Multi-Utilitarianism in Two-Agent Quasilinear Social Choice

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Abstract

We introduce a new class of rules for resolving quasilinear social choice problems. These rules extend those of Green [7]. We call such rules multi-utilitarian rules. Each multi-utilitarian rule is associated with a probability measure over the set of weighted utilitarian rules, and is derived as the expectation of this probability. These rules are characterized by the axioms *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *additivity*. By adding *recursive invariance*, we obtain a class of asymmetric rules generalizing those Green characterizes. A multi-utilitarian rule satisfying *strong monotonicity* has an associated probability measure with full support.

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1 Introduction

Building on the foundational work of Green [6, 7], this paper studies a social choice model where agents can make transfers among themselves. Imagine two agents who must decide on some social alternative. There is an infinitely divisible good, and agents have preferences which are quasilinear over the social alternative and good. The question is which alternative should be chosen, and which transfers should be recommended? We take a normative approach to this problem.

We imagine that randomization over social alternatives is permitted, and that all agents are expected utility maximizers. Moreover, we assume that agents are risk-neutral in the infinitely divisible good. Under these assumptions, we may uniquely (up to translation) represent each agent's preference by a utility function which is additively separable in the infinitely divisible good, taking the form $U(p, x) = u(p) + x$.

We do not work with the underlying space of alternatives. Instead, the primitive of the model is the utility possibility set that agents can achieve without making transfers. Thus, the theory is "welfarist," at least in terms of the social alternatives. Any two scenarios which induce the same utility possibility set *before transfers* are identified. This utility possibility set is referred to as a "problem." For any given problem, a rule specifies a pair of utilities for the agents. We assume this pair of utilities is achievable through transfers.

We discuss properties that rules for solving such problems should satisfy. Minimally, we require that a rule select *efficient* utility pairs. However; we ask that a rule satisfies several other properties. One such requirement is tied to the underlying utility representation. The utility representation derived for any given preference relation is *almost* unique. However, by adding a constant to a given utility representation, we obtain a new utility representation for a preference which is equally as valid as the first. We will require that a rule is robust to utility specification. *Translation invariance* states that the addition of a constant vector to a problem (equivalent to adding a constant to each agent's utility function) should induce an equivalent addition of this vector to its solution.

Suppose that a problem is altered, so that alternatives are added which "favor" agent 1, and some alternatives which "favor" agent 2 are removed. How should the solution respond to such a change? As the scenario becomes more favorable toward agent 1, a natural requirement is that the solution should not make agent 1 worse off. This is the axiom of *monotonicity.*

We further require that the solution to a problem should vary continuously with the problem itself. A rule is *continuous* if it is continuous with respect to the Hausdorff topology appropriately defined for this model.

Our last axiom states that given two problems, if we know the solution recommended by the rule for each problem, then we can compute the solution for the Minkowski sum of the two problems as the sum of the solutions of the original problems. This axiom is called *additivity.*

Green [7] investigates the implications of all of the axioms we have discussed, in addition to two other axioms. One of his other axioms is a basic *symmetry* condition. His other axiom, *recursive invariance*, is motivated as follows. Suppose a utility pair is selected by a rule for a given problem. Suppose this utility pair is added to the problem, resulting in a modified problem. The rule applied to the modified problem should again select this utility pair. Green characterizes the family of all rules satisfying his axioms. Our result is more general than his, but the rules we characterize share several important characteristics with the rules he characterizes.

Our main result is a characterization of the family of rules satisfying the axioms *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *additivity.* To understand how these rules work, we first discuss the concept of weighted utilitarianism. A weighted utilitarian rule is a rule in which each agent is assigned a nonnegative weight; at least one of which is positive. The agents' weights are not the same. A natural social welfare function over utility space is that which computes the weighted sum of the agents' utilities. A weighted utilitarian rule then works as follows: find the utility pair lying in the problem which induces the maximal social utility. When transfers are possible, this pair is not necessarily efficient. However, there is a remedy for this inefficiency. There exists a unique efficient utility pair (*i.e.* after transfers) whose social utility is the same as the original utility pair. The weighted utilitarian rule selects this efficient utility pair.

The class of rules satisfying our five axioms is convex. This leads us to a natural conjecture. Suppose we have given a probability distribution over the weighted utilitarian rules. For any given problem, we can compute the expected solution according to this distribution. This expected solution is itself a utility pair; thus, we can naturally identify a rule with the probability distribution. Such a rule will be called a "multi-utilitarian rule." Our main contribution is to show that the multi-utilitarian rules are the only rules satisfying the five properties.

Another result that we discuss concerns a weakening of *additivity*. Suppose that we require that the solution for the "average" of two problems is the average of the solutions. This condition is called *mixture linearity*, and it plays a fundamental role in the work of Myerson [8]. *Mixture linearity* is a requirement that precludes a rule from depending on when certain decisions are made. Suppose that a fair coin is tossed to decide which of two problems is to be faced. *Mixture linearity* requires that the ex-ante expected payoffs to agents do not depend on whether or not the rule is applied before or after the coin toss. We characterize the class of rules satisfying *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *mixture linearity*. These rules are multi-utilitarian rules in which an additional exogenous transfer is made between the agents.

Green's rules are multi-utilitarian rules which feature a probability measure placing positive probability on exactly two weighted utilitarian rules. The two weighted utilitarian rules are symmetric of each other, and have a probability of one-half. We characterize all the multi-utilitarian rules satisfying *recursive invariance*. We do not require *symmetry*. A multi-utilitarian rule satisfying *recursive invariance* places probability on *at most* two weighted utilitarian rules–one of which favors agent 1, and the other of which favors agent 2. The weighted utilitarian rules need not be symmetric of each other, and they need not be given equal probability. We call such a rule a "bi-utilitarian rule." It is obvious that by adding *symmetry*, we obtain Green's rules.

In order to establish that our generalization is useful, we discuss the axiom of *strong monotonicity*. This axiom states that as a problem becomes more favorable toward agent 1, then the solution should become more favorable toward agent 1. None of the multi-utilitarian rules satisfying *recursive invariance* satisfy *strong monotonicity*. The class of all multi-utilitarian rules satisfying *strong monotonicity* is characterized as the set of multi-utilitarian rules whose associated probability measure has full support–what we call a "full multi-utilitarian rule."

Lastly, we discuss a version of our main theorem which holds in environments for which problems need not be convex. Theorem 2 in Appendix B establishes that in such an environment, a rule is a multi-utilitarian rule if and only if it satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, *mixture linearity*, and *selection of singletons*.

Section 2 introduces the formal model. Section 3 includes the main results and proofs. Section 4 concludes.

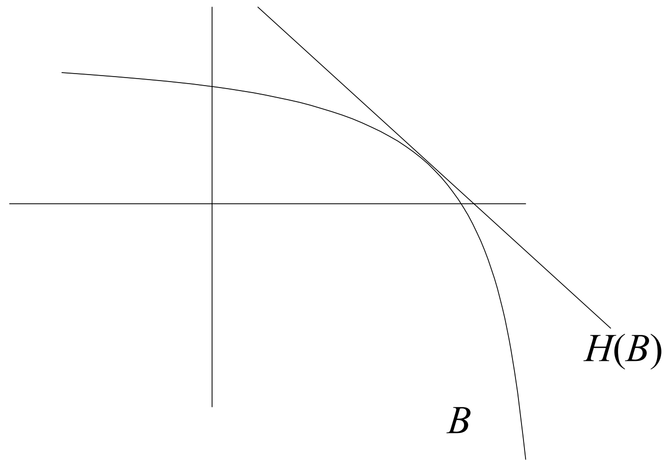

Figure 1: A problem

2 The model

2.1 Preliminaries

Let $N \equiv \{1,2\}$ be a set of **agents**. Say that a subset **B** $\subset\mathbb{R}^2$ **is bounded above** if there exists some $x \in \mathbb{R}^2$ such that $B \subset \{y : y_1 \leq x_1$ and $y_2 \leq x_2\}$. Say it is **comprehensive** if for all $x \in B$, if $y \leq x$, then $y \in B$. (Here inequality is defined pointwise). A **problem** is a nonempty subset of $\mathbb{R}^2$ which is closed, convex, comprehensive, and bounded above. By $\mathcal{B}$, we mean the set of all problems.

Let $\overline{x} : \mathcal{B} \to \mathbb{R}$ be defined as $\overline{x}(B) \equiv \max_{x\in B} x_1 + x_2$. We say x is a **solution** to a problem B if $x_1 + x_2 \leq \overline{x}(B)$. Let H be a function defined on the set of problems which maps to the set of hyperplanes of $\mathbb{R}^2$. Specifically, let $H(B)$ be defined as $H(B) \equiv \{x \in \mathbb{R}^2 : x_1 + x_2 = \overline{x}(B)\}$. Thus, $H(B)$ is the set of efficient points that the agents can achieve by making transfers. Figure 1 illustrates a typical problem.

A **rule** is a function $f : \mathcal{B} \to \mathbb{R}^2$ such that for all $B \in \mathcal{B}$, $f(B)$ is a solution for B. We could conceivably generalize the class of rules to be multi-valued, but for our purposes, single-valued rules will suffice.

2.2 Properties of rules

We discuss several normative properties that rules may satisfy. The first is the standard concept of *efficiency.*

Efficiency: For all $B \in \mathcal{B}$, $f(B) \in H(B)$.

Our next property is to be interpreted as robustness of the rule to the underlying utility specification. Formally, any two problems $B, B' \in \mathcal{B}$ such that $B' = B + x$ for some $x \in \mathbb{R}^2$ can be viewed as arising from the same underlying preferences. Hence, a rule should recommend the same social alternative and transfers in the new problem as in the old problem. But the utility value induced by this solution for the new problem is simply the old utility value, translated by x.

Translation invariance: For all $B \in \mathcal{B}$ and all $x \in \mathbb{R}^2$, $f(B + x) = f(B) + x$.

Our next property states that for problems which are the convex and comprehensive hull of singletons, the rule should select that singleton. This axiom is extremely weak and should be interpreted as saying that when there is a single feasible action, this action should be chosen and transfers should not be made.

Formally, $\mathcal{K}$ is a mapping which takes each set into its convex, comprehensive hull.

Selection of singletons: Let $x \in \mathbb{R}^2$. Then $f(\mathcal{K}(\{x\})) = x$.

Suppose that we are given two problems $B, B' \in \mathcal{B}$. Say that **$\mathbf{B'}$ dominates B for agent 1** if $H(B) = H(B')$ and the following two conditions are satisfied:

$$\begin{aligned} & B \cap \left\{x \in \mathbb{R}^2 : x_1 \geq \sup\{x_1 : x \in B' \cap H(B')\}\right\} \\ \subset \quad & B' \cap \left\{x \in \mathbb{R}^2 : x_1 \geq \sup\{x_1 : x \in B' \cap H(B')\}\right\} \end{aligned}$$

and

$$\begin{aligned} & B' \cap \left\{x \in \mathbb{R}^2 : x_2 \geq \sup\{x_2 : x \in B \cap H(B)\}\right\} \\ \subset \quad & B \cap \left\{x \in \mathbb{R}^2 : x_2 \geq \sup\{x_2 : x \in B \cap H(B)\}\right\}. \end{aligned}$$

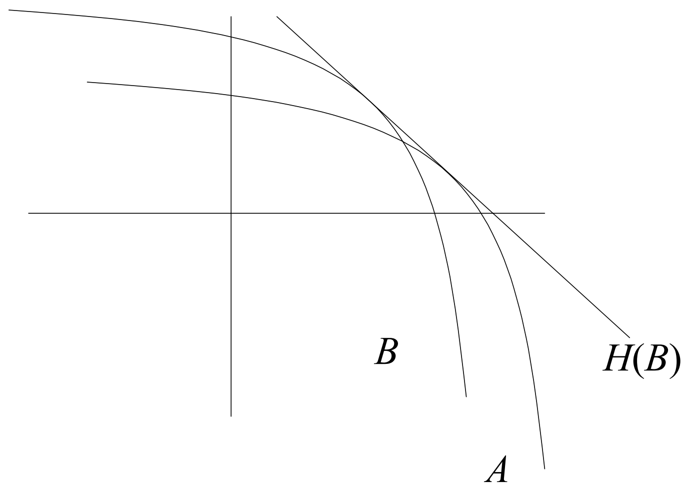

Figure 2: The problem A dominates the problem B for agent 1

A problem B' dominates B for agent 2 if B' gives agent 1 "better possibilities" than B. Note that it says nothing about what happens to the possibilities for agent 2. We could also define a notion of B' dominating B for agent 2; the definition is symmetric. Figure 2 illustrates a scenario in which A dominates the set B for agent 1.

We formulate this condition of domination so that we may discuss a simple *monotonicity* condition. Thus, imagine B' dominates B for agent 1; then it is reasonable to require that agent 1 should benefit from this domination.

Monotonicity: Let $B, B' \in \mathcal{B}$ and suppose that B' dominates B for agent 1. Then $f_1(B) \leq f_1(B')$.

Monotonicity could also be described using the language of set domination for agent 2; such variants are equivalent under *efficiency*. Green [7] introduces *monotonicity*; although his version is weaker.

The next property states that if two problems are "close," then their solutions should be "close." In order to define this, we first define the **Hausdorff extended metric** on the space $\mathcal{C}$ of closed subsets of $\mathbb{R}^2$.[1] Let $d : \mathbb{R}^2 \times \mathbb{R}^2$

[1] For d to be an **extended metric**, the following must be true:

be the Euclidean metric. Define the **distance** $d^* : \mathbb{R}^2 \times \mathcal{C} \to \mathbb{R}_+$ as

$$d^*(x, B) \equiv \inf_{y \in B} d(x, y).$$

Finally, the Hausdorff extended metric, $d_{\text{Haus}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$, is defined as

$$d_{\text{Haus}}(B, B') \equiv \max\left\{\sup_{x \in B'} d^*(x, B), \sup_{x \in B} d^*(x, B')\right\}.$$

We verify that d_{Haus} is a metric when restricted to $\mathcal{B}$.[2]

Proposition 1: The function d_{Haus} is a metric when restricted to $\mathcal{B}$.

Proof: By Lemma 3.57 of Aliprantis and Border, we know that d_{Haus} is an extended metric. Thus, we only need establish that for all $B, B' \in \mathcal{B}$, $d_{\text{Haus}}(B, B') < \infty$. Let $B, B' \in \mathcal{B}$. Let $y(B) \equiv (\sup_{x \in B} x_1, \sup_{x \in B} x_2)$ and $y(B') \equiv (\sup_{x \in B'} x_1, \sup_{x \in B'} x_2)$. We claim that $B \cap B' \neq \varnothing$. Thus, let $x_B \in B$ and let $x_{B'} \in B'$. Then $x_B \wedge x_{B'} \in B \cap B'$, as $x_B \wedge x_{B'} \leq x_B, x_{B'}$, and by comprehensivity of B and B'.[3]

Thus, let $x^* \in B \cap B'$. For all $x \leq x^*$, $x \in B \cap B'$, and hence $d^*(x, B) = 0$ and $d^*(x, B') = 0$.

We claim that there exists $c_1 > 0$ such that for all $x \in B$ such that $x_1 \geq x_1^*$ and $x_2 \leq x_2^*$, $d^*(x, B') \leq c_1$. Thus, let $x \in B$ satisfy the hypotheses. Set $c_1 \equiv y_1(B) - x_1^* > 0$. Then $x_1 \leq y_1(B)$. Moreover, $(x_1^*, x_2) \in B'$ by comprehensivity of B', so that $d((x_1, x_2), (x_1^*, x_2)) = x_1 - x_1^* \leq c_1$.

A similar argument establishes that there exists c_2 such that for all $x \in B'$ such that $x_1 \geq x_1^*$ and $x_2 \leq x_2^*$, $d^*(x, B) \leq c_2$. Moreover, there exists c_3 such that for all $x \in B$ such that $x_2 \geq x_2^*$ and $x_1 \leq x_1^*$, $d^*(x, B') \leq c_3$. Finally, there exists c_4 such that for all $x \in B'$ such that $x_2 \geq x_2^*$ and $x_1 \leq x_1^*$, $d^*(x, B) \leq c_4$.

Lastly, $A \equiv \{x \in B : x \geq x^*\}$ and $A' \equiv \{x \in B' : x \geq x^*\}$ are compact sets; hence, $d_{\text{Haus}}(A, A') < \infty$. By checking the various regions of B and B',

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i) For all $B, B' \in \mathcal{K}$, $d(B, B') \geq 0$ with equality if and only if $B = B'$

ii) For all $B, B' \in \mathcal{K}$, $d(B, B') = d(B', B)$

iii) For all $A, B, C \in \mathcal{K}$, $d(A, C) \leq d(A, B) + d(B, C)$.

The function d is a **metric** if it only takes real values.

[2] We slightly abuse notation by referring to d_{Haus} on $\mathcal{B}$ as d_{Haus}, when it should really be written $d_{\text{Haus}}|_{\mathcal{B}}$.

[3] Here, '$\wedge$' refers to the **meet** of two elements, or the pointwise infimum.

it is readily verified that

$$d_{\text{Haus}}(B, B') \leq \max\left\{c_1, c_2, c_3, c_4, \max_{x \in A} d^*(x, B'), \max_{x \in A'} d^*(x, B)\right\},$$

which is in turn less than or equal to $\max\{c_1, c_2, c_3, c_4, \max_{x\in A} d(x, A'), \max_{x\in A'} d(x, A)\} < \infty$. ■

The space $\mathcal{B}$ is endowed with the topology generated by d_{Haus}. Our next requirement is that a rule is continuous in this topology, called the **Hausdorff topology**.

Continuity: The rule f is continuous in the Hausdorff topology.

Continuity is a property which is very restrictive in this model. As a natural example of a rule satisfying all of our axioms except for *continuity*, let f the rule which selects the midpoint of the optimal efficient point for agent 1 and the optimal efficient point for agent 2.

Lastly, we discuss *additivity*. For all $A, B \in \mathcal{B}$, define $A + B \equiv \{x + y : x \in A, y \in B\}$.[4] Note that $A+B \in \mathcal{B}$. Most authors view *additivity* as a condition which states that a rule is invariant under the sequencing of when problems are faced.

Additivity: For all $A, B \in \mathcal{B}$, $f(A + B) = f(A) + f(B)$.

Under very mild conditions, *additivity* is equivalent to the following weaker condition. We use the *additivity* condition so that the parallels between our work and Green's work are clear.

Mixture linearity: For all $A, B \in \mathcal{B}$, $f\left(\frac{A+B}{2}\right) = \frac{f(A)+f(B)}{2}$.

Mixture linearity is the requirement that a rule should be invariant to "timing effects." Suppose that the two problems A and B are faced with equal probabilities. Such a scenario induces a natural utility possibility set; $\frac{A+B}{2}$. Applying the rule at this ex-ante stage results in a solution of $f\left(\frac{A+B}{2}\right)$. Waiting until after the randomization to solve the problem results in an ex-ante expected solution of $\frac{f(A)+f(B)}{2}$. *Mixture linearity* requires that there is no ex-ante benefit to either agent from either procedure.

[4]The operator '+' is referred to as the **Minkowski sum.**

Proposition 2: If a rule f satisfies $f(\mathcal{K}(\{0\})) = 0$, then it satisfies *additivity* if and only if it satisfies *mixture linearity.*

Proof: Let f be a rule satisfying $f(\mathcal{K}(\{0\})) = 0$. Suppose that f satisfies *additivity*. Let $A, B \in \mathcal{B}$. We will show that $f\left(\frac{A+B}{2}\right) = \frac{f(A)+f(B)}{2}$. By *additivity*, $f\left(\frac{A+B}{2}\right) + f\left(\frac{A+B}{2}\right) = f(A+B)$. By *additivity*, $f(A+B) = f(A) + f(B)$. Thus, $f\left(\frac{A+B}{2}\right) = \frac{f(A)+f(B)}{2}$. Conversely, suppose that f satisfies *mixture linearity*. Let $A, B \in \mathcal{B}$. We will show that $f(A+B) = f(A) + f(B)$. Thus, $f\left(\frac{A+B}{2}\right) = f\left(\frac{(A+B)}{2} + \frac{\mathcal{K}(\{0\})}{2}\right)$. By *mixture linearity*, $f\left(\frac{(A+B)}{2} + \frac{\mathcal{K}(\{0\})}{2}\right) = \frac{f(A+B)+f(\mathcal{K}(\{0\}))}{2}$. By assumption, $\frac{f(A+B)+f(\mathcal{K}(\{0\}))}{2} = \frac{f(A+B)}{2}$. By *mixture linearity*, $f\left(\frac{A+B}{2}\right) = \frac{f(A)+f(B)}{2}$. Hence, $\frac{f(A+B)}{2} = \frac{f(A)+f(B)}{2}$, so that $f(A+B) = f(A) + f(B)$. ■

We establish another connection between our axioms which will be useful for the proof of the main result.

Proposition 3: If a rule satisfies *additivity*, then it satisfies *translation invariance* if and only if it satisfies *selection of singletons.*

Proof: Suppose f is *additive*, and that it satisfies *translation invariance.* Let $x \in \mathbb{R}^2$. Then, by *additivity*, $f(2\mathcal{K}(\{0\})) = 2f(\mathcal{K}(\{0\}))$. But by definition, $2\mathcal{K}(\{0\}) = \mathcal{K}(\{0\})$. Thus, $f(\mathcal{K}(\{0\})) = 2f(\mathcal{K}(\{0\}))$, so that $f(\mathcal{K}(\{0\})) = 0$. Thus, $f(\mathcal{K}(\{x\})) = f(\mathcal{K}(\{0\}) + x)$. By *translation invariance*, $f(\mathcal{K}(\{0\}) + x) = f(\mathcal{K}(\{0\})) + x$. By the preceding statement, $f(\mathcal{K}(\{0\})) + x = x$, so that $f(\mathcal{K}(\{x\})) = x$.

Next, suppose that f satisfies *selection of singletons.* Then for all $B \in \mathcal{B}$ and all $x \in \mathbb{R}^2$, $f(B + x) = f(B + \mathcal{K}(\{x\}))$. By *additivity*, $f(B + \mathcal{K}(\{x\})) = f(B) + f(\mathcal{K}(\{x\}))$. By *selection of singletons*, $f(\mathcal{K}(\{x\})) = x$. Thus, $f(B + x) = f(B) + x$. ■

2.3 Multi-utilitarianism

We now define the set of rules which will be the focus of our study. Let $\lambda \in [0, 1]$. For $\lambda \neq 1/2$, define the **$\boldsymbol{\lambda}$-utilitarian rule** $U^\lambda : \mathcal{B} \to \mathbb{R}^2$ as follows: for all $B \in \mathcal{B}$,

$$U^\lambda(B) \equiv \left\{ y \in H(B) : \lambda y_1 + (1 - \lambda) y_2 = \sup_{x \in B} \lambda x_1 + (1 - \lambda) x_2 \right\}.$$

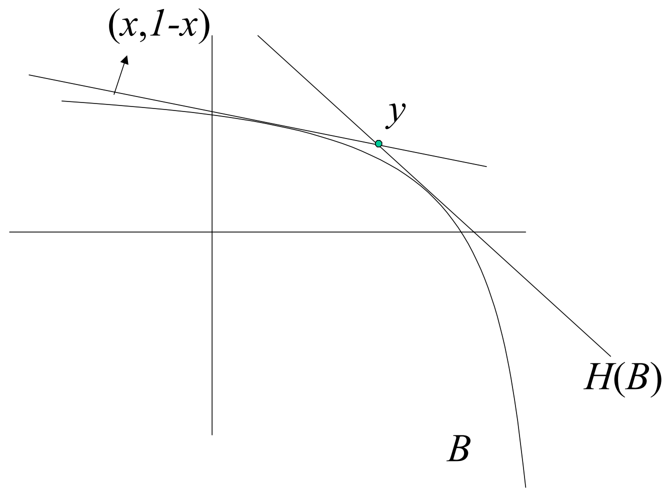

Figure 3: A weighted utilitarian rule

Call this class of rules the **weighted utilitarian rules**. Clearly, a rule corresponding to $\lambda = 1/2$ is not well-defined. In standard normative economics, the weighted utilitarian rules select a feasible alternative which maximizes a weighted sum of agents' utilities. Here, such a rule is generally not *efficient*. Thus, a weighted utilitarian rule specifies an efficient transfer which gives the same aggregate weighted utility as the maximal aggregate weighted utility which is feasible before transfers. Figure 3 illustrates a typical weighted utilitarian rule, here the point $y \equiv U^x(B)$.

A distinguishing feature of the quasilinear model is that the set of efficient solutions is convex for all problems. This feature allows us to construct many efficient rules out of old rules. Thus, let ν be a probability measure on the measurable space $([0,1] \setminus \{1/2\}, \Sigma)$, where Σ are the Borel sets restricted to $[0,1] \setminus \{1/2\}$. Define the **$\boldsymbol{\nu}$-utilitarian rule** $U^\nu : \mathcal{B} \to \mathbb{R}^2$ as follows: for all $B \in \mathcal{B}$,

$$U^\nu(B) \equiv \int_{[0,1]\setminus\{1/2\}} U^\lambda(B)\, d\nu(\lambda).$$

Call this class of rules the **multi-utilitarian rules**.

3 Results

3.1 The main result

The main result states that a rule satisfies the five properties if and only if it is a multi-utilitarian rule.

Theorem 1: A rule satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *additivity* if and only if it is a multi-utilitarian rule.

Theorem 1 is tight; we provide the independence of the axioms in Appendix A.

Theorem 1 is stated for an environment in which the domain of a rule is $\mathcal{B}$; however, there are many subclasses of $\mathcal{B}$ for which the theorem still holds. Call a problem $B \in \mathcal{B}$ **compactly generated** if there exists a compact set K such that $B = \mathcal{K}(K)$. The set of compactly generated problems is written $\mathcal{B}^K$. Theorem 1 holds if the axioms are imposed on the restricted set of problems $\mathcal{B}^K$. Another example of an important domain for which Theorem 1 holds is the class of **polytope problems**, which are those problems for which there exists a finite set $\{x_1, ..., x_m\}$ such that $B = \mathcal{K}(\{x_1, ..., x_m\})$.

A version of Theorem 1 holds for even larger domains of problems. On the domain of problems which are convex, closed, and bounded above, but not necessarily convex, the multi-utilitarian rules are characterized by *efficiency*, *translation invariance*, *monotonicity*, *continuity*, *mixture linearity*, and *selection of singletons*. Nonconvex environments are discussed formally in Appendix B, where the preceding result is proved (Theorem 2).

3.2 A discussion of proof strategy

By *translation invariance* and *continuity*, we may restrict ourselves to the class of problems whose efficient set consists of points whose aggregate utility is zero *and* which have a unique efficient utility pair. We can embed any such problem into the space of continuous functions on $[0, 1]$. Specifically, each such problem is *uniquely determined* by the solutions recommended for each of the weighted utilitarian rules. Thus, for a given problem B, $U^\lambda(B)$ is a continuous function in λ (with the value for $\lambda = 1/2$ given by $H(B) \cap B$). We work on the space of continuous functions which are induced by problems. We define an induced "rule" on this space of functions. It is easily verified that this rule is additive and monotonic (and hence continuous in the sup-norm topology). The rule may be extended to the linear hull of this class of functions, preserving additivity and monotonicity. By using a lattice-theoretic version of the Hahn-Banach theorem, we extend this monotonic functional to the entire space of continuous functions on $[0, 1]$ to obtain a monotonic linear functional. Applying the Riesz representation theorem, we conclude that the rule on the space of continuous functions is represented by integration with respect to a measure. *Monotonicity* guarantees that the measure is positive, and *translation invariance* guarantees that it assigns measure one to $[0, 1]$; hence it is a probability measure. Translating back into the space of problems results in a multi-utilitarian rule on the restricted class of problems. We then show how to extend the characterization on this restricted class to the class of all problems. A similar proof strategy is used in the work of Dekel, Lipman, and Rustichini [4], in the context of choice with unforeseen contingencies.

3.3 Proof of Theorem 1

We will not prove that a multi-utilitarian rule satisfies the axioms; this can be easily verified. The opposite direction is proved below. We note that all of the steps are equally valid on the subdomains mentioned above.

Proof: Step 1: Establishing the homogeneity of f.

We claim that for all $B \in \mathcal{B}$ and all $\alpha \geq 0$, $f(\alpha B) = \alpha f(B)$. Let $\alpha \in \mathbb{Q}$. Then $\alpha = \frac{m}{n}$ for some $m, n \in \mathbb{N}$. Let $x = f\left(\frac{1}{n}B\right)$. Then by *additivity*, as $B = n\left(\frac{1}{n}B\right)$, $nx = f(B)$. Therefore, $f\left(\frac{1}{n}B\right) = \frac{1}{n}f(B)$. By *additivity*, $f\left(\frac{m}{n}B\right) = \frac{m}{n}f(B)$, so that $f(\alpha B) = \alpha f(B)$. The result obtains by the density of the rationals and *continuity*.

Step 2: Embedding rules and problems into the space of continuous functions, and establishing properties on the induced functional.

Define $\mathcal{B}^* \subset \mathcal{B}$ to be the class of problems which have a single efficient alternative. Formally, $B \in \mathcal{B}^*$ if $H(B) \cap B$ is a singleton. Let $\mathcal{B}^0 \subset \mathcal{B}^*$ so that $B \in \mathcal{B}^0$ if $\overline{x}(B) = 0$. Most of the work of the proof is done in $\mathcal{B}^0$.

Let $C([0, 1])$ be the class of continuous, real-valued functions defined on the unit interval endowed with the sup-norm topology.

Define the function $\sigma : \mathcal{B}^0 \to C([0, 1])$ by $\sigma(B)(\lambda) \equiv U_1^\lambda(B)$. It is easily verified that for all B, $\sigma(B)$ is a continuous function on $[0, 1]$.

Moreover, σ is one-to-one between $\mathcal{B}^0$ and $C([0,1])$. It is a simple exercise to verify that for all $B, B' \in \mathcal{B}^0$, $\sigma(B+B') = \sigma(B) + \sigma(B')$. Further, for all $\alpha \geq 0$ and all $B \in \mathcal{B}^0$, $\sigma(\alpha B) = \alpha\sigma(B)$. (These properties are easily verified, similarly to Rockafellar [9], 16.4).

Moreover, B' dominates B for agent 1 if and only if $\sigma(B') \geq \sigma(B)$. To see this, let $B, B' \in \mathcal{B}^0$ such that B' dominates B for agent 1. It is obvious by definition that $\sigma(B')(1/2) \geq \sigma(B)(1/2)$. We will show that for all $\lambda \in [0,1/2)$, $\sigma(B')(\lambda) \geq \sigma(B)(\lambda)$. A symmetric argument will establish the proof for $\lambda \in (1/2,1]$. Thus, let $\lambda \in [0,1/2)$. Let

$$\begin{aligned} A' &\equiv B' \cap \left\{x \in \mathbb{R}^2 : x_2 \geq \sup\{x_2 : x \in B \cap H(B)\}\right\} \\ A &\equiv B \cap \left\{x \in \mathbb{R}^2 : x_2 \geq \sup\{x_2 : x \in B \cap H(B)\}\right\}. \end{aligned}$$

As B' dominates B for agent 1, $A' \subset A$. Defining $\sup_{x\in\varnothing} f(x)$ as $-\infty$, as $A' \subset A$,

$$\sup_{x\in A'} \lambda x_1 + (1-\lambda)x_2 \leq \sup_{x\in A} \lambda x_1 + (1-\lambda)x_2.$$

Let $x^* \equiv x \in H(B)$ such that $x_2^* = \sup\{x_2 : x \in B \cap H(B)\}$. We claim that for all $x \in B'\backslash A'$, $\lambda x_1 + (1-\lambda)x_2 \leq \lambda x_1^* + (1-\lambda)x_2^*$. Suppose that this statement is false, so that there exists $x \in B'\backslash A'$ such that $\lambda x_1 + (1-\lambda)x_2 > \lambda x_1^* + (1-\lambda)x_2^*$. As $x \notin A'$, $x_2 \leq x_2^*$. Thus,

$$\lambda(x_1 - x_1^*) > (1-\lambda)(x_2^* - x_2),$$

where the right hand side is nonnegative, as $\lambda < 1/2$ and $x_2 \leq x_2^*$. Conclude that $\lambda(x_1 - x_1^*) > 0$; in particular, $\lambda > 0$, so that

$$x_1 - x_1^* > \left(\frac{1-\lambda}{\lambda}\right)(x_2^* - x_2).$$

But $\left(\frac{1-\lambda}{\lambda}\right) \geq 1$, so that

$$x_1 - x_1^* > x_2^* - x_2,$$

and

$$x_1 + x_2 > x_1^* + x_2^*,$$

contradicting the fact that $H(B) = H(B')$. It is then clear that

$$\sup_{x\in B'} \lambda x_1 + (1-\lambda)x_2 \leq \max\left\{\sup_{x\in A} \lambda x_1 + (1-\lambda)x_2, \lambda x_1^* + (1-\lambda)x_2^*\right\},$$

from which we conclude (using the fact that $A \subset B$ and $x^* \in B$)

$$\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 \leq \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2.$$

Next, we claim that for all $\lambda < 1/2$, $U_1^\lambda(B) \leq U_1^\lambda(B')$ if and only if $\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 \leq \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$. To this end, we need only establish that $U_1^\lambda(B) \leq U_1^\lambda(B')$ implies $\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 \leq \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$ and that $U_1^\lambda(B) < U_1^\lambda(B')$ implies $\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 < \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$. We will show the first statement; the second follows by replacing the appropriate weak inequalities by strict inequalities. By definition,

$$\lambda U_1^\lambda(B') + (1-\lambda)\left(-U_1^\lambda(B')\right) = \sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2$$

and

$$\lambda U_1^\lambda(B) + (1-\lambda)\left(-U_1^\lambda(B)\right) = \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2.$$

As $\lambda < 1/2$, $\lambda\left(U_1^\lambda(B') - U_1^\lambda(B)\right) \leq (1-\lambda)\left(U_1^\lambda(B') - U_1^\lambda(B)\right)$. Hence $\lambda U_1^\lambda(B') + (1-\lambda)\left(-U_1^\lambda(B')\right) \leq \lambda U_1^\lambda(B) + (1-\lambda)\left(-U_1^\lambda(B)\right)$.

Thus, for all $\lambda < 1/2$, we conclude that $\sigma(B)(\lambda) \leq \sigma(B')(\lambda)$. The argument for $\lambda > 1/2$ is symmetric. Thus, if B' dominates B for agent 1, then $\sigma(B) \leq \sigma(B')$.

Next, we show that if $\sigma(B) \leq \sigma(B')$, then B' dominates B for agent 1. For all $\lambda \neq 1/2$, by the argument above, if $\sigma(B) \leq \sigma(B')$, we conclude $\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 \leq \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$. Suppose, by means of contradiction that B' does not dominate B for agent 1. Without loss of generality, that there exists some $y^* \in B'$ such that $y_2^* \geq x_2^*$ and $y^* \notin B$. By a version of the Separating Hyperplane Theorem (Corollary 5.59 of Aliprantis and Border [1]), there exists a pair $(\lambda, 1-\lambda) \in \mathbb{R}^2$ such that $\lambda y_1^* + (1-\lambda) y_2^* > \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$. We claim that $\lambda \in [0, 1/2]$. Clearly, $\lambda \in [0,1]$, or else the supremum over B does not exist. So, suppose that $\lambda > 1/2$. In particular, $\lambda y_1^* + (1-\lambda) y_2^* > \lambda x_1^* + (1-\lambda) x_2^*$, from which we conclude that $\lambda(y_1^* - x_1^*) > (1-\lambda)(x_2^* - y_2^*)$. As $\lambda > 0$, this implies that $y_1^* - x_1^* > \left(\frac{1-\lambda}{\lambda}\right)(x_2^* - y_2^*)$. Next, as $y_2^* > x_2^*$, and as $\left(\frac{1-\lambda}{\lambda}\right) < 1$, we conclude $\left(\frac{1-\lambda}{\lambda}\right)(x_2^* - y_2^*) > x_2^* - y_2^*$. Hence $y_1^* - x_1^* > x_2^* - y_2^*$, or $y_1^* + y_2^* > x_1^* + x_2^*$, contradicting $H(B) = H(B')$. Thus, $\lambda \in [0, 1/2]$. If $\lambda = 1/2$, $\sup_{x \in B'} \lambda x_1 + (1-\lambda) x_2 \geq \lambda y_1^* + (1-\lambda) y_2^* > \sup_{x \in B} \lambda x_1 + (1-\lambda) x_2$ contradicts $H(B) =$

$H(B')$. So $\lambda < 1/2$. But then $\sup_{x\in B'} \lambda x_1 + (1-\lambda) x_2 \geq \lambda y_1^* + (1-\lambda) y_2^* > \sup_{x\in B} \lambda x_1 + (1-\lambda) x_2$, so that $\sigma(B')(\lambda) > \sigma(B)(\lambda)$, a contradiction.

Define $T : \sigma(\mathcal{B}^0) \to \mathbb{R}$ as $T(\sigma(B)) \equiv f_1(B)$. The function T is then additive and homogeneous on $\sigma(\mathcal{B}^0)$. As f is *monotonic*, then by the preceding statements, T is monotonic, and hence continuous in the sup-norm topology.

Step 3: Extending the linear functional to the space of continuous functions.

We extend T to a vector subspace of $C([0,1])$. Thus, let $\mathcal{H} \equiv \{g-h : g,h \in \sigma(\mathcal{B}^0)\}$. Clearly, $\mathcal{H}$ is now a vector subspace. Define $T^* : \mathcal{H} \to \mathbb{R}$ by $T^*(g-h) = T(g) - T(h)$. We claim that T^* is well-defined, linear, and continuous. To see that it is well-defined, suppose that $g-h \in \mathcal{H}$ can be written as $g-h = g'-h'$. Thus, $g+h' = g'+h$. We conclude that $T(g+h') = T(g'+h)$; moreover, by additivity of T, $T(g+h') = T(g) + T(h')$ and $T(g'+h) = T(g') + T(h)$. Therefore, $T(g) + T(h') = T(g') + T(h)$. Hence, $T(g) - T(h) = T(g') - T(h')$. Therefore, T^* is well-defined. As T is linear, so is T^*. As T is *monotonic*, T^* is monotonic. To see this, suppose that $g-h \geq 0$. Then $T^*(g-h) = T(g) - T(h)$. Since $g \geq h$, and as T is monotonic, $T(g) - T(h) \geq 0$. Hence T^* is monotonic, and hence continuous.

We extend T^* to all of $C([0,1])$. We can extend T^* to all of $C([0,1])$ so that the extension is monotonic (Corollary III.9.12 of Conway [3], using the fact that $1 \in \mathcal{H}$, where $1 = \sigma(\mathcal{K}(\{(1,-1)\}))$). We refer to this continuous linear extension as T^{**}.

Step 4: Obtaining the measure representation of the rule for a restricted class of problems.

By the Riesz representation theorem (for example, see Corollary 13.15 of Aliprantis and Border [1]), there exists a countably additive measure ν on $([0,1], \Sigma)$ such that

$$T^{**}(f) \equiv \int_{[0,1]} f(\lambda)\, d\nu(\lambda).$$

Further, ν is positive if T^{**} is monotonic.

We claim that for all $c \in \mathbb{R}$, $T^{**}(c) = c$.[5] It is clear by definition that $\sigma(\mathcal{K}(\{(c,-c)\}))$ is the constant function c. Moreover, we know by *selection*

[5] We abuse notation in a standard way by identifying a constant function with the value that constant function takes.

of singletons that $f(\mathcal{K}(\{(c,-c)\}))=(c,-c)$. Thus, by definition of T, $T(c)=c$ and hence $T^{**}(c)=c$. As for all constant functions c, $T^{**}(c)=c$, we conclude that $T^{**}(c)=\nu([0,1])\,c=c$, so that $\nu([0,1])=1$.

By definition of T, for all $B\in\mathcal{B}^0$, $f(B)=(T(\sigma(B)),-T(\sigma(B)))=\left(\int_{[0,1]}\sigma(B)(\lambda)\,d\nu(\lambda),-\int_{[0,1]}\sigma(B)(\lambda)\,d\nu(\lambda)\right)$. Rewriting,

$$f(B)=\int_{[0,1]}(\sigma(B)(\lambda),-\sigma(B)(\lambda))\,d\nu(\lambda).$$

For all λ, $(\sigma(B)(\lambda),-\sigma(B)(\lambda))=U^{\lambda}(B)$. Thus,

$$f(B)=\int_{[0,1]}U^{\lambda}(B)\,d\nu(\lambda).$$

We show that this formula holds for all $B\in\mathcal{B}^*$. Let $B\in\mathcal{B}^*$ and let x satisfy $B+x\in\mathcal{B}^0$.[6] Then $f(B+x)=\int_{[0,1]}U^{\lambda}(B+x)\,d\nu(\lambda)$. For all λ, U^{λ} is *translation invariant* (it can easily be shown to hold for $\lambda=1/2$), so that $U^{\lambda}(B+x)=U^{\lambda}(B)+x$. Hence $f(B+x)=\int_{[0,1]}\left(U^{\lambda}(B)+x\right)d\nu(\lambda)$. As $\nu([0,1])=1$, the preceding is equal to $\int_{[0,1]}U^{\lambda}(B)\,d\nu(\lambda)+x$. By *translation invariance* of f, $f(B+x)=f(B)+x$. Hence, $f(B)+x=\int_{[0,1]}U^{\lambda}(B)\,d\nu(\lambda)+x$, so that $f(B)=\int_{[0,1]}U^{\lambda}(B)\,d\nu(\lambda)$.

Step 5: Verifying that $1/2$ has measure zero, and completing the characterization.

We extend the representation to all of $\mathcal{B}$. First, we establish that $\nu(\{1/2\})=0$. Let $\{B_n\}$,$\{B'_n\}\subset\mathcal{B}$ be the following sequences of problems: for all n, $B_n\equiv\mathcal{K}\left(\left\{\left(1-\frac{1}{n},-1\right),(0,0)\right\}\right)$ and $B'_n\equiv\mathcal{K}\left(\left\{(1,-1),\left(0,-\frac{1}{n}\right)\right\}\right)$. Then, note that each of B_n and B'_n converge to $\mathcal{K}(\{(1,-1),(0,0)\})$ in the Hausdorff topology. Thus, by *continuity*, $\lim_{n\to\infty}f(B_n)=\lim_{n\to\infty}f(B'_n)$. In particular, we can identify each B_n and B'_n with its induced continuous function, $\sigma(B_n)$ and $\sigma(B'_n)$. It is simple to verify that the sequence $\sigma(B_n)$ converges pointwise to

$$F(\lambda)\equiv\begin{cases}0\text{ for }\lambda\leq 1/2\\ 1\text{ for }\lambda>1/2\end{cases}$$

and that the sequence $\sigma(B'_n)$ converges pointwise to

$$F'(\lambda)\equiv\begin{cases}0\text{ for }\lambda<1/2\\ 1\text{ for }\lambda\geq 1/2\end{cases}.$$

[6] For example, let $x=(-\overline{x}(B),0)$.

In particular,

$$\lim_{n\to\infty}\int_{[0,1]}\sigma\left(B_n\right)(\lambda)\,d\nu\left(\lambda\right)=\lim_{n\to\infty}\int_{[0,1]}\sigma\left(B_n'\right)(\lambda)\,d\nu\left(\lambda\right).$$

Each of $\sigma\left(B_n\right)$ and $\sigma\left(B_n'\right)$ are bounded sequences; hence, we may apply the Lebesgue dominated convergence theorem (for example, see Theorem 11.20 of Aliprantis and Border [1]). We conclude

$$\int_{[0,1]}F\left(\lambda\right)d\nu\left(\lambda\right)=\int_{[0,1]}F'\left(\lambda\right)d\nu\left(\lambda\right).$$

Moreover, $\int_{[0,1]}F\left(\lambda\right)d\nu\left(\lambda\right)=\nu\left(\left(1/2,1\right]\right)$ and $\int_{[0,1]}F'\left(\lambda\right)d\nu\left(\lambda\right)=\nu\left(\left[1/2,1\right]\right)$. Thus $\nu\left(\left(1/2,1\right]\right)=\nu\left(\left[1/2,1\right]\right)$, or $\nu\left(\{1/2\}\right)=0$.

As the set $\mathcal{B}^*$ is dense in $\mathcal{B}$ in the Hausdorff topology, we establish that for all $B\in\mathcal{B}$,

$$f\left(B\right)=\int_{[0,1]}U^{\lambda}\left(B\right)d\nu\left(\lambda\right),$$

independently of how $U^{1/2}$ is defined. We may thus write

$$f\left(B\right)=\int_{[0,1]\setminus\{1/2\}}U^{\lambda}\left(B\right)d\nu\left(\lambda\right).$$

■

3.4 A characterization on the basis of *mixture linearity*

We establish a characterization of a class of rules based on *mixture linearity.* Fix a multi-utilitarian rule, U^{ν}. Fix some exogenous transfer from agent 2 to agent 1, say, c. For any problem B, such a rule recommends whatever is recommended by U^{ν}, plus the transfer from agent 2 to agent 1.

Corollary 1: A rule f satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *mixture linearity* if and only if there exist $c\in\mathbb{R}$ and a multi-utilitarian rule U^{ν} such that for all $B\in\mathcal{B}$, $f\left(B\right)=\left(c,-c\right)+U^{\nu}\left(B\right)$.

Proof: It is simple to verify that any such rule satisfies the axioms.

Conversely, let f be a rule satisfying the axioms. By *efficiency*, the sum of the elements of $f(\mathcal{K}(\{0\}))$ is equal to zero. Thus, let $(c, -c) \equiv f(\mathcal{K}(\{0\}))$.

Let f^* be the rule defined so that for all $B \in \mathcal{B}$, $f^*(B) \equiv f(B) - (c, -c)$. Then it is trivial to verify that f^* satisfies the axioms listed in the hypothesis of the corollary. Moreover, $f^*(\mathcal{K}(\{0\})) = 0$. Thus, by Proposition 2, we may conclude that f^* is *additive*. Hence f^* is a multi-utilitarian rule, say U^ν. Therefore, $f(B) \equiv (c, -c) + U^\nu(B)$. ■

3.5 On Green's Theorem and *recursive invariance*

Green's theorem invokes all of the axioms we discuss (except for a weaker version of *monotonicity*), in addition to the following. It states that if a solution for a problem is determined by a rule, then adding this solution to the utility possibilities set should not change the solution selected by the rule.

Recursive invariance: For all $B \in \mathcal{B}$, $f(\mathcal{K}(B \cup \{f(B)\})) = f(B)$.

Green attributes this axiom to Chun [2], although Chun never actually uses it in any characterization. Together with a basic *symmetry* condition, Green characterizes a one-parameter subset of the multi-utilitarian rules, which for lack of better terminology, we call class $\mathcal{G}$. Members of $\mathcal{G}$ are described as follows. Fix a parameter $\lambda \in [0, 1/2)$. Let ν^λ be the probability measure such that $\nu^\lambda(\{\lambda\}) = \nu^\lambda(\{1-\lambda\}) = 1/2$. The elements of $\mathcal{G}$ are the multi-utilitarian rules corresponding to such probability measures.

We will show how to derive a result related to Green's from ours as a corollary, without using the *symmetry* axiom. The proof of Green's main result relies on a beautiful functional equations argument. Here, our argument is primarily measure-theoretic.

Define a generalization of Green's rules as follows. Say a multi-utilitarian rule f is a **bi-utilitarian rule** if its associated probability measure ν has a support of *at most* two elements, one of which lies above $1/2$ and the other of which lies below $1/2$ (recall that the **support** of a probability measure is the intersection of all closed sets having probability one).[7] A bi-utilitarian rule

[7] An important feature of the definition of a bi-utilitarian rule is that there can be at most one weighted utilitarian rule favoring agent 1 and at most one favoring agent 2. In other words, it cannot put support on two weighted utilitarian rules, each of which favor agent 1. This is not indicated in the simple terminology "bi-utilitarian."

need not place equal probability on each of the two weighted utilitarian rules with which it is associated. In fact, it might place positive probability on only one weighted utilitarian rule; such a rule is itself a weighted utilitarian rule. It is clear that requiring *symmetry* will force a bi-utilitarian rule to be an element of $\mathcal{G}$.

Corollary 2: A rule satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, *additivity*, and *recursive invariance* if and only if it is a bi-utilitarian rule.

Proof: To show that a bi-utilitarian rule satisfies the axioms is simple. Conversely, suppose f is a rule satisfying the six axioms. By means of contradiction, suppose that f is not a bi-utilitarian rule. By Theorem 1, f is a multi-utilitarian rule. Let ν be the probability measure associated with f. As f is not a bi-utilitarian rule, we may assume without loss of generality that the support of ν contains more than one element which is greater than 1/2.

We will show that the support of ν contains at most one point greater than 1/2. To this end, suppose by means of contradiction that it contains at least two. Let $\lambda^* > 1/2$ be an element in the support which is strictly less than the supremal element.

We now construct a problem which is the intersection of the hyperplanes in the directions $(\lambda^*, 1-\lambda^*)$, $(1,0)$, and $(0,1)$. Thus, let $B \equiv \{x \in \mathbb{R}^2 : \lambda_1^* x_1 + (1-\lambda_1^*)\, x_2 \leq 0\} \cap \{x \in \mathbb{R}^2 : x_1 \leq 1\} \cap \{x \in \mathbb{R}^2 : x_2 \leq 0\}$. Then for all $\lambda \leq \lambda^*$, (including $\lambda < 1/2$), $U^\lambda(B) = 0$, and for all $\lambda > \lambda^*$, $U_1^\lambda(B) > 0$. As the support of ν contains points which are greater than λ^* (as λ^* was less than the supremal element), we conclude that $U_1^\nu(B) > 0$. In fact, there exists some $\lambda' > \lambda^*$ such that for all $\lambda \in (1/2, \lambda')$, $U_1^\lambda(\mathcal{K}(B \cup \{U^\nu(B)\})) = U_1^\nu(B) > U_1^\lambda(B)$. For all other λ, $U^\lambda(\mathcal{K}(B \cup \{U^\nu(B)\})) = U^\lambda(B)$. The set $(1/2, \lambda')$ has positive measure according to ν, so that this implies $U_1^\nu(\mathcal{K}(B \cup \{U^\nu(B)\})) > U_1^\nu(B)$, contradicting *recursive invariance.* ■

3.6 On *strongly monotonic* multi-utilitarian rules

A natural question is whether or not there are interesting multi-utilitarian rules which do not belong to $\mathcal{G}$. The following axiom, which is a strengthening of *monotonicity*, is violated by all members of $\mathcal{G}$.

Strong monotonicity: Let $B, B' \in \mathcal{B}$ and suppose that B' dominates B for agent 1 and that $B \neq B'$. Then $f_1(B) < f_1(B')$.

Multi-utilitarian rules satisfying *strong monotonicity* exist, and in fact a characterization of this family is possible. We demonstrate that a multi-utilitarian rule satisfies *strong monotonicity* if and only if the associated probability measure has a support of $[0,1] \setminus \{1/2\}$ in the relative topology on $[0,1] \setminus \{1/2\}$. Say a rule is a **full multi-utilitarian rule** if it is a multi-utilitarian rule whose associated probability measure has full support.

Corollary 3: A rule satisfies *efficiency*, *translation invariance*, *strong monotonicity*, *continuity*, and *additivity* if and only if it is a full multi-utilitarian rule.

Proof: It is simple to show that a full multi-utilitarian rule satisfies *strong monotonicity*.

We prove the other direction. Let f be a rule satisfying the five axioms listed in the theorem. *Strong monotonicity* implies *monotonicity*, so by Theorem 1, f is a multi-utilitarian rule. Let ν be the probability measure associated with f. It is enough to show that for all open intervals (λ_1, λ_2) with $\lambda_1 < \lambda_2$, which do not include $1/2$, $\nu((\lambda_1, \lambda_2)) > 0$. Let (λ_1, λ_2) be such an interval, and without loss of generality, suppose that $\lambda_1 > 1/2$.

We construct two problems, B' and B. Let

$$\begin{aligned} B' \equiv & \left\{x \in \mathbb{R}^2 : \lambda_1 x_1 + (1-\lambda_1) x_2 \leq 0\right\} \cap \left\{x \in \mathbb{R}^2 : \lambda_2 x_1 + (1-\lambda_2) x_2 \leq 0\right\} \\ & \cap \left\{x \in \mathbb{R}^2 : x_1 \leq 1\right\} \cap \left\{x \in \mathbb{R}^2 : x_2 \leq 1\right\}. \end{aligned}$$

Let $B \equiv \mathcal{K}\left(\left\{\left(1, \frac{\lambda_1 - 1}{\lambda_1}\right), \left(\frac{\lambda_2 - 1}{\lambda_2}, 1\right)\right\}\right)$. The problem B' is the convex, comprehensive hull of B with the origin. The important point (which is also easily verified) is that B' dominates B for agent 1 and for all $\lambda \notin (\lambda_1, \lambda_2)$, $U^\lambda(B) = U^\lambda(B')$. By *strong monotonicity*, $f_1(B') > f_1(B)$. Thus, by definition of f,

$$\int_{[0,1]\setminus\{1/2\}} U_1^\lambda(B')\, d\nu(\lambda) > \int_{[0,1]\setminus\{1/2\}} U_1^\lambda(B)\, d\nu(\lambda).$$

Rewriting,

$$\int_{[0,1]\setminus\{1/2\}} U_1^\lambda(B') - U_1^\lambda(B)\, d\nu(\lambda) > 0,$$

and as for all $\lambda \notin (\lambda_1, \lambda_2)$, $U^\lambda(B) = U^\lambda(B')$, conclude

$$\int_{(\lambda_1,\lambda_2)} U_1^\lambda(B') - U_1^\lambda(B)\, d\nu(\lambda) > 0,$$

establishing that $\nu((\lambda_1, \lambda_2)) > 0$. Thus f is a full multi-utilitarian rule. ■

4 Conclusion

A last point that bears mentioning: Green discusses a strengthening of *continuity* which involves the "bounded convergence topology." A sequence $\{B_n\}$ converges to B in the bounded convergence topology if and only if for all compact sets K, $K \cap B_n$ converges to $K \cap B$. *Continuity* with respect to the bounded convergence topology is stronger than *continuity* with respect to the Hausdorff topology; for example, the sequence $\mathcal{K}(\{(0,0),(1,-n)\})$ converges to $\mathcal{K}(\{(0,0)\})$ as $n \to \infty$ in the bounded convergence topology, but not in the Hausdorff topology. Strengthening *continuity* in Theorem 1 in this sense results in the additional implication that $\nu(\{0\}) = \nu(\{1\}) = 0$.

Extending the families of rules characterized in this work to environments involving many agents is the subject of ongoing research.

5 Appendix A: On the independence of the axioms in Theorem 1

In this Appendix, for each axiom used in the characterization provided in Theorem 1, we provide an example of a rule (or family of rules) which violates this axiom, yet satisfies the remaining axioms of Theorem 1.

Example 1: A rule that satisfies *translation invariance*, *monotonicity*, *continuity*, and *additivity* but not *efficiency*. Let $f(B) \equiv U^0(B) \wedge U^1(B)$. As is stated in the text, *monotonicity* has a parallel statement for agent 2 (which is equivalent to the original statement of *monotonicity* under the *efficiency* axiom). The rule f satisfies the alternative version of *monotonicity*.

Example 2: A rule that satisfies *efficiency*, *monotonicity*, *continuity*, and *additivity* but not *translation invariance*. Let $f(B)$ be the point of equal coordinates on $H(B)$. Thus, $x = f(B)$ if and only if $x_1 = x_2$ and $x \in H(B)$.

Example 3: A rule that satisfies *efficiency*, *translation invariance*, *continuity*, and *additivity* but not *monotonicity*. This class of rules can be characterized (by strengthening *continuity* to Lipschitz continuity), and we will call them the **generalized multi-utilitarian rules**. Formally, let ν be a countably additive signed measure of bounded variation on $([0,1] \setminus \{1/2\}, \Sigma)$ satisfying $\nu([0,1] \setminus \{1/2\}) = 1$. Define U^{ν}such that for all $B \in \mathcal{B}$,

$$U^{\nu}(B) \equiv \int_{[0,1]\setminus\{1/2\}} U^{\lambda}(B)\, d\nu(\lambda).$$

Example 4: A rule that satisfies *efficiency*, *translation invariance*, *monotonicity*, *additivity* but not *continuity*. An example of this type was already provided in the text. For another example, give agent 1 his supremal utility in the efficient set:

$$f(B) \equiv \left\{x \in \overline{H(B)} : x_1 \geq y_1 \text{ for all } y \in H(B)\right\}.$$

Example 5: A rule that satisfies *efficiency*, *translation invariance*, *monotonicity*, and *continuity* but not *additivity*. This is perhaps the easiest example to think of. A simple example, inspired by the decision theory literature (see Gilboa and Schmeidler [5]) is the following. Let Π be a convex and weak* compact set of Borel probability measures over $[0,1] \setminus \{1,2\}$. Define

$$f(B) \equiv \left(\min_{p\in\Pi} \int_{[0,1]\setminus\{1/2\}} U_1^{\lambda}(B)\, dp(\lambda), \overline{x}(B) - \min_{p\in\Pi} \int_{[0,1]\setminus\{1/2\}} U_1^{\lambda}(B)\, dp(\lambda)\right).$$

Then this rule satisfies all of the axioms but *additivity*.

6 Appendix B: Nonconvexities

The analysis above relies on the convex structure of problems. However, we can think of many instances where problems need not be convex. It is therefore of interest that a version of Theorem 1 still holds when the class of problems is expanded to cover this possibility.

Formally, define a **generalized problem** as a subset of $\mathbb{R}^2$ which is closed, comprehensive and bounded above. By $\mathcal{B}^G$, we mean the class of generalized problems. Note that $\mathcal{B} \subset \mathcal{B}^G$. All of the axioms and definitions discussed in the main body of the text are meaningful as stated. The main point of this Appendix is that Corollary 1 to Theorem 1 is also meaningful.

The proof will rely on the following Lemma and Proposition. It is important to note that for all $B, B' \in \mathcal{B}^G$, $B + B' \in \mathcal{B}^G$. Let $\mathcal{C}$ be a mapping which takes each set into its comprehensive hull.

Lemma: Let K, K' be compact sets. Then $d_{\text{Haus}}(\mathcal{C}(K), \mathcal{C}(K')) \leq d_{\text{Haus}}(K, K')$.

Proof: Clearly, for all $x \in K$, $d^*(x, \mathcal{C}(K')) \leq d^*(x, K')$. A similar statement holds for all $x \in K'$. For all $x \in \mathcal{C}(K) \cap \mathcal{C}(K')$, $d^*(x, \mathcal{C}(K')) = d^*(x, \mathcal{C}(K)) = 0$. Lastly, suppose that $x \in \mathcal{C}(K) \backslash \mathcal{C}(K')$. It is a simple matter to verify that for all $y \in \mathbb{R}^2$, $d(x, x \wedge y) \leq d(x, y)$. There exists some $x' \in K$ such that $x \leq x'$. For all $y \in K'$, $d(x', x' \wedge y) \leq d(x', y)$. Then, $d(x \wedge x', x' \wedge y) \leq d(x', x' \wedge y)$, or $d(x, x' \wedge y) \leq d(x', y)$. Lastly, $d(x, x \wedge (x' \wedge y)) \leq d(x, x' \wedge y)$. But $d(x, x \wedge (x' \wedge y)) = d(x, x \wedge y)$. Note that $x \wedge y \in \mathcal{C}(K')$. Thus, $d(x, x \wedge y) \leq d(x', y) \leq d_{\text{Haus}}(K, K')$. Taken together, these results imply that $d_{\text{Haus}}(\mathcal{C}(K), \mathcal{C}(K')) \leq d_{\text{Haus}}(K, K')$. ■

Proposition 2: Let $B \in \mathcal{B}^G$ be compactly generated. Then $\frac{\sum_{k=1}^n B}{n} \to \mathcal{K}(B)$ in the Hausdorff topology.

Proof: Let $B \in \mathcal{B}^G$ be compactly generated, in the sense that there exists some compact K so that $B = \mathcal{C}(K)$. It is a direct implication of the Shapley-Folkman Theorem (see Starr [10], p. 36) that $\frac{\sum_{k=1}^n K}{n}$ converges to its convex hull $\text{conv}(K)$ in the Hausdorff topology as $n \to \infty$. Note that $\mathcal{K}(B) = \mathcal{C}(\text{conv}(K))$. Thus, $d_{\text{Haus}}\left(\frac{\sum_{k=1}^n B}{n}, \mathcal{K}(B)\right) \leq d_{\text{Haus}}\left(\frac{\sum_{k=1}^n K}{n}, \text{conv}(K)\right)$ by the preceding Lemma, so that $\frac{\sum_{k=1}^n B}{n} \to \mathcal{K}(B)$. ■

Theorem 2: A rule f defined on $\mathcal{B}^G$ satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, and *mixture linearity* if and only if there exist $c \in \mathbb{R}$ and a multi-utilitarian rule U^ν such that for all $B \in \mathcal{B}$, $f(B) = (c, -c) + U^\nu(B)$.

Proof: Recall that Theorem 1 and Corollary 1 held on the domain of compactly generated problems. Thus, restricted to the domain of convex and compactly generated problems, f has the desired representation, with constant c and measure ν. Next, *mixture linearity* can be applied inductively to show that for all k, $f\left(\sum_{l=1}^{2^k} \frac{B}{2^k}\right) = \frac{\sum_{l=1}^{2^k} f(B)}{2^k}$. But the right hand side of this expression is precisely $f(B)$, and by Proposition 2, $\sum_{l=1}^{2^k} \frac{B}{2^k}$ tends to $\mathcal{K}(B)$, which is a convex problem. By *continuity*, then, $f(B) = f(\mathcal{K}(B))$. Thus, $f(B) = (c, -c) + U^{\nu}(B)$. The result then concludes by *continuity* and the fact that the compactly generated generalized problems are dense in the space of all generalized problems. ■

Although in this context, *additivity* need not be implied by *mixture linearity* and $f(\mathcal{K}(\{0\})) = 0$, it is clear that adding the requirement that $f(\mathcal{K}(\{0\})) = 0$ to the preceding Theorem provides a characterization of the multi-utilitarian rules in this nonconvex environment.

Corollary 4: A rule f defined on $\mathcal{B}^G$ satisfies *efficiency*, *translation invariance*, *monotonicity*, *continuity*, *mixture linearity*, and *selection of singletons* if and only if it is a multi-utilitarian rule.

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