ANNALES
POLONICI MATHEMATICI
XLI (1983)

Multi-valued solutions of a functional equation

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Abstract. The existence of a solution Φ_0 of the inequality

$$\Phi(x) \supset H(x, \Phi[f(x)])$$

implies the existence of a minimal solution $\Phi \subset \Phi_0$ of the equation

$$\Phi(x) = H(x, \Phi[f(x)]) \tag{1}$$

in some classes of multi-valued mappings. Moreover, if f and H are single-valued, then (under suitable conditions) the minimal continuum-valued solution of (1) is almost everywhere a single-valued mapping.

Let X be a non-empty set and let Y be a topological Hausdorff space. We introduce the following classes of subsets of Y:

$$n(Y) = \{A \subset Y: A \neq \emptyset\},$$

$$c(Y) = \{A \subset Y: A \neq \emptyset \text{ and } A \text{ is compact}\},$$

$$C(Y) = \{A \subset Y: A \neq \emptyset \text{ and } A \text{ is a continuum}\}.$$

Let Z be a topological space. We say that $\Phi\colon Z \to n(Y)$ is a closed mapping if, whenever $z_0 \in Z$, $y_0 \in Y$, $y_0 \notin \Phi(z_0)$, there exist two open neighbourhoods $U(z_0)$ and $V(y_0)$ such that $\Phi(z) \cap V(y_0) = \emptyset$ for every $z \in U(z_0)$.

Let $z_0 \in Z$. We say that Φ is *upper semi-continuous* (u.s.c.) at z_0 if for each open set G containing $\Phi(z_0)$ there is an open neighbourhood $U(z_0)$ such that $\Phi(z) \subset G$ for every $z \in U(z_0)$.

A mapping $\Phi\colon Z \to n(Y)$ is u.s.c. if it is u.s.c. at every $z_0 \in Z$.

We consider the following equation:

$$\Phi(x) = H(x, \Phi[f(x)]), \tag{1}$$

where the functions $f\colon X \to n(X)$, $H\colon X \times Y \to n(Y)$ are given and $H(x, \cdot)$ is closed for every $x \in X$, and $\Phi\colon X \to n(Y)$ is the unknown function.

LEMMA 1. *Let F be a family of functions $\Phi\colon X \to c(Y)$ such that:*

(I) *if $\Phi \in F$, then $H(\cdot, \Phi[f(\cdot)]) \in F$ and $\Phi[f(x)] \in c(Y)$ for $x \in X$;*

(II) *if $\Phi_n \in F$ and $\Phi_{n+1} \subset \Phi_n$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} \Phi_n \in F$.*

If a function

(2) $$\Phi_0 \in F$$

fulfils the inequality

(3) $$\Phi_0(x) \supset H(x, \Phi_0[f(x)]),$$

then the formulas

(4) $$\Phi_{n+1} = H(\cdot, \Phi_n[f(\cdot)])$$

and

(5) $$\bar{\Phi} = \bigcap_{n=1}^{\infty} \Phi_n$$

define a solution $\bar{\Phi} \subset \Phi_0$ *of* (1) *in the family* F. *If* $\Psi\colon X \to n(Y)$ *is a solution of* (1) *such that* $\Psi \subset \Phi_0$, *then* $\Psi \subset \bar{\Phi}$.

Proof. It follows by (2), (I) and (4) that $\Phi_n \in F$ for $n = 0, 1, 2, \ldots$ According to (2), (3) and (4) we have $\Phi_{n+1}(x) \subset \Phi_n(x)$ for $n = 0, 1, 2, \ldots$ and for every $x \in X$. Therefore by (II) and (5) we get $\bar{\Phi} \in F$.

It is obvious that $H(x, \bar{\Phi}[f(x)]) \subset H(x, \Phi_n[f(x)]) \subset \Phi_n(x)$ for $n = 0, 1, 2, \ldots$ and for $x \in X$. This implies the inclusion

$$H(x, \bar{\Phi}[f(x)]) \subset \bar{\Phi}(x)$$

for every $x \in X$.

Now, we take $z \in \bar{\Phi}(x)$. Then $z \in \Phi_{n+1}(x) = H(x, \Phi_n[f(x)])$ for $n = 0, 1, 2, \ldots$ There exists a sequence $\{y_n\}$ such that

$$y_n \in \Phi_n[f(x)] \subset \Phi_0[f(x)], \quad n = 0, 1, 2, \ldots$$

and

(6) $$z \in H(x, y_n).$$

If $m \geqslant n$, then $y_m \in \Phi_n[f(x)]$. Thus

(7) $$A_n := \mathrm{cl}\{y_n, y_{n+1}, \ldots\} \subset \Phi_n[f(x)], \quad n = 0, 1, 2, \ldots,$$

for the sets $\Phi_n[f(x)]$ are closed (being compact). We have $A_{n+1} \subset A_n$, $n = 0, 1, 2, \ldots$, and so there exists

(8) $$y \in \bigcap_{n=0}^{\infty} A_n.$$

We shall prove that

(9) $$z \in H(x, y).$$

Suppose, conversely, that $z \notin H(x, y)$. Then there exist two neighbourhoods U and V of points y and z, respectively, such that

$$H(x, U) \cap V = \emptyset.$$

It follows by (8) that there exists k, $k = 0, 1, 2, \ldots,$ such that $y_k \in U$. Since by (6)

$$z \in H(x, y_k) \cap V \subset H(x, U) \cap V,$$

we have a contradiction. To prove the first part of the lemma it is sufficient to notice that according to (9), (8), (7) and (5) we have

$$z \in H(x, y) \subset H\Big(x, \bigcap_{n=0}^{\infty} A_n\Big) \subset H\Big(x, \bigcap_{n=0}^{\infty} \Phi_n[f(x)]\Big) = H\big(x, \bar{\Phi}[f(x)]\big).$$

The second part of the lemma is obvious.

The set of all functions $\Phi\colon X \to n(Y)$ is partially ordered by the relation:

$$\Phi_1 \subset \Phi_2 \quad \text{iff} \quad \Phi_1(x) \subset \Phi_2(x) \quad \text{for every } x \in X.$$

LEMMA 2. *Let F be a family of mappings $\Phi\colon X \to c(Y)$ such that conditions* (I) *of Lemma* 1 *and*

(II′) *if $\Phi_\iota \in F$ for every $\iota \in I$ and the family $\{\Phi_\iota\}_{\iota \in I}$ is a chain, then $\bigcap_{\iota \in I} \Phi_\iota \in F$*

are fulfilled. Let a mapping $\Phi_0 \in F$ fulfil inequality (3). *Then there exists a minimal solution $\underline{\Phi} \subset \Phi_0$ of equation* (1) *in the family F.*

Proof. Let

$$F_0 := \{\Phi \in F\colon \Phi \subset \Phi_0, \Phi \text{ is a solution of } (1)\}.$$

It follows by Lemma 1 that the family F_0 is not empty. Let T be a chain in F_0, and let $\Phi_T = \bigcap_{\Phi \in T} \Phi$. According to (II′), $\Phi_T \in F$. Moreover, if $\Phi \in T$, then we have

$$\Phi(x) = H\big(x, \Phi[f(x)]\big) \supset H\big(x, \Phi_T[f(x)]\big)$$

and

$$\Phi_T(x) \supset H\big(x, \Phi_T[f(x)]\big).$$

Lemma 1 guarantees the existence of a solution $\bar{\Phi}_T \subset \Phi_T$ of (1) in the family F. This shows that $\bar{\Phi}_T$ is a minorant of T in F_0. The family F_0 possesses a minimal element in virtue of Kuratowski–Zorn lemma.

LEMMA 3. *Let X be a topological space and let $\Phi\colon X \to n(Y)$ be u.s.c. at $x_0 \in X$. Then the mapping $x \to (x, \Phi(x))$ is u.s.c. at x_0.*

Proof. Let G be an open set in Y such that $\Phi(x_0) \subset G$ and let V be an open set in X such that $x_0 \in V$. Then there exists an open neighbourhood

$U(x_0)$ of x_0 such that if $x \in U(x_0)$, then $\Phi(x) \subset G$, because Φ is u.s.c. at x_0. For every $x \in U(x_0) \cap V$ we have $(x, \Phi(x)) \subset V \times G$.

LEMMA 4. *Let X be a topological space. If $\Phi\colon X \to n(Y)$ is a closed mapping, then the image $\Phi(K)$ of a compact subset K of X is a closed set.*

Proof. We shall prove that the set $Y \setminus \Phi(K)$ is open. Let $y \notin \Phi(K)$. Then there exist neighbourhoods $V^x(y)$ and $U(x)$ such that

$$\Phi(U(x)) \cap V^x(y) = \emptyset$$

for every $x \in K$. Since the set K is compact, there exist elements $x_1, x_2, \ldots, x_n$ in K such that $K \subset U(x_1) \cup \ldots \cup U(x_n)$. Putting $V(y) = V^{x_1}(y) \cap \ldots \cap V^{x_n}(y)$ we have

$$\Phi(K) \cap V(y) = \emptyset.$$

Therefore $Y \setminus \Phi(K)$ is open.

DEFINITION. We say that equation (1) has *property* (P) in the family F iff the following statement is true:

If $\Phi_0 \in F$ is a solution of (3), then formulas (4) and (5) define a solution $\bar{\Phi} \subset \Phi_0$ of (1) in F and there exists a minimal solution $\underline{\Phi} \subset \Phi_0$ of (1) in F.

THEOREM 1.1. *If $f\colon X \to X$, then equation* (1) *has property* (P) *in the family of all functions $\Phi\colon X \to c(Y)$.*

2. *If $f\colon X \to X$ and H with connected values is u.s.c. with respect to the second variable, then equation* (1) *has property* (P) *in the family of all functions $\Phi\colon X \to C(Y)$.*

3. *If X is a topological Hausdorff space, $f\colon X \to c(X)$ is u.s.c. and H is u.s.c., then equation* (1) *has property* (P) *in the family of all u.s.c. functions $\Phi\colon X \to c(Y)$.*

4. *If X is a topological Hausdorff space, $f\colon X \to C(Y)$ is u.s.c. and H with connected values is u.s.c., then equation* (1) *has property* (P) *in the family of all u.s.c. functions $\Phi\colon X \to C(Y)$.*

Proof. In order to prove this theorem we must test that suitable families fulfil conditions (I) and (II′) of Lemmas 1 and 2.

1. Let $F_1 = \{\Phi\colon \Phi \subset \Phi_0,\ \Phi\colon X \to c(Y)\}$, where $\Phi_0\colon X \to c(Y)$ is a given solution of (3). For every $\Phi \in F_1$ and $x \in X$ the sets $\Phi[f(x)]$ are compact and $H(x, \Phi[f(x)])$ are closed, as the images of compact sets by closed mappings. The inclusions

$$H(x, \Phi[f(x)]) \subset H(x, \Phi_0[f(x)]) \subset \Phi_0(x) \tag{10}$$

imply that $H(x, \Phi[f(x)]) \in c(Y)$. Hence

$$H(\cdot, \Phi[f(\cdot)]) \in F_1.$$

We suppose that $\{\Phi_\iota\}_{\iota\in I}$ is a chain in F_1. Then $\Phi(x) = \bigcap_{\iota\in I} \Phi_\iota(x) \in c(Y)$ and $\Phi \subset \Phi_0$. Therefore conditions (I) and (II') of Lemmas 1 and 2 are fulfilled. The first part of the theorem is proved.

2. Let $f\colon X \to X$ and let H with connected values be u.s.c. with respect to the second variable. Let $F_2 = \{\Phi\colon \Phi \subset \Phi_0,\ \Phi\colon X \to C(Y)\}$, where $\Phi_0\colon X \to C(Y)$ is a solution of (3). The image of a connected set by an u.s.c. mapping with connected values is connected (cf. [4]). The set $H(x, \Phi[f(x)])$ is closed by Lemma 4, and by the inclusions (10) it is compact. Thus we have $H(\cdot, \Phi[f(\cdot)]) \in F_2$. The family F_2 has property (II'), because an intersection of a chain of continuums is a continuum (cf. [3]).

3. We suppose that X is a topological Hausdorff space, $f\colon X \to c(X)$ is u.s.c. and H is u.s.c.. Let $F_3 = \{\Phi\colon \Phi \subset \Phi_0,\ \Phi\colon X \to c(Y),\ \Phi \text{ is u.s.c.}\}$, where $\Phi_0\colon X \to c(Y)$ is an u.s.c. solution of (3). The set $\Phi[f(x)]$ is compact for all x because Φ is an u.s.c. mapping with compact values and $f(x)$ is compact. Since the composition of two u.s.c. mappings is an u.s.c. mapping (cf. [2]), then by Lemmas 3 and 4 and by (10) we get $H(\cdot, \Phi[f(\cdot)]) \in F_3$. The family F_3 has property (II') because the intersection of a chain of compact sets is compact and the intersection of a family of u.s.c. mappings with compact values is u.s.c. (cf. [2]).

4. We suppose that X is a topological Hausdorff space, $f\colon X \to C(X)$ is u.s.c. and H with connected values is u.s.c. Let $F_4 = \{\Phi\colon \Phi \subset \Phi_0,\ \Phi\colon X \to C(Y),\ \Phi \text{ is u.s.c.}\}$, where $\Phi_0\colon X \to C(Y)$ is an u.s.c. solution of (3). The family F_4 is included in F_3. It follows by the connectedness of $H(x, \Phi[f(x)])$ that $H(\cdot, \Phi[f(\cdot)]) \in F_4$. The family F_4 has property (II'), because the intersection of a chain of continuums is a continuum.

THEOREM 2. *Let $f\colon X \to X$ and $H\colon X \times \mathbf{R} \to \mathbf{R}$ be continuous and increasing with respect to the second variable. The function $\Phi\colon X \to C(\mathbf{R})$ is a solution of* (1) *if and only if $\varphi(x) = \min\Phi(x)$ and $\psi(x) = \max\Phi(x)$ fulfil* (1).

Proof. *Necessity.* The following equalities hold according to continuity and monotonicity of H with respect to the second variable and by (1):

$$\begin{aligned}[H(x, \varphi[f(x)]), H(x, \psi[f(x)])] &= H(x, [\varphi[f(x)], \psi[f(x)]]) \\ &= H(x, \Phi[f(x)]) = \Phi(x) = [\varphi(x), \psi(x)].\end{aligned}$$

Thus $H(x, \varphi[f(x)]) = \varphi(x)$ and $H(x, \psi[f(x)]) = \psi(x)$. Hence φ and ψ are solutions of (1).

The proof of the sufficiency is also easy.

As a corollary we get the following

THEOREM 3. *Let $f\colon X \to X$ and $H\colon X \times \mathbf{R} \to \mathbf{R}$ be an increasing and continuous function with respect to the second variable. If $\Phi_0\colon X \to C(\mathbf{R})$ is a solution of* (3), *then there exist a minimal solution $\Phi\colon X \to C(\mathbf{R})$ of* (1), *$\Phi \subset \Phi_0$, and this solution is a single-valued mapping.*

We have by a direct verification

LEMMA 5. *Let X be a topological space. If a mapping Φ: $X \to c(\boldsymbol{R})$ is u.s.c., then $\varphi(x) = \min \Phi(x)$ is lower semi-continuous and $\psi(x) = \max \Phi(x)$ is upper semi-continuous. A mapping Φ: $X \to C(\boldsymbol{R})$ is u.s.c. if and only if $\varphi(x) = \min \Phi(x)$ is lower semi-continuous and $\psi(x) = \max \Phi(x)$ is upper semi-continuous.*

DEFINITION. Let X and Y be topological spaces and let Φ: $X \to n(Y)$ be a mapping. The *closure of* Φ is the mapping $\mathrm{cl}\,\Phi$ defined by

$$\mathrm{cl}\,\Phi(x) = \{y \in Y\colon (x, y) \in \mathrm{cl}\,\Gamma_\Phi\} \quad \text{for } x \in X,$$

where Γ_Φ is the graph of Φ.

If X and Y are metric spaces, the above definition in equivalent to the condition:

$y \in \mathrm{cl}\,\Phi(x)$ if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$, $y_n \to y$ and $y_n \in \Phi(x_n)$ for every n.

LEMMA 6. *Let X be a topological space. Then the mapping* $\mathrm{cl}\,\Phi$ *is closed for every mapping Φ: $X \to n(Y)$.*

Proof. Let $y_0 \notin \mathrm{cl}\,\Phi(x_0)$. Then $(x_0, y_0) \notin \mathrm{cl}\,\Gamma_\Phi$. There exist neighbourhoods $U(x_0)$ and $V(y_0)$ such that

$$U(x_0) \times V(y_0) \cap \Gamma_\Phi = \emptyset. \tag{11}$$

We shall prove that

$$\mathrm{cl}\,\Phi(x) \cap V(y_0) = \emptyset$$

for every $x \in U(x_0)$. Let $z \in \mathrm{cl}\,\Phi(x) \cap V(y_0)$ for some $x \in U(x_0)$. Then $(x, z) \in \mathrm{cl}\,\Gamma_\Phi$. There exist points $x' \in U(x_0)$ and $z' \in V(y_0)$ such that $(x', z') \in \Gamma_\Phi$. Thus $(x', z') \in U(x_0) \times V(y_0) \cap \Gamma_\Phi$, but this is imposible by (11).

LEMMA 7. *Let X be a topological space and let Y be compact. If H: $X \times Y \to Y$ is continuous, f is an open and continuous function from X into X and if Φ: $X \to n(Y)$ is a solution of* (1), *then* $\mathrm{cl}\,\Phi$ *is an u.s.c. solution of* (1) *with compact values.*

Proof. The set $\mathrm{cl}\,\Phi(x)$ is closed, for the mapping $\mathrm{cl}\,\Phi$ is closed. Thus $\mathrm{cl}\,\Phi$ has compact values and is u.s.c. The mapping $\mathrm{cl}\,\Phi \circ f$ is u.s.c. as the composition of u.s.c. mappings. Further, $H(\cdot, \mathrm{cl}\,\Phi[f(\cdot)])$ is u.s.c. by Lemma 3 and by the continuity of H. The inclusion

$$\Phi(x) \subset H(x, \mathrm{cl}\,\Phi[f(x)])$$

implies

$$\mathrm{cl}\,\Phi(x) \subset \mathrm{cl}\,H(x, \mathrm{cl}\,\Phi[f(x)]) = H(x, \mathrm{cl}\,\Phi[f(x)]),$$

since an u.s.c. mapping with compact values is closed. Now, we shall prove, the inclusion

$$H(x, \operatorname{cl}\Phi[f(x)]) \subset \operatorname{cl}\Phi(x). \tag{12}$$

Let

$$y \in H(x, \operatorname{cl}\Phi[f(x)]) \tag{13}$$

and let U and V be any neighbourhoods of points x and y, respectively. It follows by (13) that there exists

$$z \in \operatorname{cl}\Phi[f(x)] \tag{14}$$

such that

$$y = H(x, z).$$

There exist neighbourhoods $U_1 \subset U$ and W of points x and z, respectively, such that

$$H(U_1 \times W) \subset V.$$

The function f is open; thus the set $f(U_1)$ is a neighbourhood of $f(x)$. Condition (14) implies

$$\Phi[f(U_1)] \cap W \neq \emptyset.$$

Hence, there exist $u \in U_1$ and $w \in W$ such that $w \in \Phi[f(u)]$. Consequently

$$H(u, w) \in H(u, \Phi[f(u)]) = \Phi(u)$$

and

$$H(u, w) \in H(U_1 \times W) \subset V.$$

Thus

$$\Phi(U) \cap V \neq \emptyset,$$

which completes the proof of (12).

LEMMA 8. *Let X be a topological space, $f\colon X \to X$ and let $H\colon X \times \mathbf{R} \to \mathbf{R}$ be continuous and increasing with respect to the second variable. If $\Phi\colon X \to c(\mathbf{R})$ is a solution of (1), then $\hat{\Phi}(x) := \operatorname{conv}[\Phi(x)]$ is a solution of (1) with compact and connected values. Moreover, if Φ is u.s.c., then $\hat{\Phi}$ is u.s.c., too.*

Proof. We define $\varphi(x) = \min\Phi(x)$, $\psi(x) = \max\Phi(x)$. By (1) and by the monotonicity of H with respect to the second variable we have

$$\begin{aligned} \hat{\Phi}(x) &= \operatorname{conv}[\Phi(x)] = \operatorname{conv}[H(x, \Phi[f(x)])] \\ &= [H(x, \varphi[f(x)]), H(x, \psi[f(x)])]. \end{aligned} \tag{15}$$

Since $\hat{\Phi}(x) = [\varphi(x), \psi(x)]$, we have

$$\begin{aligned} H(x, \hat{\Phi}[f(x)]) &= H(x, [\varphi[f(x)], \psi[f(x)]]) \\ &= [H(x, \varphi[f(x)]), H(x, \psi[f(x)])], \end{aligned} \tag{16}$$

by the continuity and monotonicity of H with respect to the second variable. From (15) and (16) we get

$$\hat{\Phi}(x) = H(x, \hat{\Phi}[f(x)]).$$

Lemma 5 implies that, if Φ is an u.s.c. function, then φ is lower semi-continuous and ψ is an upper semi-continuous numerical function. Thus $\hat{\Phi}(x) = [\varphi(x), \psi(x)]$ is u.s.c. by Lemma 5.

THEOREM 4. *Let X be a metric space and let $f\colon X \to X$ be a continuous and open function X into X. Moreover, let a continuous function $H\colon X \times R \to R$ be increasing with respect to the second variable. If $\Phi_0\colon X \to C(R)$ is an u.s.c. solution of inequality* (3), *and if there exists a compact set $K \subset R$ such that $\Phi_0(X) \subset K$, then a minimal u.s.c. solution $\Phi\colon X \to C(R)$ of* (1) *such that $\Phi \subset \Phi_0$ is almost everywhere (everywhere except a set of the first category) a single-valued mapping.*

Proof. The existence of a minimal u.s.c. solution $\Phi\colon X \to C(R)$ with $\Phi \subset \Phi_0$ is guaranteed by Theorem 1. If $\Phi(x) = [\varphi(x), \psi(x)]$, then (by Theorem 2) the functions φ and ψ fulfil equation (1). According to Lemma 7 the functions $\operatorname{cl}\varphi$ and $\operatorname{cl}\psi$ are u.s.c. solutions of (1). Lemma 8 implies that $\Phi_1(x) = \operatorname{conv}[\operatorname{cl}\varphi(x)]$ and $\Phi_2(x) = \operatorname{conv}[\operatorname{cl}\psi(x)]$ are u.s.c. solution of (1) and $\Phi_1\colon X \to C(R)$, $\Phi_2\colon X \to C(R)$. By the minimality of Φ we have $\Phi_1 = \Phi_2 = \Phi$. The equality $\operatorname{conv}[\operatorname{cl}\varphi(x)] = [\varphi(x), \psi(x)]$ implies that $\psi(x) \in \operatorname{cl}\varphi(x)$. This means that there exist sequences $\{y_n\}$, $\{x_n\}$ such that $y_n \to \psi(x)$, $x_n \to x$, and $y_n = \varphi(x_n)$. Thus there exists a sequence $\{x_n\}$ such that $x_n \to x$ and $\varphi(x_n) \to \psi(x)$. If φ is continuous at x, then $\varphi(x) = \psi(x)$. In virtue of the Baire theorem the set of all x at which φ is not continuous is of the first category.

Let Φ_0 be an u.s.c. solution of (3) with values in $C(R)$. K. Baron in [1] proved (under suitable assumptions) the existence of a single-valued u.s.c. solution Φ of (1) such that $\Phi \subset \Phi_0$. The following example indicates that the assumptions of Theorem 4 do not ensure the existence of an u.s.c. single-valued solution Φ of (1) such that $\Phi \subset \Phi_0$.

EXAMPLE. Let $X = Y = R$, $f(x) = 2x$, $H(x, y) = y$ and let Φ_0 be given by the formula

$$\Phi_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ [0, 1] & \text{for } x = 0, \\ 1-|2^{n+2}x-3| & \text{for } x \in [2^{-(n+1)}, 2^{-n}],\ n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

This mapping is an u.s.c. solution of (1) with compact and connected values. Thus Φ_0 fulfils (3), but equation (1) does not possess any u.s.c. single-valued solution.

References

[1] K. Baron, *On the continuous solutions of a non-linear functional equation of the first order*, Ann. Polon. Math. 28 (1973), p. 201–205.
[2] C. Berge, *Topological spaces*, Edinburgh and London 1963.
[3] R. Engelking, *General topology*, Warszawa 1977.
[4] R. E. Smithson, *Multifunction*, Nieuw. Arch. Wisk. 20 (1972), p. 31–53.

Reçu par la Rédaction le 1.12.1977