ISSN 2229-5518

Multi-Stage Retrial Queueing System with Bernoulli Feedback

J.Ebenesar Anna Bagyam, Dr.K.Udaya Chandrika

Abstract — An M/G/1 retrial queueing system with k stages of heterogeneous services and feedback is considered. Primary customers get into the system according to Poisson process. If the server is free, an arriving customer receives first stage service immediately otherwise, he enters a retrial orbit. After the completion of the first stage, the customer may proceed to second stage with probability θ_1, or feedback to the retrial group with probability p_1 or depart the system with probability $1 – \theta_1 – p_1 = q_1$. In general, after the completion of the i^{th}, (i = 1, 2, ..., k–1) stage, the customer may opt $(i+1)^{th}$ stage with probability θ_1, or feedback to the retrial group with probability p_i or depart the system with probability $1 – \theta_1 – p_i = q_i$. The customer in final stage will feedback to the retrial group with probability p_k or depart the system with probability $1 – p_k = q_k$. It is assumed that the service times and the retrial times are arbitrarily distributed. The condition under which the steady state exists is investigated. Performance measures are obtained. A stochastic decomposition is presented.

Key Words — Feedback, Heterogeneous Service, Multi Stage, Retrial Queue, Stochastic Decomposition.

1 Introduction

The retrial queueing system has been studied extensively due to its wide applicability in telephone switching system, telecommunication and computer networks. These systems are characterized by the feature that arrivals who find the server busy join the retrial queue (orbit) to try again for their requests or leave the service area immediately. For comprehensive survey on retrial queues refer [1, 2, 4] and references therein.

Recently considerable attention has been devoted to the queueing system with two or more stages of heterogeneous service. Choudhury and Paul [3] have inspected the M/G/1 system with two stages of heterogeneous service and Bernoulli feedback.

Shahkar and Badamchizadeh [7] have studied a single server general service queue with Poisson input and k-stages of service. Salehirad and Badamchizadeh [6] have analysed multi stage M/G/1 queueing system with feedback.

In many examples such as production system, bank services, computer and communication networks, the service of customers may be repeated. In this paper with this motivation a single server retrial queue with Poisson input, k stages of heterogeneous service and feedback is analyzed.

2 Model Description

Assume that the customers arrive at the system in accordance with a Poisson process with rate λ. If an arriving customer finds the server idle, the customer enters the service immediately for first stage service. If the server is found to be blocked, the arriving customer enters a retrial group. The retrial time of customer is generally distributed with distribution function A(x) and Laplace Stieltjes transform $A^*(s)$.

The server provides k stages of heterogeneous service in succession. The service discipline is assumed to be first come first served. Service time of i^{th} stage is denoted by the random variable B_i having Laplace Stieltjes transform $B_i^*(s)$ and first two moments μ_{1i} and μ_{2i}, i = 1, 2, . . . k.

After completion the i^{th} stage the customer may move to $i+1^{th}$ stage with probability θ_i or feedback to the retrial queue with probability p_i or depart the system with probability $q_i = 1 – \theta_i – p_i$, for i = 1, 2, . . . k–1. The customer in the final stage k may feedback to the retrial queue with probability p_k or depart the system with complementary probability. According to the model, the time required by a customer to complete the service cycle is a random variable B given by

$$B = \sum_{i=1}^{k} \Theta_{i-1} B_i$$

having Laplace Steltjes transform $B^*(s) = \prod_{i=1}^{k} \Theta_{i-1} \, B_i^*(s)$

and expected value $E(B) = \sum_{i=1}^{k} \Theta_{i-1} E(B_i)$

where $\Theta_i = \theta_1 \theta_2 \ldots \theta_i$ and $\Theta_0 = 1$.

The functions $\eta(x) = \dfrac{dA(x)}{1-A(x)}$ and $\mu_i(x) = \dfrac{dB_i(x)}{1-B_i(x)}$,

i = 1, 2, . . . k are the conditional completion rates (at time x) for repeated attempts and for services.

Define $\Lambda_i^* = B_1^* \, B_2^* \ldots . B_i^*$ and $\Lambda_0^* = 0$.

The first moment M_{1i} of Λ_i^* is given by

$$M_{1i} = \lim_{z\to 1} d\Lambda_i^*(\lambda(1-z))/\,dz = \lambda \sum_{j=1}^{i} E(B_j) = \lambda \sum_{j=1}^{i} \mu_{1j}$$

The second moment M_{2i} of Λ_i^* is

$$M_{2i} = \lim_{z\to 1} d^2 \Lambda_i^*(\lambda(1-z))/\,dz^2$$

- *J. Ebenesar Anna Bagyam, Assistant Professor, Department of Mathematics, SNS College of Technology, Coimbatore, Tamil Nadu, India, PH-641035, E-mail: ebenesar.j@gmail.com*
- *Dr.K.Udaya Chandrika, Professor, Department of Mathematics, Avinashilingam Deemed University for Women, Coimbatore, Tamil Nadu, India, PH- 641043, E-mail: udayakchandrika@gmail.com*

$$= \lambda^2 \left[\left(\sum_{j=1}^{i} E(B_j) \right)^2 + \sum_{j=1}^{i} Var\, B_j \right]$$

$$= \lambda^2 \left[\left(\sum_{j=1}^{i} \mu_{1j} \right)^2 + \sum_{j=1}^{i} (\mu_{2j} - \mu_{1j}^2) \right]$$

The stage of the system at time t can be described by the Markov process $\{N(t)\ ;\ t \geq 0\} = \{J(t), X(t), \xi_0(t), \xi_i(t)\ ;\ t \geq 0, i = 1, 2, \ldots, k\}$ where J(t) denotes the server state 0 or i (i = 1, 2, . . . , k) according as the server being idle or busy in i^{th} stage of service. Let X(t) denote the number of customers in the retrial queue at time t. If J(t) = 0 and X(t) > 0, then $\xi_0(t)$ represents the elapsed retrial time, if J(t) = i, (i = 1, 2, . . . , k), $\xi_i(t)$ corresponds to the elapsed time of the customer being provided i^{th} stage of service at time t.

3 Stability Condition

Let X_n ($n \geq 1$) be the orbit length at the departure epoch of n^{th} customer departure, then $\{X_n : n \geq 1\}$ is ergodic if and only if

$$\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} < A^*(\lambda)$$

Proof

Let X_n be the orbit length at the time of departure of n^{th} customer $n \geq 1$. Then $\{X_n, n \in N\}$ is irreducible and aperiodic. We now prove that the embedded Markov chain $\{X_n, n \in N\}$ is ergodic if and only if

$$\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} < A^*(\lambda).$$

Foster's criterion states that an irreducible and aperiodic Markov chain is ergodic if there exists a nonnegative function f(j), $j \in N$, and $\varepsilon > 0$, such that the mean drift $\psi_j = E[f(X_{n-1}) - f(X_n) \mid X_n = j]$ is finite for all $j \in N$ and $\psi_j < -\varepsilon$ for all $j \in N$, except perhaps for a finite number of j.

In our case $\{X_n : n \in N\}$ is irreducible and aperiodic. Now considering the function f(j) = j, we have

$$\psi_j = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} - A^*(\lambda),\ j \geq 1.$$

Hence if $\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} < A^*(\lambda)$ then $\{X_n\}$ is ergodic.

The necessary condition follows from Kaplan's condition namely $\psi_j < \infty$ for all $j \geq 0$ and there exists $j_n \in N$ such that $\psi_j \geq 0$ for $j \geq j_0$. Since the arrival stream is a Poisson process. Burke's theorem establishes the existence of the steady state probabilities of $\{J(t), X(t), t \geq 0\}$ and they are positive if and only if

$$\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} < A^*(\lambda)$$

From the mean drift

$$\psi_j = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} - A^*(\lambda) \text{ for all } j \geq 1,$$

where j denotes the number of customers in the orbit, we have the reasonable conclusion.

The term $\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i}$ represents the mean number of customers entering orbit due to the server being busy with i stage service, i = 1, 2, . . . , k. The second term, $\sum_{i=1}^{k} p_i\, \Theta_{i-1}$ is arrival due to feedback. Further $A^*(\lambda)$ provides the expected number of orbiting customers who enter service successfully. For stability the new customers arrive during a service time more slowly than customers from the orbit who enters service successfully. That is $\sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i\, M_{1i} + \sum_{i=1}^{k} p_i\, \Theta_{i-1} < A^*(\lambda)$ implying $\psi_j < 0$ for $j \geq 1$.

4 Steady State Distribution

For the process, $\{N(t)\ ;\ t \geq 0\}$, define the probabilities,

$I_0(t) = P\{J(t) = 0, X(t) = 0\}$

$I_n(t, x)\,dx = P\{J(t) = 0, X(t) = n, x < \xi_0(t) < x + dx\}, n \geq 1$

$W_{n,i}(t, x)\,dx = P\{J(t) = i, X(t) = n, x < \xi_i(t) < x + dx\},$ $n \geq 0\ ;\ i = 1, 2, \ldots k$

Let I_0, $I_n(x)$ and $W_{n,i}(x)$ are the limiting densities of $I_0(t)$, $I_n(t, x)$ and $W_{n,i}(t, x)$.

5 Steady State Probability Generating Function

The steady state equations for the model under consideration are,

$$\lambda I_0 = \sum_{i=1}^{k-1} q_i \int_0^\infty \mu_i(x)\, W_{0,i}(x)\, dx + (1 - p_k) \int_0^\infty \mu_k(x)\ W_{0,k}(x)\, dx \quad (1)$$

$$\frac{dI_n(x)}{dx} = -(\lambda + \eta(x))\, I_n(x), \qquad n \geq 1 \quad (2)$$

$$\frac{dW_{0,i}(x)}{dx} = -(\lambda + \mu_i(x))\, W_{0,i}(x),\ i = 1, 2, \ldots k \quad (3)$$

$$\frac{dW_{n,i}(x)}{dx} = -(\lambda + \mu_i(x))\, W_{n,i}(x) + \lambda\, W_{n-1,i}(x), \qquad n \geq 1, i = 1, 2, \ldots k \quad (4)$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{k-1} q_i \int_0^\infty \mu_i(x)\, W_{n,i}(x)\, dx + (1 - p_k) \int_0^\infty \mu_k(x)\ W_{n,i}(x)\, dx + \sum_{i=1}^{k} p_i \int_0^\infty \mu_i(x)\, W_{n-1,i}(x)\, dx, \qquad n \geq 1 \quad (5)$$

$$W_{0,1}(x) = \lambda I_0 + \int_0^\infty I_1(x)\,\eta(x)\,dx \quad (6)$$

$$W_{n,1}(x) = \lambda \int_0^\infty I_n(x)\,dx + \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx, \quad n \ge 1 \quad (7)$$

$$W_{n,i}(x) = \theta_{i-1} \int_0^\infty \mu_{i-1}(x)\, W_{n,i-1}(x)\,dx,\ n \ge 0, \quad i = 2, 3, \ldots k \quad (8)$$

The normalizing condition is given by the equation

$$I_0 + \sum_{n=1}^{\infty} \int_0^\infty I_n(x)\,dx + \sum_{i=1}^{k} \sum_{n=0}^{\infty} \int_0^\infty W_{n,i}(x)\,dx = 1 \quad (9)$$

Define probability generating functions,

$$I(z, x) = \sum_{n=1}^{\infty} I_n(x) z^n \text{ and}$$

$$W_i(z, x) = \sum_{n=0}^{\infty} W_{n,i}(x) z^n,\ i = 1, 2, 3, \ldots, k$$

Multiplying the equations (1) – (5) with suitable powers of z, taking the sum and solving the differential equations so obtained, we get

$$I(z, x) = I(z, 0)\, e^{-\lambda x}\,[1 - A(x)] \quad (10)$$

$$W_i(z, x) = W_i(z, 0)\, e^{-\lambda(1-z)x}\,[1 - B_i(x)],\ i = 1, 2, \ldots k \quad (11)$$

$$I(z, 0) = \sum_{i=1}^{k} [q_i + p_i z]\, W_i(z, 0)\, B_i^*(\lambda(1 - z)) - \lambda I_0 \quad (12)$$

Using equations (10) and (12), the equations (6) and (7) yield

$$W_1(z, 0)\,[[z + (1 - z) A^*(\lambda)]\,[q_1 + p_1 z]\, B_1^*(\lambda(1 - z)) - z]$$
$$= \lambda I_0 (1 - z) A^*(\lambda) - [z + (1 - z) A^*(\lambda)]$$
$$[\sum_{i=2}^{k} [q_i + p_i z]\, W_i(z, 0)\, B_i^*(\lambda(1 - z)) \quad (13)$$

and equation (8) yields

$$W_i(z, 0) = \theta_{i-1} W_{i-1}(z, 0)\, B_i^*(\lambda(1 - z))$$
$$= \theta_{i-1}\theta_{i-2} W_{i-2}(z, 0)\, \Lambda_{i-1}^*(\lambda(1 - z))\, B_i^*(\lambda(1 - z))$$
$$= \Theta_{i-1}\, \Lambda_{i-1}^*(\lambda(1-z))\, W_1(z, 0), \quad i = 2, 3, \ldots k \quad (14)$$

Substituting the expressions in equation (13) in to equation (14), and solving we get

$$W_i(z, 0) = \Theta_{i-1}\, \Lambda_{i-1}^*(\lambda(1-z))\, \lambda I_0 (1 - z) A^*(\lambda)] / D(z) \quad i = 1, 2, \ldots, k \quad (15)$$

where $D(z) = [z + (1 - z)A^*(\lambda)] \sum_{i=1}^{k} [q_i + p_i z]\, \Theta_{i-1}\, \Lambda_i^*(\lambda(1-z)) - z$

Substituting the expression of $W_i(z, 0)$, i = 1, 2, . . . , k in equation (12), we have

$$I(z, 0) = \lambda I_0 z\, [1 - \sum_{i=1}^{k} [q_i + p_i z]^*(\lambda)]\, \Theta_{i-1}\, \Lambda_i^*(\lambda(1-z))\,] / D(z) \quad (16)$$

The partial generating function of the orbit size when the server is busy in providing i[th] stage service is given by,

$$W_i(z) = \int_0^\infty W_i(z, x)\,dx$$
$$= \frac{I_0 A^*(\lambda)}{D(z)}\, \Theta_{i-1}\, \Lambda_{i-1}^*(\lambda(1-z))\,[1 - B_1^*(\lambda(1 - z)) - z]$$
$$i = 1, 2, 3, \ldots k$$

The steady state probability that the server is in i[th] stage service, is

$$W_i(1) = \frac{\Theta_{i-1}\, A^*(\lambda)\, \lambda \mu_{1i} I_0}{T_1} \quad i = 1, 2, 3, \ldots, k \quad (17)$$

where $T_1 = A^*(\lambda) - \sum_{i=1}^{k} p_i\, \Theta_{i-1} - \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k-1} \Theta_i M_{1i}$

The partial generating function of the orbit size when the server is idle as

$$I(z) = \int_0^\infty I(z, x)\,dx$$
$$= I_0 z\,[1 - A^*(\lambda)]\,[1 - \sum_{i=1}^{k} [q_i + p_i z]\, \Theta_{i-1}\, \Lambda_i^*(\lambda(1-z))] / D(z)$$

The steady state probability that the server idle in the non-empty system is

$$I(1) = \frac{[1 - A^*(\lambda)]}{T_1}\,[\sum_{i=1}^{k} p_i\, \Theta_{i-1} + \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i M_{1i}]\, I_0 \quad (18)$$

Substituting the expressions of $I(1)$ and $W_i(1)$ for $i = 1, 2, \ldots k$ in the normalizing condition $I_0 + I(1) + \sum_{i=1}^{k} W_i(1) = 1$, We get

$$I_0 = T_1 / [A^*(\lambda) T_2]$$

where $T_2 = 1 - \sum_{i=1}^{k} \Theta_{i-1}[p_i + M_{1i}] + \sum_{i=1}^{k-1} \Theta_i M_{1i} + \sum_{i=1}^{k} \Theta_{i-1} \lambda \mu_{1i}$

The probability generating function for the number of customers in the system is

$$K(z) = I_0 + I(z) + z \sum_{i=1}^{k} W_i(z)$$
$$= I_0 A^*(\lambda)\,[\sum_{i=1}^{k} [q_i + p_i z]\, \Theta_{i-1}\, \Lambda_i^*(\lambda(1-z))$$
$$+ z \sum_{i=1}^{k} [1 - B_i^*(\lambda(1-z))]\, \Theta_{i-1}\, \Lambda_1^*(\lambda(1-z)) - z] / D(z) \quad (19)$$

The mean number of customers in the system

$$L_s = K'(1) = \frac{N_2}{T_2} + \frac{N_1 T_3}{T_1 T_2} \quad (20)$$

where

$$N_1 = 1 - \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \sum_{i=1}^{k-1} \Theta_i M_{1i} - \sum_{i=1}^{k} p_i\, \Theta_{i-1} + \sum_{i=1}^{k} \lambda \mu_{1i}\, \Theta_{i-1}$$

$$N_2 = \sum_{i=1}^{k} \lambda \mu_{1i}\, \Theta_{i-1} + \sum_{i=1}^{k} \lambda \mu_{1i} M_{1i}\, \Theta_{i-1} - \sum_{i=1}^{k} p_i\, \Theta_{i-1} M_{1i}$$
$$+ \frac{1}{2}\left[\sum_{i=1}^{k} \Theta_i M_{2i} + \sum_{i=1}^{k} \mu_{2i}\, \Theta_{i-1} - \sum_{i=1}^{k} \Theta_{i-1} M_{2i}\right]$$

ISSN 2229-5518

$$T_3 = [1 - A^*(\lambda)]\,[A^*(\lambda) - T_1] + \sum_{i=1}^{k} p_i\,\Theta_{i-1}\,M_{1i} + \frac{1}{2}\left[\sum_{i=1}^{k} \Theta_{i-1}\,M_{2i} - \sum_{i=1}^{k-1} \Theta_i\,M_{2i}\right]$$

The probability generating function for the number of customers in the queue is

$$H(z) = I_0 + I(z) + \sum_{i=1}^{k} W_i(z)$$

$$= I_0\,A^*(\lambda)\,\Big[\sum_{i=1}^{k} [q_i + p_i z]\,\Theta_{i-1}\,\Lambda_i^*(\lambda(1-z)) + \sum_{i=1}^{k} [1 - B_i^*(\lambda(1-z))]\,\Theta_{i-1}\,\Lambda_{i-1}^*(\lambda(1-z)) - z\Big] \,/ D(z)$$

(21)

The mean number of customers in the queue is

$$L_q = H'(1) = L_s - \sum_{i=1}^{k} \Theta_{i-1}\,\lambda\,\mu_{1i}]\,/\,T_2 \qquad (22)$$

6 Stochastic Decomposition

Stochastic decomposition has been widely observed among M/G/1 type queues. The decomposition property states that the number of customers in the system in steady state at a random point of time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system in steady state at a random point in time, the other random variable may have different probabilistic interpretation in specific cases depending on the vacation scheduled.

Let $\pi(z)$ be the probability generating function of the number of customers in the classical M/G/1 queue with k-stages of service facility and feedback. Then

$$\pi(z) = [1 - T_1 - A^*(\lambda)]\Big[\sum_{i=1}^{k} [q_i + p_i z]\,\Theta_{i-1}\,\Lambda_i^*(\lambda(1-z)) + z\sum_{i=1}^{k} [1 - B_i^*(\lambda(1-z))]\,\Theta_{i-1}\,\Lambda_{i-1}^*(\lambda(1-z)) - z\Big] \,/\, D_1(z)T_2]$$

where $D_1(z) = \sum_{i=1}^{k} [q_i + p_i z]\,\Theta_{i-1}\,\Lambda_i^*(\lambda(1-z)) - z$

If the server is idle either due to retrial of customers from the orbit or due to empty system, we say that the server is on vacation. Let $\psi(z)$ be the probability generating function of the number of customers in the system at a random point of time given that the server is on vacation. Then

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= T_2\Big[\sum_{i=1}^{k} [q_i + p_i z]\,\Theta_{i-1}\,\Lambda_i^*(\lambda(1-z)) - z\Big] \,/\, [D(z)\,[1 - T_1 - A^*(\lambda)]]$$

From equation (19) it is observed that the probability generating function of the number of customers in the system K(z) is decomposed as

$$K(z) = \pi(z)\;\psi(z).$$

This shows that the decomposition law is valid for the model under consideration.

References

[1] Artalejo, J.R., "A classified bibliography of research on retrial queues : Progress in 1990-1999", Top, Vol. 7, No. 2, pp. 187-211, 1999.

[2] Artalejo, J.R., "Accessible bibliography on retrial queues", Mathematical Computer Modelling, Vol. 30, No. 3-4, pp. 1-6, 1999.

[3] Choudhury, G., Paul, M., "A two phase queueing system with Bernoulli feedback", Inf. Manage Sci., Vol. 16, No. 1, pp. 35-52, 2005.

[4] Falin, G.I., Templeton, J.G.C., "Retrial queues", Chapman and Hall, London, 1997.

[5] Madan, K.C., Choudhury, G., "A single server queue with two phases of heterogeneous service under Bernoulli schedule and a general vacation time", Inf. Manage. Sci., Vol. 16, No. 2, pp. 1-16, 2005.

[6] Shahkar, G.H., Badamchizadeh, A., "On $M/(G_1, G_2, \ldots, G_k)/V/1(BS)$", Far East J. Theor. Stat., Vol. 20, No. 2, pp. 151-162, 2006.

[7] Salehirad, M.R., Badamchizadeh, A., "On the multi-phase M/G/1 queueing system with random feedback", EJOR, Vol. 17, pp. 131-139, 2009.

[8] Wang, J., "An M/G/1 queue with second optional service and server breakdowns", Com. Math. Appl., Vol. 47, pp. 1713-1723, 2004.