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MULTI-TIME SYSTEMS OF CONSERVATION LAWS

By

ALDO BAZAN (*Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530, Cidade Universitária 21945-970, Rio de Janeiro, Brazil*),

PAOLA LORETI (*Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, via A. Scarpa n.16 00161 Roma*),

AND

WLADIMIR NEVES (*Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530, Cidade Universitária 21945-970, Rio de Janeiro, Brazil*)

Abstract. Motivated by the work of P. L. Lions and J.-C. Rochet (1986) concerning multi-time Hamilton-Jacobi equations, we introduce the theory of multi-time systems of conservation laws. We show the existence and uniqueness of solution to the Cauchy problem for a system of multi-time conservation laws with two independent time variables in one space dimension. Our proof relies on a suitable generalization of the Lax-Oleinik formula.

1. Introduction. This paper introduces the theory of multi-time systems of conservation laws. Since to our knowledge nothing has been done in this direction, we first give the statement of the theory in Section 1.1. In order to show that the theory is well-introduced, we prove in the final section the solvability of the Cauchy problem for a system of multi-time conservation laws with two independent time variables in one space dimension. The solvability relies on a generalization of the Lax-Oleinik formula for two independent times; see Definition 3.2. Therefore, we exploit in this paper the explicit Lax formula (2.5) as solution for the multi-time Hamilton-Jacobi system (2.1), a concept introduced by Rochet [18] in the context of mathematical economic problems.

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E-mail address: aabp2003@pg.im.ufrj.br
E-mail address: paolaloreti@gmail.com
E-mail address: wladimir@im.ufrj.br

Besides the philosophical question of the existence of multiple-time dimensions, multi-time phenomena are rather common. For instance, networks in communication theory, including traffic models with the potential to consider traffic jams, lead to the use of different time scales. In this direction, we address the work of Gu, Chung and Hui [8], which is related to traffic flow problems in inhomogeneous lattices. In fact, traffic flow seems to be one of the prelude sources of conservation laws, leading, for instance, to the Burgers equation. Another source of interesting physical problems where multi-time phenomena is present comes from general relativity and electromagnetism. In this direction, we point the reader to the works of Neagu and Udriste [15] and Stickforth [19]; the last one is concerned with the Kepler problem. One of the most amazing examples which leads to multiple dimensions, even more than two time scales, is given by string theory; we point the reader to the books of Steven [9] and Zwiebach [22]. Most of these physical problems are modelled by systems of conservation laws, here with two or more time independent scales. Finally, we have to mention that one of the motivations to introduce multi-time conservation laws comes from Lions-Rochet's paper [12], concerning multi-time Hamilton-Jacobi equations.

The mathematical theory of multi-time Hamilton-Jacobi equations was developed by P. L. Lions and J.-C. Rochet [12]. In that paper Lions and Rochet showed the existence of solution for (2.1). Since then, many works have been written in the context of multi-time Hamilton-Jacobi equations to extend the results of Lions and Rochet. The existing literature shows existence and uniqueness of the solution for a more general class of Hamiltonians and gives weaker regularity conditions on the initial data. For instance, see the works of Barles and Tourin [3] for Lipschitz initial-data as well as Plaskacz and Quincampoix [16] for initial data bounded by a semi-continuous function; they present existence and uniqueness under the hypotheses (H1), (H2), (H3) in [3] and Assumption A in [16]. See Remark 2.1. We also address the paper of Imbert and Volle [10], which considers a more general class of vectorial Hamilton-Jacobi equations.

For our multi-time conservation laws, we were here more interested in explicit Lipschitz regular solutions for (2.1). Then, under the condition that the initial data is Lipschitz and the Hamiltonians are convex and coercive, we give an explicit and new proof of existence for the multi-time Hamilton-Jacobi equations, using the inf-convolution and Γ-convolution operations. We show that Lax formula (2.5) is a Lipschitz function, which solves the Cauchy problem (2.1); see Theorem 2.6. The same strategy used to prove Theorem 2.6, with small modifications, also shows that the Lax formula is a viscosity solution of (2.1) in the sense presented in Definition 2.7. Although the section on viscosity solutions of Hamilton-Jacobi equations gives known results in literature, here we organize the topics in order to give the correspondence with multi-time conservation laws. To make the paper complete on its own, we prefer to give statements and proofs adapted to this context. By the doubling variables technique, we show that there exists at most one Lipschitz bounded solution for (2.1); see Theorem 2.8. Hence Section 3 presents the existence and uniqueness solution to the Cauchy problem (3.1). First, we differentiate the Lax formula with respect to the spatial variable and formally show that it is the best candidate to solve (3.1). After that we establish in Lemma 3.1 a generalization of the Lax-Oleinik formula for multi-time variables. Then, we give in Definition 3.3 the exact notion of solution to (3.1) and prove the existence of an integral solution in Theorem 3.4. After that, by the BV regularity property obtained by the Lax-Oleinik formula, we show that the integral solution is an entropy solution to the Cauchy problem (3.1) in the sense of Definition 1.2. Finally, we prove the uniqueness result in Theorem 3.6.

1.1. *Statement of the theory.* The aim of this section is to provide the basic theory for multi-time systems of conservation laws in multidimensional space dimensions. We are going to formulate the initial-value problem, where the systems of equations are complemented by an initial data, that is, the Cauchy problem.

Fix n, d and s to be positive natural numbers. Let $t_1, t_2, ..., t_n$ be n-time independent scales, and consider the points $(t_1, \dots, t_n, x_1, \dots, x_d) \in \mathbb{R}^n \times \mathbb{R}^d$. In fact, for simplicity of exposition and without loss of generality, we consider only two time scales. Moreover, we denote the spatial variable $(x_1, \dots, x_d) = x$.

Let U be an open subset of $\mathbb{R}^s$, usually called the set of states, where for each (t_1, t_2, x)

$$u(t_1, t_2, x) \in U, \qquad \big(u = (u^1, \dots, u^s)\big).$$

Now, let $f_i : U \to (\mathbb{R}^s)^d$, $(i = 1, 2)$, be two smooth maps called flux functions. In general, we postulate that there exist at most $f_i^\prime s$ different flux functions as the number of time-independent variables. Then, we are in position to establish the following multi-time system of conservation laws in general form:

$$\begin{aligned} \frac{\partial u^i}{\partial t_1} + \frac{\partial f^i_{1j}(u)}{\partial x_j} &= 0, \\ \frac{\partial u^i}{\partial t_2} + \frac{\partial f^i_{2j}(u)}{\partial x_j} &= 0, \end{aligned} \tag{1.1}$$

where $(t_1, t_2, x) \in (0, \infty)^2 \times \mathbb{R}^d$, $u(t_1, t_2, x) \in U$ is the unknown and f_1, f_2 are given. Moreover, we remark that the summation convention is used; that is, whenever an index is repeated once, and only once, a summation over the range of this index is performed.

Definition 1.1. The system (1.1) is said to be hyperbolic, when for any $u \in U$ and any direction $\xi \in S^{d-1}$, each matrix

$$A^i_{1k} := \frac{\partial f^i_{1j}(u)}{\partial u_k}\, \xi_j \quad \text{and} \quad A^i_{2k} := \frac{\partial f^i_{2j}(u)}{\partial u_k}\, \xi_j \qquad (1 \le i, k \le s),$$

has s real eigenvalues $\lambda_{i1}(u, \xi) \le \lambda_{i2}(u, \xi) \le \dots \le \lambda_{is}(u, \xi)$, $(i = 1, 2)$ and is diagonalizable. Therefore, there exist $2s$ linearly independent right and left corresponding eigenvectors, respectively: $r_i(u, \xi)$, $l_i(u, \xi)$, $(i = 1, 2)$, and

$$A_i(u, \xi)\, r_i(u, \xi) = \lambda_i\, r_i(u, \xi) \quad \text{and} \quad l_i^T(u, \xi)\, A_i(u, \xi) = \lambda_i\, l_i(u, \xi).$$

Moreover, when the eigenvalues are all distinct, the system (1.1) is said to be strictly hyperbolic.

Hence, we formulate the Cauchy Problem: Find $u(t_1, t_2, x) \in U$ to be a function in $(0, \infty)^2 \times \mathbb{R}^d$ that satisfies the system (1.1) and also the initial data

$$u(0, 0, x) = u_0(x) \qquad \text{for all } x \in \mathbb{R}^d, \tag{1.2}$$

where $u_0 : \mathbb{R}^d \to U$ is a given function.

Therefore, we have established the Cauchy problem (1.1)-(1.2) for multi-time systems of conservation laws in general form and so, many questions are in order at this point. First of all, one could ask if (1.1)-(1.2) is well-defined, since this problem seems to be overdetermined. In this direction, we show in Section 3 well-posedness to the Cauchy problem (1.1)-(1.2) for d and s equal to one, Lipschitz initial data and smooth convex flux functions.

Last but not least, let us write $y = (t_1, t_2, x)$ and for $u(y) \in \mathbb{R}$, we define

$$F(u) := \begin{pmatrix} u & 0 & f_1(u) \\ 0 & u & f_2(u) \end{pmatrix}.$$

Then, from equation (1.1) we have

$$\operatorname{div}_y F(u) \equiv \frac{\partial F_{ij}(u)}{\partial y_j} = 0, \qquad \Big(i = 1, 2;\ j = 1, \dots, d+2\Big). \tag{1.3}$$

One could expect to apply the standard conservation laws theory. In this way, we have the following

DEFINITION 1.2. A field $q(u)$ is called a convex entropy flux associated with the conservation law (1.3) if there exists a continuous differentiable convex function $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$q_{ij}(\lambda) = \int_0^\lambda \partial_u \eta(s)\, \partial_u F_{ij}(s)\, ds, \qquad \text{for each } \lambda \in \mathbb{R}.$$

Moreover, a measurable and bounded scalar function $u = u(y)$ is called an entropy solution of the conservation law (1.3) associated with initial data $u_0 \in L^\infty(\mathbb{R}^d)$, if the following entropy inequality

$$\iint_{\mathbb{R}^{d+2}} q_{ij}(u)\, \partial_{y_j} \phi \, dy \geq 0$$

holds for each convex entropy flux q and all smooth test function ϕ compactly supported in $(0,T)^2 \times \mathbb{R}^d$, for all $T > 0$, and also the initial data

$$\operatorname*{ess\,lim}_{t_1, t_2 \to 0^+} \int_{\mathbb{R}} |u(t_1, t_2, x) - u_0(x)|\, dx = 0. \tag{1.4}$$

The main issue of the paper will be the existence and uniqueness result as mentioned before when $s = d = 1$. For that, we exploit the well-known idea established to study conservation laws (at least in one spatial dimension) from the Hamilton-Jacobi equations.

1.2. *Functional notation and some results.* Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. The Legendre-Fenchel conjugate of f, that is, the function $f^* : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, is defined by the formula

$$f^*(x) := \sup_{y \in \mathbb{R}^d} \{x \cdot y - f(y)\},$$

where $x \cdot y$ is the scalar product of vectors $x, y \in \mathbb{R}^d$. We recall that f^* is a convex function even if f is not, and we put $f^{**} = (f^*)^*$. If f is convex, the Fenchel-Moreau theorem establishes an important duality result between f and its conjugate: if f is lower semicontinuous and convex, then $f^{**} = f$. In the following we consider proper functions. If $f : \mathbb{R}^d \to \mathbb{R}$ is coercive, i.e.

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$, then f^* is also coercive.

For a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $\operatorname{Lip}(f)$ the Lipschitz constant of f; that is, for each $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \leq \operatorname{Lip}(f)\, \|x - y\|.$$

Given $f, g : \mathbb{R}^d \to \mathbb{R}$, we define (for a more general context, see Moreau [13])

$$f \bigtriangledown g : \mathbb{R}^d \to \mathbb{R} \quad \text{and} \quad f \,\Box\, g : \mathbb{R}^d \to \mathbb{R},$$

respectively the infimal-convolution (or inf-convolution) and gamma-convolution (or Γ-convolution) of f, g, by

$$\big(f \bigtriangledown g\big)(x) = \inf_{y \in \mathbb{R}^d} \{f(x-y) + g(y)\} \tag{1.5}$$

and

$$\big(f \,\Box\, g\big)(x) = \big(f^*(x) + g^*(x)\big)^*. \tag{1.6}$$

These operations are dual in the following sense.

THEOREM 1.3. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be two convex functions. Then,

$$f \bigtriangledown g = f \,\Box\, g.$$

The proof can be seen in Rockafellar's book [17], page 145, Theorem 16.4. In fact, there are also more general conditions on f and g such that these operations are identical; we point the reader to [13]. One recalls further that infimal-convolution and gamma-convolution possess the properties of commutativity and associativity.

Finally, just for completeness of the paper, let us recall the Moreau-Yosida approximation, which will be mentioned a posteriori. For each $\tau > 0$, the Moreau-Yosida approximation of $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$f^\tau(x) := \inf_{y \in \mathbb{R}^d} \Big\{ \frac{\|x-y\|^2}{2\,\tau} + f(y) \Big\}.$$

2. Multi-time Hamilton-Jacobi equations. We begin this section by studying some interesting features of the multi-time Hamilton-Jacobi equations. For simplicity of explanation, we consider only two independent times. So, we will focus on the following problem: Find $w : (0,\infty)^2 \times \mathbb{R}^d \to \mathbb{R}$, satisfying

$$\begin{aligned} w_{t_1} + H_1(Dw) = 0 \qquad & \text{in } (0,\infty)^2 \times \mathbb{R}^d, \\ w_{t_2} + H_2(Dw) = 0 \qquad & \text{in } (0,\infty)^2 \times \mathbb{R}^d, \\ w(0,0,x) = g(x) \quad & \text{on } \mathbb{R}^d, \end{aligned} \tag{2.1}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a given initial datum and $H_i : \mathbb{R}^d \to \mathbb{R}$ $(i = 1, 2)$ are given functions, usually called Hamiltonians. Here, we are mostly interested in explicit solutions for (2.1) given by formulas with $\mathbb{R}^d$ domains, since they will be exploited a posteriori in order to show solvability of multi-time conservation laws.

When $t_1 = t_2 =: t$ and hence $H_1 = H_2 =: H$, the system (2.1) is the usual Hamilton-Jacobi equations. In this context, we recall some well-known facts and discuss new viewpoints. We point, for instance, to Alvarez, Barron and Ishii [1], Bardi and Evans [2], and also Lions and Rochet [12], and references therein.

1. If H is convex and coercive and g is Lipschitz, then we have an explicit solution called the Lax formula; that is

$$\begin{aligned} w_L(t,x) &= \inf_{y\in\mathbb{R}^d} \Big\{ tH^*\Big(\frac{x-y}{t}\Big) + g(y)\Big\} \\ &= \inf_{y\in\mathbb{R}^d} \big\{ (tH)^*(x-y) + g(y)\big\} \\ &= \big((tH)^* \ \triangledown\ g\big)(x). \end{aligned} \tag{2.2}$$

Therefore, the Lax formula is given by the inf-convolution operation.

2. If g is convex and H is at least continuous, satisfying

$$\lim_{\|p\|\to\infty} \frac{tH(p)+g^*(p)}{\|p\|} = \infty \tag{2.3}$$

uniformly with respect to any bounded t, then we have an explicit solution called the Hopf formula; that is

$$w_H(t,x) = \big(tH + g^*\big)^*(x), \tag{2.4}$$

which is clearly a convex function.

These two formulas (2.2) and (2.4) are well known in the literature as Hopf-Lax formulas, despite the fact that they are not equal. For instance, a necessary condition to have both formulas defined is that H and g should be convex (assuming that we have enough regularity). Moreover, for convex Hamiltonian, the Hopf formula could be written as

$$w_H(x) = \big((tH)^* \,\Box\, g\big)(x).$$

Hence by Theorem 1.3, we see that

$$w_L(x) = \big((tH)^* \ \triangledown\ g\big)(x) = \big((tH)^* \,\Box\, g\big)(x) = w_H(x).$$

Consequently, H and g being convex is a necessary and sufficient condition to have $w_L = w_H$; besides that, H coercive is equivalent to condition (2.3).

Now, we turn our attention back to the (vectorial) multi-time Hamilton-Jacobi problem (2.1) and, hereafter we do not use the subscripts L and H respectively to denote the Lax and Hopf formulas. Under the assumptions that g is convex and continuous on $\mathbb{R}^d$ and that H_i $(i=1,2)$ are continuous and satisfy (2.3), Proposition 4 of Lions-Rochet's paper [12] presents an explicit Hopf formula, that is to say

$$w(t_1,t_2,x) = \big(t_1H_1 + t_2H_2 + g^*\big)^*(x),$$

which solves (2.1) *a.e.* in $[0,T]^2\times\mathbb{R}^d$, for $T>0$. We should note that they do not present in that paper an explicit Lax formula. Indeed, considering that H_i $(i=1,2)$ are convex, g is bounded and uniformly continuous, and further that Dg is measurable and bounded or H_i $(i=1,2)$ are coercive, they show in Proposition 5 the following:

$$w(t_1,t_2,x) = S_{H_1}(t_1)\,S_{H_2}(t_2)g(x) = S_{H_2}(t_2)\,S_{H_1}(t_1)g(x),$$

which solves (2.1) *a.e.* and is Lipschitz on $\mathbb{R}^d\times[\,\varepsilon,T]^2$ for all $\varepsilon>0$.

On the other hand, following our discussion above, we propose here to study the following (called) Lax formula, that is

$$\begin{aligned} w(t_1,t_2,x) &= \big((t_1H_1+t_2H_2)^* \;\triangledown\; g\big)(x) \\ &= \inf_{y\in\mathbb{R}^d}\big\{(t_1H_1+t_2H_2)^*(x-y)+g(y)\big\}, \end{aligned} \tag{2.5}$$

where, for our purposes, we assume that g is Lipschitz in $\mathbb{R}^d$.

REMARK 2.1. Some remarks are in order:

1. The regularity of g, i.e. Lipschitz continuous, is a natural assumption in order to show solvability of the multi-time system of conservation laws. In fact, this condition could be relaxed using the Moreau-Yosida approximation g^τ of g and then applying the same strategy used in Alvarez, Barron and Ishii [1].

2. The Lax formula (2.5) already appears, as well, in Imbert and Vollet's paper (see [10]) to study the vectorial Hamilton-Jacobi equations. Completely different from that paper, here we are interested in showing existence and uniqueness of (2.1) and, further, Lipschitz regularity of (2.5) in an explicit and computationally way, which will be exploited in the multi-time conservation laws section.

3. If we agree with the notation $w(t,x)=\big(S_H(t)\,g\big)(x)$ for (2.2), then we observe that

$$\begin{aligned} w(t_1,t_2,x) &= \big((t_1H_1+t_2H_2)^* \;\triangledown\; g\big)(x) \\ &= \big((t_1H_1)^*\,\square\,(t_2H_2)^* \;\triangledown\; g\big)(x) \\ &= \big((t_1H_1)^* \;\triangledown\; (t_2H_2)^* \;\triangledown\; g\big)(x), \end{aligned}$$

which justifies the notation and commutativity in Proposition 5 in Lions and Rochet's paper [12].

4. For simplicity, we sometimes denote $t_1H_1+t_2H_2 =: \mathbf{t}\cdot\mathbf{H}$ (as obvious notation) and, the Lax formula (2.5) becomes

$$w(t_1,t_2,x) = \big((\mathbf{t}\cdot\mathbf{H})^* \;\triangledown\; g\big)(x).$$

5. Finally, we give respectively hypotheses $(H1)-(H3)$ in Barles and Tourin [3] and Assumption A in Plaskacz and Quincampoix [16]:

$(H1)$ For any $R>0$, there exists a constant $K_R>0$, such that

$$\begin{aligned} &|H_i(x,p)|\le K_R && \text{in } \mathbb{R}^d\times\{|p|\le R\}, i=1,2, \\ &|D_pH_i(x,p)|\le K_R\,(1+|x|) && \text{a.e. in } \mathbb{R}^d\times\{|p|\le R\}, i=1,2. \end{aligned}$$

$(H2)$ H_1,H_2 are coercive uniformly with respect to $x\in\mathbb{R}^d$.

$(H3)$ H_1,H_2 are C^1 in $\mathbb{R}^d\times\mathbb{R}^d$ and satisfy

$$D_xH_1(x,p)\,D_pH_2(x,p)-D_xH_2(x,p)\,D_pH_1(x,p)=0,$$

for each $x,p\in\mathbb{R}^d$. The equality above is always satisfied if H_1,H_2 do not depend on x; furthermore, the Hamiltonians could be assumed to be locally Lipschitz.

Assumption A: $H(u,p)=\tilde{H}(u,p)+\lambda(u)$, where $\lambda(u)$ is a C^1-real scalar non-negative and non-increasing function, and $\tilde{H}:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ satisfy

$$\begin{aligned} &\tilde{H}(u,\cdot) && \text{is a concave and positively homogeneous function,} \\ &\tilde{H}(\cdot,p) && \text{is a non-increasing } C^1 \text{ function.} \end{aligned}$$

2.1. *Existence.* First, we show that the infimum in (2.5) is in fact a minimum; hence the infimal convolution is said to be exact. Moreover, w is a continuous function.

LEMMA 2.2. Assume that $g : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function, and let w be defined by (2.5). Then,

$$w(t_1, t_2, x) = \min_{y \in \mathbb{R}^d} \big\{ (\mathbf{t} \cdot \mathbf{H})^*(x - y) + g(y) \big\}.$$

Moreover, w is a continuous function.

Proof. By definition of infimum, there exists $\{y_n\}$ on $\mathbb{R}^d$ such that

$$w(t_1, t_2, x) = \lim_{n \to \infty} \big\{ (\mathbf{t} \cdot \mathbf{H})^*(x - y_n) + g(y_n) \big\}.$$

If $\{y_n\}$ has at least one convergent subsequence, we are done. Otherwise, $\{y_n\}$ should be unbounded, which is not the case. Indeed, recall that H_i^* $(i = 1, 2)$ are coercive; hence $(\mathbf{t} \cdot \mathbf{H})^*$ is also coercive. Therefore, there exist a non-negative real arbitrary number λ and a constant β such that, for n sufficiently large,

$$(\mathbf{t} \cdot \mathbf{H})^*(x - y_n) \geq \lambda \, \|x - y_n\| - \beta - 1/n.$$

Moreover, since the function g is Lipschitz continuous, we have

$$g(y_n) \geq -\operatorname{Lip}(g) \, \|y_n\| + g(0).$$

Then, it follows by the above inequalities that

$$\begin{aligned}
(\mathbf{t} \cdot \mathbf{H})^*(x - y_n) + g(y_n) &\geq \lambda \, \|x - y_n\| - \operatorname{Lip}(g) \, \|y_n\| + g(0) - \beta - 1/n \\
&\geq \lambda \big(\|y_n\| - \|x\| \big) - \operatorname{Lip}(g) \, \|y_n\| + g(0) - \beta - 1/n \\
&\geq C \, \|y_n\| + g(0) - \beta - 1/n,
\end{aligned}$$

where C is a positive constant (take $\lambda > \operatorname{Lip}(g)$). Then, passing to the limit as $n \to \infty$, we have a contradiction, since the infimum in (2.5) is finite. □

The next lemma establishes the semigroup property of the Lax formula.

LEMMA 2.3. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous function and w defined by (2.5). Then, for each $0 \leq s_i < t_i$ $(i = 1, 2)$ and all $x \in \mathbb{R}^d$, it follows that

$$w(t_1, t_2, x) = \min_{y \in \mathbb{R}^d} \big\{ \big((\mathbf{t} - \mathbf{s}) \cdot \mathbf{H} \big)^*(x - y) + w(s_1, s_2, y) \big\}. \tag{2.6}$$

Proof. The proof is a simple application of the inf-convolution and Γ-convolution operations. Indeed, we have

$$\begin{aligned}
w(t_1, t_2, x) &= \big((t_1 H_1 + t_2 H_2)^* \ \triangledown \ g \big)(x) \\
&= \big(((t_1 - s_1) H_1 + s_1 H_1 + (t_2 - s_2) H_2 + s_2 H_2)^* \ \triangledown \ g \big)(x) \\
&= \Big(\big((t_1 - s_1) H_1 + (t_2 - s_2) H_2 \big)^* \, \square \, \big(s_1 H_1 + s_2 H_2 \big)^* \ \triangledown \ g \Big)(x) \\
&= \Big(\big((t_1 - s_1) H_1 + (t_2 - s_2) H_2 \big)^* \ \triangledown \ \big(s_1 H_1 + s_2 H_2 \big)^* \ \triangledown \ g \Big)(x),
\end{aligned}$$

where we have used Theorem 1.3. □

Now, we prove that w defined by (2.5) is a Lipschitz continuous function. Therefore, by Rademacher's Theorem (see [6]), w is differentiable almost everywhere in $\mathbb{R}^d$ and for almost all $t_1, t_2 > 0$.

LEMMA 2.4. The function w defined by (2.5) is Lipschitz in $[0,\infty)^2 \times \mathbb{R}^d$. Moreover, we have

$$\lim_{t_1,t_2 \to 0} w(t_1,t_2,x) = g(x) \quad \text{on } \mathbb{R}^d. \tag{2.7}$$

Proof. 1. First, fix $t_1, t_2 > 0$ and $x, x_0 \in \mathbb{R}^d$. Choose $y \in \mathbb{R}^d$ such that

$$w(t_1,t_2,x) = (t_1 H_1 + t_2 H_2)^*(x-y) + g(y).$$

Thus we have

$$\begin{aligned} w(t_1,t_2,x_0) - w(t_1,t_2,x) &= \min_{z \in \mathbb{R}^d} \big\{ (t_1 H_1 + t_2 H_2)^*(x-z) + g(z) \big\} \\ &\quad - (t_1 H_1 + t_2 H_2)^*(x-y) - g(y) \\ &\le g(x_0 - x + y) - g(y) \le \operatorname{Lip}(g) \|x_0 - x\|, \end{aligned}$$

where we have used $z = x_0 - x + y$. Now, reverting x_0 and x in the above, we obtain

$$|w(t_1,t_2,x) - w(t_1,t_2,x_0)| \le \operatorname{Lip}(g)\, \|x - x_0\|; \tag{2.8}$$

that is, $w(t_1,t_2,x)$ is Lipschitz with respect to the spatial variable $x \in \mathbb{R}^d$.

2. Since g is Lipschitz continuous, for each $x, y \in \mathbb{R}^d$, we have

$$g(y) \ge g(x) - \operatorname{Lip}(g)\, \|x - y\|.$$

Therefore, by definition of $w(t_1,t_2,x)$, we obtain

$$\begin{aligned} g(x) - w(t_1,t_2,x) &\le \max_{y \in \mathbb{R}^d} \big\{ \operatorname{Lip}(g)\, \|x-y\| - \big(t_1 H_1 + t_2 H_2\big)^*(x-y) \big\} \\ &\le \max_{z \in \mathbb{R}^d} \big\{ \max_{\xi \in B_{\operatorname{Lip}(g)}(0)} z \cdot \xi - \big(t_1 H_1 + t_2 H_2\big)^*(z) \big\} \\ &= \max_{\xi \in B_{\operatorname{Lip}(g)}(0)} \big(t_1 H_1 + t_2 H_2\big)(\xi). \end{aligned} \tag{2.9}$$

On the other hand, taking $x = y$ in the definition of $w(t_1,t_2,x)$, it follows that

$$w(t_1,t_2,x) - g(x) \le \big(t_1 H_1 + t_2 H_2\big)^*(0).$$

Consequently, we obtain

$$\inf_{\xi \in \mathbb{R}^d} \big(\mathbf{t} \cdot \mathbf{H}\big)(\xi) \le g(x) - w(t_1,t_2,x) \le \max_{\xi \in B_{\operatorname{Lip}(g)}(0)} \big(\mathbf{t} \cdot \mathbf{H}\big)(\xi). \tag{2.10}$$

Furthermore, passing to the limit as $t_1, t_2 \to 0$, we obtain (2.7).

3. Finally, we show that w is Lipschitz continuous with respect to the time variables. Fix $0 < s_i < t_i$ $(i = 1,2)$ and $x \in \mathbb{R}^d$. By (2.8) for each t_1, t_2, we have

$$\operatorname{Lip}(w(t_1,t_2,\cdot)) \le \operatorname{Lip}(g).$$

Then, we apply the semigroup property of the Lax formula given by Lemma 2.3 and proceed similarly as we have done in step 2 above. Hence the result follows. □

To end up this section, let us show that (2.5) solves the multi-time Hamilton-Jacobi partial differential equation in (2.1) wherever w is differentiable. One recalls that the initial data is shown by Lemma 2.4.

LEMMA 2.5. Let $(t_1, t_2, x) \in (0,\infty)^2 \times \mathbb{R}^d$ be a differentiable point for the multi-time Lax formula given by (2.5). Then,

$$\partial_{t_1} w(t_1,t_2,x) + H_1\big(Dw(t_1,t_2,x)\big) = 0,$$

$$\partial_{t_2} w(t_1,t_2,x) + H_2\big(Dw(t_1,t_2,x)\big) = 0.$$

Proof. Let us show the first differential equality; the second is similar. First, by the semigroup property, we have

$$w(t_1,t_2,x) \leq \big((t_1-s_1)H_1\big)^*(x-y) + w(s_1,t_2,y), \tag{2.11}$$

where we have used $0 < s_2 = t_2$, $0 < s_1 < t_1$ and $y \in \mathbb{R}^d$. Take $\delta > 0$, $q \in \mathbb{R}^d$ fixed, and replace in (2.11) $s_1 \mapsto t_1$, $\quad t_1 \mapsto t_1 + \delta$, $\quad y \mapsto x \quad$ and $\quad x \mapsto x + \delta\, q$. Thus, we have

$$w(t_1+\delta, t_2, x+\delta q) - w(t_1,t_2,x) \leq \delta\, H_1^*(q).$$

Then, dividing by δ and letting δ go to 0^+, we obtain

$$w_{t_1}(t_1,t_2,x) + q \cdot Dw(t_1,t_2,x) - H_1^*(q) \leq 0.$$

Consequently, by the above inequality, it follows that

$$w_{t_1}(t_1,t_2,x) + \max_{p\in\mathbb{R}^d}\{p \cdot Dw(t_1,t_2,x) - H_1^*(p)\} \leq 0,$$

which implies

$$w_{t_1}(t_1,t_2,x) + H_1\big(Dw(t_1,t_2,x)\big) \leq 0.$$

Now choose $z \in \mathbb{R}^d$ such that

$$w(t_1,t_2,x) = \big(t_1 H_1 + t_2 H_2\big)^*(x-z) + g(z).$$

Fix $\delta > 0$ and conveniently set $t_1 = s_1 + \delta$,

$$y = \frac{t_1-\delta}{t_1}\,x + \frac{\delta}{t_1}\,z, \quad \text{so} \quad \frac{x-z}{t_1} = \frac{y-z}{s_1}.$$

Therefore, by definition of $w(s_1,t_2,y)$, we obtain

$$\begin{aligned} w(t_1,t_2,x) - w(s_1,t_2,y) &\geq \big(t_1H_1 + t_2H_2\big)^*(x-z) - \big(s_1H_1 + t_2H_2\big)^*(y-z) \\ &\geq \delta\, H_1^*\Big(\frac{x-z}{t_1}\Big). \end{aligned}$$

Then, passing to the limit as $\delta \to 0^+$ after dividing by δ, we obtain

$$w_{t_1}(t_1,t_2,x) + \frac{x-z}{t_1} \cdot Dw(t_1,t_2,x) - H_1^*\Big(\frac{x-z}{t_1}\Big) \geq 0. \tag{2.12}$$

Finally, we have by (2.12)

$$
\begin{aligned}
w_{t_1}(t_1,t_2,x) + H_1\big(Dw(t_1,t_2,x)\big) &= w_{t_1}(t_1,t_2,x) \\
&\quad + \max_{q\in\mathbb{R}^d}\big\{q\cdot Dw(t_1,t_2,x) - H_1^*(q)\big\} \\
&\geq w_{t_1}(t_1,t_2,x) \\
&\quad + \frac{x-z}{t_1}\cdot Dw(t_1,t_2,x) - H_1^*\Big(\frac{x-z}{t_1}\Big) \geq 0.
\end{aligned}
$$

□

Consequently, we have proved in this section the following

THEOREM 2.6. Let w be the Lax formula given by (2.5). Then, w is Lipschitz continuous, differentiable a.e. in $(0,\infty)^2\times\mathbb{R}^d$, and solves the multi-time Hamilton-Jacobi initial-value problem

$$
\begin{aligned}
w_{t_1} + H_1(Dw) = 0 &\qquad \text{a.e. in } (0,\infty)^2\times\mathbb{R}^d, \\
w_{t_2} + H_2(Dw) = 0 &\qquad \text{a.e. in } (0,\infty)^2\times\mathbb{R}^d, \\
w(0,0,x) = g(x) &\quad \text{ on } \mathbb{R}^d.
\end{aligned}
$$

DEFINITION 2.7. A continuous function $w:(0,\infty)^2\times\mathbb{R}^d\to\mathbb{R}$ is called:

- A viscosity subsolution of the initial-value problem (2.1), provided

$$w(0,0,\cdot) = g(\cdot) \quad \text{on } \mathbb{R}^d$$

and for each $\phi\in C^1((0,\infty)^2\times\mathbb{R}^d)$ if $w-\phi$ has a local maximum in $(\tau_1,\tau_2,\xi)\in(0,\infty)^2\times\mathbb{R}^d$, then

$$
\begin{aligned}
\phi_{t_1}(\tau_1,\tau_2,\xi) + H_1\big(D\phi(\tau_1,\tau_2,\xi)\big) \leq 0, \\
\phi_{t_2}(\tau_1,\tau_2,\xi) + H_2\big(D\phi(\tau_1,\tau_2,\xi)\big) \leq 0.
\end{aligned}
$$

- A viscosity supersolution of the initial-value problem (2.1), provided

$$w(0,0,\cdot) = g(\cdot) \quad \text{on } \mathbb{R}^d$$

and for each $\phi\in C^1((0,\infty)^2\times\mathbb{R}^d)$ if $w-\phi$ has a local minimum in $(\tau_1,\tau_2,\xi)\in(0,\infty)^2\times\mathbb{R}^d$, then

$$
\begin{aligned}
\phi_{t_1}(\tau_1,\tau_2,\xi) + H_1\big(D\phi(\tau_1,\tau_2,\xi)\big) \geq 0, \\
\phi_{t_2}(\tau_1,\tau_2,\xi) + H_2\big(D\phi(\tau_1,\tau_2,\xi)\big) \geq 0.
\end{aligned}
$$

Moreover, w is said to be a viscosity solution of (2.1) when it is both a viscosity supersolution and a viscosity subsolution of (2.1).

One observes that, with a similar strategy used before, it is not difficult to show that w given by (2.5) is a viscosity subsolution and also a viscosity supersolution of (2.1). Then, by definition it is a viscosity solution of (2.1).

2.2. *Uniqueness.* In this section using the idea of doubling variables (see for instance Kruzkov [11], Crandall, Evans and Lions [5]), we show the uniqueness of bounded Lipschitz solutions for the initial-value problem (2.1).

THEOREM 2.8. Assume that the initial data g is a bounded Lipschitz function and that H_i $(i = 1, 2)$ are convex and coercive. Then, there exists at most one Lipschitz bounded viscosity solution of (2.1).

Proof. 1. Let α be a positive real number, defined as

$$\alpha := \sup_{[0,+\infty)^2\times\mathbb{R}^d} \big(w - \tilde{w}\big), \tag{2.13}$$

where w and $\tilde{w}$ are two Lipschitz bounded solutions of (2.1) with the same initial data. Now, we choose $0 < \epsilon, \lambda_1, \lambda_2 < 1$ and define the function Θ as

$$\begin{aligned}\Theta(t_1, t_2, s_1, s_2, x, y) := w(t_1, t_2, x) &- \tilde{w}(s_1, s_2, y)\\ &- \rho_{\epsilon,\lambda_1,\lambda_2}(t_1, t_2, s_1, s_2, x, y)\end{aligned}$$

for each $t_i, s_i \geq 0$ $(i = 1, 2)$ and $x, y \in \mathbb{R}^d$, where

$$\begin{aligned}\rho_{\epsilon,\lambda_1,\lambda_2}(t_1, t_2, s_1, s_2, x, y) := {}& \frac{\lambda_1}{2}(t_1 + s_1) + \frac{\lambda_2}{2}(t_2 + s_2)\\ &+ \epsilon^{-2}\Big((t_1 - s_1)^2 + (t_2 - s_2)^2 + \|x - y\|^2\Big)\\ &+ \epsilon\,(\|x\|^2 + \|y\|^2).\end{aligned}$$

So, as

$$\lim_{\|(t_1,t_2,s_1,s_2,x,y)\|\to+\infty} \rho_{\epsilon,\lambda_1,\lambda_2}(t_1, t_2, s_1, s_2, x, y) = +\infty,$$

we have

$$\lim_{\|(t_1,t_2,s_1,s_2,x,y)\|\to+\infty} \Theta(t_1, t_2, s_1, s_2, x, y) = -\infty$$

and, as the function Θ is continuous in its domain and proper (not indentically $\pm\infty$), there must be a point of maximum, i.e., there must exist a point $(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \in [0, +\infty)^4 \times \mathbb{R}^{2d}$, such that

$$\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) = \max_{[0,+\infty)^4\times\mathbb{R}^d} \Theta(t_1, t_2, s_1, s_2, x, y). \tag{2.14}$$

2. From (2.14), the map

$$(t_1, t_2, x) \longmapsto \Theta(t_1, t_2, \hat{s_1}, \hat{s_2}, x, \hat{y})$$

has a maximum in $(\hat{t}_1, \hat{t}_2, \hat{x})$. If we write Θ as

$$\Theta(t_1, t_2, \hat{s_1}, \hat{s_2}, x, \hat{y}) = w(t_1, t_2, x) - v(t_1, t_2, x),$$

where

$$v(t_1, t_2, x) := \tilde{w}(\hat{s}_1, \hat{s}_2, \hat{y}) + \rho_{\epsilon,\lambda_1,\lambda_2}(t_1, t_2, \hat{s}_1, \hat{s}_2, x, \hat{y}),$$

then $(w - v)$ has a maximum in $(\hat{t}_1, \hat{t}_2, \hat{x})$. Since w is a viscosity solution of (2.1), it follows that

$$v_{t_1}(\hat{t}_1, \hat{t}_2, \hat{x}) + H_1\big(Dv(\hat{t}_1, \hat{t}_2, \hat{x})\big) \leq 0,$$

$$v_{t_2}(\hat{t}_1, \hat{t}_2, \hat{x}) + H_2\big(Dv(\hat{t}_1, \hat{t}_2, \hat{x})\big) \leq 0.$$

Now, using the definition of v we obtain

$$\begin{aligned} &\frac{\lambda_1}{2} + \epsilon^{-2}(\hat{t}_1 - \hat{s}_1) + H_1\left(\frac{2}{\epsilon^2}(\hat{x} - \hat{y}) + 2\epsilon\hat{x}\right) \leq 0, \\ &\frac{\lambda_2}{2} + \epsilon^{-2}(\hat{t}_2 - \hat{s}_2) + H_2\left(\frac{2}{\epsilon^2}(\hat{x} - \hat{y}) + 2\epsilon\hat{x}\right) \leq 0. \end{aligned} \tag{2.15}$$

Analogously, the map

$$(s_1, s_2, y) \longmapsto -\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, y)$$

has a minimum in $(\hat{s}_1, \hat{s}_2, \hat{y})$. We write $-\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, y)$ as

$$-\Theta(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, y) := \tilde{w}(s_1, s_2, y) - \tilde{v}(s_1, s_2, y).$$

Hence, $(\tilde{w} - \tilde{v})$ has a minimum in $(\hat{s}_1, \hat{s}_2, y)$, where

$$\tilde{v}(s_1, s_2, y) := w(\hat{t}_1, \hat{t}_2, \hat{x}) - \rho_{\epsilon,\lambda_1,\lambda_2}(\hat{t}_1, \hat{t}_2, s_1, s_2, \hat{x}, y).$$

Similarly to (2.15), we have

$$\begin{aligned} &-\frac{\lambda_1}{2} + \epsilon^{-2}(\hat{t}_1 - \hat{s}_1) + H_1\left(2\epsilon^{-2}(\hat{x} - \hat{y}) - 2\epsilon\hat{y}\right) \geq 0, \\ &-\frac{\lambda_2}{2} + \epsilon^{-2}(\hat{t}_2 - \hat{s}_2) + H_2\left(2\epsilon^{-2}(\hat{x} - \hat{y}) - 2\epsilon\hat{y}\right) \geq 0. \end{aligned} \tag{2.16}$$

3. Finally, making the difference between (2.16) and (2.15) with respect to the first line, we have

$$\lambda_1 \leq H_1\left(2\epsilon^{-2}(\hat{x} - \hat{y}) - 2\epsilon\hat{y}\right) - H_1\left(2\epsilon^{-2}(\hat{x} - \hat{y}) + 2\epsilon\hat{x}\right).$$

Since H_1 is locally Lipschitz continuous (and the maximum point $(\hat{t_1}, \hat{t_2}, \hat{s_1}, \hat{s_2}, \hat{x}, \hat{y})$ is attained in a compact ball), we have

$$\lambda_1 \leq 2\epsilon \left\|\hat{y} + \hat{x}\right\|. \tag{2.17}$$

At this point, we need an estimate of $\|\hat{y} + \hat{x}\|$ to conclude that $\lambda_1 = 0$, since $\epsilon > 0$ is arbitrary. It will be obtained thanks to the definition of $\rho_{\epsilon,\lambda_1,\lambda_2}$. In fact, we can fix $0 < \epsilon, \lambda_1, \lambda_2 < 1$ so small that (2.13) implies

$$\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \geq \sup_{[0,T]^2 \times \mathbb{R}^{2d}} \Theta(t_1, t_2, t_1, t_2, x, x) \geq \frac{\alpha}{2}. \tag{2.18}$$

Moreover, since

$$\Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \geq \Theta(0, 0, 0, 0, 0, 0),$$

it follows that

$$\begin{aligned} \rho_{\epsilon,\lambda_1,\lambda_2}(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \leq{}& [w(\hat{t}_1, \hat{t}_2, \hat{x}) - w(0, 0, 0)] \\ &- [\tilde{w}(\hat{s}_1, \hat{s}_2, \hat{y}) - \tilde{w}(0, 0, 0)]. \end{aligned}$$

Since w and $\tilde{w}$ are bounded, we obtain as $\epsilon \to 0^+$

$$\begin{aligned} |\hat{t}_1 - \hat{s}_1|, |\hat{t}_2 - \hat{s}_2|, \|\hat{x} - \hat{y}\| &= O(\epsilon), \\ \epsilon\left(\|\hat{x}\|^2 + \|\hat{y}\|^2\right) &= O(1). \end{aligned} \tag{2.19}$$

The last equation of (2.19) implies that

$$\begin{aligned} \epsilon\left(\|\hat{x}\| + \|\hat{y}\|\right) &= \epsilon^{\frac{1}{4}}\ \epsilon^{\frac{3}{4}}\left(\|\hat{x}\| + \|\hat{y}\|\right) \\ &\le \epsilon^{\frac{1}{2}} + C\,\epsilon^{\frac{3}{2}}(\|\hat{x}\|^2 + \|\hat{y}\|^2) \le C\,\epsilon^{\frac{1}{2}} \end{aligned} \tag{2.20}$$

for some positive constant C. To complete the proof, we use (2.20) in (2.17) and get

$$\lambda_1 \le 2C\epsilon^{\frac{1}{2}}.$$

Similarly, we obtain that $\lambda_2 = 0$, and this contradiction completes the proof. □

REMARK 2.9. Note that in the proof, the points $\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2$ could be zero, and in that case, with respect to the time, the function Θ would be constant. To see that this does not happen, we recall that

$$\Theta(\hat{t}_1, \hat{t}_2, \hat{t}_1, \hat{t}_2, \hat{x}, \hat{x}) \le \Theta(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y})$$

and from this, we get

$$\begin{aligned} &w(\hat{t}_1, \hat{t}_2, \hat{x}) - \tilde{w}(\hat{s}_1, \hat{s}_2, \hat{y}) - \rho_{\epsilon,\lambda_1,\lambda_2}(\hat{t}_1, \hat{t}_2, \hat{s}_1, \hat{s}_2, \hat{x}, \hat{y}) \\ &\qquad \ge w(\hat{t}_1, \hat{t}_2, \hat{x}) - \tilde{w}(\hat{t}_1, \hat{t}_2, \hat{x}) - \rho_{\epsilon,\lambda_1,\lambda_2}(\hat{t}_1, \hat{t}_2, \hat{t}_1, \hat{t}_2, \hat{x}, \hat{x}). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \epsilon^{-2}((\hat{t}_1 - \hat{s}_1)^2 + (\hat{t}_2 - \hat{s}_2)^2 + \|\hat{x} - \hat{y}\|^2) &\le w(\hat{x}, \hat{t}_1, \hat{t}_2) - \tilde{w}(\hat{y}, \hat{s}_1, \hat{s}_2) \\ &\quad - \frac{\lambda_1}{2}(\hat{t}_1 - \hat{s}_1) + \frac{\lambda_2}{2}(\hat{t}_2 - \hat{s}_2) + \epsilon(\hat{x} - \hat{y})(\hat{x} + \hat{y}). \end{aligned}$$

Then, by (2.19), (2.20) and the Lipschitz continuity of $\tilde{w}$, we have

$$|\hat{t}_1 - \hat{s}_1|, |\hat{t}_2 - \hat{s}_2|, \|\hat{x} - \hat{y}\| = o(\epsilon). \tag{2.21}$$

Now, let ω be the modulus of continuity of w; that is,

$$|w(t_1, t_2, x) - \tilde{w}(s_1, s_2, y)| \le \omega(|t_1 - s_1| + |t_2 - s_2| + \|x - y\|)$$

for all $x, y \in \mathbb{R}^n$, $0 \le t,\ s \le T$, and $\omega(r) \to 0$ as $r \to 0$. Similarly, $\tilde{\omega}(\cdot)$ will denote the modulus of continuity of $\tilde{w}$. Then (2.18) implies

$$\begin{aligned} \frac{\alpha}{2} \le w(\hat{t}_1, \hat{t}_2, \hat{x}) - \tilde{w}(\hat{s}_1, \hat{s}_2, \hat{y}) &= [w(\hat{t}_1, \hat{t}_2, \hat{x}) - w(\hat{t}_1, 0, \hat{x})] + [w(\hat{t}_1, 0, \hat{x}) - w(0, 0, \hat{x})] \\ &\quad + [w(0, 0, \hat{x}) - \tilde{w}(0, 0, \hat{x})] + [\tilde{w}(0, 0, \hat{x}) - \tilde{w}(\hat{t}_1, 0, \hat{x})] \\ &\quad + [\tilde{w}(\hat{t}_1, 0, \hat{x}) - \tilde{w}(\hat{t}_1, \hat{t}_2, \hat{x})] + [\tilde{w}(\hat{t}_1, \hat{t}_2, \hat{x}) - \tilde{w}(\hat{s}_1, \hat{s}_2, \hat{y})]. \end{aligned}$$

Therefore, using (2.19), (2.21) and the initial condition, we have

$$\frac{\alpha}{2} \le \omega(\hat{t}_2) + \omega(\hat{t}_1) + \tilde{\omega}(\hat{t}_1) + \tilde{\omega}(\hat{t}_2) + \tilde{\omega}(o(\epsilon)).$$

As ϵ is a positive arbitrary number, we can take it as small as necessary to obtain

$$\frac{\alpha}{4} \le \omega(\hat{t}_2) + \omega(\hat{t}_1) + \tilde{\omega}(\hat{t}_1) + \tilde{\omega}(\hat{t}_2),$$

and this implies for some constant $\mu > 0$,

$$\hat{t}_1, \hat{t}_2 \geq \mu > 0.$$

Analogously, we have $\hat{s}_1, \hat{s}_2 \geq \mu > 0$.

3. Multi-time conservation laws. Once we have established existence and uniqueness for the multi-time Hamilton-Jacobi system, we are going to use it in this section in order to show solvability of the multi-time system of conservation laws. Therefore, we fix d, s equal to one and for given $H_i (i = 1, 2)$ two smooth (uniformly) convex flux functions, we consider the following Cauchy problem: Find $u : (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}$, satisfying

$$\begin{aligned} u_{t_1} + \partial_x H_1(u) &= 0 \qquad \text{in } (0,\infty)^2 \times \mathbb{R}, \\ u_{t_2} + \partial_x H_2(u) &= 0 \qquad \text{in } (0,\infty)^2 \times \mathbb{R}, \\ u(0,0,x) &= u_0(x) \quad \text{on } \mathbb{R}, \end{aligned} \tag{3.1}$$

where $u_0 \in L^\infty(\mathbb{R})$ is given initial data. With no loss of generality, we assume $H_i(0) = 0$ $(i = 1, 2)$. Following the usual strategy for $1D$ scalar conservation laws, we define

$$g(x) := \int_0^x u_0(y)\, dy \qquad (x \in \mathbb{R}); \tag{3.2}$$

thus, g is a Lipschitz function with $\mathrm{Lip}(g) = \|u_0\|_\infty$. Recall the multi-time Lax formula given by (2.5). Thus by Theorem 2.6, w solves the multi-time Hamilton-Jacobi system (2.1) and, if we assume that w is smooth, then we can differentiate that system with respect to x to deduce

$$\begin{aligned} w_{x\,t_1} + \partial_x H_1(w_x) &= 0 \qquad \text{in } (0,\infty)^2 \times \mathbb{R}, \\ w_{x\,t_2} + \partial_x H_2(w_x) &= 0 \qquad \text{in } (0,\infty)^2 \times \mathbb{R}, \\ w_x(0,0,x) &= u_0(x) \quad \text{on } \mathbb{R}. \end{aligned} \tag{3.3}$$

Now, setting $u = w_x$ we obtain that u solves the system (3.1). Certainly, the computation is only formal. Indeed, even though the function w is differentiable a.e., we are not allowed to differentiate $H_1(w_x)$ with respect to x, similarly to H_2. Although,

$$\begin{aligned} u(t_1, t_2, x) :&= \partial_x \big(\min_{y \in \mathbb{R}} \big\{ \big(t_1 H_1 + t_2 H_2\big)^*(x - y) + g(y) \big\} \big) \\ &= \partial_x \big((\mathbf{t} \cdot \mathbf{H})^* \ \triangledown\ g \big)(x) \end{aligned} \tag{3.4}$$

seems to be the best candidate for a solution to the Cauchy problem (3.1). In fact, we will show that such a function u as defined above is a (weak integral) solution, but before that, let us first show a more useful formula.

LEMMA 3.1. (**Multi-time Lax-Oleinik formula**). Assume $H_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are smooth uniformly convex, with $u_0 \in L^\infty(\mathbb{R})$ and g given by (3.2). Then, for each $t_1, t_2 > 0$, there exists for all but at most countably many values $x \in \mathbb{R}$, such that (3.4) has the following form:

$$u(t_1, t_2, x) = \Big((t_1 H_1)^* \ \triangledown\ (t_2 H_2)^* \Big)' \big(x - y(t_1, t_2, x)\big), \tag{3.5}$$

where the mapping $x \mapsto y(t_1,t_2,x)$ is nondecreasing. Moreover, for each $z>0$,

$$u(t_1,t_2,x+z)-u(t_1,t_2,x) \le \operatorname{Lip}\big(\big((t_1H_1+t_2H_2)^*\big)'\big)\, z. \tag{3.6}$$

DEFINITION 3.2. Equation (3.5) is called the multi-time Lax-Oleinik formula.

Proof. 1. Fix $t_1,t_2>0$, $x_1<x_2$. There exists at least one point $y_1\in\mathbb{R}$, such that

$$w(t_1,t_2,x_1)=\big(t_1H_1+t_2H_2\big)^*(x_1-y_1)+g(y_1). \tag{3.7}$$

Now, we claim that, for each $y<y_1$,

$$\big(t_1H_1+t_2H_2\big)^*(x_2-y_1)+g(y_1)<\big(t_1H_1+t_2H_2\big)^*(x_2-y)+g(y).$$

Indeed, let $\tau\in(0,1)$, given by

$$\tau=\frac{y_1-y}{(x_2-x_1)+(y_1-y)},$$

and for convenience, we write

$$\begin{aligned} x_2-y_1&=\tau\,(x_1-y_1)+(1-\tau)\,(x_2-y),\\ x_1-y&=(1-\tau)\,(x_1-y_1)+\tau\,(x_2-y). \end{aligned}$$

Therefore, since $(H_i^*)''>0$ $(i=1,2)$, it follows that

$$\begin{aligned} \big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_2-y_1)&<\tau\,\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_1-y_1)+(1-\tau)\,\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_2-y),\\ \big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_1-y)&<(1-\tau)\,\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_1-y_1)+\tau\,\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_2-y). \end{aligned}$$

Then, combining the two above inequalities, we obtain

$$\begin{aligned} &\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_2-y_1)+\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_1-y)\\ &\qquad\qquad<\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_1-y_1)+\big(\mathbf{t}\cdot\mathbf{H}\big)^*(x_2-y). \end{aligned} \tag{3.8}$$

Moreover, by the definition of $w(t_1,t_2,x_1)$, we have

$$-\big(t_1H_1+t_2H_2\big)^*(x_1-y)-g(y)\le-\big(t_1H_1+t_2H_2\big)^*(x_1-y_1)-g(y_1). \tag{3.9}$$

Then, from (3.8) and (3.9),

$$\big(t_1H_1+t_2H_2\big)^*(x_2-y_1)+g(y_1)<\big(t_1H_1+t_2H_2\big)^*(x_2-y)+g(y),$$

and so the claim is proved.

2. From the claim proved before, we observe that to compute the minimum below, i.e.

$$\min_{y\in\mathbb{R}}\Big\{\big(t_1H_1+t_2H_2\big)^*(x_2-y)+g(y)\Big\},$$

we only need to consider those $y\ge y_1$, where y_1 satisfies (3.7). Therefore, for each $t_1,t_2>0$ and $x\in\mathbb{R}$, we could define the point $y(t_1,t_2,x)$ equal to the smallest value of those points y giving the minimum of

$$\big(t_1H_1+t_2H_2\big)^*(x-y)+g(y).$$

Consequently, for each $t_1,t_2>0$, the mapping $x\mapsto y(t_1,t_2,x)$ is nondecreasing and thus continuous for all but at most countably many $x\in\mathbb{R}$. Moreover, at such a point x, the value $y(t_1,t_2,x)$ are those unique y yielding the minimum.

3. Since the function w is Lipschitz and thus differentiable a.e. and the mapping $x \mapsto y(t_1,t_2,x)$ is monotone and so differentiable a.e. as well, given $t_1,t_2>0$ for a.e. $x\in\mathbb{R}$, the mappings

$$\begin{aligned}
&x \mapsto \big(t_1H_1+t_2H_2\big)^*(x-y(t_1,t_2,x)),\\
&x \mapsto g(y(t_1,t_2,x))
\end{aligned}$$

are also differentiable for a.e. $x\in\mathbb{R}$. Then, we have for such a differentiable point x,

$$\begin{aligned}
u(t_1,t_2,x) &= \partial_x\Big(\big(t_1H_1+t_2H_2\big)^*(x-y(t_1,t_2,x))+g(y)\Big)\\
&= \Big(\big(t_1H_1+t_2H_2\big)^*\Big)'(x-y(t_1,t_2,x))\,\Big(1-y_x(t_1,t_2,x)\Big)\\
&+\partial_x\Big(g(y(t_1,t_2,x))\Big).
\end{aligned}$$

But since the mapping $y\mapsto (t_1H_1+t_2H_2)^*+g$ has a minimum at $y=y(t_1,t_2,x)$, it follows that

$$-\Big(\big(t_1H_1+t_2H_2\big)^*\Big)'(x-y(t_1,t_2,x))\,y_x(t_1,t_2,x)+\partial_x\big(g(y(t_1,t_2,x))\big)=0,$$

and thus we obtain (3.5).

4. Finally, by equation (3.5), the monotonicity of $\big((t_1H_1+t_2H_2)^*\big)'$ and $y(t_1,t_2,\cdot)$ as well, we have for each $z>0$

$$\begin{aligned}
u(t_1,t_2,x) &= \big((t_1H_1+t_2H_2)^*\big)'(x-y(t_1,t_2,x))\\
&\geq \big((t_1H_1+t_2H_2)^*\big)'(x-y(t_1,t_2,x+z))\\
&\geq \big((t_1H_1+t_2H_2)^*\big)'(x+z-y(t_1,t_2,x+z))\\
&\quad-\operatorname{Lip}\big(\big((t_1H_1+t_2H_2)^*\big)'\big)\,z\\
&= u(t_1,t_2,x+z)-\operatorname{Lip}\big(\big((t_1H_1+t_2H_2)^*\big)'\big)\,z.
\end{aligned}$$

Therefore, we obtain

$$u(t_1,t_2,x+z)-u(t_1,t_2,x)\leq \operatorname{Lip}\big(\big((t_1H_1+t_2H_2)^*\big)'\big)\,z.$$

□

3.1. *Existence.* Now we are ready to show the solvability of the multi-time system of conservation laws in $1D$ for two independent times. First, let us define in which sense a bounded and measurable real function u defined in $(0,\infty)^2\times\mathbb{R}$ is a weak (integral) solution of (3.1).

DEFINITION 3.3. Given $u_0\in L^\infty(\mathbb{R})$, a function $u\in L^\infty((0,\infty)^2\times\mathbb{R})$ is said to be a weak integral solution of the Cauchy problem (3.1) if it satisfies

• Multi-time conservation laws: For all $\varphi\in C_0^\infty((0,\infty)^2\times\mathbb{R})$,

$$\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}}\big(u\,\varphi_{t_1}+H_1(u)\,\varphi_x\big)\,dxdt_1dt_2=0, \tag{3.10}$$

$$\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}}\big(u\,\varphi_{t_2}+H_2(u)\,\varphi_x\big)\,dxdt_1dt_2=0. \tag{3.11}$$

- Initial condition: For any $\gamma \in L^1(\mathbb{R})$,

$$\operatorname*{ess\,lim}_{t_1,t_2\to 0^+} \int_{\mathbb{R}} \big(u(t_1,t_2,x) - u_0(x)\big)\,\gamma(x)\,dx = 0. \tag{3.12}$$

THEOREM 3.4. The function $u \in L^\infty((0,\infty)^2 \times \mathbb{R})$ given by Lemma 3.1 and equation (3.5) is a weak solution of the Cauchy problem (3.1).

Proof. First, we define for $t_1, t_2 > 0$ and $x \in \mathbb{R}$,

$$w(t_1,t_2,x) = \min_{y\in\mathbb{R}} \left\{ (t_1 H_1 + t_2 H_2)^*(x-y) + g(y) \right\},$$

which by Theorem 2.6 is a Lipschitz continuous function, differentiable a.e in $(0,\infty)^2\times\mathbb{R}$, and solves

$$\begin{aligned} w_{t_1} + H_1(w_x) = 0 \qquad &\text{a.e. in } (0,\infty)^2 \times \mathbb{R}, \\ w_{t_2} + H_2(w_x) = 0 \qquad &\text{a.e. in } (0,\infty)^2 \times \mathbb{R}, \\ w(0,0,x) = g(x) \quad &\text{on } \mathbb{R}. \end{aligned} \tag{3.13}$$

Now, we take $\varphi \in C_0^\infty((0,\infty)^2 \times \mathbb{R})$, multiply the first equation in (3.13) by φ_x and integrate over $(0,\infty)^2 \times \mathbb{R}$ to obtain

$$\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} \big(w_{t_1}\,\varphi_x + H_1(w_x)\,\varphi_x\big)\,dxdt_1dt_2 = 0.$$

Then, we observe that

$$\begin{aligned} \int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} w_{t_1}\,\varphi_x\,dxdt_1dt_2 &= -\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} w\,\varphi_{t_1x}\,dxdt_1dt_2 \\ &= \int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} w_x\,\varphi_{t_1}\,dxdt_1dt_2, \end{aligned}$$

where we are allowed to integrate by parts, since the mapping $x \mapsto w(t_1,t_2,x)$ is Lipschitz continuous and then absolutely continuous for each $t_1, t_2 > 0$. Moreover, for each $t_2 > 0$ and $x \in \mathbb{R}$, the mapping $t_1 \mapsto w(t_1,t_2,x)$ is also absolutely continuous. Therefore, we have

$$\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} \big(w_x\,\varphi_{t_1} + H_1(w_x)\,\varphi_x\big)\,dxdt_1dt_2 = 0,$$

and by a similar argument, we obtain

$$\int_0^\infty\!\!\int_0^\infty\!\!\int_{\mathbb{R}} \big(w_x\,\varphi_{t_2} + H_2(w_x)\,\varphi_x\big)\,dxdt_1dt_2 = 0.$$

Finally, we recall that $u = w_x$ *a.e.* as precisely defined by (3.5). Then, the multi-time conservation laws condition of Definition 3.3 is satisfied.

To show the initial condition, we apply the same strategy before and the result follows using (2.7). $\square$

3.2. *Uniqueness.* We show the existence of a weak integral solution u to the problem (3.1), where u is given by (3.5). Recall that the integral solution is slightly different from the entropy solution given by Definition 1.2; that is, a measurable and bounded function $u(t_1, t_2, x)$ is an entropy solution to (3.1) if for all entropy pairs $(\eta(u), q_i(u))$ $(i = 1, 2)$ and for each $T > 0$, the following holds true:

$$\int_0^T\int_0^T\int_{\mathbb{R}} \big(\eta(u)\,\varphi_{t_1} + q_1(u)\,\varphi_x\big)\,dxdt_1dt_2 \geq 0, \tag{3.14}$$

$$\int_0^T\int_0^T\int_{\mathbb{R}} \big(\eta(u)\,\varphi_{t_2} + q_2(u)\,\varphi_x\big)\,dxdt_1dt_2 \geq 0, \tag{3.15}$$

for each non-negative test function $\varphi \in C_0^\infty((0,T)^2 \times \mathbb{R})$, and also the initial condition (3.12) is satisfied. It follows by (3.6) that, for each $t_1, t_2 \in (0, T)$ fixed, $u(t_1, t_2, \cdot)$ has locally bounded variation. Indeed, we know that for each $z > 0$,

$$\frac{u(t_1, t_2, x+z) - u(t_1, t_2, x)}{z} \leq c,$$

where $c := \operatorname{Lip}\big(\big((t_1H_1 + t_2H_2)^*\big)'\big)$. Let $\tilde{u}(t_1, t_2, x) = u(t_1, t_2, x) - \tilde{c}\,x$ for $\tilde{c} > c$. Then, we have for each $z > 0$

$$\tilde{u}(t_1, t_2, x+z) - \tilde{u}(t_1, t_2, x) < 0;$$

that is, $\tilde{u}(t_1, t_2, \cdot)$ is a decreasing function and hence has locally bounded total variation. Since this is also true for $\tilde{c}\,x$, we obtain that $u(t_1, t_2, \cdot)$ has locally bounded variation. Therefore, the well-known theory of Vol′pert [20] allow us to apply the chain rule for BV functions and write for a.e. $x \in \mathbb{R}$, $i = 1, 2$

$$\partial_x H_i(u(t_1, t_2, x)) = H_i'(u(t_1, t_2, x))\;(u(t_1, t_2, x))_x,$$

and thus since u is an integral solution, we have in the sense of measures

$$|u_{t_i}| \leq \max_{\xi \in B_{\|u\|_\infty}(0)} |H_i'(\xi)|\;\;|u_x|; \tag{3.16}$$

that is to say, u_{t_1}, u_{t_2} are locally Radon measures.

Now, let η be a smooth convex function. Again, with no loss of generality, we may as well also take $\eta(0) = 0$. Then, we multiply (3.16) by $\eta'(u)$ and apply again the chain rule for BV functions to obtain in the measure sense

$$\begin{aligned} \eta(u)_{t_1} + \partial_x q_1(u) = 0, \\ \eta(u)_{t_2} + \partial_x q_2(u) = 0. \end{aligned} \tag{3.17}$$

Consequently, it is not difficult to see that the integral solution u is in fact an entropy solution, where the estimate (3.16) is crucial in order to show the initial data (1.4). Moreover, by a standard approximation procedure, we may assume that the pair (η, q_i) $(i = 1, 2)$ are the Kruzkov entropies, that is,

$$\int_0^T\int_0^T\int_{\mathbb{R}} \big(|u-v|\,\varphi_{t_1} + \operatorname{sgn}(u-v)\big(H_1(u) - H_1(v)\big)\,\varphi_x\big)\,dxdt_1dt_2 = 0, \tag{3.18}$$

$$\int_0^T\int_0^T\int_{\mathbb{R}} \big(|u-v|\,\varphi_{t_2} + \operatorname{sgn}(u-v)\big(H_2(u) - H_2(v)\big)\,\varphi_x\big)\,dxdt_1dt_2 = 0, \tag{3.19}$$

for each $v \in \mathbb{R}$ fixed and all test functions $\varphi \in C_0^\infty((0,T)^2 \times \mathbb{R})$. Therefore, we are in position to apply the doubling variables technique due to Kruzkov; see [11]. In fact, this is nowadays a standard procedure, and thus adapted to our case leads to the following result.

LEMMA 3.5. Let u and v be two entropy solutions to the problem (3.1) corresponding to initial data u_0, v_0 respectively. Then, we have the L^1-contraction type inequalities

$$\begin{aligned} &\int_0^T\int_{B_R(0)} |u(t_1,\tau,x)-v(t_1,\tau,x)|\,\zeta_2(\tau)\,dxd\tau \\ &\qquad\qquad \le \int_0^T\int_{B_{R_1}(0)} |u(0,\tau,x)-v(0,\tau,x)|\,\zeta_2(\tau)\,dxd\tau, \\ &\int_0^T\int_{B_R(0)} |u(\tau,t_2,x)-v(\tau,t_2,x)|\,\zeta_1(\tau)\,dxd\tau \\ &\qquad\qquad \le \int_0^T\int_{B_{R_2}(0)} |u(\tau,0,x)-v(\tau,0,x)|\,\zeta_1(\tau)\,dxd\tau, \end{aligned} \tag{3.20}$$

which holds for all $B_R(0)$, $R > 0$ and almost all $t_1, t_2 > 0$, where for $i = 1,2$, $\zeta_i \in C_0^\infty(0,T)$, $B_{R_i} = B_{R+M_i t_i}(0)$, and M_i denotes the Lipschitz constant of H_i.

THEOREM 3.6. Let u and v be two entropy solutions to the problem (3.1) corresponding to initial data u_0, v_0 respectively. If $u_0 = v_0$ almost everywhere, then $u = v$ almost everywhere.

Proof. For $\delta > 0$, we take $\zeta_1(\tau) = \chi_{(0,\delta)}(\tau)$ in the second inequality of (3.20). Then, dividing both sides of the inequality by δ and passing to the limit as $\delta \to 0^+$, we obtain

$$\int_{\mathbb{R}} |u(0,t_2,x)-v(0,t_2,x)|\,dx = 0.$$

Similarly, for $\theta > 0$ sufficiently small, we take $\zeta_2(\tau) = \chi_{(t_2-\theta,t_2+\theta)}(\tau)$ in the first inequality of (3.20). Again, dividing the inequality by θ and passing to the limit as θ goes to 0^+, the uniqueness result follows; that is, $u \equiv v$ almost everywhere. $\square$

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