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Multi-term fractional differential equations with nonlocal boundary conditions

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Abstract: We introduce and study a new kind of nonlocal boundary value problems of multi-term fractional differential equations. The existence and uniqueness results for the given problem are obtained by applying standard fixed point theorems. We also construct some examples for demonstrating the application of the main results.

Keywords: Caputo fractional derivative; multi-term fractional derivatives; existence; uniqueness; fixed point theorems

MSC: 34A08; 34B10; 34B15

1 Introduction

Non-integer (arbitrary) order calculus has been extensively studied by many researchers in the recent years. The literature on the topic is now much enriched and contains a variety of results. The overwhelming interest in this branch of mathematical analysis results from its extensive applications in modeling several real world problems occurring in natural and social sciences. The mathematical models based on the tools of fractional calculus provide more insight into the characteristics of the associated phenomena in view of the nonlocal nature of fractional order operators in contrast to integer order operators. Examples include bioengineering [14], physics [9], thermoelasticity [15], etc. Boundary value problems of fractional order differential equations and inclusions have also attracted a significant attention and one can find a great deal of work on the topic involving different kinds of boundary conditions, for instance, see [1–7] and the references cited therein.

Besides the equations involving only one differential operator, there are certain equations containing more than one differential operators. Such equations are called multi-term differential equations, see [5, 12, 13, 16].

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In this paper we investigate a new type of boundary value problems of multi-term fractional differential equations and nonlocal three-point boundary conditions. Precisely, we consider the following problem:

$$(a_2\,{}^cD^{q+2} + a_1\,{}^cD^{q+1} + a_0\,{}^cD^q)x(t) = f(t, x(t)),\ \ 0 < q < 1,\ \ 0 < t < 1, \tag{1}$$

$$x(0) = 0,\ \ x(\eta) = 0,\ \ x(1) = 0,\ \ 0 < \eta < 1, \tag{2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function and a_i $(i = 0, 1, 2)$ are real positive constants.

We prove the existence of solutions for the problem (1)-(2) by means of Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative, while the uniqueness of solutions is established by Banach fixed point theorem. These results are presented in Section 3. An auxiliary lemma concerning the linear variant of (1)-(2) and some definitions are given in Section 2. Section 4 contains illustrative examples for the main results.

2 Basic results

We begin this Section with some definitions [10].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\tau > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is defined by*

$$I^\tau h(u) = \int_0^u \frac{(u - v)^{\tau-1}}{\Gamma(\tau)} h(v)dv, \quad u > 0,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the Gamma function.

Definition 2.2. *The Caputo derivative of order τ for a function $h : [0, \infty) \to \mathbb{R}$ with $h(x) \in C^n[0, \infty)$ is defined by*

$${}^cD^\tau h(u) = \frac{1}{\Gamma(n - \tau)} \int_0^u \frac{h^{(n)}(v)}{(u - v)^{\tau+1-n}} dv = I^{n-\tau} h^{(n)}(u),\ t > 0,\ n - 1 < \tau < n,$$

Property 2.1. *With the given notations, the following equality holds:*

$$I^\tau({}^cD^\tau h(u)) = h(u) - c_0 - c_1u \ldots - c_{n-1}u^{n-1},\ u > 0,\ n - 1 < \tau < n, \tag{3}$$

where c_i $(i = 1, \ldots, n - 1)$ are arbitrary constants.

The following lemma facilitates the transformation of the problem (1)-(2) into a fixed point problem.

Lemma 2.1. *For any $y \in C([0, 1], \mathbb{R})$, the solution of linear multi-term fractional differential equation*

$$(a_2\,{}^cD^{q+2} + a_1\,{}^cD^{q+1} + a_0\,{}^cD^q)x(t) = y(t),\ \ 0 < q < 1,\ \ 0 < t < 1, \tag{4}$$

supplemented with the boundary conditions (2) is given by

$$\begin{aligned}(i)\quad x(t) = \frac{1}{a_2(m_2 - m_1)} \Bigg\{ &\int_0^t \int_0^s \Phi(t) \frac{(s - u)^{q-1}}{\Gamma(q)} y(u)du\, ds \\ &+\sigma_1(t) \int_0^1 \int_0^s \Phi(1) \frac{(s - u)^{q-1}}{\Gamma(q)} y(u)du\, ds \\ &+\sigma_2(t) \int_0^\eta \int_0^s \Phi(\eta) \frac{(s - u)^{q-1}}{\Gamma(q)} y(u)du\, ds \Bigg\}, \qquad if\ \ a_1^2 - 4a_0a_2 > 0,\end{aligned} \tag{5}$$

$$(ii)\quad x(t) = \frac{1}{a_2}\Bigg\{ \int_0^t \int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds$$
$$+\psi_1(t)\int_0^1 \int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds \tag{6}$$
$$+\psi_2(t)\int_0^\eta \int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds\Bigg\}, \quad \text{if } a_1^2 - 4a_0a_2 = 0,$$

$$(iii)\quad x(t) = \frac{1}{a_2\beta}\Bigg\{ \int_0^t \int_0^s \Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds$$
$$+\varphi_1(t)\int_0^1 \int_0^s \Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds \tag{7}$$
$$+\varphi_2(t)\int_0^\eta \int_0^s \Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du\, ds\Bigg\}, \quad \text{if } a_1^2 - 4a_0a_2 < 0,$$

where

$$\begin{cases} \Phi(\kappa) = e^{m_2(\kappa-s)} - e^{m_1(\kappa-s)}, \quad \kappa = t,\ 1,\ and\ \eta, \\ m_1 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2}, \quad m_2 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_2}, \\ \sigma_1(t) = \dfrac{\gamma_2\rho_2(t) - \gamma_4\rho_1(t)}{\mu}, \quad \sigma_2(t) = \dfrac{\gamma_3\rho_1(t) - \gamma_1\rho_2(t)}{\mu}, \\ \mu = \gamma_1\gamma_4 - \gamma_2\gamma_3 \neq 0, \\ \rho_1(t) = \dfrac{m_2(1-e^{m_1 t}) - m_1(1-e^{m_2 t})}{a_2 m_1 m_2(m_2 - m_1)}, \quad \rho_2(t) = e^{m_1 t} - e^{m_2 t}, \\ \gamma_1 = \dfrac{m_2(1-e^{m_1}) - m_1(1-e^{m_2})}{a_2 m_1 m_2(m_2-m_1)}, \quad \gamma_2 = \dfrac{m_2(1-e^{m_1\eta}) - m_1(1-e^{m_2\eta})}{a_2 m_1 m_2(m_2-m_1)}, \\ \gamma_3 = e^{m_1} - e^{m_2}, \quad \gamma_4 = e^{m_1\eta} - e^{m_2\eta}, \end{cases} \tag{8}$$

$$\begin{cases} \Psi(\kappa) = (\kappa - s)e^{m(\kappa-s)}, \quad \kappa = t,\ 1,\ and\ \eta, \\ \psi_1(t) = \dfrac{(t-\eta)e^{m(t+\eta)} - te^{mt} + \eta e^{m\eta}}{\Lambda}, \\ \psi_2(t) = \dfrac{(1-t)e^{m(t+1)} + te^{mt} - e^{m}}{\Lambda}, \\ \Lambda = (\eta - 1)e^{m(\eta+1)} - \eta e^{m\eta} + e^{m} \neq 0, \ m = \dfrac{-a_1}{2a_2}, \end{cases} \tag{9}$$

$$\begin{cases} \Omega(\kappa) = e^{-\alpha(\kappa-s)}\sin\beta(\kappa - s), \quad \kappa = t,\ 1,\ and\ \eta, \\ \alpha = \dfrac{a_1}{2a_2}, \quad \beta = \dfrac{\sqrt{4a_0a_2 - a_1{}^2}}{2a_2}, \\ \varphi_1(t) = \dfrac{\omega_4\varrho_1(t) - \omega_2\varrho_2(t)}{\Omega}, \quad \varphi_2(t) = \dfrac{\omega_1\varrho_2(t) - \omega_3\varrho_1(t)}{\Omega}, \\ \varrho_1(t) = \dfrac{\beta - \beta e^{-\alpha t}\cos\beta t - \alpha e^{-\alpha t}\sin\beta t}{\alpha^2+\beta^2}, \quad \varrho_2(t) = a_2\beta e^{-\alpha t}\sin\beta t, \\ \omega_1 = \dfrac{\beta - \beta e^{-\alpha}\cos\beta - \alpha e^{-\alpha}\sin\beta}{\alpha^2+\beta^2}, \\ \omega_2 = \dfrac{\beta - \beta e^{-\alpha\eta}\cos\beta\eta - \alpha e^{-\alpha\eta}\sin\beta\eta}{\alpha^2+\beta^2}, \\ \omega_3 = a_2\beta e^{-\alpha}\sin\beta, \quad \omega_4 = a_2\beta e^{-\alpha\eta}\sin\beta\eta, \\ \Omega = \omega_2\omega_3 - \omega_1\omega_4 \neq 0. \end{cases} \tag{10}$$

Proof. Case (i): $a_1^2 - 4a_0a_2 > 0$.

Applying the operator I^q on (4) and using (3), we get

$$(a_2D^2 + a_1D + a_0)x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + c_1, \tag{11}$$

where c_1 is an arbitrary constant. By the method of variation of parameters, the solution of (11) can be written as

$$\begin{aligned} x(t) &= c_2 e^{m_1 t} + c_3 e^{m_2 t} - \frac{1}{a_2(m_2-m_1)} \int_0^t e^{m_1(t-s)} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du + c_1 \right) ds \\ &\quad + \frac{1}{a_2(m_2-m_1)} \int_0^t e^{m_2(t-s)} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du + c_1 \right) ds, \end{aligned} \tag{12}$$

where m_1 and m_2 are given by (8). Using $x(0) = 0$ in (12), we get

$$\begin{aligned} x(t) &= c_1 \left[\frac{m_2(1-e^{m_1 t}) - m_1(1-e^{m_2 t})}{a_2 m_1 m_2 (m_2-m_1)} \right] + c_2 \left(e^{m_1 t} - e^{m_2 t} \right) \\ &\quad - \frac{1}{a_2(m_2-m_1)} \left[\int_0^t \left(e^{m_1(t-s)} - e^{m_2(t-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds \right], \end{aligned} \tag{13}$$

which together with the conditions $x(1) = 0$ and $x(\eta) = 0$ yields the following system of equations in the unknown constants c_1 and c_2:

$$\begin{aligned} c_1\gamma_1 + c_2\gamma_3 &= \frac{1}{a_2(m_2-m_1)} \int_0^1 \left(e^{m_1(1-s)} - e^{m_2(1-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds, \\ c_1\gamma_2 + c_2\gamma_4 &= \frac{1}{a_2(m_2-m_1)} \int_0^\eta \left(e^{m_1(\eta-s)} - e^{m_2(\eta-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds. \end{aligned}$$

Solving the above system together with the notations (8), we find that

$$\begin{aligned} c_1 = \frac{1}{a_2\mu(m_2-m_1)} &\left[\gamma_4 \int_0^1 \left(e^{m_1(1-s)} - e^{m_2(1-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds \right. \\ &\left. - \gamma_3 \int_0^\eta \left(e^{m_1(\eta-s)} - e^{m_2(\eta-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds \right], \end{aligned}$$

and

$$\begin{aligned} c_2 = \frac{1}{a_2\mu(m_2-m_1)} &\left[\gamma_1 \int_0^\eta \left(e^{m_1(\eta-s)} - e^{m_2(\eta-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds \right. \\ &\left. - \gamma_2 \int_0^1 \left(e^{m_1(1-s)} - e^{m_2(1-s)} \right) \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)du \right) ds \right]. \end{aligned}$$

Substituting the value of c_1 and c_2 in (13), we obtain the solution (5). The converse of the lemma follows by direct computation.

The other two cases can be treated in a similar manner. This completes the proof. □

3 Existence and Uniqueness Results

Let $\mathcal{C} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1]$ to $\mathbb{R}$ equipped with the norm defined by$\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$.

By Lemma 2.1, we transform the problem (1)-(2) into equivalent fixed point problems according to the cases $(i) - (iii)$ as follows.

(i) For ${a_1}^2 - 4a_0a_2 > 0$, we define

$$x = \mathcal{J}x, \tag{14}$$

where the operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned}(\mathcal{J}x)(t) = \frac{1}{a_2(m_2 - m_1)}\Bigg\{ &\int_0^t \int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\sigma_1(t)\int_0^1 \int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\sigma_2(t)\int_0^\eta \int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds\Bigg\},\end{aligned} \tag{15}$$

where $\Phi(\cdot)$, $\sigma_1(t)$ and $\sigma_2(t)$ are defined by (8).

(ii) In case ${a_1}^2 - 4a_0a_2 = 0$, we have

$$x = \mathcal{H}x, \tag{16}$$

where the operator $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned}(\mathcal{H}x)(t) = \frac{1}{a_2}\Bigg\{ &\int_0^t \int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\psi_1(t)\int_0^1 \int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\psi_2(t)\int_0^\eta \int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds\Bigg\},\end{aligned} \tag{17}$$

where $\Psi(\cdot)$, $\psi_1(t)$ *and* $\psi_2(t)$ are given by (9).

(iii) When ${a_1}^2 - 4a_0a_2 < 0$, let us define

$$x = \mathcal{K}x, \tag{18}$$

where the operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned}(\mathcal{K}x)(t) = \frac{1}{a_2\beta}\Bigg\{ &\int_0^t \int_0^s \Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\varphi_1(t)\int_0^1 \int_0^s \Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds \\ &+\varphi_2(t)\int_0^\eta \int_0^s \Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u, x(u))du\,ds\Bigg\},\end{aligned} \tag{19}$$

where $\Omega(\cdot)$, $\varphi_1(t)$ *and* $\varphi_2(t)$ are defined by (10).

In the sequel, for the sake of computational convenience, we set

$$\begin{cases} \widehat{\sigma}_1 = \max\limits_{t\in[0,1]} |\sigma_1(t)|, \ \ \widehat{\sigma}_2 = \max\limits_{t\in[0,1]} |\sigma_2(t)|, \\ \varepsilon = \max\limits_{t\in[0,1]} \left| \dfrac{m_2(1-e^{m_1 t}) - m_1(1-e^{m_2 t})}{a_2 m_1 m_2 (m_2 - m_1)} \right|, \\ \lambda = \dfrac{1}{\Gamma(q+1)} \left\{ \varepsilon + \widehat{\sigma}_1 \gamma_1 + \eta^q \widehat{\sigma}_2 \gamma_2 \right\}, \ \ \lambda_1 = \lambda - \dfrac{\varepsilon}{\Gamma(q+1)}, \end{cases} \tag{20}$$

$$\begin{cases} \widehat{\psi}_1 = \max\limits_{t\in[0,1]} |\psi_1(t)|, \ \ \widehat{\psi}_2 = \max\limits_{t\in[0,1]} |\psi_2(t)|, \\ \mu = \dfrac{1}{a_2 m^2 \Gamma(q+1)} \left\{ (1+\widehat{\psi}_1)\Big((m-1)e^m + 1\Big) + \widehat{\psi}_2 \eta^q \Big((m\eta - 1)e^{m\eta} + 1\Big) \right\}, \\ \mu_1 = \mu - \dfrac{\Big((m-1)e^m + 1\Big)}{a_2 m^2 \Gamma(q+1)}, \end{cases} \tag{21}$$

$$\begin{cases} \widehat{\varphi}_1 = \max\limits_{t\in[0,1]} |\varphi_1(t)|, \ \ \widehat{\varphi}_2 = \max\limits_{t\in[0,1]} |\varphi_2(t)|, \\ \rho = \dfrac{1}{a_2(\alpha^2+\beta^2)\Gamma(q+1)} \Big\{ (1+\widehat{\varphi}_1)\Big(1 - e^{-\alpha}\cos\beta - (\alpha/\beta)e^{-\alpha}\sin\beta\Big) \\ \quad + \widehat{\varphi}_2 \eta^q \Big(1 - e^{-\alpha\eta}\cos\beta\eta - (\alpha/\beta)e^{-\alpha\eta}\sin\beta\eta\Big) \Big\}, \\ \rho_1 = \rho - \dfrac{(1 - e^{-\alpha}\cos\beta - (\alpha/\beta)e^{-\alpha}\sin\beta)}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}. \end{cases} \tag{22}$$

Before presenting our first existence result for the problem (1)-(2), let us state Krasnoselskii's fixed point theorem [11] that plays a key role in its proof.

Theorem 3.1. (Krasnoselskii's fixed point theorem). *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let F_1 and F_2 be the operators satisfying the conditions: (i) $F_1 y_1 + F_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) F_1 is compact and continuous; (iii) F_2 is a contraction mapping. Then there exists $y \in Y$ such that $y = F_1 y + F_2 y$.*

In the forthcoming analysis, we need the following assumptions:
(A_1) $|f(t,x) - f(t,y)| \le \ell |x-y|$, for all $t \in [0,1]$, $x, y \in \mathbb{R}$, $\ell > 0$.
(A_2) $|f(t,x)| \le \vartheta(t)$, for all $(t,x) \in [0,1] \times \mathbb{R}$ and $\vartheta \in C([0,1], \mathbb{R}^+)$.

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions (A_1) and (A_2). Then the problem (1)-(2) has at least one solution on $[0,1]$ provided that*
(i) $\ell\lambda_1 < 1$ for ${a_1}^2 - 4a_0 a_2 > 0$, where λ_1 is given by (20);
(ii) $\ell\mu_1 < 1$ for ${a_1}^2 - 4a_0 a_2 = 0$, where μ_1 is given by (21);
(iii) $\ell\rho_1 < 1$ for If ${a_1}^2 - 4a_0 a_2 < 0$, where ρ_1 is given by (22).

Proof. (i) Setting $\sup_{t\in[0,1]} |\vartheta(t)| = \|\vartheta\|$ and choosing

$$r_1 \ge \frac{\|\vartheta\|}{\Gamma(q+1)} \left\{ \varepsilon + \widehat{\sigma}_1 \gamma_1 + \eta^q \widehat{\sigma}_2 \gamma_2 \right\}, \tag{23}$$

we consider a closed ball $B_{r_1} = \{x \in \mathcal{C} : \|x\| \le r_1\}$. Introduce the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ on B_{r_1} as follows:

$$(\mathcal{J}_1 x)(t) = \frac{1}{a_2(m_2 - m_1)} \int_0^t \int_0^s \Phi(t) \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du\, ds, \tag{24}$$

$$(\mathcal{J}_2 x)(t) = \frac{1}{a_2(m_2 - m_1)} \Bigg\{ \sigma_1(t) \int_0^1 \int_0^s \Phi(1) \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du\, ds$$

$$+\sigma_2(t)\int_0^t\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Bigg\}. \tag{25}$$

Observe that $\mathcal{J}=\mathcal{J}_1+\mathcal{J}_2$. For $x,y\in B_{r_1}$, we have

$$\begin{aligned}
&\|\mathcal{J}_1x+\mathcal{J}_2y\|\\
&=\sup_{t\in[0,1]}|(\mathcal{J}_1x)(t)+(\mathcal{J}_2y)(t)|\\
&\le \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&+|\sigma_1(t)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds\\
&+|\sigma_2(t)|\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds\Bigg\}\\
&\le \frac{\|\vartheta\|}{a_2(m_2-m_1)\Gamma(q+1)}\sup_{t\in[0,1]}\Bigg\{t^q\int_0^t\left(e^{m_2(t-s)}-e^{m_1(t-s)}\right)ds\\
&+|\sigma_1(t)|\int_0^1\left(e^{m_2(1-s)}-e^{m_1(1-s)}\right)ds+\eta^q|\sigma_2(t)|\int_0^\eta\left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)ds\Bigg\}\\
&\le \frac{\|\vartheta\|}{\Gamma(q+1)}\left\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\right\}\le r_1,
\end{aligned}$$

where we used (23). Thus $\mathcal{J}_1x+\mathcal{J}_2y\in B_{r_1}$. Using the assumption (A_1) together with the condition $\ell\lambda_1<1$, we can show that $\mathcal{J}_2$ is a contraction as follows:

$$\begin{aligned}
&\|\mathcal{J}_2x-\mathcal{J}_2y\|\\
&=\sup_{t\in[0,1]}|(\mathcal{J}_2x)(t)-(\mathcal{J}_2y)(t)|\\
&\le \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{|\sigma_1(t)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\,ds\\
&+|\sigma_2(t)|\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\,ds\Bigg\}\\
&\le \frac{\ell}{a_2(m_2-m_1)\Gamma(q+1)}\sup_{t\in[0,1]}\Bigg\{|\sigma_1(t)|\int_0^1\left(e^{m_2(1-s)}-e^{m_1(1-s)}\right)ds\\
&+\eta^q|\sigma_2(t)|\int_0^\eta\left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)ds\Bigg\}\|x-y\|\\
&\le \frac{\ell}{\Gamma(q+1)}\left\{\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\right\}\|x-y\|=\ell\lambda_1\|x-y\|.
\end{aligned}$$

Note that continuity of f implies that the operator $\mathcal{J}_1$ is continuous. Also, $\mathcal{J}_1$ is uniformly bounded on B_{r_1} as

$$\|\mathcal{J}_1x\|=\frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\left|\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\right|\le\frac{\|\vartheta\|\varepsilon}{\Gamma(q+1)}.$$

Now we prove the compactness of operator $\mathcal{J}_1$. We define $\sup_{(t,x)\in[0,1]\times B_{r_1}} |f(t,x)| = \bar{f}$. Then, for $0 < t_1 < t_2 < 1$, we have

$$
\begin{aligned}
&|(\mathcal{J}_1x)(t_2) - (\mathcal{J}_1x)(t_1)| \\
&= \frac{1}{a_2(m_2-m_1)}\Bigg|\int_0^{t_1}\int_0^{s}\Big[\Phi(t_2)-\Phi(t_1)\Big]\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds \\
&\quad + \int_{t_1}^{t_2}\int_0^{s}\Phi(t_2)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Bigg| \\
&\le \frac{\bar{f}}{a_2m_1m_2(m_2-m_1)\Gamma(q+1)}\Big\{\Big(t_1^q - t_2^q\Big)\Big(m_1(1-e^{m_2(t_2-t_1)}) - m_2(1-e^{m_1(t_2-t_1)})\Big) \\
&\quad + t_1^q\Big(m_1(e^{m_2t_2}-e^{m_2t_1}) - m_2(e^{m_1t_2}-e^{m_1t_1})\Big)\Big\} \to 0 \ \ as\ \ t_2 - t_1 \to 0,
\end{aligned}
$$

independent of x. Thus $\mathcal{J}_1$ is relatively compact on B_{r_1}. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{J}_1$ is compact on B_{r_1}. Thus all the assumptions of Theorem 3.1 are satisfied. So, by the conclusion of Theorem 3.1, the problem (1)-(2) has at least one solution on [0, 1].

(ii) Let us consider $B_{r_2} = \{x \in \mathcal{C} : \|x\| \le r_2\}$, where $\sup_{t\in[0,1]}|\vartheta(t)| = \|\vartheta\|$ and

$$
r_2 \ge \frac{\|\vartheta\|}{a_2m^2\Gamma(q+1)}\Big\{(1+\widehat{\psi}_1)\Big((m-1)e^m+1\Big) + \widehat{\psi}_2\eta^q\Big((m\eta-1)e^{m\eta}+1\Big)\Big\}. \tag{26}
$$

Introduce the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on B_{r_2} as follows:

$$
(\mathcal{H}_1x)(t) = \frac{1}{a_2}\int_0^t\int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds, \tag{27}
$$

$$
\begin{aligned}
(\mathcal{H}_2x)(t) = \frac{1}{a_2}\Bigg\{&\psi_1(t)\int_0^1\int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds \\
&+\psi_2(t)\int_0^\eta\int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Bigg\}.
\end{aligned} \tag{28}
$$

Observe that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. For $x, y \in B_{r_2}$, we have

$$
\begin{aligned}
\|\mathcal{H}_1x + \mathcal{H}_2y\| &= \sup_{t\in[0,1]}|(\mathcal{H}_1x)(t) + (\mathcal{H}_2y)(t)| \\
&\le \frac{1}{a_2}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds \\
&\quad +\psi_1(t)\int_0^1\int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds \\
&\quad +\psi_2(t)\int_0^\eta\int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds\Bigg\} \\
&\le \frac{\|\vartheta\|}{a_2\Gamma(q+1)}\sup_{t\in[0,1]}\Bigg\{t^q\int_0^t (t-s)e^{m(t-s)}ds \\
&\quad +|\psi_1(t)|\int_0^1(1-s)e^{m(1-s)}ds + |\psi_2(t)|\eta^q\int_0^\eta(\eta-s)e^{m(\eta-s)}ds\Bigg\}
\end{aligned}
$$

$$\leq \frac{\|\vartheta\|}{a_2 m^2 \Gamma(q+1)}\Big\{(1+\widehat{\psi}_1)\Big((m-1)e^m+1\Big)+\widehat{\psi}_2\eta^q\Big((m\eta-1)e^{m\eta}+1\Big)\Big\} \leq r_2,$$

where we used (26). Thus $\mathcal{H}_1 x + \mathcal{H}_2 y \in B_{r_2}$. Using the assumption (A_1) together with $\ell\mu_1 < 1$, it is easy to show that $\mathcal{H}_2$ is a contraction. Note that continuity of f implies that the operator $\mathcal{H}_1$ is continuous. Also, $\mathcal{H}_1$ is uniformly bounded on B_{r_2} as

$$\|\mathcal{H}_1 x\| = \sup_{t\in[0,1]} |(\mathcal{H}_1 x)(t)| \leq \frac{\|\vartheta\|}{a_2 m^2 \Gamma(q+1)}\Big\{(m-1)e^m+1\Big\}.$$

Now we prove the compactness of operator $\mathcal{H}_1$. For that, let $\sup_{(t,x)\in[0,1]\times B_{r_2}} |f(t,x)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
&|(\mathcal{H}_1 x)(t_2) - (\mathcal{H}_1 x)(t_1)|\\
&= \frac{1}{a_2}\Bigg|\int_0^{t_1}\int_0^{s}\Big(\Psi(t_2)-\Psi(t_1)\Big)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\\
&+\int_{t_1}^{t_2}\int_0^{s}\Psi(t_2)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Bigg|\\
&\leq \frac{\overline{f}}{a_2 m^2\Gamma(q+1)}\Big\{t_1^q\Big(mt_2e^{mt_2}-mt_1e^{mt_1}+e^{mt_1}-e^{mt_2}-m(t_2-t_1)e^{m(t_2-t_1)}+e^{m(t_2-t_1)}-1\Big)\\
&+t_2^q\Big(m(t_2-t_1)(e^{m(t_2-t_1)}-e^{m(t_2-t_1)})+1\Big)\Big\} \to 0 \ \ as \ \ t_2-t_1 \to 0,
\end{aligned}$$

independent of x. Thus, $\mathcal{H}_1$ is relatively compact on B_{r_2}. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{H}_1$ is compact on B_{r_2}. Thus all the assumption of Theorem 3.1 are satisfied. So the conclusion of Theorem 3.1 applies and hence the problem (1)-(2) has at least one solution on [0, 1].

(*iii*) As before, letting $\sup_{t\in[0,1]} |\vartheta(t)| = \|\vartheta\|$ and

$$\begin{aligned}
r_3 \geq &\frac{\|\vartheta\|}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big\{(1+\widehat{\varphi}_1)\Big(1-e^{-\alpha}\cos\beta-(\alpha/\beta)e^{-\alpha}\sin\beta\Big)\\
&+\widehat{\varphi}_2\eta^q\Big(1-e^{-\alpha\eta}\cos\beta\eta-(\alpha/\beta)e^{-\alpha\eta}\sin\beta\eta\Big)\Big\},
\end{aligned} \tag{29}$$

we consider $B_{r_3} = \{x \in \mathcal{C} : \|x\| \leq r_3\}$. Define the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ on B_{r_3} as follows:

$$(\mathcal{K}_1 x)(t) = \frac{1}{a_2\beta}\Big\{\int_0^{t}\int_0^{s}\Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Big\}, \tag{30}$$

$$\begin{aligned}
(\mathcal{K}_2 x)(t) = &\frac{1}{a_2\beta}\Big\{\varphi_1(t)\int_0^{1}\int_0^{s}\Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))dus\,ds\\
&+\varphi_2(t)\int_0^{\eta}\int_0^{s}\Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Big\}.
\end{aligned} \tag{31}$$

Observe that $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$. For $x, y \in B_{r_3}$, as before, it can be shown that $\mathcal{K}_1 x + \mathcal{K}_2 y \in B_{r_3}$. Using the assumption (A_1) together with $\ell\rho_1 < 1$, we can show that $\mathcal{K}_2$ is a contraction. Also, $\mathcal{K}_1$ is uniformly bounded on B_{r_3} as

$$\|\mathcal{K}_1 x\| = \sup_{t\in[0,1]} |(\mathcal{K}_1 x)(t)| \leq \frac{\|\vartheta\|}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big(1-e^{-\alpha}\cos\beta-\alpha/\beta e^{-\alpha}\sin\beta\Big).$$

Fixing $\sup_{(t,x)\in[0,1]\times B_{r_3}} |f(t,x)| = \overline{f}$, we have

$$|(\mathcal{K}_1 x)(t_2) - (\mathcal{K}_1 x)(t_1)|$$

$$\leq \frac{\overline{f}}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big\{t_1^q\Big((\alpha/\beta)\sin\beta(t_2-t_1)e^{-\alpha(t_2-t_1)}-(\alpha/\beta)\sin\beta t_2e^{-\alpha t_2}+(\alpha/\beta)\sin\beta t_1e^{-\alpha t_1}$$
$$+\cos\beta(t_2-t_1)e^{-\alpha(t_2-t_1)}-\cos\beta t_2e^{-\alpha t_2}+\cos\beta t_1e^{-\alpha t_1}-1\Big)$$
$$+t_2^q\Big(1-(\alpha/\beta)\sin\beta(t_2-t_1)e^{-\alpha(t_2-t_1)}-\cos\beta(t_2-t_1)e^{-\alpha(t_2-t_1)}\Big)\Big\},$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$ $(0 < t_1 < t_2 < 1)$. Thus, employing the earlier arguments, $\mathcal{K}_1$ is compact on B_{r_3}. In view of the foregoing arguments, it follows that the problem (1)-(2) has at least one solution on $[0, 1]$. The proof is completed. □

Remark 3.1. *In the above theorem, we can interchange the roles of the operators*

(1) $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *to obtain a second result by replacing* $\ell\lambda_1 < 1$ *with the condition:* $\dfrac{\ell\varepsilon}{\Gamma(q+1)} < 1$;

(2) $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *to obtain a second result by replacing* $\ell\mu_1 < 1$ *with the condition:* $\dfrac{\ell\{(m-1)e^m+1\}}{a_2m^2\Gamma(q+1)} < 1$;

(3) $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *to obtain a second result by replacing* $\ell\rho_1 < 1$ *with the condition:* $\dfrac{\ell\{1-e^{-\alpha}\cos\beta-(\alpha/\beta)e^{-\alpha}\sin\beta\}}{a_2(\alpha^2+\beta^2)\Gamma(q+1)} < 1.$

Now we establish the uniqueness of solutions for the problem (1)-(2) by means of Banach's contraction mapping principle.

Theorem 3.3. *Assume that* $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* (A_1) *is satisfied. Then the problem (1)-(2) has a unique solution on* $[0, 1]$ *if*

(i) $\ell\lambda < 1$ *for* ${a_1}^2 - 4a_0a_2 > 0$, *where* λ *is given by* (20)*;*

(ii) $\ell\mu < 1$ *for* ${a_1}^2 - 4a_0a_2 = 0$, *where* μ *is given by* (21)*;*

(iii) $\ell\rho < 1$ *for If* ${a_1}^2 - 4a_0a_2 < 0$, *where* ρ *is given by* (22).

Proof. (i) Let us define $\sup_{t\in[0,1]}|f(t,0)| = M$ and select $\kappa_1 \geq \dfrac{\lambda M}{1-\ell\lambda}$ to show that $\mathcal{J}B_{\kappa_1} \subset B_{\kappa_1}$, where $B_{\kappa_1} = \{x \in \mathcal{C} : \|x\| \leq \kappa_1\}$ and $\mathcal{J}$ is defined by (15). Using the condition (A_1), we have

$$\begin{aligned}|f(t,x)| &= |f(t,x)-f(t,0)+f(t,0)| \leq |f(t,x)-f(t,0)|+|f(x,0)|\\ &\leq \ell\|x\|+M \leq \ell\kappa_1+M.\end{aligned} \tag{32}$$

Then, for $x \in B_{\kappa_1}$, we obtain

$$\begin{aligned}\|\mathcal{J}(x)\| &= \sup_{t\in[0,1]}|\mathcal{J}(x)(t)|\\
&\leq \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\quad+|\sigma_1(t)|\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\quad+|\sigma_2(t)|\int_0^\eta\int_0^s\Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\Bigg\}\\
&\leq \frac{(\ell\kappa_1+M)}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\Big(e^{m_2(t-s)}-e^{m_1(t-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds\end{aligned}$$

$$+|\sigma_1(t)|\int_0^1\left(e^{m_2(1-s)}-e^{m_1(1-s)}\right)\frac{s^q}{\Gamma(q+1)}ds$$
$$+|\sigma_2(t)|\int_0^\eta\left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)\frac{s^q}{\Gamma(q+1)}ds\Bigg\}$$
$$\le\frac{(\ell\kappa_1+M)}{\Gamma(q+1)}\left\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\right\}=(\ell\kappa_1+M)\lambda\le\kappa_1,$$

which clearly shows that $\mathcal{J}x\in B_{\kappa_1}$ for any $x\in B_{\kappa_1}$. Thus $\mathcal{J}B_{\kappa_1}\subset B_{\kappa_1}$. Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we have

$$\|(\mathcal{J}x)-(\mathcal{J}y)\|\le\frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s\Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\,ds$$
$$+|\sigma_1(t)|\int_0^1\int_0^s\Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\,ds$$
$$+|\sigma_2(t)|\int_0^\eta\int_0^s\Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\,ds\Bigg\}$$
$$\le\frac{\ell}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\left(e^{m_2(t-s)}-e^{m_1(t-s)}\right)\frac{s^q}{\Gamma(q+1)}ds$$
$$+|\sigma_1(t)|\int_0^1\left(e^{m_2(1-s)}-e^{m_1(1-s)}\right)\frac{s^q}{\Gamma(q+1)}ds$$
$$+|\sigma_2(t)|\int_0^\eta\left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)\frac{s^q}{\Gamma(q+1)}ds\Bigg\}\|x-y\|$$
$$\le\frac{\ell}{\Gamma(q+1)}\left\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\right\}\|x-y\|=\ell\lambda\|x-y\|,$$

where λ is given by (20) and depends only on the parameters involved in the problem. In view of the condition $\ell<1/\lambda$, it follows that $\mathcal{J}$ is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), the problem (1)-(2) with $a_1{}^2-4a_0a_2>0$ has a unique solution on [0, 1].

(*ii*) Let us define $\sup_{t\in[0,1]}|f(t,0)|=M$ and select $\kappa_2\ge\dfrac{\mu M}{1-\ell\mu}$. As in (*i*), one can show that $\mathcal{H}B_{\kappa_2}\subset B_{\kappa_2}$, where $\mathcal{H}$ is defined by (17). Also, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we can obtain

$$\|(\mathcal{H}x)-(\mathcal{H}y)\|$$
$$\le\frac{\ell}{a_2m^2\Gamma(q+1)}\left\{(1+\widehat{\psi}_1)\left((m-1)e^m+1\right)+\widehat{\psi}_2\eta^q\left((m\eta-1)e^{m\eta}+1\right)\right\}\|x-y\|$$
$$=\ell\mu\|x-y\|$$

where μ is given by (21). By the condition $\ell<1/\mu$, we deduce that the operator $\mathcal{H}$ is a contraction. Thus, by the contraction mapping principle, the problem (1)-(2) with $a_1{}^2-4a_0a_2=0$ has a unique solution on [0, 1].

(*iii*) Letting $\sup_{t\in[0,1]}|f(t,0)|=M$ and selecting $\kappa_3\ge\dfrac{\rho M}{1-\ell\rho}$, it can be shown that $\mathcal{K}B_{\kappa_3}\subset B_{\kappa_3}$, where $B_{\kappa_3}=\{x\in\mathcal{C}:\|x\|\le\kappa_3\}$ and $\mathcal{K}$ is defined by (19). Moreover, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we can find that

$$\|(\mathcal{K}x)-(\mathcal{K}y)\|$$

$$\begin{aligned}
&\le \frac{\ell}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big\{(1+\widehat{\varphi}_1)\Big(1-e^{-\alpha}\cos\beta-(\alpha/\beta)e^{-\alpha}\sin\beta\Big)\\
&\quad+\widehat{\varphi}_2\eta^q\Big(1-e^{-\alpha\eta}\cos\beta\eta-(\alpha/\beta)e^{-\alpha\eta}\sin\beta\eta\Big)\Big\}\|x-y\|\\
&=\ell\rho\|x-y\|,
\end{aligned}$$

where ρ is given by (22). Evidently, it follows by the condition $\ell < 1/\rho$ that $\mathcal{K}$ is a contraction. Thus, by the contraction mapping principle, the problem (1)-(2) with ${a_1}^2 - 4a_0a_2 < 0$ has a unique solution on $[0, 1]$. This completes the proof. □

The next existence result is based on Leray-Schauder nonlinear alternative [8], which is stated below.

Theorem 3.4. *(Nonlinear alternative for single valued maps). Let Y be a closed, convex subset of a Banach space X and V be an open subset of Y with $0 \in V$. Let $G : \overline{V} \to Y$ be a continuous and compact (that is, $G(\overline{V})$ is a relatively compact subset of Y) map. Then either G has a fixed point in $\overline{V}$ or there is a $v \in \partial V$ (the boundary of V in Y) and $\varepsilon \in (0, 1)$ with $u = \varepsilon G(u)$.*

In order to establish our last result, we need the following conditions.
(H_1)There exist a function $g \in C([0, 1], \mathbb{R}^+)$, and a nondecreasing function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, y)| \le g(t)Q(\|y\|),\ \forall (t, y) \in [0, 1] \times \mathbb{R}$.
(H_2)–(i, P) There exists a constant $K_i > 0$ such that

$$\frac{K_i}{\|g\|Q(K_i)P} > 1, \quad i = 1, 2, 3, \quad P \in \{\lambda, \mu, \rho\}.$$

Theorem 3.5. *Let $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem* (1)-(2) *has at least one solution on* $[0, 1]$ *if*
(a) (H_1) and $(H_2) - (1, \lambda)$ are satisfied for $a_1^2 - 4a_0a_2 > 0$;
(b) (H_1) and $(H_2) - (2, \mu)$ are satisfied for $a_1^2 - 4a_0a_2 = 0$;
(c) (H_1) and $(H_2) - (3, \rho)$ are satisfied for $a_1^2 - 4a_0a_2 < 0$.

Proof. (a) Let us first show that the operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ defined by (15) maps bounded sets into bounded sets in $\mathcal{C} = C([0, 1], \mathbb{R})$. For a positive number ζ_1, let $\mathcal{B}_{\zeta_1} = \{x \in \mathcal{C} : \|x\| \le \zeta_1\}$ be a bounded set in $\mathcal{C}$. Then we have

$$\begin{aligned}
\|\mathcal{J}(x)\| &= \sup_{t\in[0,1]} |\mathcal{J}(x)(t)|\\
&\le \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\quad+|\sigma_1(t)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\quad+|\sigma_2(t)|\int_0^\eta\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\Bigg\}\\
&\le \frac{\|g\|Q(\zeta_1)}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\Big(e^{m_2(t-s)}-e^{m_1(t-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds\\
&\quad+|\sigma_1(t)|\int_0^1\Big(e^{m_2(1-s)}-e^{m_1(1-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds\\
&\quad+|\sigma_2(t)|\int_0^\eta\Big(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds\Bigg\}
\end{aligned}$$

$$\leq \frac{\|g\|Q(\zeta_1)}{\Gamma(q+1)}\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\},$$

which yields

$$\|\mathcal{J}x\| \leq \frac{\|g\|Q(\zeta_1)}{\Gamma(q+1)}\{\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2\}.$$

Next we show that $\mathcal{J}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_{\zeta_1}$, where $\mathcal{B}_{\zeta_1}$ is a bounded set of $\mathcal{C}$. Then we obtain

$$\begin{aligned}
&|(\mathcal{J}x)(t_2) - (\mathcal{J}x)(t_1)| \\
&\leq \frac{1}{a_2(m_2-m_1)}\Bigg\{\Bigg|\int_0^{t_1}\int_0^s \Big[\Phi(t_2)-\Phi(t_1)\Big]\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds \\
&+\int_{t_1}^{t_2}\int_0^s \Phi(t_2)\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\,ds\Bigg| \\
&+|\sigma_1(t_2)-\sigma_1(t_1)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds \\
&+|\sigma_2(t_2)-\sigma_2(t_1)|\int_0^{\eta}\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,y(u))|du\,ds\Bigg\} \\
&\leq \frac{\bar{f}}{a_2m_1m_2(m_2-m_1)\Gamma(q+1)}\Big\{\Big(t_1^q - t_2^q\Big)\Big(m_1(1-e^{m_2(t_2-t_1)}) - m_2(1-e^{m_1(t_2-t_1)})\Big) \\
&+t_1^q\Big(m_1(e^{m_2t_2}-e^{m_2t_1}) - m_2(e^{m_1t_2}-e^{m_1t_1})\Big) \\
&+|\sigma_1(t_2)-\sigma_1(t_1)|(m_2(1-e^{m_1}) - m_1(1-e^{m_2})) \\
&+|\sigma_1(t_2)-\sigma_1(t_1)|\eta^q(m_2(1-e^{m_1\eta}) - m_1(1-e^{m_2\eta}))\Big\},
\end{aligned}$$

which tends to zero as $t_2 - t_1 \to 0$ independently of $x \in \mathcal{B}_{\zeta_1}$. From the foregoing arguments, it follows by the Arzelá-Ascoli theorem that $\mathcal{J}: \mathcal{C} \to \mathcal{C}$ is completely continuous.

The proof will be complete by virtue of Theorem 3.4 once we establish that the set of all solutions to the equation $x = \theta\mathcal{J}x$ is bounded for $\theta \in [0,1]$. To do so, let x be a solution of $x = \theta\mathcal{J}x$ for $\theta \in [0,1]$. Then, for $t \in [0,1]$, we get

$$\begin{aligned}
|x(t)| &= |\theta\mathcal{J}x(t)| \\
&\leq \frac{1}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \Phi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds \\
&\quad+|\sigma_1(t)|\int_0^1\int_0^s \Phi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds \\
&\quad+|\sigma_2(t)|\int_0^{\eta}\int_0^s \Phi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\Bigg\} \\
&\leq \frac{\|g\|Q(\|x\|)}{a_2(m_2-m_1)}\sup_{t\in[0,1]}\Bigg\{\int_0^t\Big(e^{m_2(t-s)} - e^{m_1(t-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds \\
&\quad+|\sigma_1(t)|\int_0^1\Big(e^{m_2(1-s)} - e^{m_1(1-s)}\Big)\frac{s^q}{\Gamma(q+1)}ds
\end{aligned}$$

$$+|\sigma_2(t)|\int_0^\eta \left(e^{m_2(\eta-s)}-e^{m_1(\eta-s)}\right)\frac{s^q}{\Gamma(q+1)}ds\Bigg\}$$
$$\le \frac{\|g\|Q(\|x\|)}{\Gamma(q+1)}\left\{\varepsilon+\widehat{\sigma}_1\gamma_1+\eta^q\widehat{\sigma}_2\gamma_2\right\}$$
$$=\|g\|Q(\|x\|)\lambda,$$

which on taking the norm for $t\in[0,1]$, yields

$$\frac{\|x\|}{\|g\|Q(\|x\|)\lambda}\le 1.$$

In view of $(H_2)-(1,\lambda)$, there is no solution x such that $\|x\|\neq K_1$. Let us set

$$U_1=\{x\in\mathcal{C}:\|x\|<K_1\}.$$

As the operator $\mathcal{J}:\overline{U}_1\to\mathcal{C}$ is continuous and completely continuous, we infer from the choice of U_1 that there is no $u\in\partial U_1$ such that $u=\theta\mathcal{J}(u)$ for some $\theta\in(0,1)$. Hence, by Theorem 3.4, we deduce that $\mathcal{J}$ has a fixed point $u\in\overline{U}_1$ which is a solution of the problem (1)-(2).

(b) As in part (a), it can be shown that the operator $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ defined by (17) maps bounded sets into bounded sets in $\mathcal{C}=C([0,1],\mathbb{R})$. For that, let ζ_2 be a positive number and let $\mathcal{B}_{\zeta_2}=\{x\in\mathcal{C}:\|x\|\le\zeta_2\}$ be a bounded set in $\mathcal{C}$. Then we have

$$\|\mathcal{H}(x)\|=\sup_{t\in[0,1]}|\mathcal{H}(x)(t)|$$
$$\le\frac{1}{a_2}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s\Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds$$
$$+|\psi_1(t)|\int_0^1\int_0^s\Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds$$
$$+|\psi_2(t)|\int_0^\eta\int_0^s\Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\Bigg\}$$
$$\le\frac{\|g\|Q(\zeta_2)}{a_2m^2\Gamma(q+1)}\left\{(1+\widehat{\psi}_1)\Big((m-1)e^m+1\Big)+\widehat{\psi}_2\eta^q\Big((m\eta-1)e^{m\eta}+1\Big)\right\}.$$

In order to show that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$, let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in\mathcal{B}_{\zeta_2}$, where $\mathcal{B}_{\zeta_2}$ is a bounded set of $\mathcal{C}$. Then we get

$$|(\mathcal{H}x)(t_2)-(\mathcal{H}x)(t_1)|$$
$$\le\frac{\overline{f}}{a_2m^2\Gamma(q+1)}\Big\{t_1^q\Big(mt_2e^{mt_2}-mt_1e^{mt_1}+e^{mt_1}-e^{mt_2}-m(t_2-t_1)e^{m(t_2-t_1)}+e^{m(t_2-t_1)}-1)\Big)$$
$$+t_2^q\Big(m(t_2-t_1)(e^{m(t_2-t_1)}-e^{m(t_2-t_1)})+1\Big)\Big\}$$
$$+\frac{\overline{f}|\psi_1(t_2)-\psi_1(t_1)|}{a_2}\int_0^1(1-s)e^{m(\eta-s)}\frac{s^q}{\Gamma(q+1)}ds$$
$$+\frac{\overline{f}|\psi_2(t_12)-\psi_2(t_1)|}{a_2}\int_0^\eta(\eta-s)e^{m(\eta-s)}\frac{s^q}{\Gamma(q+1)}ds,$$

which tends to zero as $t_2-t_1\to 0$ independently of $x\in\mathcal{B}_{\zeta_2}$. As argued before, $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ is completely continuous. To show that the set of all solutions to the equation $x=\theta\mathcal{H}x$ is bounded for $\theta\in[0,1]$, let x be a solution of $x=\theta\mathcal{H}x$ for $\theta\in[0,1]$. Then, for $t\in[0,1]$, we find that

$$|x(t)|=|\theta\mathcal{H}x(t)|$$

$$\begin{aligned}
&\le \frac{1}{a_2}\sup_{t\in[0,1]}\left\{\int_0^t\int_0^s \Psi(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\right.\\
&+|\psi_1(t)|\int_0^1\int_0^s \Psi(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\left.+|\psi_2(t)|\int_0^\eta\int_0^s \Psi(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\right\}\\
&\le \frac{\|g\|Q(\|x\|)}{a_2 m^2\Gamma(q+1)}\left\{(1+\widehat{\psi}_1)\Big((m-1)e^m+1\Big)+\widehat{\psi}_2\eta^q\Big((m\eta-1)e^{m\eta}+1\Big)\right\}\\
&= \|g\|Q(\|x\|)\mu.
\end{aligned}$$

Thus

$$\frac{\|x\|}{\|g\|Q(\|x\|)\mu}\le 1.$$

In view of $(H_2)-(2,\mu)$, there is no solution x such that $\|x\|\neq K_2$. Let us define

$$U_2=\{x\in\mathcal{C}:\|x\|<K_2\}.$$

Since the operator $\mathcal{H}:\overline{U}_2\to\mathcal{C}$ is continuous and completely continuous, there is no $u\in\partial U_2$ such that $u=\theta\mathcal{H}(u)$ for some $\theta\in(0,1)$ by the choice of U_2. In consequence, by Theorem 3.4, we deduce that $\mathcal{H}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (1)-(2).

(c) As in the preceding cases, one can show that the operator $\mathcal{K}$ defined by (19) is continuous and completely continuous. We only provide the outline for the last part (*a-priori bounds*) of the proof. Let x be a solution of $x=\theta\mathcal{K}x$ for $\theta\in[0,1]$, where $\mathcal{K}$ is defined by (19). Then, for $t\in[0,1]$, we have

$$\begin{aligned}
|x(t)| &= |\theta\mathcal{K}x(t)|\\
&\le \frac{1}{a_2\beta}\int_0^t\int_0^s \Omega(t)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&+|\varphi_1(t)|\int_0^1\int_0^s \Omega(1)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&+|\varphi_2(t)|\int_0^\eta\int_0^s \Omega(\eta)\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\,ds\\
&\le \frac{\|g\|Q(\|x\|)}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\left\{(1+\widehat{\varphi}_1)\Big(1-e^{-\alpha}\cos\beta-(\alpha/\beta)e^{-\alpha}\sin\beta\Big)\right.\\
&\left.+\widehat{\varphi}_2\eta^q\Big(1-e^{-\alpha\eta}\cos\beta\eta-(\alpha/\beta)e^{-\alpha\eta}\sin\beta\eta\Big)\right\}\\
&= \|g\|Q(\|x\|)\rho,
\end{aligned}$$

which yields

$$\frac{\|x\|}{\|g\|Q(\|x\|)\rho}\le 1.$$

In view of $(H_2)-(3,\rho)$, there is no solution x such that $\|x\|\neq K_3$. Let us set

$$U_3=\{x\in\mathcal{C}:\|x\|<K_3\}.$$

As before, one can show that the operator $\mathcal{K}$ has a fixed point $u\in\overline{U}_3$, which is a solution of the problem (1)-(2). This completes the proof. □

4 Examples

Example 4.1. *Consider the following boundary value problem*

$$({}^cD^{8/3} + 5\,{}^cD^{5/3} + 4\,{}^cD^{2/3})x(t) = \frac{A}{(t+5)^2}\Big(\frac{|x|}{1+|x|} + \cos t\Big), \quad 0 < t < 1, \tag{33}$$

$$x(0) = 0, \quad x(3/5) = 0, \quad x(1) = 0, \tag{34}$$

Here, $q = 2/3$, $\eta = 3/5$, $a_2 = 1$, $a_1 = 5$, $a_0 = 4$, $a_1^2 - 4a_0a_2 = 9 > 0$, A is a positive constant to be fixed later and

$$f(t,x) = \frac{A}{(t+5)^2}\Big(\frac{|x|}{1+|x|} + \cos t\Big).$$

Clearly

$$|f(t,x) - f(t,y)| \le 0.04\,A|x-y|,$$

with $\ell = 0.04\,A$. Using the given values, we find that $\lambda = [(\varepsilon + \widehat{\sigma}_1\gamma_1 + \eta^q\widehat{\sigma}_2\gamma_2)/\Gamma(q+1)] \approx 0.67232$, and $\lambda_1 = \lambda - \dfrac{\varepsilon}{\Gamma(q+1)} \approx 0.52953$. It is easy to check that $|f(t,x)| \le 2A/(t+5)^2 = \vartheta(t)$ and $\ell\lambda_1 < 1$ when $A < 47.211678$. As all the conditions of Theorem 3.2 (*i*) are satisfied, the problem (33)-(34) has at least one solution on $[0, 1]$. On the other hand, $\ell\lambda < 1$ whenever $A < 37.184674$ and thus there exists a unique solution for the problem (33)-(34) on $[0, 1]$ by Theorem 3.3 (*i*).

Example 4.2. *Consider the multi-term fractional differential equation*

$$(3\,{}^cD^{13/5} + 6\,{}^cD^{8/5} + 3\,{}^cD^{3/5})x(t) = \frac{B}{t^2+5}\Big(\sin t + \tan^{-1}x(t)\Big), \quad 0 < t < 1, \tag{35}$$

supplemented with the boundary conditions

$$x(0) = 0, \quad x(2/3) = 0, \quad x(1) = 0, \tag{36}$$

Here, $q = 3/5$, $\eta = 2/3$, $a_2 = 3$, $a_1 = 6$, $a_0 = 3$, $a_1^2 - 4a_0a_2 = 0$, B is a positive constant to be determined later and

$$f(t,x) = \frac{B}{t^2+5}\Big(\sin t + \tan^{-1}x(t)\Big).$$

Clearly

$$|f(t,x) - f(t,y)| \le 0.2\,B|x-y|,$$

where $\ell = 0.2\,B$. Using the given values, it is found that $\mu = \Big[(1 + \widehat{\psi}_1)((m-1)e^m + 1) + \widehat{\psi}_2\eta^q((m\eta - 1)e^{m\eta} + 1)/a_2m^2\Gamma(q+1)\Big] \approx 0.59146$, and $\mu_1 = \mu - \dfrac{((m-1)e^m + 1)}{a_2m^2\Gamma(q+1)} \approx 0.29573$. Further, $|f(t,x)| \le B(2+\pi)/2(t^2+5) = \vartheta(t)$ and $\ell\mu_1 < 1$ when $B < 16.907314$. As all the conditions of Theorem 3.2 (*ii*) are satisfied, the problem (35)-(36) has at least one solution on $[0, 1]$. On the other hand, $\ell\mu < 1$ whenever $B < 8.453657$. Thus there exists a unique solution for the problem (35)-(36) on $[0, 1]$ by Theorem 3.3 (*ii*).

Example 4.3. *consider the multi-term fractional boundary value problem given by*

$$({}^cD^{8/3} + {}^cD^{5/3} + {}^cD^{2/3})x(t) = \frac{C}{\sqrt{36+t^2}}\Big(\frac{|x|}{1+|x|} + \frac{1}{2}\Big), \quad 0 < t < 1, \tag{37}$$

$$x(0) = 0, \quad x(3/5) = 0, \quad x(1) = 0, \tag{38}$$

where, $q = 2/3$, $\eta = 3/5$, $a_2 = 1$, $a_1 = 1$, $a_0 = 1$, $a_1^2 - 4a_0a_2 = -3 < 0$, C is a positive constant to be fixed later and

$$f(t,x) = \frac{C}{\sqrt{36+t^2}}\Big(\frac{|x|}{1+|x|} + \frac{1}{2}\Big).$$

Clearly

$$|f(t,x) - f(t,y)| \le (C/6)\,|x-y|,$$

with $\ell = C/6$. Using the given values, we have

$$\rho = \frac{1}{a_2(\alpha^2+\beta^2)\Gamma(q+1)}\Big\{(1+\widehat{\varphi}_1)\Big(1-e^{-\alpha}\cos\beta-\alpha/\beta e^{-\alpha}\sin\beta\Big)$$
$$+\widehat{\varphi}_2\eta^q\Big(1-e^{-\alpha\eta}\cos\beta\eta-\alpha/\beta e^{-\alpha\eta}\sin\beta\eta\Big)\Big\}\approx 0.75392,$$

and $\rho_1 = \rho - \dfrac{(1-e^{-\alpha}\cos\beta-\alpha/\beta e^{-\alpha}\sin\beta)}{a_2(\alpha^2+\beta^2)\Gamma(q+1)} \approx 0.37696$. Also $|f(t,x)| \le 3C/2\sqrt{36+t^2} = \vartheta(t)$ and $\ell\rho_1 < 1$ when $C < 15.916808$. Clearly all the conditions of Theorem 3.2 (*iii*) hold true. Thus the problem (37)-(38) has at least one solution on $[0,1]$. On the other hand, $\ell\rho < 1$ whenever $C < 7.958404$. Thus there exists a unique solution for the problem (37)-(38) on $[0,1]$ by Theorem 3.3 (*iii*).

Example 4.4. *Consider the following nonlocal boundary value problem of multi-term fractional differential equation*

$$({}^cD^{8/3} + 5\,{}^cD^{5/3} + 4\,{}^cD^{2/3})x(t) = \frac{Pt}{\sqrt{t+16}}\Big(\frac{|x|}{8(1+|x|)} + \sin x + \frac{1}{8}\Big), \quad 0<t<1, \tag{39}$$
$$x(0)=0, \quad x(3/5)=0, \quad x(1)=0, \tag{40}$$

where, $q = 2/3$, $\eta = 3/5$, $a_1^2 - 4a_0a_2 = 9 > 0$, P is a positive constant and

$$f(t,x) = \frac{Pt}{\sqrt{t+16}}\Big(\frac{|x|}{8(1+|x|)} + \sin x + \frac{1}{8}\Big).$$

Clearly

$$|f(t,x)| \le \frac{Pt}{\sqrt{t+16}}\Big(\frac{1}{4} + \|x\|\Big) = g(t)Q(\|x\|),$$

with $g(t) = \dfrac{Pt}{\sqrt{t+16}}$, $Q(\|x\|) = \dfrac{1}{4} + \|x\|$. Letting $P = 2$ and using the condition $(H_2) - (1,\lambda)$, we find that $K_1 > 0.10102$ (we have used $\lambda = 0.67232$). Thus, the conclusion of Theorem 3.5 (a) applies to the problem (39)-(40).

Example 4.5. *Consider the following boundary value problem*

$$(3\,{}^cD^{13/5} + 6\,{}^cD^{8/5} + 3\,{}^cD^{3/5})x(t) = \frac{1}{2\sqrt{t+9}}\Big(\frac{|x|^3}{1+|x|^3} + e^{-t}\Big), \quad 0<t<1, \tag{41}$$
$$x(0)=0, \quad x(2/3)=0, \quad x(1)=0, \tag{42}$$

where, $q = 3/5$, $\eta = 2/3$, $a_1^2 - 4a_0a_2 = 0$,

$$f(t,x) = \frac{1}{2\sqrt{t+9}}\Big(\frac{|x|^3}{1+|x|^3} + e^{-t}\Big).$$

Clearly

$$|f(t,x)| \le \frac{1+e^{-t}}{2\sqrt{t+9}} = g(t)Q(\|x\|),$$

with $g(t) = \dfrac{1+e^{-t}}{2\sqrt{t+9}}$, $Q(\|x\|) = 1$. Using the condition $(H_2) - (2,\mu)$, we find that $K_2 > 0.3943$ (with $\mu = 0.59146$). Thus, the conclusion of Theorem 3.5 (b) applies to the problem (41)-(42).

Example 4.6. *consider the following problem*

$$({}^cD^{8/3} + {}^cD^{5/3} + {}^cD^{2/3})x(t) = \frac{3t}{t^2+6}\Big(\frac{1}{2} + \sin x\Big), \quad 0<t<1, \tag{43}$$
$$x(0)=0, \quad x(3/5)=0, \quad x(1)=0, \tag{44}$$

Here, $q = 2/3$, $\eta = 3/5$, $a_1^2 - 4a_0a_2 = -3 < 0$, and

$$f(t,x) = \frac{3t}{t^2+6}\left(\frac{1}{2} + \sin x\right).$$

Clearly

$$|f(t,x)| \leq \frac{3t}{t^2+6}\left(\frac{1}{2} + \|x\|\right),$$

with $g(t) = \frac{3t}{t^2+6}$, $Q(\|x\|) = \frac{1}{2} + \|x\|$. By the condition $(H_2) - (3,\rho)$, it is find that $K_3 > 0.30252$ (with $\rho = 0.75392$). Thus, the conclusion of Theorem 3.5 (c) applies to the problem (43)-(44).

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