# Multi-valued $F$-contractions and the solution of certain functional and integral equations

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**Abstract.** Wardowski [Fixed Point Theory Appl., 2012:94] introduced a new concept of contraction and proved a fixed point theorem which generalizes Banach contraction principle. Following this direction of research, we will present some fixed point results for closed multi-valued $F$-contractions or multi-valued mappings which satisfy an $F$-contractive condition of Hardy-Rogers-type, in the setting of complete metric spaces or complete ordered metric spaces. An example and two applications, for the solution of certain functional and integral equations, are given to illustrate the usability of the obtained results.

## 1. Introduction

It is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 [7], is one of the most important theorems in classical functional analysis. Indeed it is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics. Starting from these considerations, the study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers, see for example [3, 4, 6, 11, 14, 15, 21, 23–28]. In [16], Nadler extended the contraction mapping principle from the single-valued mappings to the multi-valued mappings. Precisely, Nadler proved the following theorem.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space and let $T : X \rightarrow CB(X)$ be a multi-valued mapping satisfying*

$$H(Tx, Ty) \leq k\, d(x, y), \tag{1}$$

*for all $x, y \in X$, where $k$ is a constant such that $k \in [0, 1)$ and $CB(X)$ denotes the family of non-empty closed bounded subsets of X. Then T has a fixed point.*

The reader can see [1, 2, 13] and references therein, for recent results in this direction. Recently, Wardowski [29] introduced a new concept of contraction and proved a fixed point theorem which generalizes Banach contraction principle. Following this direction of research, in this paper, we will present some fixed point results for closed multi-valued $F$-contractions or multi-valued mappings which satisfy an $F$-contractive condition of Hardy-Rogers-type, in the setting of complete metric spaces or complete ordered metric spaces. Moreover, an example and two applications, for the solution of certain functional and integral equations, are given to illustrate the usability of the obtained results.

## 2. Preliminaries

The aim of this section is to present some notions and results used in the paper. Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

**Definition 2.1.** *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

 *(F1)  F is strictly increasing;*
 *(F2)  for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to +\infty} F(\alpha_n) = -\infty$;*
 *(F3)  there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.*

*We denote with $\mathcal{F}$ the family of all functions $F$ that satisfy the conditions (F1)-(F3).*

**Definition 2.2 ([29]).** *Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{2}$$

*for all $x, y \in X$ with $d(Tx, Ty) > 0$.*

**Definition 2.3 ([12]).** *Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an F-contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + L d(y, Tx)),$$

*for all $x, y \in X$ with $d(Tx, Ty) > 0$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.*

Now, let $(X, d)$ be a metric space and $C(X)$ be the family of non-empty closed subsets of $X$. If $T : X \to C(X)$ is a multi-valued mapping, then we put

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

**Definition 2.4.** *Let $(X, d)$ be a metric space. A multi-valued mapping $T : X \to C(X)$ is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x, y \in X$ with $y \in Tx$ there exists $z \in Ty$ for which*

$$\tau + F(d(y, z)) \leq F(M(x, y)) \tag{3}$$

*if $d(y, z) > 0$.*

If we choose the mapping $F$ opportunely, then we obtain some classes of contractions known in the literature. See the following examples.

**Example 2.5 ([29]).** *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(x) = \ln x$. It is clear that $F$ satisfies (F1)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying (2) is an F-contraction such that*

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

*It is clear that for $x, y \in X$ such that $Tx = Ty$ the previous inequality also holds and hence $T$ is a contraction.*

**Example 2.6.** *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(x) = \ln x$. For each multi-valued mapping $T : X \to C(X)$ satisfying (3) we have*

$$d(y, z) \leq e^{-\tau} M(x, y), \text{ for all } x, y \in X, y \neq z.$$

*It is clear that for $z, y \in X$ such that $y = z$ the previous inequality also holds.*

**Definition 2.7.** *Let $T : X \to C(X)$ be a multi-valued mapping. The graph of $T$ is the subset $\{(x, y) : x \in X, y \in Tx\}$ of $X \times X$; we denote the graph of $T$ by $G(T)$. Then $T$ is a closed multi-valued mapping if the graph $G(T)$ is a closed subset of $X \times X$.*

### 3. Fixed points for *F*-contractions in complete metric spaces

In this section, we give some fixed point results for *F*-contractions in a complete metric space.

**Theorem 3.1.** *Let $(X, d)$ be a complete metric space and let $T : X \to C(X)$ be a closed F-contraction. Then T has a fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point of $X$ and choose $x_1 \in Tx_0$. If $x_1 = x_0$, then $x_1$ is a fixed point of $T$ and the proof is completed. Assume that $x_1 \neq x_0$. Since $T$ is an $F$-contraction, then there exists $x_2 \in Tx_1$ such that

$$\tau + F(d(x_1, x_2)) \leq F(M(x_0, x_1)) \text{ and } x_2 \neq x_1.$$

Also, we get that there exists $x_3 \in Tx_2$ such that

$$\tau + F(d(x_2, x_3)) \leq F(M(x_1, x_2)) \text{ and } x_3 \neq x_2.$$

Repeating this process, we find that there exists a sequence $\{x_n\}$ with initial point $x_0$ such that $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$\tau + F(d(x_n, x_{n+1})) \leq F(M(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}.$$

This implies

$$F(d(x_n, x_{n+1})) < F(M(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}$$

and consequently, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) \quad &< \quad \max\left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} \right\} \\
&= \quad \max\left\{ d(x_{n-1}, x_n), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n)}{2} \right\} \\
&\leq \quad \max\left\{ d(x_{n-1}, x_n), d(x_n, Tx_n), \frac{d(x_{n-1}, x_n) + d(x_n, Tx_n)}{2} \right\} \\
&= \quad \max\left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\}.
\end{aligned}
$$

Obviously, if $\max\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_n, x_{n+1})$, we have a contradiction and so

$$\max\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_{n-1}, x_n).$$

Consequently, we get

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \tag{4}$$

Now, let $d_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By (4), we have

$$F(d_n) \leq F(d_{n-1}) - \tau \leq \cdots \leq F(d_0) - n\tau, \text{ for all } n \in \mathbb{N} \tag{5}$$

and hence $\lim_{n \to +\infty} F(d_n) = -\infty$. By the property (F2), we get that $d_n \to 0$ as $n \to +\infty$. Now, let $k \in (0, 1)$ such that $\lim_{n \to +\infty} d_n^k F(d_n) = 0$. By (5), the following holds for all $n \in \mathbb{N}$:

$$d_n^k F(d_n) - d_n^k F(d_0) \leq d_n^k (F(d_0) - n\tau) - d_n^k F(d_0) = -n\tau d_n^k \leq 0. \tag{6}$$

Letting $n \to +\infty$ in (6), we deduce $\lim_{n \to +\infty} n d_n^k = 0$ and hence $\lim_{n \to +\infty} n^{1/k} d_n = 0$. Clearly, $\lim_{n \to +\infty} n^{1/k} d_n = 0$ ensures that the series $\sum_{n=1}^{+\infty} d_n$ is convergent. This implies that $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. Now, we prove that $u$ is a fixed point of $T$. Since $T$ is a closed multi-valued mapping and $(x_n, x_{n+1}) \to (u, u)$, we get $u \in Tu$ and hence $u$ is a fixed point of $T$. $\square$

We note that in a metric space every upper semicontinuous multi-valued mapping is closed. Precisely, we recall that $T$ is upper semicontinuous if and only if for each closed set $B \in C(X)$, we have that $T^{-1}(B) = \{x : T(X) \cap B \neq \emptyset\}$ is closed. Then, from Theorem 3.1, we obtain the following corollary.

**Corollary 3.2.** *Let $(X, d)$ be a complete metric space and let $T : X \to C(X)$ be an upper semicontinuous F-contraction. Then $T$ has a fixed point.*

**Example 3.3.** *Consider the complete metric space $(X, d)$, where $X = \{0, 1, 2, \ldots\}$ and $d : X \times X \to [0, +\infty)$ is given by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y, \end{cases}$$

*Let $T : X \to C(X)$ be defined by*

$$Tx = \begin{cases} \{0\} & \text{if } x \in \{0, 1\}, \\ \{0, \ldots, x - 1\} & \text{if } x \geq 2. \end{cases}$$

*Clearly the multi-valued mapping $T$ is closed.*

*Now, we show that $T$ satisfies (3), where $\tau = 1$ and $F(x) = \ln x + x$ for each $x \in \mathbb{R}^+$. To this aim, for all $x, y \in X$ with $y \in Tx$, we choose $z = 0 \in Ty$. First, we note that $d(y, z) > 0$ if and only if $x \geq 2$ and $y > 0$. If this holds true, then $d(y, z) = y < x + y = d(x, y)$ and hence*

$$d(y, z) - M(x, y) \leq d(y, z) - d(x, y) \leq -2.$$

*It follows easily that*

$$\frac{d(y, z)}{M(x, y)} e^{d(x, y) - M(x, y)} \leq e^{-1}$$

*and hence*

$$1 + F(d(y, z)) \leq F(M(x, y)),$$

*for all $x, y \in X$ with $d(y, z) > 0$. Then, by Theorem 3.1, $T$ has a fixed point.*
*On the other hand, it is easy to show that Theorem 1.1 is not applicable in this case. Indeed, assume there exists $k \in [0, 1)$ such that (1) holds true, then $H(Tx, T0) = x - 1 \leq kx$, for all $x \geq 1$. This implies that $\dfrac{x - 1}{x} \leq k$ and, for $x \to +\infty$, we get $1 \leq k$, a contradiction.*

Next, we give a fixed point result for multi-valued *F*-contractions of Hardy-Rogers-type in a complete metric space.

**Theorem 3.4.** *Let $(X, d)$ be a complete metric space and let $T : X \to CB(X)$. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)), \tag{7}$$

*for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point of $X$ and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then $x_1$ is a fixed point of $T$ and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since $F$ is continuous from the right, there exists a real number $h > 1$ such that

$$F(h\, H(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Now, from $d(x_1, Tx_1) < h H(Tx_0, Tx_1)$, we deduce that there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \le h H(Tx_0, Tx_1)$. Consequently, we get

$$F(d(x_1, x_2)) \le F(h H(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau,$$

which implies

$$
\begin{aligned}
2\tau + F(d(x_1, x_2)) &\le 2\tau + F(H(Tx_0, Tx_1)) + \tau \\
&\le F(\alpha d(x_0, x_1) + \beta d(x_0, Tx_0) + \gamma d(x_1, Tx_1) + \delta d(x_0, Tx_1) + L d(x_1, Tx_0)) + \tau \\
&\le F(\alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta d(x_0, x_2)) + \tau \\
&\le F((\alpha + \beta + \delta) d(x_0, x_1) + (\gamma + \delta) d(x_1, x_2)) + \tau.
\end{aligned}
$$

Since $F$ is strictly increasing, we deduce

$$d(x_1, x_2) \le (\alpha + \beta + \delta) d(x_0, x_1) + (\gamma + \delta) d(x_1, x_2)$$

and hence

$$(1 - \gamma - \delta) d(x_1, x_2) < (\alpha + \beta + \delta) d(x_0, x_1).$$

From $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \ne 1$, we deduce that $1 - \gamma - \delta > 0$ and so

$$d(x_1, x_2) < \frac{\alpha + \beta + \delta}{1 - \gamma - \delta} d(x_0, x_1) = d(x_0, x_1).$$

Consequently,

$$\tau + F(d(x_1, x_2)) \le F(d(x_0, x_1)).$$

Continuing in this manner, we can define a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$\tau + F(d(x_{n+1}, x_{n+2})) \le F(d(x_n, x_{n+1})), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Proceeding as in the proof of Theorem 3.1, we obtain that $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. Now, we prove that $u$ is a fixed point of $T$. If there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in Tu$ for all $k \in \mathbb{N}$, since $Tu$ is closed and $x_{n_k} \to u$, we get $u \in Tu$ and the proof is completed. So we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tu$ for all $n \in \mathbb{N}$ with $n \ge n_0$. This implies that $Tx_{n-1} \ne Tu$ for all $n \ge n_0$. Now, using (7) with $x = x_n$ and $y = u$, we obtain

$$2\tau + F(H(Tx_n, Tu)) \le F(\alpha d(x_n, u) + \beta d(x_n, Tx_n) + \gamma d(u, Tu) + \delta d(x_n, Tu) + L d(u, Tx_n)),$$

which implies

$$
\begin{aligned}
2\tau + F(d(x_{n+1}, Tu)) &\le 2\tau + F(H(Tx_n, Tu)) \\
&\le F(\alpha d(x_n, u) + \beta d(x_n, Tx_n) + \gamma d(u, Tu) + \delta d(x_n, Tu) + L d(u, Tx_n)) \\
&\le F(\alpha d(x_n, u) + \beta d(x_n, x_{n+1}) + \gamma d(u, Tu) + \delta d(x_n, Tu) + L d(u, x_{n+1})).
\end{aligned}
$$

Since $F$ is strictly increasing, we have

$$d(x_{n+1}, Tu) < \alpha d(x_n, u) + \beta d(x_n, x_{n+1}) + \gamma d(u, Tu) + \delta d(x_n, Tu) + L d(u, x_{n+1}).$$

Letting $n \to +\infty$ in the previous inequality, as $\gamma + \delta < 1$ we get

$$d(u, Tu) \le (\gamma + \delta) d(u, Tu) < d(u, Tu),$$

which implies $d(u, Tu) = 0$. Since $Tu$ is closed we obtain that $u \in Tu$, that is, $u$ is a fixed point of $T$. $\quad\square$

## 4. Fixed points for *F*-contractions in ordered metric spaces

The existence of fixed points of self-mappings defined on certain type of ordered sets plays an important role in the order theoretic approach. This study has been initiated in 2004 by Ran and Reurings [22], and followed by Nieto and Rodríguez-Lopez [17]. Then, several interesting and valuable results appeared in this direction [3, 18, 19, 21, 25].

Let $X$ be a non-empty set. If $(X, d)$ is a metric space and $(X, \preceq)$ is partially ordered, then $(X, d, \preceq)$ is called an ordered metric space. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Further, a self-mapping $T$ on $X$ is called non-decreasing if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x \in X$. Moreover, an ordered metric space $(X, d, \preceq)$ is regular if the following condition holds:

(r) for every non-decreasing sequence $\{x_n\}$ in $X$ convergent to some $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

**Definition 4.1 ([8]).** *Let $(X, \preceq)$ be a partially ordered set. Let $A$ and $B$ be two nonempty subsets of $X$. The relation between $A$ and $B$ is denoted and defined as follows: $A \prec_1 B$, if for each $a \in A$ there exists $b \in B$ such that $a \preceq b$. Also, we say that $A \prec_2 B$ whenever for each $a \in A$ and $b \in B$ we have $a \preceq b$.*

**Theorem 4.2.** *Let $(X, d, \preceq)$ be an ordered complete metric space and let $T : X \to CB(X)$. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)), \tag{8}$$

*for all comparable $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. If the following conditions are satisfied:*

 *(i) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$;*

 *(ii) for $x, y \in X$, $x \preceq y$ implies $Tx \prec_2 Ty$;*

 *(iii) $X$ is regular;*

*then $T$ has a fixed point.*

*Proof.* By assumption (i), there exists $x_1 \in Tx_0$ such that $x_0 \preceq x_1$. By assumption (ii), $Tx_0 \prec_2 Tx_1$. If $x_1 \in Tx_1$, then $x_1$ is a fixed point of $T$ and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since $F$ is continuous from the right, there exists a real number $h > 1$ such that

$$F(h\, H(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Now, from $d(x_1, Tx_1) < h\, H(Tx_0, Tx_1)$ and $Tx_0 \prec_2 Tx_1$, we deduce that there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq h\, H(Tx_0, Tx_1)$ and $x_1 \preceq x_2$. Again, from assumption (ii), we get $Tx_1 \prec_2 Tx_2$ and hence following the same lines in the proof of Theorem 3.4, in view of assumption (ii), we can construct a monotone increasing sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$ and

$$x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots,$$

that is, $x_n$ and $x_{n+1}$ are comparable and so $Tx_n \prec_2 Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Next, proceeding as in the proof of Theorem 3.1, we obtain that $\{x_n\}$ is a Cauchy sequence. Also, since $X$ is a complete metric space, there exists $u \in X$ such that $x_n \to u$. From assumption (iii), we deduce that $x_n \preceq u$ for all $n \in \mathbb{N} \cup \{0\}$. The rest of the proof is analogous to Theorem 3.4 and so we omit the details. $\square$

## 5. Applications

*5.1. Existence of bounded solutions of functional equations*

Mathematical optimization is one of the fields in which the methods of fixed point theory are widely used. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming. In this setting, the problem of dynamic programming related to multistage process reduces to solving the functional equation

$$q(x) = \sup_{y \in D}\{f(x, y) + G(x, y, q(\tau(x, y)))\}, \quad x \in W, \tag{9}$$

where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$. We assume that $U$ and $V$ are Banach spaces, $W \subseteq U$ is a state space and $D \subseteq V$ is a decision space. Precisely, see also [9, 10], the studied process consists of:

(a) a state space, which is the set of the initial state, actions and transition model of the process;
(b) a decision space, which is the set of possible actions that are allowed for the process.

Here, we study the existence of the bounded solution of the functional equation (9).
Let $B(W)$ denote the set of all bounded real-valued functions on $W$ and, for an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Clearly, $(B(W), \|\cdot\|)$ endowed with the metric $d$ defined by

$$d(h, k) = \sup_{x \in W} |h(x) - k(x)|, \tag{10}$$

for all $h, k \in B(W)$, is a Banach space. Indeed, the convergence in the space $B(W)$ with respect to $\|\cdot\|$ is uniform. Thus, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then $\{h_n\}$ converges uniformly to a function, say $h^*$, that is bounded and so $h^* \in B(W)$.
We also define $T : B(W) \to B(W)$ by

$$T(h)(x) = \sup_{y \in D}\{f(x, y) + G(x, y, h(\tau(x, y)))\}, \tag{11}$$

for all $h \in B(W)$ and $x \in W$. Obviously, if the functions $f$ and $G$ are bounded then $T$ is well-defined. Finally, let

$$M(h, k) = \max\left\{d(h, k), d(h, T(h)), d(k, T(k)), \frac{d(h, T(k)) + d(k, T(h))}{2}\right\}.$$

We will prove the following theorem.

**Theorem 5.1.** *Let $T : B(W) \to B(W)$ be an upper semicontinuous operator defined by* (11) *and assume that the following conditions are satisfied:*

*(i) $G : W \times D \times \mathbb{R} \to \mathbb{R}$ and $f : W \times D \to \mathbb{R}$ are continuous and bounded;*

*(ii) there exists $\tau \in \mathbb{R}^+$ such that $\left|G(x, y, h(x)) - G(x, y, k(x))\right| \le e^{-\tau} M(h, k)$, for all $h, k \in B(W)$, where $x \in W$ and $y \in D$.*

*Then the functional equation* (9) *has a bounded solution.*

*Proof.* Note that $(B(W), d)$ is a complete metric space, where $d$ is the metric given by (10). Let $\lambda$ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$
\begin{aligned}
T(h_1)(x) &< f(x, y_1) + G(x, y_1, h_1(\tau(x, y_1))) + \lambda, &\tag{12}\\
T(h_2)(x) &< f(x, y_2) + G(x, y_2, h_2(\tau(x, y_2))) + \lambda, &\tag{13}\\
T(h_1)(x) &\ge f(x, y_2) + G(x, y_2, h_1(\tau(x, y_2))), &\tag{14}\\
T(h_2)(x) &\ge f(x, y_1) + G(x, y_1, h_2(\tau(x, y_1))). &\tag{15}
\end{aligned}
$$

Therefore, by using (12) and (15), it follows that

$$
\begin{aligned}
T(h_1)(x) - T(h_2)(x) \quad &< \quad G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1))) + \lambda \\
&\leq \quad \left| G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1))) \right| + \lambda \\
&\leq \quad e^{-\tau} M(h_1, h_2) + \lambda.
\end{aligned}
$$

Then, we get

$$
T(h_1)(x) - T(h_2)(x) < e^{-\tau} M(h_1, h_2) + \lambda. \tag{16}
$$

Analogously, by using (13) and (14), we have

$$
T(h_2)(x) - T(h_1)(x) < e^{-\tau} M(h_1, h_2) + \lambda. \tag{17}
$$

Hence, from (16) and (17) we obtain

$$
|T(h_1)(x) - T(h_2)(x)| < e^{-\tau} M(h_1, h_2) + \lambda,
$$

that is,

$$
d(T(h_1), T(h_2)) \leq e^{-\tau} M(h_1, h_2) + \lambda.
$$

Since the above inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrary, then we conclude immediately that

$$
d(T(h_1), T(h_2)) \leq e^{-\tau} M(h_1, h_2),
$$

for each $x \in W$. Equivalently, by passing to logarithms, we can write this as

$$
\ln(d(T(h_1), T(h_2))) \leq \ln(e^{-\tau} M(h_1, h_2)),
$$

and, after routine calculations, we get

$$
\tau + \ln(d(T(h_1), T(h_2))) \leq \ln(M(h_1, h_2)).
$$

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(x) = \ln x$, for each $x \in W$, is in $\mathcal{F}$ and so we deduce that the operator $T$ is an $F$-contraction. Thus, due to the continuity of $T$, Corollary 3.2 applies to the operator $T$, which has a fixed point $h^* \in B(W)$, that is, $h^*$ is a bounded solution of the functional equation (9). □

## 5.2. Existence of solutions of integral equations

Finally, we discuss the application of fixed point methods to the following Volterra type integral equation:

$$
u(t) = \int_0^t K(t, s, u(s)) \, ds + g(t), \ t \in [0, \Lambda], \tag{18}
$$

where $\Lambda > 0$.
Precisely, we study the existence of the solution of the integral equation (18).
Let $C([0, \Lambda], \mathbb{R})$ denote the space of all continuous functions on $[0, \Lambda]$ and, for an arbitrary $u \in C([0, \Lambda], \mathbb{R})$, define $\|u\|_\tau = \sup_{t \in [0, \Lambda]} \{|u(t)|e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Note that $\|\cdot\|_\tau$ is a norm equivalent to the supremum norm and $C([0, \Lambda], \mathbb{R}), \|\cdot\|_\tau)$ endowed with the metric $d_\tau$ defined by

$$
d_\tau(u, v) = \sup_{t \in [0, \Lambda]} \{|u(t) - v(t)| e^{-\tau t}\}, \tag{19}
$$

for all $u, v \in C([0, \Lambda], \mathbb{R})$, is a Banach space, see also [5, 20]. Now, $C([0, \Lambda], \mathbb{R})$ can be equipped with the partial order $\preceq$ given by

$$
u, v \in C([0, \Lambda], \mathbb{R}), \qquad u \preceq v \Longleftrightarrow u(t) \leq v(t), \text{ for all } t \in [0, \Lambda].
$$

Moreover, in [17], it is showed that $(C([0, \Lambda], \mathbb{R}), \preceq)$ is regular.
We will prove the following theorem.

**Theorem 5.2.** *Assume that the following conditions are satisfied:*

*(i)* $K : [0, \Lambda] \times [0, \Lambda] \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0, \Lambda] \to \mathbb{R}$ *are continuous;*

*(ii)* $K(t, s, \cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing, for all* $t, s \in [0, \Lambda]$;

*(iii) there exists* $u_0 \in C([0, \Lambda], \mathbb{R})$ *such that, for all* $t \in [0, \Lambda]$, *we have*

$$u_0(t) \leq \int_0^t K(t, s, u_0(s)) \, ds + g(t);$$

*(iv) there exists* $\tau \in [1, +\infty)$ *such that*

$$|K(t, s, u) - K(t, s, v)| \leq \tau e^{-2\tau} |u - v|,$$

*for all* $t, s \in [0, \Lambda]$ *and* $u, v \in \mathbb{R}$ *such that* $u \leq v$.

*Then the integral equation* (18) *has a solution.*

*Proof.* Note that $(C([0, \Lambda], \mathbb{R}), d_\tau)$ is a complete metric space, where $d_\tau$ is the metric given by (19). Define $T : C([0, \Lambda], \mathbb{R}) \to C([0, \Lambda], \mathbb{R})$ by

$$T(u)(t) = \int_0^t K(t, s, u(s)) \, ds + g(t), \; t \in [0, \Lambda].$$

From assumption (ii) $T$ is increasing. Now, for all $u, v \in C([0, \Lambda], \mathbb{R})$ such that $u \leq v$, by assumption (iv), we have

$$
\begin{aligned}
|T(u)(t) - T(v)(t)| &\leq \int_0^t |K(t, s, u(s)) - K(t, s, v(s))| \, ds \\
&\leq \int_0^t \tau e^{-2\tau} |u(s) - v(s)| \, ds \\
&= \int_0^t \tau e^{-2\tau} |u(s) - v(s)| e^{-\tau s} e^{\tau s} \, ds \\
&\leq \int_0^t e^{\tau s} \tau e^{-2\tau} |u(s) - v(s)| e^{-\tau s} \, ds \\
&\leq \tau e^{-2\tau} \|u - v\|_\tau \int_0^t e^{\tau s} \, ds \\
&\leq \tau e^{-2\tau} \frac{1}{\tau} \|u - v\|_\tau e^{\tau t}.
\end{aligned}
$$

This implies that

$$|T(u)(t) - T(v)(t)| e^{-\tau t} \leq e^{-2\tau} \|u - v\|_\tau,$$

or equivalently,

$$d_\tau(T(u), T(v)) \leq e^{-2\tau} d_\tau(u, v).$$

By passing to logarithms, we can write this as

$$\ln(d_\tau(T(u), T(v))) \leq \ln(e^{-2\tau} d_\tau(u, v)),$$

and, after routine calculations, we get

$$2\tau + \ln(d_\tau(T(u), T(v))) \leq \ln(d_\tau(u, v)).$$

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(u) = \ln u$, for each $u \in C([0, \Lambda], \mathbb{R})$, is in $\mathcal{F}$ and so we deduce that the operator $T$ satisfies condition (8) with $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$. Clearly, from assumption (iii), $u_0 \leq T(u_0)$ and hence Theorem 4.2 applies to the operator $T$, which has a fixed point $u^* \in C([0, \Lambda], \mathbb{R})$, that is, $u^*$ is a solution of the integral equation (18). $\square$

# References

[1] M. Abbas, B. Ali, C. Vetro, A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces, Topology and its Applications 160 (2013) 553–563.

[2] H. Aydi, M. Abbas, C. Vetro, Common fixed points for multivalued generalized contractions on partial metric spaces, Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A (2013), doi:10.1007/s13398-013-0120-z.

[3] I. Altun, G. Durmaz, Some fixed point theorems on ordered cone metric spaces, Rendiconti del Circolo Matematico di Palermo 58 (2009) 319–325.

[4] M. Arshad, A. Azam, P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory and Applications 2009:493965 (2009).

[5] A. Augustynowicz, Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function, Serdica Mathematical Journal 23 (1997) 203–210.

[6] A. Azam, M. Arshad, I. Beg, Common fixed points of two maps in cone metric spaces, Rendiconti del Circolo Matematico di Palermo 57 (2008) 433–441.

[7] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae 3 (1922) 133–181.

[8] I. Beg, A. R. Butt, Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces, Mathematical Communications 15 (2010) 65–76.

[9] R. Bellman, Methods of Nonliner Analysis. Vol. II, vol. 61 of Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1973.

[10] R. Bellman, E. S. Lee, Functional equations in dynamic programming, Aequationes Mathematicae 17 (1978) 1–18.

[11] V. Berinde, F. Vetro, Common fixed points of mappings satisfying implicit contractive conditions, Fixed Point Theory and Applications 2012:105 (2012).

[12] M. Cosentino, P. Vetro, Fixed point results for *F*-contractive mappings of Hardy-Rogers-type, submitted.

[13] B. Damjanovic, B. Samet, C. Vetro, Common fixed point theorems for multi-valued maps, Acta Mathematica Scientia. Series B 32 (2012) 818–824.

[14] C. Di Bari, P. Vetro, $\varphi$-pairs and common fixed points in cone metric spaces, Rendiconti del Circolo Matematico di Palermo 57 (2008) 279–285.

[15] C. Di Bari, P. Vetro, Weakly $\varphi$-pairs and common fixed points in cone metric spaces, Rendiconti del Circolo Matematico di Palermo 58 (2009) 125–132.

[16] S. B. Nadler Jr, Multi-valued contraction mappings, Pacific Journal of Mathematics 30 (1969) 475–488.

[17] J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.

[18] J. J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Mathematica Sinica (English Series) 23 (2007) 2205–2212.

[19] J. J. Nieto, R. L. Pouso, R. Rodríguez-López, Fixed point theorems in ordered abstract spaces, Proceedings of the American Mathematical Society 132 (2007) 2505–2517.

[20] D. O'Regan, A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, Journal of Mathematical Analysis and Applications 341 (2008) 1241–1252.

[21] D. Paesano, P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology and its Applications 159 (2012) 911–920.

[22] A. C. M. Ran, M. C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proceedings of the American Mathematical Society 132 (2004) 1435–1443.

[23] I. A. Rus, Generalized Contractions and Applications, Cluj University Press, Cluj-Napoca, 2001.

[24] I. A. Rus, A. Petruşel, G. Petruşel, Fixed Point Theory, Cluj University Press, Cluj-Napoca, 2008.

[25] R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, Fixed point theorems in generalized partially ordered G-metric spaces, Mathematical and Computer Modelling 52 (2010) 797–801.

[26] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Analysis 71 (2009) 5313–5317.

[27] F. Vetro, S. Radenović, Nonlinear $\psi$-quasi-contractions of Ćirić-type in partial metric spaces, Applied Mathematics and Computation 219 (2012) 1594–1600.

[28] P. Vetro, Common fixed points in cone metric spaces, Rendiconti del Circolo Matematico di Palermo 56 (2007) 464–468.

[29] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Applications 2012:94 (2012).