w} \tilde{\theta}(s)^{-1+\frac{1}{4 a(1-a)}}, \quad V_{1}(s)=\frac{2 a^{2} \cos \theta}{(1-a)(2 a-1)} C_{1 w} \tilde{\theta}(s)^{-1+\frac{1}{4 a(1-a)}}
$$

Because $f_{\text {res }} \equiv 0$ and $f$ is analytic at $\left(U_{0}, V_{0}\right)=\left(\frac{W_{0}}{\cos \theta}, V_{0}\right)$, the Taylor expansion of $f$ for deriving equations for $\mathbf{Y}_{n} \equiv\left(W_{n}, V_{n}\right)$ with $n \geq 2$ yields that all possible terms appeared in $\mathbf{Y}_{n}$ are the linear combination of powers of $\theta(t)^{-m \lambda}$ with $m \geq 1$, where $\lambda=1-\frac{1}{4 a(1-a)}$. Summarizing the above arguments, we have the second order asymptotic expansion of blow-up solutions:

$$
\begin{align*}
w(s) & \sim \sqrt{\frac{1-2 a}{8 a^{2}}} \tilde{\theta}(s)^{-1 / 2}+C_{1 w} \tilde{\theta}(s)^{-\frac{3}{2}+\frac{1}{4 a(1-a)}} \\
v(s) & \sim \sqrt{\frac{1}{2(1-2 a)}} \tilde{\theta}(s)^{-1 / 2}+\frac{2 a^{2}}{(1-a)(2 a-1)} C_{1 w} \tilde{\theta}(s)^{-\frac{3}{2}+\frac{1}{4 a(1-a)}} \tag{4.34}
\end{align*}
$$

as $s \rightarrow s_{\max }$. Interestingly, the exponent of the second term $-\frac{3}{2}+\frac{1}{4 a(1-a)}$ can be both positive and negative, indicating that the divergent behavior can be enhanced depending on $a$. Nevertheless, the exponent is always larger than $-1 / 2$, because the quantity $1-\frac{1}{4 a(1-a)}$ is always negative. Reverting to the original $t$-timescale via (4.21), we obtain the following result.
Theorem 4.13. The system 4.20) with $a \in(0,1 / 2)$ admits a blow-up solution with the following second-order asymptotic expansion as $t \rightarrow t_{\max }$ :

$$
\begin{align*}
u(t) & \sim \sqrt{\frac{(1-2 a) \tan \theta}{8 a^{2}}} \theta(t)^{-1 / 2}+C \theta(t)^{-\frac{3}{2}+\frac{1}{4 a(1-a)}}, \\
v(t) & \sim \frac{1}{2} \sqrt{\frac{\sin (2 \theta)}{1-2 a}} \theta(t)^{-1 / 2}+\frac{2 a^{2} \cos \theta}{(1-a)(2 a-1)} C \theta(t)^{-\frac{3}{2}+\frac{1}{4 a(1-a)}} \tag{4.35}
\end{align*}
$$

with a constant $C=C(\theta)$ with $\theta \in(0, \pi / 2)$.

### 4.4 Andrews' system II

The next example is the following system:

$$
\left\{\begin{array}{l}
u^{\prime}=u^{2}(2 a v-b u)  \tag{4.36}\\
v^{\prime}=b u v^{2}
\end{array}\right.
$$

with parameters $a, b$ with $a>0$ and $2 a>b>0$. Our interest here is the asymptotic expansion of blow-up solutions with $u(0), v(0)>0$.

The system (4.36) comes from the following three dimensional system of ODEs expressing the motion of crystalline curvature flow for triangles (cf. [2]):

$$
\left\{\begin{array}{l}
u_{1}^{\prime}=u_{1}^{2}\left(a_{2} u_{2}+a_{3} u_{3}-a_{1} u_{1}\right)  \tag{4.37}\\
u_{2}^{\prime}=u_{2}^{2}\left(a_{3} u_{3}+a_{1} u_{1}-a_{2} u_{2}\right) \\
u_{3}^{\prime}=u_{3}^{2}\left(a_{1} u_{1}+a_{2} u_{2}-a_{3} u_{3}\right)
\end{array}\right.
$$

where $a_{1}, a_{2}, a_{3}$ are positive constants. In [29], the parameter dependence of blow-up behavior is investigated under the additional assumption $a_{2}=a_{3} \equiv a>0$. In this case, it is proved in [29] that $\left\{u_{2}=u_{3}\right\}$ is invariant under (4.37) and that, if $2 a>a_{1}>0$, the solutions with initial points satisfying $u_{2}(0)=u_{3}(0)>0$ blow up at $t=t_{\max }<\infty$ with the blow-up rate

$$
u_{1}(t)=O\left(\theta(t)^{-1 / 2}\right), \quad u_{2}(t)=u_{3}(t)=O\left(\theta(t)^{-1 / 2}\right)
$$

as $t \rightarrow t_{\text {max }}-0$.
Remark 4.14. In prior works (e.g. [1, [2]), it is also observed that, if $a_{1}=a_{2}+a_{3}, u_{1}(0)>0$ and $u_{2}(0)=u_{3}(0)>0$, then the solution blows up in a finite time $t_{\max }$ with the blow-up rate $\left(\theta(t)^{-1} \log \theta(t)^{-1}\right)^{1 / 2}$. Notice that the parameter constraint is different from our present setting.

As in previous examples, we introduce

$$
\begin{equation*}
\tilde{u}=\frac{u}{\sqrt{b}}, \quad \tilde{v}=\frac{v}{\sqrt{b}}, \quad s=b^{2} t, \quad 2 a=b(1+\sigma) \tag{4.38}
\end{equation*}
$$

with an auxiliary parameter $\sigma$, which transform (4.36) into

$$
\frac{d \tilde{u}}{d s}=\tilde{u}^{2}\{(1+\sigma) \tilde{v}-\tilde{u}\}, \quad \frac{d \tilde{v}}{d s}=\tilde{u} \tilde{v}^{2}
$$

In particular, the system becomes a one-parameter family. Our interest here is then the blow-up solution $(\tilde{u}(s), \tilde{v}(s))$ with the following blow-up rate

$$
\tilde{u}(t)=O\left(\tilde{\theta}(s)^{-1 / 2}\right), \quad \tilde{v}(t)=O\left(\tilde{\theta}(s)^{-1 / 2}\right), \quad \tilde{\theta}(s)=s_{\max }-s=b^{2} \theta(t)
$$

Expand the solution $(\tilde{u}(s), \tilde{v}(s))$ as the asymptotic series

$$
\begin{array}{ll}
\tilde{u}(s)=\tilde{\theta}(s)^{-1 / 2} U(s) \equiv \tilde{\theta}(s)^{-1 / 2} \sum_{n=0}^{\infty} U_{n}(s), & U_{n}(s) \ll U_{n-1}(s), \\
\lim _{s \rightarrow s_{\max }} U_{n}(s)=U_{0}  \tag{4.39}\\
\tilde{v}(s)=\tilde{\theta}(s)^{-1 / 2} V(s) \equiv \tilde{\theta}(s)^{-1 / 2} \sum_{n=0}^{\infty} V_{n}(s), & V_{n}(s) \ll V_{n-1}(s), \quad \lim _{s \rightarrow s_{\max }} V_{n}(s)=V_{0}
\end{array}
$$

Substituting (4.39) into (4.36), we have

$$
\begin{equation*}
\frac{d U}{d s}=\tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} U+U^{2}\{(1+\sigma) V-U\}\right\}, \quad \frac{d V}{d s}=\tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} V+U V^{2}\right\} \tag{4.40}
\end{equation*}
$$

The balance law under $\left(U_{0}, V_{0}\right) \neq(0,0)$ requires

$$
-\frac{1}{2}+U_{0}\left\{(1+\sigma) V_{0}-U_{0}\right\}=0, \quad-\frac{1}{2}+U_{0} V_{0}=0
$$

and hence

$$
\begin{equation*}
1=2 U_{0}\left\{(1+\sigma) V_{0}-U_{0}\right\}, \quad 1=2 U_{0} V_{0} \tag{4.41}
\end{equation*}
$$

which are used below. These identities yield the following consequence:

$$
U_{0}=\sqrt{\frac{\sigma}{2}}, \quad V_{0}=\frac{1}{\sqrt{2 \sigma}}
$$

and we have the first order asymptotic expansion of blow-up solutions:

$$
\tilde{u}(s) \sim \sqrt{\frac{\sigma}{2}} \tilde{\theta}(s)^{-1 / 2}, \quad \tilde{v}(s) \sim \frac{1}{\sqrt{2 \sigma}} \tilde{\theta}(s)^{-1 / 2}
$$

as $s \rightarrow s_{\text {max }}$.

Next the second term $\left(U_{1}(s), V_{1}(s)\right)$ is calculated. From the linearization of the vector field (4.40) at $\left(U_{0}, V_{0}\right)$, the governing system for $\left(U_{1}(s), V_{1}(s)\right)$ is

$$
\begin{equation*}
\frac{d}{d s}\binom{U_{1}}{V_{1}}=\tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} I_{2}+D\left(U_{0}, V_{0} ; \sigma\right)\right\}\binom{U_{1}}{V_{1}} \tag{4.42}
\end{equation*}
$$

where

$$
D\left(U_{0}, V_{0} ; \sigma\right)=\left(\begin{array}{cc}
2(1+\sigma) U V-3 U^{2} & (1+\sigma) U^{2} \\
V^{2} & 2 U V
\end{array}\right)_{(U, V)=\left(U_{0}, V_{0}\right)}=\left(\begin{array}{cc}
1-\frac{\sigma}{2} & \frac{\sigma(1+\sigma)}{2} \\
\frac{1}{2 \sigma} & 1
\end{array}\right)
$$

under the identity (4.41). In this example, the blow-up power-determining matrix $A$ is

$$
A=-\frac{1}{2} I_{2}+D\left(U_{0}, V_{0} ; \sigma\right)
$$

and blow-up power eigenvalues are easily calculated through eigenvalues of $D\left(U_{0}, V_{0} ; \sigma\right)$, which solve

$$
\left(\mu-1+\frac{\sigma}{2}\right)(\mu-1)-\frac{1+\sigma}{4}=\mu^{2}+\left(\frac{\sigma}{2}-2\right) \mu+\frac{3}{4}(1-\sigma)=0
$$

The roots of this equation are

$$
\mu=\frac{3}{2}, \quad \frac{1}{2}(1-\sigma)
$$

and hence the blow-up power eigenvalues are

$$
\lambda \equiv \mu-\frac{1}{2}=1, \quad-\frac{\sigma}{2}
$$

The value 1 is an eigenvalue of $A$. By our assumption $2 a>b>0, \sigma>0$ always holds, and hence another blow-up power eigenvalue is always negative. Therefore the second term $\left(U_{1}(s), V_{1}(s)\right)$ has the following form:

$$
U_{1}(s)=C_{1 u} \tilde{\theta}(s)^{\sigma / 2}, \quad V_{1}(s)=C_{1 v} \tilde{\theta}(s)^{\sigma / 2}
$$

while the constraint among constants are determined by (4.42):

$$
\begin{aligned}
& -\rho C_{1 u}=-\frac{1}{2} C_{1 u}+\left(1-\frac{\sigma}{2}\right) C_{1 u}+\frac{\sigma(1+\sigma)}{2} C_{1 v} \\
& -\rho C_{1 v}=-\frac{1}{2} C_{1 v}+\frac{1}{2 \sigma} C_{1 u}+C_{1 v}
\end{aligned}
$$

where $\rho=\sigma / 2>0$. We thus have

$$
C_{1 v}=-\frac{1}{\sigma(1+\sigma)} C_{1 u}
$$

and the second order asymptotic expansion of the blow-up solution $(\tilde{u}(s), \tilde{v}(s))$ is obtained as follows:

$$
\begin{aligned}
& \tilde{u}(s) \sim \sqrt{\frac{\sigma}{2}} \tilde{\theta}(s)^{-1 / 2}+C_{1 u} \tilde{\theta}(s)^{\frac{\sigma}{2}-\frac{1}{2}} \\
& \tilde{v}(s) \sim \frac{1}{\sqrt{2 \sigma}} \tilde{\theta}(s)^{-1 / 2}-\frac{1}{\sigma(1+\sigma)} C_{1 u} \tilde{\theta}(s)^{\frac{\sigma}{2}-\frac{1}{2}}
\end{aligned}
$$

as $s \rightarrow s_{\text {max }}$.
Finally, we shall derive the third term $\left(U_{2}(s), V_{2}(s)\right)$ to derive an interesting observation (Remark 4.16). From the balance law and (4.42), the governing equation for $\left(U_{2}(t), V_{2}(t)\right)$ is

$$
\begin{align*}
\frac{d}{d s}\binom{U_{2}}{V_{2}}= & \tilde{\theta}(s)^{-1}\left\{-\frac{1}{2} I_{2}+D\left(U_{0}, V_{0} ; \sigma\right)\right\}\binom{U_{2}}{V_{2}} \\
& +\tilde{\theta}(s)^{-1}\binom{2(1+\sigma) U_{0} U_{1} V_{1}+\left((1+\sigma) V_{0}-3 U_{0}\right) U_{1}^{2}+(1+\sigma) U_{1}^{2} V_{1}-U_{1}^{3}}{U_{0} V_{1}^{2}+2 V_{0} U_{1} V_{1}+U_{1} V_{1}^{2}} \tag{4.43}
\end{align*}
$$

according to (3.15) for $j=3$ with $f_{\text {res }} \equiv 0$. Here the inhomogeneous term consists of terms of order $\tilde{\theta}(s)^{2 \rho-1}$ and $\tilde{\theta}(s)^{3 \rho-1}$ with $\rho=\sigma / 2$. Because $\rho>0$ under our assumption, the terms of order $\tilde{\theta}(s)^{3 \rho-1}$ is negligible compared with those of the order $\tilde{\theta}(s)^{2 \rho-1}$, which indicates that the governing equation determining the third order term is essentially

$$
\begin{equation*}
\frac{d}{d s}\binom{U_{2}}{V_{2}} \approx \tilde{\theta}(s)^{-1}\left[\left\{-\frac{1}{2} I_{2}+D\left(U_{0}, V_{0} ; \sigma\right)\right\}\binom{U_{2}}{V_{2}}+\binom{2(1+\sigma) U_{0} U_{1} V_{1}+\left((1+\sigma) V_{0}-3 U_{0}\right) U_{1}^{2}}{U_{0} V_{1}^{2}+2 V_{0} U_{1} V_{1}}\right] \tag{4.44}
\end{equation*}
$$

Integration of the system (4.43) yields that the essential terms in $\left(U_{2}(s), V_{2}(s)\right)$ are of order $\tilde{\theta}(s)^{2 \rho}$ and hence these possess the following forms, respectively:

$$
C_{2 u} \tilde{\theta}(s)^{2 \rho}, \quad C_{2 v} \tilde{\theta}(s)^{2 \rho}
$$

Substituting them into (4.43) and comparing coefficients of $\tilde{\theta}(s)^{2 \rho}$, equivalent to coefficient balance for (4.44), we have the following constraint:

$$
-2 \rho\binom{C_{2 u}}{C_{2 v}}=\left[-\frac{1}{2} I_{2}+\left(\begin{array}{cc}
1-\frac{\sigma}{2} & \frac{\sigma(1+\sigma)}{2} \\
\frac{1}{2 \sigma} & 1
\end{array}\right)\right]\binom{C_{2 u}}{C_{2 v}}+\binom{\left.\left\{-\frac{2}{\sigma} U_{0}+(1+\sigma) V_{0}-3 U_{0}\right)\right\} C_{1 u}^{2}}{\left\{U_{0}-2 \sigma(1+\sigma) V_{0}\right\} \frac{1}{\sigma^{2}(1+\sigma)^{2}} C_{1 u}^{2}}
$$

equivalently

$$
\left[\left(2 \rho-\frac{1}{2}\right) I_{2}+\left(\begin{array}{cc}
1-\frac{\sigma}{2} & \frac{\sigma(1+\sigma)}{2} \\
\frac{1}{2 \sigma} & 1
\end{array}\right)\right]\binom{C_{2 u}}{C_{2 v}}=-\binom{\left.\left\{-\frac{2}{\sigma} U_{0}+(1+\sigma) V_{0}-3 U_{0}\right)\right\} C_{1 u}^{2}}{\left\{U_{0}-2 \sigma(1+\sigma) V_{0}\right\} \frac{1}{\sigma^{2}(1+\sigma)^{2}} C_{1 u}^{2}}
$$

The inverse of the coefficient matrix in the left hand side is

$$
\begin{aligned}
& \frac{1}{\left(2 \rho+\frac{1}{2}-\frac{\sigma}{2}\right)\left(2 \rho+\frac{1}{2}\right)-\frac{1+\sigma}{4}}\left(\begin{array}{cc}
2 \rho+\frac{1}{2} & -\frac{\sigma(1+\sigma)}{2} \\
-\frac{1}{2 \sigma} & 2 \rho+\frac{1}{2}-\frac{\sigma}{2}
\end{array}\right) \\
& =\frac{1}{\sigma(\sigma+1)}\left(\begin{array}{cc}
2 \sigma+1 & -\sigma(1+\sigma) \\
-\frac{1}{\sigma} & 1+\sigma
\end{array}\right) . \quad(\text { from } \rho=\sigma / 2)
\end{aligned}
$$

Using $U_{0}=\sigma V_{0}$ and $U_{0}=\sqrt{\sigma / 2}$, we have

$$
\begin{aligned}
\binom{C_{2 u}}{C_{2 v}} & =-\frac{1}{\sigma(\sigma+1)}\left(\begin{array}{cc}
2 \sigma+1 & -\sigma(1+\sigma) \\
-\frac{1}{\sigma} & 1+\sigma
\end{array}\right)\binom{\left.\left\{-\frac{2}{\sigma} U_{0}+(1+\sigma) V_{0}-3 U_{0}\right)\right\} C_{1 u}^{2}}{\left\{U_{0}-2 \sigma(1+\sigma) V_{0}\right\} \frac{1}{\sigma^{2}(1+\sigma)^{2}} C_{1 u}^{2}} \\
& =-\frac{1}{\sigma(\sigma+1)}\left(\begin{array}{cc}
2 \sigma+1 & -\sigma(1+\sigma) \\
-\frac{1}{\sigma} & 1+\sigma
\end{array}\right)\binom{\left.\left\{-\frac{2}{\sigma}+(1+\sigma) \frac{1}{\sigma}-3\right)\right\}}{\{1-2(1+\sigma)\} \frac{1}{\sigma^{2}(1+\sigma)^{2}}} U_{0} C_{1 u}^{2} \\
& =\frac{1+2 \sigma}{\sigma^{2}(\sigma+1)} \sqrt{\frac{\sigma}{2}}\left(\begin{array}{cc}
2 \sigma+1 & -\sigma(1+\sigma) \\
-\frac{1}{\sigma} & 1+\sigma
\end{array}\right)\binom{1}{\sigma(1+\sigma)^{2}} C_{1 u}^{2} \\
& =\frac{1+2 \sigma}{\sigma^{2}(\sigma+1)^{2}} \sqrt{\frac{\sigma}{2}\binom{\sigma(2 \sigma+3)}{-1} C_{1 u}^{2} .}
\end{aligned}
$$

As a summary, the third order asymptotic expansion of the blow-up solution $(\tilde{u}(s), \tilde{v}(s))$ is calculated as follows:

$$
\begin{aligned}
& \tilde{u}(s) \sim \sqrt{\frac{\sigma}{2}} \tilde{\theta}(s)^{-1 / 2}+C_{1 u} \tilde{\theta}(s)^{\frac{\sigma}{2}-\frac{1}{2}}+\frac{1+2 \sigma}{\sigma^{2}(\sigma+1)^{2}} \sqrt{\frac{\sigma}{2}} \sigma(2 \sigma+3) C_{1 u}^{2} \tilde{\theta}(s)^{\sigma-1 / 2} \\
& \tilde{v}(s) \sim \frac{1}{\sqrt{2 \sigma}} \tilde{\theta}(s)^{-1 / 2}-\frac{1}{\sigma(1+\sigma)} C_{1 u} \tilde{\theta}(s)^{\frac{\sigma}{2}-\frac{1}{2}}-\frac{1+2 \sigma}{\sigma^{2}(\sigma+1)^{2}} \sqrt{\frac{\sigma}{2}} C_{1 u}^{2} \tilde{\theta}(s)^{\sigma-1 / 2}
\end{aligned}
$$

as $s \rightarrow s_{\text {max }}$. Now the constant $\delta$ defined in (3.28) is estimated as follows. First, $\alpha=(1,1)$ and $k=2$. Because $f_{\text {res }} \equiv 0$, we have $\gamma_{1}=\gamma_{2}=+\infty$ and hence

$$
\delta=\min \left\{\min _{i=1,2}\left\{\frac{\alpha_{i}}{k}+\gamma_{i}\right\}+1, \frac{\sigma}{2}\right\}=\frac{a}{b}-\frac{1}{2}
$$

Proposition 3.13 indicates that $\operatorname{ord}_{\theta}\left(Y_{m}\right) \geq m\left(\frac{a}{b}-\frac{1}{2}\right)$ for $m \geq 2$ and hence there is no term $\theta(t)^{m \sigma / 2}$ with $m \leq 2$ in $Y_{n}(t), n \geq 3$. Finally we obtain the third order asymptotic expansion of the blow-up solution in the original timescale $(u(t), v(t))$ through (4.38).

Theorem 4.15. The system 4.36) with $0<b<2 a$ admits a blow-up solution with the following third order asymptotic expansion with $t \rightarrow t_{\max }$ :

$$
\begin{align*}
& u(t) \sim \frac{\sqrt{2 a-b}}{\sqrt{2} b} \theta(t)^{-1 / 2}+C \theta(t)^{\frac{a}{b}-1}+\frac{b(4 a-b)(4 a+b)}{4 \sqrt{2} a^{2} \sqrt{2 a-b}} C^{2} \theta(t)^{\frac{2 a}{b}-\frac{3}{2}} \\
& v(t) \sim \frac{1}{\sqrt{2(2 a-b)}} \theta(t)^{-1 / 2}-\frac{b^{2}}{2 a(2 a-b)} C \theta(t)^{\frac{a}{b}-1}-\frac{b^{3}(4 a-b)}{4 \sqrt{2} a^{2}(2 a-b)^{3 / 2}} C^{2} \theta(t)^{\frac{2 a}{b}-\frac{3}{2}} \tag{4.45}
\end{align*}
$$

with $C=b^{(4 a-3 b) / 2 b} C_{1}$ and a constant $C_{1} \in \mathbb{R}$.
Notice that the power of $t_{\max }-t$ in the second term is $\rho-\frac{1}{2}$ and that in the third term is $2 \rho-\frac{1}{2}$, where $\rho=\frac{\sigma}{2}=\frac{a}{b}-\frac{1}{2}>0$.

Remark 4.16. Similar to multi-order asymptotics of blow-up solutions for 4.20), powers of $\theta(t)$ can be both positive and negative, depending of the quantity $\rho=\frac{a}{b}-\frac{1}{2}$, which itself is positive. Observe that the second terms in 4.45) have the negative power of $\theta(t)$ if and only if

$$
\frac{a}{b}-1<0 \quad \Leftrightarrow \quad a<b<2 a
$$

Similarly, the third terms in 4.45) have the negative power of $\theta(t)$ if and only if

$$
\frac{2 a}{b}-\frac{3}{2}<0 \quad \Leftrightarrow \quad a<\frac{3}{4} b<\frac{3}{2} a .
$$

We know that the $n$-th terms of blow-up solutions have the negative power of $\theta(t)$ if and only if

$$
(n-1)\left(\frac{a}{b}-\frac{1}{2}\right)-\frac{1}{2}<0 \quad \Leftrightarrow \quad a<\frac{n}{2(n-1)} b<\frac{n}{n-1} a
$$

As the parameter $b(<2 a)$ approaches to $2 a$, the number

$$
\sharp\left\{n \in \mathbb{N} \left\lvert\, n \rho<\frac{1}{2}\right.\right\}
$$

determining blow-up solutions of the form

$$
u(t)=\theta(t)^{-1 / 2} \sum_{n=0}^{\infty} u_{n} \theta(t)^{\rho n}, \quad v(t)=\theta(t)^{-1 / 2} \sum_{n=0}^{\infty} v_{n} \theta(t)^{\rho n}
$$

for 4.36) increases and the expansion would "converge" to the blow-up solution with the blow-up rate $O\left(\theta^{-1 / 2}\left(\log \theta(t)^{-1}\right)^{-1 / 2}\right)$ mentioned in Remark 4.14. This expectation imply that multi-order asymptotic expansion of blow-up solutions through the present methodology could contribute to investigate the parameter dependence of blow-up rates.

### 4.5 Keyfitz-Kranser-type system

The next example is a 2 -dimensional system

$$
\begin{equation*}
u^{\prime}=u^{2}-v, \quad v^{\prime}=\frac{1}{3} u^{3}-u, \tag{4.46}
\end{equation*}
$$

originated from the Keyfitz-Kranser system [24] which is a system of conservation laws admitting a singular shock. Blow-up solutions of (4.46) are studied in [28] by means of dynamics at infinity ${ }^{14}$. Observe that the system is asymptotically quasi-homogeneous of type $\alpha=\left(\alpha_{1}, \alpha_{2}\right)=(1,2)$ and order $k+1=2$, consisting of the quasi-homogeneous part $f_{\alpha, k}$ and the lower-order part $f_{\text {res }}$ given as follows:

$$
\begin{equation*}
f_{\alpha, k}(u, v)=\binom{u^{2}-v}{\frac{1}{3} u^{3}}, \quad f_{\text {res }}(u, v)=\binom{0}{-u} . \tag{4.47}
\end{equation*}
$$

It is proved in [28] that the system (4.46) admits the following solutions blowing up as $t \rightarrow t_{\max }-0$ associated with two different equilibria at infinity:

$$
\begin{equation*}
u(t)=O\left(\theta(t)^{-1}\right), \quad v(t)=O\left(\theta(t)^{-2}\right), \quad \text { as } \quad t \rightarrow t_{\max }-0 . \tag{4.48}
\end{equation*}
$$

### 4.5.1 Blow-up asymptotics for the quasi-homogeneous part

Before treating the system (4.46), we consider a simpler system

$$
\begin{equation*}
u^{\prime}=u^{2}-v, \quad v^{\prime}=\frac{1}{3} u^{3}, \tag{4.49}
\end{equation*}
$$

namely the quasi-homogeneous part of (4.46). As in previous examples, expand the solution $(u(t), v(t))$ as the asymptotic series

$$
\begin{array}{lll}
u(t)=\theta(t)^{-1} U(t) \equiv \theta(t)^{-1} \sum_{n=0}^{\infty} U_{n}(t), & U_{n}(t) \ll U_{n-1}(t), & \lim _{t \rightarrow t_{\max }} U_{n}(t)=U_{0}, \\
v(t)=\theta(t)^{-2} V(t) \equiv \theta(t)^{-2} \sum_{n=0}^{\infty} V_{n}(t), & V_{n}(t) \ll V_{n-1}(t), & \lim _{t \rightarrow t_{\max }} V_{n}(t)=V_{0} . \tag{4.50}
\end{array}
$$

Note that the quasi-homogeneity induces the different rate of blow-ups among components. The balance law for (4.49), and hence for (4.46) is

$$
U_{0}=U_{0}^{2}-V_{0}, \quad 2 V_{0}=\frac{1}{3} U_{0}^{3},
$$

which yields

$$
\begin{equation*}
U_{0}=3 \pm \sqrt{3}, \quad V_{0}=\frac{1}{6} U_{0}^{3} . \tag{4.51}
\end{equation*}
$$

In particular, we have two different solutions of the balance law.

[^12]Next we consider the second term $\left(U_{1}(t), V_{1}(t)\right)$. Substituting (4.50) into 4.49) and using the balance law, we have

$$
\begin{align*}
& \frac{d}{d t}\left(\sum_{n=1}^{\infty} U_{n}\right)=\theta(t)^{-1}\left\{-\left(\sum_{n=1}^{\infty} U_{n}\right)+2 U_{0}\left(\sum_{n=1}^{\infty} U_{n}\right)+\left(\sum_{n=1}^{\infty} U_{n}\right)^{2}-\left(\sum_{n=1}^{\infty} V_{n}\right)\right\} \\
& \frac{d}{d t}\left(\sum_{n=1}^{\infty} V_{n}\right)=\theta(t)^{-1}\left\{-2\left(\sum_{n=1}^{\infty} V_{n}\right)+U_{0}^{2}\left(\sum_{n=1}^{\infty} U_{n}\right)+U_{0}\left(\sum_{n=1}^{\infty} U_{n}\right)^{2}+\frac{1}{3}\left(\sum_{n=1}^{\infty} U_{n}\right)^{3}\right\} \tag{4.52}
\end{align*}
$$

The governing system for $\left(U_{1}(t), V_{1}(t)\right)$ is

$$
\frac{d}{d t}\binom{U_{1}}{V_{1}}=\theta(t)^{-1}\left\{\left(\begin{array}{cc}
-1 & 0  \tag{4.53}\\
0 & -2
\end{array}\right)+\left(\begin{array}{cc}
2 U_{0} & -1 \\
U_{0}^{2} & 0
\end{array}\right)\right\}\binom{U_{1}}{V_{1}}
$$

Depending on the choice of $U_{0}$, the eigenstructure of the blow-up power-determining matrix changes.
Case 1. $U_{0}=3-\sqrt{3}$.
In this case, the blow-up power-determining matrix $A$ is

$$
A=\left(\begin{array}{cc}
2(3-\sqrt{3})-1 & -1 \\
(3-\sqrt{3})^{2} & -2
\end{array}\right)=\left(\begin{array}{cc}
5-2 \sqrt{3} & -1 \\
12-6 \sqrt{3} & -2
\end{array}\right)
$$

and the eigenvalues solve

$$
(\lambda-5+2 \sqrt{3})(\lambda+2)+(12-6 \sqrt{3})=\lambda^{2}+(-3+2 \sqrt{3}) \lambda+2-2 \sqrt{3}=0
$$

The eigenvalues are

$$
\lambda=1, \quad 2-2 \sqrt{3}
$$

and associating eigenvectors are

$$
\left.\begin{array}{rl}
\left(\begin{array}{cc}
5-2 \sqrt{3} & -1 \\
12-6 \sqrt{3} & -2
\end{array}\right)\binom{a}{b}=\binom{a}{b} & \Rightarrow\binom{a}{b}=\binom{1}{4-2 \sqrt{3}} \\
\binom{5-2 \sqrt{3}}{12-6 \sqrt{3}} \\
12
\end{array}\right)\binom{a}{b}=(2-2 \sqrt{3})\binom{a}{b} \quad \Rightarrow \quad\binom{a}{b}=\binom{1}{3} .
$$

Because $2-2 \sqrt{3}<0$, this eigenvalue contributes the power of $\theta(t)$ in the asymptotic expansion, while $\lambda=1$ makes no contribution to asymptotic expansions. Introducing the nonsingular matrix

$$
P=\left(\begin{array}{cc}
1 & 1  \tag{4.54}\\
4-2 \sqrt{3} & 3
\end{array}\right) \quad \Leftrightarrow \quad P^{-1} \equiv\left(\begin{array}{cc}
p^{11} & p^{12} \\
p^{21} & p^{22}
\end{array}\right)=\frac{1}{-1+2 \sqrt{3}}\left(\begin{array}{cc}
3 & -1 \\
-4+2 \sqrt{3} & 1
\end{array}\right)
$$

the second term $\left(U_{1}(t), V_{1}(t)\right)$ is calculated as

$$
\binom{U_{1}(t)}{V_{1}(t)}=P\binom{0}{C \theta(t)^{-2+2 \sqrt{3}}}=\binom{C \theta(t)^{-2+2 \sqrt{3}}}{3 C \theta(t)^{-2+2 \sqrt{3}}}
$$

as $t \rightarrow t_{\max }$. Letting $C_{1 u} \equiv C$, we have the second order asymptotic expansion of blow-up solutions $(u(t), v(t))$ with $U_{0}=3-\sqrt{3}$ as follows:

$$
\begin{aligned}
& u(t) \sim(3-\sqrt{3}) \theta(t)^{-1}+C_{1 u} \theta(t)^{-3+2 \sqrt{3}} \\
& v(t) \sim(9-5 \sqrt{3}) \theta(t)^{-2}+3 C_{1 u} \theta(t)^{-4+2 \sqrt{3}}
\end{aligned}
$$

as $t \rightarrow t_{\text {max }}$.
Next the third term $\left(U_{2}(t), V_{2}(t)\right)$ is considered with $U_{0}=3-\sqrt{3}$. From (4.52) and (4.53), the governing equation for $\left(U_{2}(t), V_{2}(t)\right)$ is

$$
\begin{equation*}
\frac{d}{d t}\binom{U_{2}}{V_{2}}=\theta(t)^{-1}\left\{A\binom{U_{2}}{V_{2}}+\binom{U_{1}^{2}}{U_{0} U_{1}^{2}+\frac{1}{3} U_{1}^{3}}\right\} \tag{4.55}
\end{equation*}
$$

From the asymptotic assumption $U_{1}(t) \ll U_{0}$, the essential term of $\left(U_{2}(t), V_{2}(t)\right)$ is approximately governed by

$$
\frac{d}{d t}\binom{U_{2}}{V_{2}} \approx \theta(t)^{-1}\left\{A\binom{U_{2}}{V_{2}}+\binom{U_{1}^{2}}{U_{0} U_{1}^{2}}\right\}
$$

The general solution of this approximate system is

Letting $r=2 \sqrt{3}-2(>0)$, we calculate the integrals. First,

$$
\begin{aligned}
& -\int_{t}^{t_{\max }}\left(p^{11} U_{1}(s)^{2}+p^{12} U_{0} U_{1}(s)^{2}\right) d s \\
& \quad=-p^{11} C_{1 u}^{2} \int_{t}^{t_{\max }} \theta(s)^{2 r} d s-U_{0} p^{12} C_{1 u}^{2} \int_{t}^{t_{\max }} \theta(s)^{2 r} d s \\
& \quad=\frac{-1}{2 r+1}\left\{p^{11}+U_{0} p^{12}\right\} C_{1 u}^{2} \theta(t)^{2 r+1} \\
& \quad=\frac{-(9 \sqrt{3}+10)}{143} C_{1 u}^{2} \theta(t)^{2 r+1}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \int_{0}^{t} \theta(s)^{-r-1}\left(p^{21} U_{1}(s)^{2}+p^{22} U_{0} U_{1}(s)^{2}\right) d s \\
& \quad=p^{21} C_{1 u}^{2} \int_{0}^{t} \theta(s)^{r-1} d s+U_{0} p^{22} C_{1 u}^{2} \int_{0}^{t} \theta(s)^{r-1} d s \\
& \quad=\frac{-1}{r}\left\{p^{21}+U_{0} p^{22}\right\} C_{1 u}^{2} \theta(t)^{r}+\tilde{C} \\
& \quad=-\frac{1+2 \sqrt{3}}{22} C_{1 u}^{2} \theta(t)^{r}+\tilde{C}
\end{aligned}
$$

where the integral constant $\tilde{C}$ is chosen so that $\tilde{c}_{2}+\tilde{C}=0$ (cf. (3.19)). Therefore

$$
\binom{U_{2}(t)}{V_{2}(t)} \approx P\binom{\frac{-(9 \sqrt{3}+10)}{143} C_{1 u}^{2} \theta(t)^{2 r}}{-\frac{1+2 \sqrt{3}}{22} C_{1 u}^{2} \theta(t)^{2 r}}=\binom{-\frac{3+4 \sqrt{3}}{26} C_{1 u}^{2} \theta(t)^{2 r}}{-\frac{1+10 \sqrt{3}}{26} C_{1 u}^{2} \theta(t)^{2 r}} .
$$

As a consequence, we have the third order asymptotic expansion of blow-up solutions $(u(t), v(t))$ with $U_{0}=3-\sqrt{3}$ as follows:

$$
\begin{aligned}
& u(t) \sim(3-\sqrt{3}) \theta(t)^{-1}+C_{1 u} \theta(t)^{-3+2 \sqrt{3}}-\frac{3+4 \sqrt{3}}{26} C_{1 u}^{2} \theta(t)^{-5+4 \sqrt{3}} \\
& v(t) \sim(9-5 \sqrt{3}) \theta(t)^{-2}+3 C_{1 u} \theta(t)^{-4+2 \sqrt{3}}-\frac{1+10 \sqrt{3}}{26} C_{1 u}^{2} \theta(t)^{-6+4 \sqrt{3}}
\end{aligned}
$$

as $t \rightarrow t_{\text {max }}$.
Case 2. $U_{0}=3+\sqrt{3}$.
In this case, the blow-up power-determining matrix $A$ is

$$
A=\left(\begin{array}{cc}
2(3+\sqrt{3})-1 & -1 \\
(3+\sqrt{3})^{2} & -2
\end{array}\right)=\left(\begin{array}{cc}
5+2 \sqrt{3} & -1 \\
12+6 \sqrt{3} & -2
\end{array}\right)
$$

and the eigenvalues solve

$$
\begin{aligned}
(\lambda-5-2 \sqrt{3})(\lambda+2)+(12+6 \sqrt{3}) & =\lambda^{2}+(-3-2 \sqrt{3}) \lambda-2(5+2 \sqrt{3})+12+6 \sqrt{3} \\
& =\lambda^{2}+(-3-2 \sqrt{3}) \lambda+2+2 \sqrt{3}=0
\end{aligned}
$$

The eigenvalues are

$$
\lambda=1, \quad 2+2 \sqrt{3}
$$

both of which are positive. Therefore these eigenvalues make no contributions to multi-order asymptotic expansions. In particular, we know that $U_{n}(t) \equiv 0, V_{n}(t) \equiv 0$ for all $n \geq 1$ and hence the first order asymptotic expansion

$$
u(t) \sim(3+\sqrt{3}) \theta(t)^{-1}, \quad v(t) \sim(9+5 \sqrt{3}) \theta(t)^{-2}
$$

indeed expresses the exact solution of (4.49). As a summary, we obtain the following result.
Theorem 4.17. The quasi-homogeneous system 4.49) admits blow-up solutions with the following third-order asymptotic expansions as $t \rightarrow t_{\text {max }}$ :

$$
\begin{aligned}
& u(t) \sim(3-\sqrt{3}) \theta(t)^{-1}+C \theta(t)^{-3+2 \sqrt{3}}-\frac{3+4 \sqrt{3}}{26} C^{2} \theta(t)^{-5+4 \sqrt{3}} \\
& v(t) \sim(9-5 \sqrt{3}) \theta(t)^{-2}+3 C \theta(t)^{-4+2 \sqrt{3}}-\frac{1+10 \sqrt{3}}{26} C^{2} \theta(t)^{-6+4 \sqrt{3}}
\end{aligned}
$$

with a free parameter $C$, and

$$
u(t) \sim(3+\sqrt{3}) \theta(t)^{-1}, \quad v(t) \sim(9+5 \sqrt{3}) \theta(t)^{-2}
$$

The latter solution is indeed the exact solution of 4.49.

### 4.5.2 Blow-up asymptotics for (4.46)

We go back to the original system (4.46). Notice first that the balance law for (4.46) is identical with that of (4.49) because the quasi-homogeneous components are identical. Substituting the asymptotic form (4.50) into (4.46), we have

$$
\begin{align*}
\frac{d}{d t}\left(\sum_{n=1}^{\infty} U_{n}\right)= & \theta(t)^{-1}\left\{-\left(\sum_{n=1}^{\infty} U_{n}\right)+2 U_{0}\left(\sum_{n=1}^{\infty} U_{n}\right)+\left(\sum_{n=1}^{\infty} U_{n}\right)^{2}-\left(\sum_{n=1}^{\infty} V_{n}\right)\right\} \\
\frac{d}{d t}\left(\sum_{n=1}^{\infty} V_{n}\right)= & \theta(t)^{-1}\left\{-2\left(\sum_{n=1}^{\infty} V_{n}\right)+U_{0}^{2}\left(\sum_{n=1}^{\infty} U_{n}\right)+U_{0}\left(\sum_{n=1}^{\infty} U_{n}\right)^{2}+\frac{1}{3}\left(\sum_{n=1}^{\infty} U_{n}\right)^{3}\right\}  \tag{4.56}\\
& -\theta(t)\left(\sum_{n=0}^{\infty} U_{n}\right),
\end{align*}
$$

where the balance law (4.51) is applied to eliminating the principal terms. The governing system for $\left(U_{1}(t), V_{1}(t)\right)$ is

$$
\frac{d}{d t}\binom{U_{1}}{V_{1}}=\theta(t)^{-1}\left\{\left(\begin{array}{cc}
-1 & 0  \tag{4.57}\\
0 & -2
\end{array}\right)+\left(\begin{array}{cc}
2 U_{0} & -1 \\
U_{0}^{2} & 0
\end{array}\right)\right\}\binom{U_{1}}{V_{1}}-\theta(t)\binom{0}{U_{0}}
$$

where the lower order term $-\theta(t)\left(0, U_{0}\right)^{T}$ is added to 4.53). Because the linear (homogeneous) part is determined by the blow-up power-determining matrix $A$, the nonsingular matrix $P$ diagonalizing $A$ can be applied to (4.57). The concrete forms of solutions depend on the choice of $U_{0}$, like (4.49).

Case 1. $U_{0}=3-\sqrt{3}$.
In this case, the matrix $P$ is given in (4.54), and $\rho=2 \sqrt{3}-2(>0)$ is used to determine the order of $\theta(t)$. The general solution of (4.57) is then

$$
\binom{U_{1}(t)}{V_{1}(t)}=P\binom{\theta(t)^{-1}\left\{-\int_{t}^{t_{\max }} \theta(s)^{1+1}\left(p^{11} 0+p^{12}\left(-U_{0}\right)\right) d s\right\}}{\theta(t)^{r}\left\{\tilde{c}_{2}+\int_{0}^{t} \theta(s)^{-\rho+1}\left(p^{21} 0+p^{22}\left(-U_{0}\right)\right) d s\right\}}
$$

The integrals are calculated to obtain

$$
\begin{aligned}
& -\int_{t}^{t_{\max }} \theta(s)^{1+1}\left(p^{11} 0+p^{12}\left(-U_{0}\right)\right) d s=-p^{12}\left(-U_{0}\right) \int_{t}^{t_{\max }} \theta(s)^{2} d s \\
& \quad=-\frac{3-\sqrt{3}}{3(-1+2 \sqrt{3})} \theta(t)^{3} \\
& \quad=-\frac{-3+5 \sqrt{3}}{33} \theta(t)^{3}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{t} \theta(s)^{-\rho+1}\left(p^{21} 0+p^{22}\left(-U_{0}\right)\right) d s=p^{22}\left(-U_{0}\right) \int_{0}^{t} \theta(s)^{-\rho+1} d s \\
& \quad=\frac{3-\sqrt{3}}{(-1+2 \sqrt{3})(-\rho+2)} \theta(t)^{-\rho+2}+C \\
& \quad=\frac{3-\sqrt{3}}{(-1+2 \sqrt{3})(4-2 \sqrt{3})} \theta(t)^{-\rho+2}+C \\
& \quad=\frac{9+7 \sqrt{3}}{22} \theta(t)^{-r+2}+C
\end{aligned}
$$

with the integral constant $C$. Therefore

$$
\binom{U_{1}(t)}{V_{1}(t)}=P\binom{\theta(t)^{-1}\left\{-\frac{-3+5 \sqrt{3}}{33} \theta(t)^{3}\right\}}{\theta(t)^{\rho}\left\{C_{1 u}+\frac{9+\sqrt{3}}{22} \theta(t)^{-\rho+2}\right\}}
$$

with a constant $C_{1 u}$. In particular,

$$
\begin{align*}
\binom{U_{1}(t)}{V_{1}(t)} & =\left(\begin{array}{cc}
1 & 1 \\
4-2 \sqrt{3} & 3
\end{array}\right)\binom{\frac{3-5 \sqrt{3}}{33} \theta(t)^{2}}{C_{1 u} \theta(t)^{\rho}+\frac{9+7 \sqrt{3}}{22} \theta(t)^{2}}  \tag{4.58}\\
& =\binom{C_{1 u} \theta(t)^{\rho}+\left(\frac{3-5 \sqrt{3}}{33}+\frac{9+7 \sqrt{3}}{22}\right) \theta(t)^{2}}{3 C_{1 u} \theta(t)^{\rho}+\left(\frac{(3-5 \sqrt{3})(4-2 \sqrt{3})}{32}+\frac{3(9+7 \sqrt{3})}{22}\right) \theta(t)^{2}} \\
& =\binom{C_{1 u} \theta(t)^{\rho}+\left(\frac{2(3-5 \sqrt{3})}{66}+\frac{3(9+7 \sqrt{3})}{66}\right) \theta(t)^{2}}{3 C_{1 u} \theta(t)^{\rho}+\left(\frac{2(42-26 \sqrt{3})}{66}+\frac{9(9+7 \sqrt{3})}{66}\right) \theta(t)^{2}} \\
& =\binom{C_{1 u} \theta(t)^{\rho}+\left(\frac{(6-10 \sqrt{3})}{66}+\frac{(27+21 \sqrt{3})}{66}\right) \theta(t)^{2}}{3 C_{1 u} \theta(t)^{\rho}+\left(\frac{(8-52 \sqrt{3})}{66}+\frac{(81+63 \sqrt{3})}{66}\right) \theta(t)^{2}} \\
& =\binom{C_{1 u} \theta(t)^{\rho}+\frac{3+\sqrt{3}}{6} \theta(t)^{2}}{3 C_{1 u} \theta(t)^{\rho}+\frac{15+\sqrt{3}}{6} \theta(t)^{2}} .
\end{align*}
$$

As a consequence, we have the second order asymptotic expansion of blow-up solutions $(u(t), v(t))$ with $U_{0}=3-\sqrt{3}$ as follows:

$$
\begin{align*}
& u(t) \sim(3-\sqrt{3}) \theta(t)^{-1}+C_{1 u} \theta(t)^{-3+2 \sqrt{3}}+\frac{3+\sqrt{3}}{6} \theta(t), \\
& v(t) \sim(9-5 \sqrt{3}) \theta(t)^{-2}+3 C_{1 u} \theta(t)^{-4+2 \sqrt{3}}+\frac{15+\sqrt{3}}{6} \tag{4.59}
\end{align*}
$$

as $t \rightarrow t_{\max }$ with a constant $C_{1 u}$. In the present case, the additional terms of $O(\theta(t))$ in $u$ and of $O(1)$ in $v$, respectively, are added as the contribution of the lower-order term $f_{\text {res }}$. We observe that $\rho$ and $2=k+\alpha_{2}-1$ are exponents of $\theta(t)$ determining $(U(t), V(t))$. The exponent $\rho$ comes from blow-up power eigenvalues (recall that $-\rho$ is one of such eigenvalues), while 2 comes from $f_{\text {res }}$. Because $\rho$ and 2 are $\mathbb{Z}$-linearly independent, coefficients of all terms in (4.59) are uniquely
determined. In fact, when we consider the governing system for the third term $\left(U_{2}(t), V_{2}(t)\right)$ given by

$$
\begin{equation*}
\frac{d}{d t}\binom{U_{2}}{V_{2}} \approx \theta(t)^{-1}\left\{A\binom{U_{2}}{V_{2}}+\binom{U_{1}^{2}}{U_{0} U_{1}^{2}}\right\}-\theta(t)\binom{0}{U_{1}} \tag{4.60}
\end{equation*}
$$

we see that the solution has the form

$$
\begin{aligned}
& P^{-1}\binom{U_{2}(t)}{V_{2}(t)} \\
& =\binom{\theta(t)^{-1}\left[-\int_{t}^{t_{\max }} \theta(s)^{1-1}\left(c_{11} \theta(s)^{2 \rho}+c_{12} \theta(s)^{2+\rho}+c_{13} \theta(s)^{4}\right) d s-\int_{t}^{t_{\max }} \theta(s)^{1+1}\left(c_{14} \theta(s)^{\rho}+c_{15} \theta(s)^{2}\right) d s\right]}{\theta(t)^{\rho}\left[\tilde{c}_{2}+\int_{0}^{t} \theta(s)^{-\rho-1}\left(c_{21} \theta(s)^{2 \rho}+c_{22} \theta(s)^{2+\rho}+c_{23} \theta(s)^{4}\right) d s+\int_{0}^{t} \theta(s)^{-\rho+1}\left(c_{24} \theta(s)^{\rho}+c_{25} \theta(s)^{2}\right) d s\right]} \\
& =\binom{\tilde{c}_{11} \theta(t)^{2 \rho}+\tilde{c}_{12} \theta(t)^{2+\rho}+\tilde{c}_{13} \theta(t)^{4}+\tilde{c}_{14} \theta(t)^{2+\rho}+\tilde{c}_{15} \theta(t)^{4}}{\tilde{c}_{21} \theta(t)^{2 \rho}+\tilde{c}_{22} \theta(t)^{2+\rho}+\tilde{c}_{23} \theta(t)^{4}+\tilde{c}_{24} \theta(t)^{2+\rho}+\tilde{c}_{25} \theta(t)^{4}}
\end{aligned}
$$

where the constant $\tilde{c}_{2}$ is chosen so that the constant term vanishes in the second component (cf. (3.19) ). All possible exponents appeared in the above solution does not contain 2 and $\rho$. In fact, all components appeared here has the form $2 \beta_{1}+\rho \beta_{2}$, where $\beta_{1}, \beta_{2} \in \mathbb{Z}_{\geq 0}$ satisfying $\beta_{1}+\beta_{2}=2$. By induction, we can prove that all possible exponents appeared in $\mathbf{Y}_{n}(t) \equiv\left(U_{n}(t), V_{n}(t)\right)$ as the form $2 \beta_{1}+\rho \beta_{2}$, where $\beta_{1}, \beta_{2} \in \mathbb{Z}_{\geq 0}$ satisfying $\beta_{1}+\beta_{2}=n$. For example, the governing system determining $\mathbf{Y}_{3}(t) \equiv\left(U_{3}(t), V_{3}(\bar{t})\right.$ ) is, combining the neglected terms in the derivation of $\mathbf{Y}_{2}(t)=\left(U_{2}(t), V_{2}(t)\right)$,

$$
\begin{equation*}
\frac{d}{d t}\binom{U_{3}}{V_{3}} \approx \theta(t)^{-1}\left\{A\binom{U_{3}}{V_{3}}+\binom{2 U_{1} U_{2}+U_{2}^{2}}{U_{1}^{3}+2 U_{0} U_{1} U_{2}+U_{0} U_{2}^{2}+\frac{1}{3} U_{1}^{3}+U_{1}^{2} U_{2}+U_{1} U_{2}^{2}}\right\}-\theta(t)\binom{0}{U_{2}} \tag{4.61}
\end{equation*}
$$

and all integrands appeared in solving $\mathbf{Y}_{3}(t)$ consist of $\theta(t)^{\mu}$ with $\mu=2 \beta_{1}+\rho \beta_{2}-1$, where $\beta_{1}, \beta_{2} \in \mathbb{Z}_{\geq 0}$ with $\beta_{1}+\beta_{2}=3$. Therefore the integration of (4.61) yields our claim with $n=3$. The general case can also be treated in the similar way. By the $\mathbb{Z}$-linear independence of 2 and $\rho$, all terms appeared in $\left(U_{n}(t), V_{n}(t)\right)$ are uniquely determined for each $n \geq 0$. In particular, (4.59) is the second order exact asymptotic expansion of $(u(t), v(t))$ as $t \rightarrow t_{\max }$.

Case 2. $U_{0}=3+\sqrt{3}$.
In the similar way to the case $U_{0}=3-\sqrt{3}$, the asymptotic expansion of the blow-up solution with $U_{0}=3+\sqrt{3}$ can be calculated. Unlike the fully quasi-homogeneous case, the lower-order part $f_{\text {res }}$ makes a contribution to determine the (nontrivial) asymptotic behavior.

To this end, calculate the nonsingular matrix $P_{+}$diagonalizing $A$ with $U_{0}=3+\sqrt{3}$, which is constructed through calculations of eigenvectors:

$$
\left.\begin{array}{rl}
\left(\begin{array}{cc}
5+2 \sqrt{3} & -1 \\
12+6 \sqrt{3} & -2
\end{array}\right)\binom{a}{b}=\binom{a}{b} & \Rightarrow\binom{a}{b}=\binom{1}{4+2 \sqrt{3}}, \\
\left(\begin{array}{c}
5+2 \sqrt{3} \\
12+6 \sqrt{3}
\end{array}-2\right.
\end{array}\right)\binom{a}{b}=(2+2 \sqrt{3})\binom{a}{b} \quad \Rightarrow \quad\binom{a}{b}=\binom{1}{3} .
$$

Let $P_{+}$be then the following matrix:

$$
P_{+}=\left(\begin{array}{cc}
1 & 1 \\
4+2 \sqrt{3} & 3
\end{array}\right) \quad \Leftrightarrow \quad P_{+}^{-1} \equiv\left(\begin{array}{cc}
p_{+}^{11} & p_{+}^{12} \\
p_{+}^{21} & p_{+}^{22}
\end{array}\right)=\frac{-1}{1+2 \sqrt{3}}\left(\begin{array}{cc}
3 & -1 \\
-(4+2 \sqrt{3}) & 1
\end{array}\right)
$$

The general solution of (4.57) is written by

$$
\binom{U_{1}(t)}{V_{1}(t)}=P_{+}\binom{\theta(t)^{-1}\left\{-\int_{t}^{t_{\max }} \theta(s)^{1+1}\left(p_{+}^{11} 0+p_{+}^{12}\left(-U_{0}\right)\right) d s\right\}}{\theta(t)^{-\rho_{+}}\left\{-\int_{t}^{t_{\max }} \theta(s)^{\rho_{+}+1}\left(p_{+}^{21} 0+p_{+}^{22}\left(-U_{0}\right)\right) d s\right\}}
$$

where $\rho_{+}=2+2 \sqrt{3}$. The integrals are calculated to obtain

$$
\begin{aligned}
& -\int_{t}^{t_{\max }} \theta(s)^{1+1}\left(p_{+}^{11} 0+p_{+}^{12}\left(-U_{0}\right)\right) d s=-p_{+}^{12}\left(-U_{0}\right) \int_{t}^{t_{\max }} \theta(s)^{2} d s \\
& \quad=\frac{(3+\sqrt{3})}{3(1+2 \sqrt{3})} \theta(t)^{3} \\
& \quad=\frac{3+5 \sqrt{3}}{33} \theta(t)^{3}
\end{aligned}
$$

and

$$
\begin{aligned}
& -\int_{t}^{t_{\max }} \theta(s)^{\rho_{+}+1}\left(p_{+}^{21} 0+p_{+}^{22}\left(-U_{0}\right)\right) d s=-p_{+}^{22}\left(-U_{0}\right) \int_{t}^{t_{\max }} \theta(s)^{\rho_{+}+1} d s \\
& \quad=-\frac{3+\sqrt{3}}{(1+2 \sqrt{3})\left(\rho_{+}+2\right)} \theta(t)^{\rho_{+}+2} \\
& \quad=-\frac{3+\sqrt{3}}{(1+2 \sqrt{3})(4+2 \sqrt{3})} \theta(t)^{\rho_{+}+2} \\
& \quad=\frac{9-7 \sqrt{3}}{22} \theta(t)^{\rho_{+}+2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \binom{U_{1}(t)}{V_{1}(t)}=P_{+}\binom{\theta(t)^{-1}\left\{\frac{3+5 \sqrt{3}}{33} \theta(t)^{3}\right\}}{\theta(t)^{-\rho_{+}}\left\{\frac{9-7 \sqrt{3}}{22} \theta(t)^{\rho_{+}+2}\right\}} \\
& =\left(\begin{array}{cc}
1 & 1 \\
4+2 \sqrt{3} & 3
\end{array}\right)\binom{\frac{3+5 \sqrt{3}}{33} \theta(t)^{2}}{\frac{9-7 \sqrt{3}}{22} \theta(t)^{2}} \\
& =\binom{\left(\frac{3+5 \sqrt{3}}{33}+\frac{9-7 \sqrt{3}}{22}\right) \theta(t)^{2}}{\left(-\frac{(3+5 \sqrt{3})(4+2 \sqrt{3})}{33}+\frac{3(9-7 \sqrt{3})}{22}\right) \theta(t)^{2}} \\
& =\binom{\frac{3-\sqrt{3}}{6} \theta(t)^{2}}{\frac{15-\sqrt{3}}{6} \theta(t)^{2}} \text {. }
\end{aligned}
$$

As a consequence, we have the second order asymptotic expansion of blow-up solutions $(u(t), v(t))$ with $U_{0}=3+\sqrt{3}$ as follows:

$$
u(t) \sim(3+\sqrt{3}) \theta(t)^{-1}+\frac{3-\sqrt{3}}{6} \theta(t), \quad v(t) \sim(9+5 \sqrt{3}) \theta(t)^{-2}+\frac{15-\sqrt{3}}{6}
$$

as $t \rightarrow t_{\text {max }}$.
Theorem 4.18. The system 4.46) admits the following two families of blow-up solutions as $t \rightarrow t_{\text {max }}$ :

$$
\begin{align*}
& u(t) \sim(3-\sqrt{3}) \theta(t)^{-1}+C \theta(t)^{-3+2 \sqrt{3}}+\frac{3+\sqrt{3}}{6} \theta(t) \\
& v(t) \sim(9-5 \sqrt{3}) \theta(t)^{-2}+3 C \theta(t)^{-4+2 \sqrt{3}}+\frac{15+\sqrt{3}}{6} \tag{4.62}
\end{align*}
$$

with a free parameter $C \in \mathbb{R}$ and

$$
\begin{equation*}
u(t) \sim(3+\sqrt{3}) \theta(t)^{-1}+\frac{3-\sqrt{3}}{6} \theta(t), \quad v(t) \sim(9+5 \sqrt{3}) \theta(t)^{-2}+\frac{15-\sqrt{3}}{6} \tag{4.63}
\end{equation*}
$$

Remark 4.19. Define $\iota: \mathbb{Q}[\sqrt{3}] \rightarrow \mathbb{Q}[\sqrt{3}]$ by

$$
\iota(1)=1, \quad \iota(\sqrt{3})=-\sqrt{3}
$$

and extend linearly on $\mathbb{Q}[\sqrt{3}] \equiv\{a+b \sqrt{3} \mid a, b \in \mathbb{Q}\}$. Then $\iota^{2}=\mathrm{id}$ holds. That is, $\iota$ is an involution. We first observe in 4.46$)$ that roots of the balance law $\left(U_{0}, V_{0}\right)=(3 \pm \sqrt{3}, 9 \pm 5 \sqrt{3})$ are mapped to each other via $\iota$. 4.62) with $C=0$ and 4.63) indicate that the second order asymptotic expansions are also mapped via ८ by mapping individual coefficients, which would expect the involution, or more general symmetry correspondence among different blow-up solutions, possibly with a specific choice of free parameters.

### 4.6 An artificial system in the presence of Jordan blocks

The next example concerns with an artificial system such that the blow-up power-determining matrix has a non-trivial Jordan block. We see through this example that multi-order asymptotic expansion of blow-ups can be executed regardless of the presence of Jordan blocks.

### 4.6.1 The presence of terms of order $k+\alpha_{i}-1$

First we consider

$$
\begin{equation*}
u^{\prime}=u^{2}+v, \quad v^{\prime}=a u^{3}+3 u v-u^{2} \tag{4.64}
\end{equation*}
$$

where $a \in \mathbb{R}$ is a parameter. This system is asymptotically quasi-homogeneous of type $\alpha=(1,2)$ and order $k+1=2$, consisting of the quasi-homogeneous part $f_{\alpha, k}$ and the lower-order part $f_{\text {res }}$ given as follows:

$$
\begin{equation*}
f_{\alpha, k}(u, v)=\binom{u^{2}+v}{a u^{3}+3 u v}, \quad f_{\mathrm{res}}(u, v)=\binom{0}{-u^{2}} . \tag{4.65}
\end{equation*}
$$

Assume that the system (4.64) admits a blow-up solution with the following asymptotic behavior:

$$
u(t)=O\left(\theta(t)^{-1}\right), \quad v(t)=O\left(\theta(t)^{-2}\right) \quad \text { as } \quad t \rightarrow t_{\max }-0
$$

Under this assumption and the associated ansatz

$$
\begin{array}{ll}
u(t)=\theta(t)^{-1} U(t) \equiv \theta(t)^{-1} \sum_{n=0}^{\infty} U_{n}(t), & U_{n}(t) \ll U_{n-1}(t), \\
\lim _{t \rightarrow t_{\max }} U_{n}(t)=U_{0}  \tag{4.66}\\
v(t)=\theta(t)^{-2} V(t) \equiv \theta(t)^{-2} \sum_{n=0}^{\infty} V_{n}(t), & V_{n}(t) \ll V_{n-1}(t), \quad \lim _{t \rightarrow t_{\max }} V_{n}(t)=V_{0}
\end{array}
$$

we solve the balance law and investigate the eigenstructure of the associated blow-up power eigenvalues. The balance law is

$$
\binom{U_{0}}{2 V_{0}}=\binom{U_{0}^{2}+V_{0}}{a U_{0}^{3}+3 U_{0} V_{0}}
$$

Our particular interest here is the case $a=0$, in which case the root is $\left(U_{0}, V_{0}\right)=(1,0)$. We fix $a=0$ for a while. The blow-up power-determining matrix at $\left(U_{0}, V_{0}\right)$ is

$$
A=\left(\begin{array}{cc}
-1 & 0 \\
0 & -2
\end{array}\right)+\left(\begin{array}{cc}
2 U_{0} & 1 \\
3 V_{0} & 3 U_{0}
\end{array}\right)=\left(\begin{array}{cc}
1 & 1 \\
0 & 1
\end{array}\right)
$$

that is, the matrix $A$ has nontrivial Jordan block. The governing system for determining $\left(U_{1}(t), V_{1}(t)\right)$ is therefore

$$
\frac{d}{d t}\binom{U_{1}}{V_{1}}=\theta(t)^{-1}\left(\begin{array}{ll}
1 & 1  \tag{4.67}\\
0 & 1
\end{array}\right)\binom{U_{1}}{V_{1}}-\binom{0}{U_{0}^{2}}=\theta(t)^{-1}\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\binom{U_{1}}{V_{1}}-\binom{0}{1}
$$

The fundamental matrix of the homogeneous part and its inverse are

$$
\Phi_{2}(t ; 1)=\left(\begin{array}{cc}
\theta(t)^{-1} & \theta(t)^{-1} \ln \left(\theta(t)^{-1}\right) \\
0 & \theta(t)^{-1}
\end{array}\right), \quad \Phi_{2}(t ; 1)^{-1}=\left(\begin{array}{cc}
\theta(t) & -\theta(t) \ln \left(\theta(t)^{-1}\right) \\
0 & \theta(t)
\end{array}\right)
$$

respectively. Because all eigenvalues of $A$ are positive, $\operatorname{Spec}(A)$ makes no contribution to determine the order of $\theta(t)$, while the combination of $\Phi_{2}(t ; 1)$ and the inhomogeneous term $(0,-1)^{T}$ induce the successive terms of $(U(t), V(t))$. The general solution of (4.67) satisfying the asymptotic assumption (4.66) is

$$
\begin{aligned}
\binom{U_{1}(t)}{V_{1}(t)} & =-\Phi_{2}(t ; 1) \int_{t}^{t_{\max }} \Phi_{2}(s ; 1)^{-1}\binom{0}{-1} d s \\
& =-\Phi_{2}(t ; 1) \int_{t}^{t_{\max }}\binom{\theta(s) \ln \left(\theta(s)^{-1}\right)}{-\theta(s)} d s \\
& =\left(\begin{array}{cc}
\theta(t)^{-1} & \theta(t)^{-1} \ln \left(\theta(t)^{-1}\right) \\
0 & \theta(t)^{-1}
\end{array}\right)\binom{-\frac{1}{4} \theta(t)^{2}\left\{2 \ln \left(\theta(t)^{-1}\right)+1\right\}}{\frac{1}{2} \theta(t)^{2}} \\
& =\binom{-\frac{1}{4} \theta(t)\left\{2 \ln \left(\theta(t)^{-1}\right)+1\right\}+\frac{1}{2} \theta(t) \ln \left(\theta(t)^{-1}\right)}{\frac{1}{2} \theta(t)} \\
& =\frac{1}{4}\binom{-\theta(t)}{2 \theta(t)} .
\end{aligned}
$$

Now the constant $\delta$ defined in (3.28) is estimated as follows. First, $\alpha=(1,2)$ and $k=1$. From (4.65) we know that $\left(\gamma_{1}, \gamma_{2}\right)=(+\infty,-2)$. Because there are no negative blow-up power eigenvalues, $\delta$ is evaluated as

$$
\delta=\min \left\{\frac{1}{1}+\infty, \frac{2}{1}-2\right\}+1=1
$$

Proposition 3.13 indicates that $\operatorname{ord}_{\theta}\left(\mathbf{Y}_{m}\right) \geq m$ for $m \geq 2$ and hence there is no term $\theta(t)^{m}$ with $m \leq 1$ in $\mathbf{Y}_{n}(t)=\left(U_{n}(t), V_{n}(t)\right), n \geq 2$. In particular, we have the second order asymptotic expansion of $(u(t), v(t))$ as $t \rightarrow t_{\max }-0$ :

$$
u(t) \sim \theta(t)^{-1}-\frac{1}{4}, \quad v(t) \sim \frac{1}{2} \theta(t)^{-1}
$$

The governing equation for the third term $\left(U_{2}(t), V_{2}(t)\right)$ is

$$
\begin{align*}
\frac{d}{d t}\binom{U_{2}}{V_{2}} & =\theta(t)^{-1}\left\{\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\binom{U_{2}}{V_{2}}+\binom{U_{1}^{2}}{3 U_{1} V_{1}}\right\}-\binom{0}{2 U_{1}} \\
& =\theta(t)^{-1}\left\{\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\binom{U_{2}}{V_{2}}+\binom{\frac{1}{16} \theta(t)^{2}}{-\frac{3}{8} \theta(t)^{2}}\right\}+\frac{1}{2}\binom{0}{\theta(t)} \\
& =\theta(t)^{-1}\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\binom{U_{2}}{V_{2}}+\frac{1}{16}\binom{\theta(t)}{2 \theta(t)} . \tag{4.68}
\end{align*}
$$

The solution is therefore

$$
\begin{aligned}
\binom{U_{2}(t)}{V_{2}(t)} & =-\frac{1}{16} \Phi_{2}(t ; 1) \int_{t}^{t_{\max }}\left(\begin{array}{cc}
\theta(s) & -\theta(s) \ln \left(\theta(s)^{-1}\right) \\
0 & \theta(s)
\end{array}\right)\binom{\theta(s)}{2 \theta(s)} d s \\
& =-\frac{1}{16} \Phi_{2}(t ; 1) \int_{t}^{t_{\max }}\binom{\theta(s)^{2}-2 \theta(s)^{2} \ln \left(\theta(s)^{-1}\right)}{2 \theta(s)^{2}} d s \\
& =\frac{1}{16}\left(\begin{array}{cc}
\theta(t)^{-1} & \theta(t)^{-1} \ln \left(\theta(t)^{-1}\right) \\
0 & \theta(t)^{-1}
\end{array}\right)\binom{-\frac{1}{3} \theta(t)^{3}-2\left[-\frac{1}{9} \theta(t)^{3}\left\{3 \ln \left(\theta(t)^{-1}\right)+1\right\}\right]}{-\frac{2}{3} \theta(t)^{3}} \\
& =\frac{1}{16}\binom{-\frac{1}{3} \theta(t)^{2}-2\left[-\frac{1}{9} \theta(t)^{2}\left\{3 \ln \left(\theta(t)^{-1}\right)+1\right\}\right]-\frac{2}{3} \theta(t)^{2} \ln \left(\theta(t)^{-1}\right)}{-\frac{2}{3} \theta(t)^{2}} \\
& =\frac{-1}{144}\binom{\theta(t)^{2}}{6 \theta(t)^{2}} .
\end{aligned}
$$

As a summary, we obtain the following result.
Theorem 4.20. The system 4.64) with $a=0$ admits a blow-up solution with the following thirdorder asymptotic expansion as $t \rightarrow t_{\max }-0$ :

$$
u(t) \sim \theta(t)^{-1}-\frac{1}{4}-\frac{1}{144} \theta(t), \quad v(t) \sim \frac{1}{2} \theta(t)^{-1}-\frac{1}{24} .
$$

### 4.6.2 The absence of terms of order $k+\alpha_{i}-1$

Here we consider

$$
\begin{equation*}
u^{\prime}=u^{2}+v, \quad v^{\prime}=a u^{3}+3 u v-u \tag{4.69}
\end{equation*}
$$

with a real parameter $a \in \mathbb{R}$, instead of (4.64). The difference from (4.64) is the replacement of $-u^{2}$ by $-u$ in the second component of the vector fields. Because the quasi-homogeneous part is unchanged, the balance law and blow-up power-determining matrix are the same as those of (4.64). Now we consider the asymptopic expansion (4.66) for (4.69):

$$
\frac{d}{d t}\binom{U_{1}}{V_{1}}=\theta(t)^{-1}\left(\begin{array}{ll}
1 & 1  \tag{4.70}\\
0 & 1
\end{array}\right)\binom{U_{1}}{V_{1}}-\binom{0}{\theta(t)}
$$

The solution satisfying the asymptotic assumption (4.66) is

$$
\begin{aligned}
\binom{U_{1}(t)}{V_{1}(t)} & =-\Phi_{2}(t ; 1) \int_{t}^{t_{\max }} \Phi_{2}(s ; 1)^{-1}\binom{0}{-\theta(s)} d s \\
& =-\Phi_{2}(t ; 1) \int_{t}^{t_{\max }}\binom{\theta(s)^{2} \ln \left(\theta(s)^{-1}\right)}{-\theta(s)^{2}} d s \\
& =\left(\begin{array}{cc}
\theta(t)^{-1} & \theta(t)^{-1} \ln \left(\theta(t)^{-1}\right) \\
0 & \theta(t)^{-1}
\end{array}\right)\binom{-\frac{1}{9} \theta(t)^{3}\left\{3 \ln \left(\theta(t)^{-1}\right)+1\right\}}{\frac{1}{3} \theta(t)^{3}} \\
& =\binom{-\frac{1}{9} \theta(t)^{2}\left\{3 \ln \left(\theta(t)^{-1}\right)+1\right\}+\frac{1}{3} \theta(t)^{2} \ln \left(\theta(t)^{-1}\right)}{\frac{1}{3} \theta(t)^{2}} \\
& =\frac{1}{9}\binom{-\theta(t)^{2}}{3 \theta(t)^{2}} .
\end{aligned}
$$

Similar to (4.64), the constant $\delta$ in (3.28) is estimated as

$$
\delta=\min \left\{\frac{1}{1}+\infty, \frac{2}{1}-1\right\}+1=2
$$

where we have used $\left(\gamma_{1}, \gamma_{2}\right)=(+\infty,-1)$. Proposition 3.13 indicates that $\operatorname{ord}_{\theta}\left(\mathbf{Y}_{m}\right) \geq 2 m$ for $m \geq 2$ and hence there is no term $\theta(t)^{m}$ with $m \leq 2$ in $\mathbf{Y}_{n}(t)=\left(U_{n}(t), V_{n}(t)\right), n \geq 2$. As a summary, we have the following result.
Theorem 4.21. The system 4.69) with $a=0$ admits a blow-up solution with the following second-order asymptotic expansion as $t \rightarrow t_{\max }-0$ :

$$
u(t) \sim \theta(t)^{-1}-\frac{1}{9} \theta(t), \quad v(t) \sim \frac{1}{3}
$$

In contrast to Theorem 4.20 for (4.64), $v(t)$ remains bounded as $t \rightarrow t_{\max }$, while $u(t)$ blows up 15. Another interesting observation here is the absence of logarithmic terms in the asymptotic expansion, although the fundamental matrix $\Phi_{2}(t ; 1)$ includes the logarithmic terms. In practical calculations, there are cancellations of logarithmic terms in integration of inhomogeneous terms, which can also occur in calculations of higher-order terms $\left(U_{n}(t), V_{n}(t)\right), n \geq 3$.

## Concluding Remarks

In this paper, we have provided a systematic methodology to derive asymptotic expansions of blow-up solutions in arbitrary orders for autonomous ODEs possessing quasi-homogeneity in an

[^13]asymptotic sense, as well as their with their justification, and parameter dependence under mild assumptions. Through our proposed methodology with several examples, we see that

- only fundamental solver of linear ODEs is necessary to derive asymptotic expansions of an arbitrary order, at least in the formal sense;
- all possible powers of $\theta(t)$ can be extracted even if the series is not a power series;
- dependence of powers of $\theta(t)$ on parameters in systems can be clearly extracted;
- low singular or regular behavior of solutions, corresponding to 0 -valued components in roots of the balance law, can be also extracted through the present methodology, while they have not been extracted from dynamics at infinity in preceding results (cf. [28] whose details are briefly reviewed in Part II [23]).

We end this paper by providing several discussions towards the extension of our present arguments and perspectives of blow-up solutions, as well as a short introduction to Part II [23].

## Beyond Type-I Blow-Ups

The present methodology for asymptotic expansions is based on Assumption 3.1. that is, the blowup is assumed to be type-I. There are another type of blow-up behavior whose singularity as a function of $\theta(t)$ is stronger than type-I blow-ups, which are often called type-II blow-up\& ${ }^{16}$. In [29], dynamical aspects of type-II blow-ups and grow-ups; divergence of solutions with $t_{\max }=\infty$, are studied, where non-hyperbolic equilibria or invariant sets at infinity play key roles in characterization of such asymptotic behavior. We have seen in the present arguments that hyperbolicity of linearized matrices for systems of our interests are essential to determine type-I blow-up behavior and their asymptotic expansions, indicating that non-hyperbolicity can induce completely different situation in the study of asymptotic expansions. Moreover, according to [29], type-II blow-up mechanism has various aspects and hence studies for individual systems should be collected towards a comprehensive understandings of asymptotic expansions of blow-ups beyond type-I.

## Short Introduction to Part II [23]

We briefly introduce arguments in Part II 23. As mentioned in Introduction, a dynamical characterization of blow-up solutions in terms of dynamics at infinity is proposed by the third author and his collaborators. In the present paper, on the other hand, a systematic methodology for calculating multi-order asymptotic expansions of blow-up solutions is proposed, implying characterization of blow-up solutions from different viewpoints (e.g., [28]. See Introduction for more references). Now we have two characterizations for identical blow-up solutions. It is then natural to ask the correspondence between these two characterizations.

In Part II [23], we address a correspondence of blow-up characterizations. More precisely, we correspond roots of the balance law (Definition (3.2) and eigenstructures of the blow-up powerdetermining matrices (Definition (3.3) to "equilibria at infinity" and the associated eigenstructure of linearized systems. The correspondence implies that algebraic objects determining asymptotic expansions characterize dynamical properties of blow-ups, such as existence, persistence under

[^14]perturbation of initial points, and blow-up rates. As a corollary, we shall prove that the balance law and blow-up power-determining matrix themselves provide a criterion of the existence of blow-up solutions which we are interested in (remark that, throughout our arguments in the present paper, the existence of blow-ups is assumed). The dynamical interpretation of our present methodology through arguments in Part II [23] will give a new insight into characterizations of blow-up solutions.

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## A Tools

## A. 1 Right-sided compactifications for asymptotically autonomous systems

In several results, time-variable compactifications for asymptotically autonomous systems introduced in e.g. [34] are applied to completing our proofs discussed in the next section. These advanced notions in dynamical systems are briefly summarized in the present section. Note that simpler proofs can be constructed, which will yield the same statement in more general setting.

Recall that our main issue is (3.4). Now we introduce the new time variable

$$
\begin{equation*}
s=-\ln \theta(t) \quad \Leftrightarrow \quad \theta(t)=e^{-s}, \quad \frac{d f}{d t}=\frac{d f}{d s} \frac{d s}{d t}=\theta(t)^{-1} \frac{d f}{d s} \tag{A.1}
\end{equation*}
$$

Then the system (3.4) is transformed into

$$
\begin{equation*}
\frac{d}{d s} \mathbf{Y}=-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y})+e^{-s\left(I+\frac{1}{k} \Lambda_{\alpha}\right)} f_{\mathrm{res}}\left(e^{\frac{s}{k} \Lambda_{\alpha}} \mathbf{Y}\right) \tag{A.2}
\end{equation*}
$$

By the asymptotic quasi-homogeneity of $f$, the nonautonomous system A.2) is asymptotically autonomous as $s \rightarrow+\infty$ (corresponding to $t \rightarrow t_{\max }-0$ ) in the sense that A.2) admits the future limit system as $s \rightarrow \infty$ (cf. [34]):

$$
\begin{equation*}
\frac{d}{d s} \mathbf{Y}=-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y}) \tag{A.3}
\end{equation*}
$$

In particular, $e^{-s\left(I+\frac{1}{k} \Lambda_{\alpha}\right)} f_{\text {res }}\left(e^{\frac{s}{k} \Lambda_{\alpha}} \mathbf{Y}\right) \rightarrow 0$ as $s \rightarrow+\infty$, which is locally uniform in $\mathbf{Y}$. The root of the balance law $\mathbf{Y}_{0}$ is an equilibrium of the future limit system A.3). Here we briefly summarize the technique of compactification of for asymptotically autonomous systems in the time variable introduced in 34. In particular, right-sided compactification is reviewed.

Assumption A. 1 ([34, Assumption 2.2). The general system

$$
\frac{d \mathbf{y}}{d s}=f(\mathbf{y}, \Gamma(s)), \quad \Gamma: \mathbb{R} \rightarrow V: C^{1}
$$

is considered with an open interval $V \subset \mathbb{R}$, where $f: U \times V \rightarrow \mathbb{R}^{n}$ with an open subset $U \subset \mathbb{R}^{n}$ is $C^{r}$ with $r \geq 1$. The function $\Gamma$ is assumed to be asymptotically constant in the sense that the future limit $\lim _{t \rightarrow+\infty} \Gamma(t)=\Gamma_{+} \in V$ exists. A coordinate transform $\tilde{s}=\phi(s)$ satisfying all the following properties is chosen:

$$
\begin{aligned}
& \phi:\left[s_{-}, \infty\right) \rightarrow\left[\tilde{s}_{-}, 1\right), \quad \phi \in C^{k \geq 2}, \quad \lim _{s \rightarrow+\infty} \phi(s)=1 \\
& \frac{d \phi}{d s}(s)>0 \quad\left(s \geq s_{-}\right), \quad \lim _{s \rightarrow+\infty} \frac{d \phi}{d s}(s)=0
\end{aligned}
$$

with the inverse $h(\tilde{s})=\phi^{-1}(\tilde{s})$.
The resulting autonomous right-sided compactified system is the following:

$$
\begin{equation*}
\frac{d \mathbf{y}}{d s}=f(\mathbf{y}, \Gamma(h(\tilde{s}))), \quad \frac{d \tilde{s}}{d s}=\gamma(\tilde{s}) \tag{A.4}
\end{equation*}
$$

where

$$
\begin{aligned}
f(\mathbf{y}, \Gamma(h(\tilde{s}))) & = \begin{cases}f(\mathbf{y}, \Gamma(h(\tilde{s}))) & \tilde{s} \in\left[\tilde{s}_{-}, 1\right) \\
f\left(\mathbf{y}, \Gamma^{+}\right) & \tilde{s}=1\end{cases} \\
\gamma(\tilde{s}) & = \begin{cases}1 / h^{\prime}(\tilde{s}) & \tilde{s} \in\left[\tilde{s}_{-}, 1\right) \\
0 & \tilde{s}=1\end{cases}
\end{aligned}
$$

Then it follows from Proposition 2.1 and Theorem 2.2 in (34 that A.4) is $C^{1}$-smooth on $U \times\left(\tilde{s}_{-}, 1\right.$ ] if and only if the following limits exist:

$$
\begin{equation*}
\lim _{s \rightarrow+\infty} \frac{\frac{d}{d s} \Gamma(s)}{\frac{d}{d s} \phi(s)}=\lim _{\tilde{s} \rightarrow+1-0} \frac{d}{d \tilde{s}} \Gamma(h(\tilde{s})), \quad \lim _{s \rightarrow+\infty} \frac{\frac{d^{2}}{d s^{2}} \phi(s)}{\frac{d}{d s} \phi(s)}=-\lim _{\tilde{s} \rightarrow+1-0} \frac{\frac{d^{2}}{d s^{2}} h(\tilde{s})}{\left(\frac{d}{d s} h(\tilde{s})\right)^{2}} \tag{A.5}
\end{equation*}
$$

In particular, under the existence of the above limits, the system (A.2) is $C^{1}$-extended to the future-limit system (A.3) through the compactified system (A.4).

By assumption of the blow-up power-determining matrix $A$, we have

$$
\sharp\{\lambda \in \operatorname{Spec}(A) \mid \operatorname{Re} \lambda<0\}=m_{A}
$$

and hence $\mathbf{Y}_{0}$ is a hyperbolic saddle of a.3) admitting the $m_{A}$-dimensional stable manifold. Here we apply the right-sided compactification in $s$ to (A.2). As a simple one, we apply the right-side exponential compactification (cf. Section 4.2 in [34])

$$
\begin{equation*}
\tilde{s}=\phi(s) \equiv \phi_{(\nu)}(s)=1-e^{-\nu s}, \quad s=h(\tilde{s}) \equiv \phi_{(\nu)}^{-1}(\tilde{s})=-\frac{1}{\nu} \ln (1-\tilde{s}) \tag{A.6}
\end{equation*}
$$

with $\nu \in(0,1)$, and let $\Gamma(s)=e^{-s}$. In particular, functions $\Gamma, \phi$ and $h$ are $C^{\infty}$-smooth for $s<\infty$, equivalently $\tilde{s}<1$. Then limits in A.5) exist and the one-sided compactified autonomous system

$$
\begin{align*}
\frac{d}{d s} \mathbf{Y} & =-\frac{1}{k} \Lambda_{\alpha} \mathbf{Y}+f_{\alpha, k}(\mathbf{Y})+(1-\tilde{s})^{\nu^{-1}\left(I+\frac{1}{k} \Lambda_{\alpha}\right)} f_{\text {res }}\left((1-\tilde{s})^{-\frac{1}{k \nu} \Lambda_{\alpha}} \mathbf{Y}\right)  \tag{A.7}\\
\frac{d \tilde{s}}{d s} & =\nu(1-\tilde{s})
\end{align*}
$$

is $C^{1}$-smooth on $\mathbb{R}^{n} \times(-\infty, 1]$ jointly in $(\mathbf{Y}, \tilde{s})$ because $\nu<1$, which is in fact $C^{r}$ in the present case by choosing $\nu$ sufficiently small, if necessary, where $r$ is the differentiability of $f$. This transformation can be applied to asymptotically autonomous systems admitting the future limit system, such as (3.11).

## B Proofs of results

Proofs of statements in Section 3 are collected here.

## B. 1 Proof of Proposition 3.4

Apply the right-sided compactification to (3.11) in the similar way to what we have discussed in Appendix A. 1 which yields

$$
\begin{align*}
\frac{d}{d s} \mathbf{Y}_{1} & =A \mathbf{Y}_{1}+(1-\tilde{s})^{\nu^{-1}\left(I+\frac{1}{k} \Lambda_{\alpha}\right)} f_{\mathrm{res}}\left((1-\tilde{s})^{-\frac{1}{k \nu} \Lambda_{\alpha}} \mathbf{Y}_{0}\right)  \tag{B.1}\\
\frac{d \tilde{s}}{d s} & =\nu(1-\tilde{s})
\end{align*}
$$

From the asymptotic quasi-homogeneity of $f$, this system possesses an equilibrium $\tilde{\mathbf{Y}}_{1}\left(\equiv\left(\mathbf{Y}_{1}, \tilde{s}\right)\right)=$ $(\mathbf{0}, 1)$ on the future limit space $\{\tilde{s}=1\}$, which is the only possible equilibrium so that the asymptotic relation (3.7) is satisfied. Moreover, we see from the form of the system that eigenvalues of the linearized matrix at the equilibrium consists of $\operatorname{Spec}(A)$ and $\{-\nu\}$. By assumption, the equilibrium $\tilde{\mathbf{Y}}_{1}$ is hyperbolic and hence, by the Stable Manifold Theorem (e.g. [32]), the ( $m_{A}+1$ )-dimensional local stable manifold $W_{\mathrm{loc}}^{s}\left(\tilde{\mathbf{Y}}_{1}\right)$ for (B.1) is constructed. Now one variable generating $W_{\mathrm{loc}}^{s}\left(\tilde{\mathbf{Y}}_{1}\right)$ is $\tilde{s}$, which is identical with the time $s$ in the nonautonomous system

$$
\begin{equation*}
\frac{d}{d s} \mathbf{Y}_{1}=A \mathbf{Y}_{1}+\tilde{\mathbf{g}}_{1}(s) \tag{B.2}
\end{equation*}
$$

before applying the right-sided compactification to obtaining (B.1), where

$$
\tilde{\mathbf{g}}_{1}(s)=e^{-s\left(I+\frac{1}{k} \Lambda_{\alpha}\right)} f_{\text {res }}\left(e^{\frac{s}{k} \Lambda_{\alpha}} \mathbf{Y}_{0}\right)
$$

Therefore, the remaining $m_{A}$ parameters generate the family of solutions $\mathbf{Y}_{1}(s)$.
On the other hand, the solution $\mathbf{Y}_{1}(s)$ of ( (B.2) possesses the expression

$$
\begin{equation*}
\mathbf{Y}_{1}(s)=e^{\left(s-s_{0}\right) A}\left\{\mathbf{Y}_{1}^{0}+\int_{s_{0}}^{s} e^{\left(s_{0}-\eta\right) A} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\} \tag{B.3}
\end{equation*}
$$

by the variation of constants formula, where $s_{0}=-\ln \theta\left(t_{0}\right)$. (B.3) is also written as

$$
\begin{align*}
\mathbf{Y}_{1}(s) & =e^{\left(s-s_{0}\right) A}\left\{\left(P_{-}+P_{+}\right) \mathbf{Y}_{1}^{0}+\int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A}\left(P_{-}+P_{+}\right) \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\} \\
& =e^{\left(s-s_{0}\right) A}\left\{\left(P_{-}+P_{+}\right) \mathbf{Y}_{1}^{0}+\left(P_{-}+P_{+}\right) \int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A_{\mathbf{g}}} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\} \tag{B.4}
\end{align*}
$$

where we have used the $A$-invariance of $P_{ \pm}$and the fact that $P_{ \pm}$are independent of the time variable $s$ in the second equality. Note that solutions $\mathbf{Y}_{1}(s)$ determining trajectories on $W_{\text {loc }}^{s}\left(\tilde{\mathbf{Y}}_{1}\right)$ are bounded on $s \in\left[s_{0}, \infty\right)$. In particular, such solutions $\mathbf{Y}_{1}(s)$ of the system (B.2) satisfy all requirements so that the Lyapunov-Perron's method (cf. Section III. 6 in [16] or Proposition 6.4 in [22] for linear inhomogeneous systems) can be applied. Consequently, a bounded solution $\mathbf{Y}_{1}(s)$ on $\left[s_{0}, \infty\right)$ also admits the form

$$
\mathbf{Y}_{1}(s)=e^{\left(s-s_{0}\right) A}\left\{P_{-} \mathbf{Y}_{1}^{0}+\int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A} P_{-} \tilde{\mathbf{g}}_{1}(\eta) d \eta-\int_{s}^{\infty} e^{-\left(\eta-s_{0}\right) A} P_{+} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\}
$$

for any $\mathbf{Y}_{1}^{0} \in \mathbb{R}^{n}$. This is also written by

$$
\begin{aligned}
\mathbf{Y}_{1}(s)= & e^{\left(s-s_{0}\right) A}\left\{\left(P_{-}+P_{+}\right) \mathbf{Y}_{1}^{0}+\int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right. \\
& \left.-\left(P_{+} \mathbf{Y}_{1}^{0}+P_{+} \int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right)-\int_{s}^{\infty} e^{-\left(\eta-s_{0}\right) A} P_{+} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\}
\end{aligned}
$$

which follows from ( $(\overline{\text { B. } 3)}$ ) that

$$
P_{+} \mathbf{Y}_{1}^{0}=-\int_{s_{0}}^{\infty} e^{-\left(\eta-s_{0}\right) A} P_{+} \tilde{\mathbf{g}}_{1}(\eta) d \eta
$$

Substituting this into (B.4), we have

$$
\mathbf{Y}_{1}(s)=e^{\left(s-s_{0}\right) A}\left\{P_{-}\left(\mathbf{Y}_{1}^{0}+\int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right)-P_{+} \int_{s}^{\infty} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{1}(\eta) d \eta\right\}, \quad \mathbf{Y}_{1}^{0} \in \mathbb{R}^{n}
$$

Back to the original time-scale $t$, we have

$$
\mathbf{Y}_{1}(t)=\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A}\left\{P_{-}\left(\mathbf{Y}_{1}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right\}
$$

where $t_{0}=t_{\max }-e^{-s_{0}}$. By the uniqueness of solutions, this solution corresponds to a solution trajectory on $W_{\text {loc }}^{s}\left(\tilde{\mathbf{Y}}_{1}\right)$ for (B.1) we have obtained first, whenever $\left\|P_{-} \mathbf{Y}_{1}^{0}\right\|$ is sufficiently small.

## B. 2 Proof of Proposition 3.5

The basic idea for proving the existence of a solution converging to $\mathbf{0} \in \mathbb{R}^{n}$ is the same as the proof of Proposition 3.4. That is, the right-sided compactification to the time-transformed system

$$
\begin{equation*}
\frac{d}{d s} \mathbf{Y}_{j}=A \mathbf{Y}_{j}+\tilde{\mathbf{g}}_{j}(s) \tag{B.5}
\end{equation*}
$$

of (3.15) through $s=-\ln \theta(t)$ is discussed, where

$$
\begin{aligned}
\tilde{\mathbf{g}}_{j}(s)= & \left\{R_{\alpha, k}\left(\mathbf{S}_{j-1}(\mathbf{Y})(s)\right)-R_{\alpha, k}\left(\mathbf{S}_{j-2}(\mathbf{Y})(s)\right)\right\} \\
& +e^{-s\left(I+\frac{1}{k} \Lambda_{\alpha}\right)}\left\{f_{\mathrm{res}}\left(e^{\frac{s}{k} \Lambda_{\alpha}} \mathbf{S}_{j-1}(\mathbf{Y})(s)\right)-f_{\mathrm{res}}\left(e^{\frac{s}{k} \Lambda_{\alpha}} \mathbf{S}_{j-2}(\mathbf{Y})(s)\right\} .\right.
\end{aligned}
$$

The corresponding compactified system becomes

$$
\begin{align*}
\frac{d}{d s} \mathbf{Y}_{j}= & A \mathbf{Y}_{j}+\left\{R_{\alpha, k}\left(\mathbf{S}_{j-1}(\mathbf{Y})(s)\right)-R_{\alpha, k}\left(\mathbf{S}_{j-2}(\mathbf{Y})(s)\right)\right\} \\
& +(1-\tilde{s})^{\nu^{-1}\left(I+\frac{1}{k} \Lambda_{\alpha}\right)}\left\{f_{\mathrm{res}}\left((1-\tilde{s})^{-\frac{1}{k \nu} \Lambda_{\alpha}} \mathbf{S}_{j-1}(\mathbf{Y})(s)\right)-f_{\mathrm{res}}\left((1-\tilde{s})^{-\frac{1}{k \nu} \Lambda_{\alpha}} \mathbf{S}_{j-2}(\mathbf{Y})(s)\right)\right\} \tag{B.6}
\end{align*}
$$

$$
\frac{d \tilde{s}}{d s}=\nu(1-\tilde{s}) .
$$

In particular, the $\left(m_{A}+1\right)$-dimensional local stable manifold $W_{\text {loc }}^{s}\left(\tilde{\mathbf{Y}}_{j}\right)$ of the hyperbolic saddle $\tilde{\mathbf{Y}}_{j}=\left(\mathbf{Y}_{j}, \tilde{s}\right)=(\mathbf{0}, 1)$ for B.6) is constructed. Here recall that the second term $\mathbf{Y}_{1}(t)$ includes $m_{A}$ free parameters chosen from $\mathbb{E}_{A}^{s}$ and the power decay term $\theta(\theta)^{-A} P_{-} \mathbf{Y}_{1}^{0}$. This power decay term also appears in $\mathbf{Y}_{j}(t)$ through the same arguments in those obtaining $\mathbf{Y}_{1}(t)$. In contrast, this term has to vanish so that the asymptotic relation (3.7) holds.

Now go back to the proof of Proposition 3.5. By the same arguments as Section B.1 we know that (B.5) possesses the solution

$$
\begin{equation*}
\mathbf{Y}_{j}(s)=e^{\left(s-s_{0}\right) A}\left\{P_{-}\left(\mathbf{Y}_{j}^{0}+\int_{s_{0}}^{s} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{j}(\eta) d \eta\right)-P_{+} \int_{s}^{\infty} e^{-\left(\eta-s_{0}\right) A} \tilde{\mathbf{g}}_{j}(\eta) d \eta\right\} \tag{B.7}
\end{equation*}
$$

where $\mathbf{Y}_{j}^{0} \in \mathbb{R}^{n}$, but the exponentially decaying terms characterized by $\operatorname{Spec}(A) \cap\{\operatorname{Re} \lambda<0\}$ should be vanished by the asymptotic relation

$$
\mathbf{Y}_{j}(s) \ll \mathbf{Y}_{1}(s) \quad \text { as } s \rightarrow \infty
$$

and hence the constant vector $\mathbf{Y}_{j}^{0}$ should be specified so that this requirement is satisfied. This constraint is considered below.

Back to the original time-scale $t$, we have

$$
\begin{equation*}
\mathbf{Y}_{j}(t)=\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A}\left\{P_{-}\left(\mathbf{Y}_{j}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta\right\} \tag{B.8}
\end{equation*}
$$

where $t_{0}=t_{\max }-e^{-s_{0}}$. We shall determine the appropriate $\mathbf{Y}_{j}^{0}=\left(Y_{j, 1}^{0}, \ldots, Y_{j, n}^{0}\right)^{T}$. To this end, consider the primitive function $\mathbf{H}_{j}(t)$ satisfying (3.20). Let

$$
\nu_{j, i}:=\operatorname{deg}_{\theta}\left(\left\{\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(t)\right\}_{i}\right)
$$

Now our determination of $\mathbf{Y}_{j}^{0}$ consists of two cases.
Case 1. $\nu_{j, i} \leq-1$.
In this case, define $Z_{j, i}:=\left(\mathbf{H}_{j}\left(t_{0}\right)\right)_{i}$.
Case 2. $\nu_{j, i}>-1$.
In this case, similar to arguments in the proof of Proposition 2.13 we rewrite the rightmost integral in (3.20) by

$$
\begin{align*}
& \left\{\int_{t_{0}}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(\eta) d \eta-\int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(\eta) d \eta\right\}_{i}  \tag{B.9}\\
& \equiv I_{0 ; j, i}-\left\{\int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} P_{-} \mathbf{g}_{j}(\eta) d \eta\right\}_{i} \\
& =I_{0 ; j, i}+\left(\mathbf{H}_{j}(t)\right)_{i}
\end{align*}
$$

where we have used the fact that $\mathbf{H}_{j}(t)$ is chosen so that $\lim _{t \rightarrow t_{\text {max }}-0}\left(\mathbf{H}_{j}(t)\right)=0$. By the similar arguments to Proposition 2.10 the first term converges to a finite value $I_{0 ; j, i}$ as the improper integral. By definition of improper integrals and (3.20), we have

$$
I_{0 ; j, i}=\lim _{t \rightarrow t_{\max }-0}\left(\mathbf{H}_{j}(t)\right)_{i}-\left(\mathbf{H}_{j}\left(t_{0}\right)\right)_{i}=-\left(\mathbf{H}_{j}\left(t_{0}\right)\right)_{i}
$$

In this case, define $Z_{j, i}:=-I_{0 ; j, i}=\mathbf{H}_{j}\left(t_{0}\right)$. Finally, define the vector $\mathbf{Z}_{j}$ by $\mathbf{Z}_{j}:=\left(Z_{j, 1}, \ldots, Z_{j, n}\right)^{T} \equiv$ $\mathbf{H}_{j}\left(t_{0}\right)$.

Because $\mathbf{Z}_{j}$, namely $\mathbf{H}_{j}\left(t_{0}\right)$, is uniquely determined once $t_{0}, \mathbf{g}_{j}(\eta)$ is, in particular the lower terms $\left\{\mathbf{Y}_{l}(t)\right\}_{l=0}^{j-1}$ are determined, and hence letting $\mathbf{Y}_{j}^{0}$ so that

$$
P_{-} \mathbf{Y}_{j}^{0}=P_{-}\left(P \mathbf{H}_{j}\left(t_{0}\right)\right)
$$

the $\mathbb{E}_{A}^{s}$-component of the constant term in (B.8) becomes 0 for all $t \in\left(t_{0}, t_{\max }\right)$. The initial point at $t=t_{0}$ is uniquely determined by

$$
\mathbf{Y}_{j}\left(t_{0}\right)=P_{-} \mathbf{Y}_{j}^{0}-P_{+} \int_{t_{0}}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{j}(\eta) d \eta
$$

which contains free parameters determined by $\mathbf{Y}_{1}(t)$ in $\mathbf{g}_{j}(t)$. The proof is therefore completed.

## B. 3 Proof of Theorem 3.8

By the same argument as Theorem 3.2 in [34, we know that the point $\tilde{\mathbf{Y}}_{0} \equiv\left(\mathbf{Y}_{0}, 1\right)$ is a hyperbolic saddle for the one-sided compactified system (A.7). Then the saddle $\mathbf{Y}_{0}$ admits the $\left(m_{A}+1\right)$ dimensional local stable manifold $W_{\text {loc }}^{s}\left(\tilde{\mathbf{Y}}_{0}\right)$ in $\mathbb{R}^{n} \times(-\infty, 1]$ by the Stable Manifold Theorem (e.g. [32]). From the $C^{\infty}$-smoothness of $\phi_{(\nu)}, \Gamma(s)=e^{-s}$ on $\mathbb{R}$ and $f$ on $\mathbb{R}^{n}$, the smooth dependence of solutions of (A.2) on initial points and time as well as the uniqueness of solutions yield that $W_{\text {loc }}^{s}\left(\tilde{\mathbf{Y}}_{0}\right) \cap\left(\mathbb{R}^{n} \times(-\infty, 1)\right)$ is indeed $C^{r}$-smooth in the $s$-timescale. Moreover, the Stable Manifold Theorem implies that the convergence of solutions to $\mathbf{Y}_{0}$ is exponential, that is, for each $i=$ $1, \ldots, n$, we have

$$
\begin{equation*}
Y_{i}(s)-Y_{i, 0}=C_{i} e^{-\mu_{i} s}(1+o(1)) \quad \text { as } \quad s \rightarrow+\infty \tag{B.10}
\end{equation*}
$$

with a constant $C_{i}$ and an exponent $\mu_{i}$ with $\operatorname{Re} \mu_{i}>0$. Finally we go back to the system in the original $t$-timescale. The exponential decay (B.10) implies that

$$
Y_{i}(t)-Y_{i, 0}=\tilde{C}_{i}\left(t_{\max }-t\right)^{\mu_{i}}(1+o(1)) \quad \text { as } \quad t \rightarrow t_{\max }
$$

which is continuous in $t$ including $t \rightarrow t_{\max }$, and $C^{r}$-smooth for $t<t_{\max }$.

## B. 4 Proof of Proposition 3.10

The basic idea is the same as the proof of Theorem 3.8 except the system we consider is

$$
\begin{align*}
\frac{d}{d t} \mathbf{Y}_{N}^{c}= & \theta(t)^{-1}\left[\left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \mathbf{Y}_{N}^{c}+\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right)-R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)\right\}\right] \\
& +\theta(t)^{\frac{1}{k} \Lambda_{\alpha}}\left\{f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}}\left\{\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right\}\right)-f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})\right)\right\} \tag{B.11}
\end{align*}
$$

instead of (3.8). Using (A.1), (B.11) is transformed into

$$
\begin{align*}
\frac{d}{d s} \mathbf{Y}_{N}^{c}= & \left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \mathbf{Y}_{N}^{c}+\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right)-R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)\right\} \\
& +e^{-s\left(I+\frac{1}{k} \Lambda_{\alpha}\right)}\left\{f_{\mathrm{res}}\left(e^{-\frac{s}{k} \Lambda_{\alpha}}\left\{\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right\}\right)-f_{\mathrm{res}}\left(e^{-\frac{s}{k} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})\right)\right\} \tag{B.12}
\end{align*}
$$

Applying the right-side exponential compactification (A.6) to (B.12), the following system is obtained:

$$
\begin{aligned}
\frac{d}{d s} \mathbf{Y}_{N}^{c}= & \left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \mathbf{Y}_{N}^{c}+\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right)-R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)\right\} \\
& +(1-\tilde{s})^{\nu^{-1}\left(I+\frac{1}{k} \Lambda_{\alpha}\right)}\left\{f_{\mathrm{res}}\left((1-\tilde{s})^{\frac{1}{k \nu} \Lambda_{\alpha}}\left\{\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right\}\right)-f_{\mathrm{res}}\left((1-\tilde{s})^{\frac{1}{k \nu} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})\right)\right\} \\
\frac{d \tilde{s}}{d s}= & \nu(1-\tilde{s})
\end{aligned}
$$

By the definition of $R_{\alpha, k}, \mathbf{Y}_{N}^{c}=\mathbf{0} \in \mathbb{R}^{n}$ is an equilibrium of the future limit system

$$
\frac{d}{d s} \mathbf{Y}_{N}^{c}=\left(-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right) \mathbf{Y}_{N}^{c}+\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})+\mathbf{Y}_{N}^{c}\right)-R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)\right\}
$$

whose linearized matrix at $\mathbf{Y}_{N}^{c}=\mathbf{0}$ is $A=-\frac{1}{k} \Lambda_{\alpha}+D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)$. Now $\mathbf{Y}_{j}(t) \ll \mathbf{Y}_{0} \in \mathbb{R}^{n}$ as $t \rightarrow t_{\max }$ is required for all $j \geq 1$ by our construction of (3.21), which indicates that $\mathbf{0} \in \mathbb{R}^{n}$ is the only possible limit of the remainder $\mathbf{Y}_{N}^{c}$. By assumption, $\left(\mathbf{Y}_{N}^{c}, \tilde{s}\right)=(\mathbf{0}, 1)$ is saddle and hence it admits $\left(m_{A}+1\right)$-dimensional local stable manifold. In particular, any solution $\mathbf{Y}_{N}^{c}$ on the manifold converges to $\mathbf{0} \in \mathbb{R}^{n}$ as $s \rightarrow \infty$, equivalently $t \rightarrow t_{\max }$ for small free parameters $\left(C_{1}, \ldots, C_{m_{A}}\right)$.

## B. 5 Proof of Theorem 3.11

Note that there is a nonsingular matrix $P$ such that $P^{-1} A P=\Lambda$ is the Jordan normal form. It follows from this transformation that

$$
\theta(t)^{A} \equiv e^{(\ln \theta(t)) A}=e^{(\ln \theta(t)) P \Lambda P^{-1}}=P e^{(\ln \theta(t)) \Lambda} P^{-1}=P \theta(t)^{\Lambda} P^{-1}
$$

First assume that $A$ is diagonalizable, in which case

$$
\theta(t)^{\Lambda}=\operatorname{diag}\left(\theta(t)^{\lambda_{1}}, \ldots, \theta(t)^{\lambda_{n}}\right), \quad\left\{\lambda_{i}\right\}_{i=1}^{n}=\operatorname{Spec}(A)
$$

All possible orders of $\theta(t)$ are obviously listed as $-\beta \cdot{ }_{A} \lambda$ when $\beta \in \mathbb{Z}_{\geq 0}^{n}$ with $|\beta|=1$. Assume that, for given $N \geq 1$, all possible orders of $\theta(t)$ are listed as the collection $\left\{-\beta \cdot{ }_{A} \lambda\right\}$ for $\beta \in \mathbb{Z}_{\geq 0}^{n}$. The system for determining $\mathbf{Y}_{N+1}$ is

$$
\frac{d}{d t} \mathbf{Y}_{N+1}=\theta(t)^{-1}\left[A \mathbf{Y}_{N+1}+\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(t)\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)\right)\right\}\right]
$$

Because $R_{\alpha, k}$ is the remainder of quasi-homogeneous vector field, then the remainder $R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)-$ $R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})\right)$ consists of powers of the form $\theta(t)^{-\beta \cdot A \lambda}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$. The solution $\mathbf{Y}_{N+1}=$ $\mathbf{Y}_{N+1}(t)$ is

$$
\begin{align*}
& \mathbf{Y}_{N+1}(t) \\
& =P\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\Lambda} P^{-1}\left\{P_{-}\left(\mathbf{Y}_{N+1}^{0}+\int_{t_{0}}^{t} P\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} \mathbf{g}_{N+1}(\eta) d \eta\right)-P_{+} \int_{t}^{t_{\max }} P\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1} \mathbf{g}_{N+1}(\eta) d \eta\right\} \\
& =P\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\Lambda} P^{-1} P_{-}\left(\mathbf{Y}_{N+1}^{0}+\int_{t_{0}}^{t} P\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1}\left\{\theta(\eta)^{-1}\left(R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(\eta)\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(\eta)\right)\right)\right\} d \eta\right) \\
&  \tag{B.13}\\
& \quad-P\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-\Lambda} P^{-1}\left(P_{+} \int_{t}^{t_{\max }} P\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{\Lambda} P^{-1}\left\{\theta(\eta)^{-1}\left(R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(\eta)\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(\eta)\right)\right)\right\} d \eta\right)
\end{align*}
$$

Integrating all terms individually, we know that the resulting functions also consist of powers of the form $\theta(t)^{-\beta \cdot A \lambda}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$.

Next consider the general case where $A$ admits non-trivial Jordan blocks. From the form of the fundamental matrix $\theta(t)^{A}$, we know that all components of $\mathbf{Y}_{1}(t)$ are linear combinations of functions of the form $\theta(t)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(t))^{M}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$ satisfying $|\beta|=1$ and $M \in \mathbb{Z}_{\geq 0}$ not greater than $\sum_{i=1}^{d} \sum_{l=1}^{\mu_{i}}\left(m_{i, l}-1\right)$. Recall that $\left\{m_{i, l}\right\}_{l=1}^{\mu_{i}}$ denotes the collection of natural numbers corresponding to the size of Jordan blocks $J_{*}\left(\tilde{\lambda}_{i}\right)$, for $i=1, \ldots, d$, associated with mutually disjoint eigenvalues $\left\{\tilde{\lambda}_{i}\right\}_{i=1}^{d}$ of $A$, and $\mu_{i}$ denotes the geometric multiplicity of $\tilde{\lambda}_{i}$. The exponent
$M$ is determined by the number of eigenvalues admitting non-trivial Jordan blocks and their size. Assume that, for $N \geq 1, \mathbf{Y}_{N}(t)$ consists of the linear combination of functions of the form $\theta(t)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(t))^{M}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$ and $M \in \mathbb{Z}_{\geq 0}$. Then the remainder $R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)-$ $R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})\right)$ also consists of linear combinations of powers of the form

$$
\theta(t)^{-\beta \cdot A \lambda}(\ln \theta(t))^{M}, \quad \beta \in \mathbb{Z}_{\geq 0}^{n}, \quad M \in \mathbb{Z}_{\geq 0}
$$

Similar to the diagonalizable case, the solution $\mathbf{Y}_{N+1}(t)$ is written by (B.13) and we see that it is sufficient to verify integrals of $\theta(\eta)^{-\beta \cdot A \lambda-1}(\ln \theta(\eta))^{M}$ in $\eta$, which are

$$
\begin{aligned}
\int_{t_{0}}^{t} \theta(\eta)^{-\beta \cdot A_{A} \lambda-1}(\ln \theta(\eta))^{M} d \eta= & {\left[\frac{1}{\beta \cdot{ }_{A} \lambda} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(\eta))^{M}\right]_{t_{0}}^{t}-\frac{1}{\beta \cdot{ }_{A} \lambda} \int_{t_{0}}^{t} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda}\left\{\frac{d}{d \eta}(\ln \theta(\eta))^{M}\right\} d \eta } \\
& +\frac{M(M-1)}{\left(\beta \cdot{ }_{A} \lambda\right)^{2}} \int_{t_{0}}^{t} \theta(\eta)^{-\beta \cdot A \lambda-1}(\ln \theta(\eta))^{M-2} d \eta \\
= & \cdots \\
= & \left\{\sum_{l=0}^{M} \frac{M!}{(M-l)!}\left(\frac{1}{\beta \cdot{ }_{A} \lambda}\right)^{l+1}(\ln \theta(t))^{M-l}\right\} \theta(t)^{-\beta \cdot A \lambda}+(\text { constant })
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t}^{t_{\max }} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda-1}(\ln \theta(\eta))^{M} d s & =\left[\frac{1}{\beta \cdot{ }_{A} \lambda} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(\eta))^{M}\right]_{t}^{t_{\max }}-\frac{1}{\beta \cdot{ }_{A} \lambda} \int_{t}^{t_{\max }} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda}\left\{\frac{d}{d \eta}(\ln \theta(\eta))^{M}\right\} d \eta \\
& =\frac{-1}{\beta \cdot{ }_{A} \lambda} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(\eta))^{M}+\frac{M}{\beta \cdot{ }_{A} \lambda} \int_{t}^{t_{\max }} \theta(\eta)^{-\beta \cdot{ }_{A} \lambda-1}(\ln \theta(\eta))^{M-1} d \eta \\
& =\cdots \\
& =-\left\{\sum_{l=0}^{M} \frac{M!}{(M-l)!}\left(\frac{1}{\beta \cdot{ }_{A} \lambda}\right)^{l+1}(\ln \theta(t))^{M-l}\right\} \theta(t)^{-\beta \cdot{ }_{A} \lambda}
\end{aligned}
$$

whenever $\beta \cdot{ }_{A} \lambda \neq 0$, where we have used

$$
\lim _{t \rightarrow t_{\max }-0} \frac{\log \left(t_{\max }-t\right)}{\left(t_{\max }-t\right)^{a}}=0 \quad \text { for } a<0
$$

When $\beta \cdot{ }_{A} \lambda=0$ with $\beta \not \equiv 0$, this indicates that $\operatorname{Spec}(A) \subset\{\operatorname{Re} \lambda>0\}$ and only integrals over $\left[t, t_{\max }\right)$ are involved. The integral is then calculated as follows:

$$
\begin{aligned}
\int_{t}^{t_{\max }} \theta(\eta)^{\beta \cdot{ }_{A} \lambda-1}(\ln \theta(\eta))^{M} d s & =\int_{t}^{t_{\max }} \theta(\eta)^{-1}(\ln \theta(\eta))^{M} d \eta \\
& =\int_{t}^{t_{\max }}-\frac{d}{d \eta}(\ln \theta(\eta))(\ln \theta(\eta))^{M} d \eta \\
& =\frac{-1}{M+1}(\ln \theta(t))^{M+1}
\end{aligned}
$$

In every cases, the integral is described by the linear combination of functions of the form $\theta(\eta)^{-\beta \cdot A} \lambda(\ln \theta(\eta))^{M}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$ and $M \in \mathbb{Z}_{\geq 0}$ and hence all components of $\mathbf{Y}_{N+1}(t)$ are again linear combinations of functions of the form $\theta(t)^{-\beta \cdot{ }_{A} \lambda}(\ln \theta(\eta))^{M}$ with $\beta \in \mathbb{Z}_{\geq 0}^{n}$ and $M \in \mathbb{Z}_{\geq 0}$. Exponents of $\ln \theta(t)$ in $\mathbf{Y}_{N+1}(t)$ is determined by $N+1, \alpha, k$ and $\left\{m_{i, l}\right\}_{l=1}^{\mu_{i}}, i=1, \ldots, d$.

## B. 6 Proof of Proposition 3.13

First we have $\operatorname{deg}_{\theta}\left(Y_{0, i}\right)=0$ for all $i=1, \ldots, n$ when $Y_{0, i} \neq 0$, otherwise $\operatorname{deg}_{\theta}\left(Y_{0, i}\right)=+\infty$, by Example 2.7. These identities imply (3.27) with $N=0$.

Next consider $\mathbf{Y}_{1}(t)$. By definition of $\gamma_{i}$, we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\left\{\mathbf{g}_{1}(t)\right\}_{i}\right) \equiv \operatorname{deg}_{\theta}\left(\theta(t)^{\alpha_{i} / k} f_{i ; \operatorname{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{0}\right)\right) \geq \frac{\alpha_{i}}{k}+\gamma_{i} \tag{B.14}
\end{equation*}
$$

and hence $\operatorname{deg}_{\theta}\left(\mathrm{g}_{1}(t)\right)>-1$ by assumption. By (2.16) in Proposition 2.13, we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{-} \mathbf{g}_{1}(\eta) d \eta\right) \geq \min \left\{\min _{l=1, \ldots, n}\left\{\frac{\alpha_{l}}{k}+\gamma_{l}\right\}+1, \min _{\substack{\lambda \in \operatorname{Spec}(A) \\ \operatorname{Re} \lambda<0}}(-\operatorname{Re} \lambda)\right\} \tag{B.15}
\end{equation*}
$$

as the estimate of $\operatorname{deg}_{\theta}$ for vector-valued functions. Therefore we have

$$
\begin{aligned}
& \operatorname{deg}_{\theta}\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} P_{-}\left(\mathbf{Y}_{1}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right)\right) \\
& \quad \geq \min \left\{\operatorname{deg}_{\theta}\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} P_{-} \mathbf{Y}_{1}^{0}\right), \quad \operatorname{deg}_{\theta}\left(\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{-} \mathbf{g}_{1}(\eta) d \eta\right)\right\}
\end{aligned}
$$

(from Proposition 2.10,2)

$$
\begin{equation*}
\geq \min \left\{\min _{i=1, \ldots, n}\left\{\frac{\alpha_{i}}{k}+\gamma_{i}\right\}+1, \min _{\substack{i=1, \ldots, n \\ \operatorname{Re} \lambda_{i}<0}} \operatorname{Re}\left(-\lambda_{i}\right)\right\}=\delta \quad(\text { from }(\overline{\mathrm{B} .15})) \tag{B.16}
\end{equation*}
$$

where we have used the commutativity $A P_{-}=P_{-} A$. Similarly, from (2.17) in Proposition 2.13, we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{+} \mathbf{g}_{1}(\eta) d \eta\right) \geq \min _{l=1, \ldots, n}\left\{\frac{\alpha_{l}}{k}+\gamma_{l}\right\}+1 \geq \delta \tag{B.17}
\end{equation*}
$$

Combining (B.16) and (B.17), we have

$$
\begin{aligned}
& \operatorname{deg}_{\theta}\left(\mathbf{Y}_{1}(t)\right) \\
& \quad \geq \min \left\{\operatorname{deg}_{\theta}\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} P_{-}\left(\mathbf{Y}_{1}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{1}(\eta) d \eta\right)\right), \operatorname{deg}_{\theta}\left(-\int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta(t)}\right)^{A} P_{+} \mathbf{g}_{1}(\eta) d \eta\right)\right\} \\
& \quad \geq \delta
\end{aligned}
$$

and hence our claim (3.27) is proved when $N=1$.
Now we assume that (3.27) holds true for some $N \geq 1$. Our next issue is the case $N+1$. The main investigation is the magnitude of $\theta(t)$ in the following two vector-valued functions:

$$
\begin{aligned}
& \mathbf{g}_{N+1,1}(t)=\theta(t)^{-1}\left\{R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(t)\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)\right)\right\} \\
& \mathbf{g}_{N+1,2}(t)=\theta(t)^{\frac{1}{k} \Lambda_{\alpha}}\left\{f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})(t)\right)-f_{\text {res }}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N-1}(\mathbf{Y})(t)\right)\right\}
\end{aligned}
$$

First note that, by the Taylor's theorem, there are constants $\eta_{1}, \eta_{2} \in(0,1)$ such that

$$
\begin{aligned}
& R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(t)\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)\right) \\
& =f_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})(t)\right)-f_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)\right)-D f_{\alpha, k}\left(\mathbf{Y}_{0}\right) \mathbf{Y}_{N}(t) \\
& =\left\{D f_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)+\eta_{1} \mathbf{Y}_{N}(t)\right)-D f_{\alpha, k}\left(\mathbf{Y}_{0}\right)\right\} \mathbf{Y}_{N}(t) \\
& =\left[\frac{1}{2} D^{2} f_{\alpha, k}\left(\mathbf{Y}_{0}+\eta_{2}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)+\eta_{1} \mathbf{Y}_{N}(t)-\mathbf{Y}_{0}\right)\right)\left\{\mathbf{S}_{N-1}(\mathbf{Y})(t)+\eta_{1} \mathbf{Y}_{N}(t)-\mathbf{Y}_{0}\right\}\right] \mathbf{Y}_{N}(t) .
\end{aligned}
$$

By assertions in (2.3), we observe that $D^{2} f_{\alpha, k}\left(\mathbf{Y}_{0}+\eta_{2}\left(\mathbf{S}_{N-1}(\mathbf{Y})(t)+\eta_{1} \mathbf{Y}_{N}(t)-\mathbf{Y}_{0}\right)\right)$ is $O(1)$ as $t \rightarrow t_{\max }$ as a bilinear map. By the assumption of induction and Proposition2.10-2 and 3, we have

$$
\operatorname{deg}_{\theta}\left(R_{\alpha, k}\left(\mathbf{S}_{N}(\mathbf{Y})\right)-R_{\alpha, k}\left(\mathbf{S}_{N-1}(\mathbf{Y})\right)\right) \geq(N+1) \delta
$$

and hence

$$
\operatorname{deg}_{\theta}\left(\mathbf{g}_{N+1,1}(t)\right) \geq(N+1) \delta-1
$$

Next consider $\mathbf{g}_{N+1,2}(t)$. First we have

$$
\begin{aligned}
& f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})(t)\right)-f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N-1}(\mathbf{Y})(t)\right) \\
& =D f_{\mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N-1}(\mathbf{Y})(t)+\eta_{3} \theta^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{N}(t)\right) \theta^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{Y}_{N}(t)
\end{aligned}
$$

by the Taylor's theorem for some constant $\eta_{3} \in(0,1)$. Under the constraint (3.26) for $f_{\text {res }}$, we have

$$
\operatorname{deg}_{\theta}\left(f_{i, \mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N}(\mathbf{Y})\right)-f_{i, \mathrm{res}}\left(\theta(t)^{-\frac{1}{k} \Lambda_{\alpha}} \mathbf{S}_{N-1}(\mathbf{Y})\right)\right) \geq \gamma_{i}+N \delta
$$

where we have also used the assumption of the induction $\operatorname{deg}_{\theta}\left(\mathbf{Y}_{N}(t)\right) \geq N \delta$. In particular, we have

$$
\begin{aligned}
\operatorname{deg}_{\theta}\left(\mathbf{g}_{N+1,2}(t)\right) & \geq \min _{l=1, \ldots, n}\left\{\frac{\alpha_{l}}{k}+\gamma_{l}\right\}+N \delta \\
& \geq(N+1) \delta-1
\end{aligned}
$$

In particular, we have

$$
\begin{equation*}
\operatorname{deg}_{\theta}\left(\mathbf{g}_{N+1}(t)\right) \geq \min \left\{\operatorname{deg}_{\theta}\left(\mathbf{g}_{N+1,1}(t)\right), \operatorname{deg}_{\theta}\left(\mathbf{g}_{N+1,2}(t)\right)\right\} \geq(N+1) \delta-1 \tag{B.18}
\end{equation*}
$$

which is larger than -1 because $\delta>0$ by assumption. Now the inequality (2.19) in Proposition 2.13 with $\mathbf{g}_{N+1,1}(t), \mathbf{g}_{N+1,2}(t)$, and the projector $P_{-}$yields

$$
\operatorname{deg}_{\theta}\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} P_{-}\left(\mathbf{Y}_{N+1}^{0}+\int_{t_{0}}^{t}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} \mathbf{g}_{N+1}(\eta) d \eta\right)\right) \geq(N+1) \delta
$$

where we have used the commutativity $A P_{-}=P_{-} A$. Note that the constant $\mathbf{Y}_{N+1}^{0}$ is chosen so that the requirements stated in Proposition 3.5 are satisfied.

Finally, similar to the case $N=1$, we have

$$
\operatorname{deg}_{\theta}\left(\left(\frac{\theta(t)}{\theta\left(t_{0}\right)}\right)^{-A} \int_{t}^{t_{\max }}\left(\frac{\theta(\eta)}{\theta\left(t_{0}\right)}\right)^{A} P_{+} \mathbf{g}_{N+1}(\eta) d \eta\right) \geq(N+1) \delta
$$

As a summary, we have

$$
\operatorname{deg}_{\theta}\left(\mathbf{Y}_{N+1}(t)\right) \geq(N+1) \delta
$$

and the proof is completed.


[^0]:    *(Corresponding author, kmatsue@imi.kyushu-u.ac.jp)

[^1]:    ${ }^{1}$ In the linear homogeneous system, it is proved by Fuchs that the system admits solutions expressed by the Frobenius series around regular singular points. This ansatz is typically applied to linear inhomogeneous equations or nonlinear equations (cf. [14, 25] 26]). There is also a technique for calculating asymptotic expansions of solutions around irregular singular points (e.g. 4]), which is omitted here since it is beyond our scope in the present study.

[^2]:    ${ }^{2}$ In preceding studies, all $\alpha_{i}$ 's and $k$ are typically assumed to be natural numbers. In the present study, on the other hand, the above generalization is valid.
    ${ }^{3}$ Throughout the rest of this paper, real positive numbers or functions to the matrices are denoted in the similar manner.

[^3]:    ${ }^{4}$ This consistency is also valid for exponentially decaying functions, which can be applied to extending the present results in future works.

[^4]:    ${ }^{5}$ If $\gamma_{f_{0}}>-1$, the improper integral converges to a finite value. In particular, the limit $\lim _{t \rightarrow t_{\max }-0} \mathbf{F}_{f_{0}}(t)$ exists.

[^5]:    ${ }^{6}$ The invariant projector onto the invariant subspace associated with purely imaginary eigenvalues is also provided in general, but it is out of our interest in the present argument.

[^6]:    ${ }^{7}$ By Theorem 3.8 this asymptotic relation is proved to be valid. See Remark 3.9
    ${ }^{8}$ We would try to express $\tilde{\mathbf{Y}}(t)$ as an asymptotic power series, but there is no guarantee that $\tilde{\mathbf{Y}}(t)$ can be expressed as power series, as mentioned in Introduction.

[^7]:    ${ }^{9}$ If our interests are only several lower terms, the governing part from the remaining term can be eliminated.

[^8]:    ${ }^{10}$ The terminology formal in this sentence means that neither the convergence of the infinite series (3.21) nor the asymptotic relation (3.7) are considered in detail.

[^9]:    ${ }^{11}$ This situation includes the case that $Y_{N, i}(t) \equiv 0$.

[^10]:    ${ }^{12}$ Our assumption $0<a<1$ is essential for this correspondence.

[^11]:    ${ }^{13}$ There is a typo of the system in 29 (Equation (4.7)). The correct form is 4.20) below. The rest of arguments in [29] is developed for the correct system 4.20].

[^12]:    ${ }^{14}$ Detailed treatments are also reviewed in [23].

[^13]:    ${ }^{15}$ In some context in partial differential equations, this kind of blow-up behavior is referred to as nonsimultaneous blow-up (cf. 17, 18).

[^14]:    ${ }^{16}$ This terminology is well known in blow-up studies for partial differential equations (e.g. 19]).

