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Multi-variable Conformable Fractional Calculus

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Abstract. Conformable fractional derivative is introduced by the authors Khalil et al. In this study we develop their concept and introduce multi-variable conformable derivative for a vector valued function with several variables.

1. Introduction

For many years, many definitions of fractional derivative have been introduced by various researchers. One of them is the Riemann-Liouville fractional derivative and the second one is the so-called Caputo derivative. But these are not all purpose definitions. Recently, a new fractional derivative has been introduced in [5] and one can see that the new derivative suggested in this paper satisfies all the properties of the standard one. Applications of new defined fractional derivative called conformable fractional derivative are studied in papers [2–4]. However, definitions given in the literature are only for the real valued functions. In this paper, we introduce conformable fractional derivative definition for the vector valued functions with several real variables. Our paper has the following order: In Section 2, some basic definitions and theorems appeared in the literature are given. In Section 3, $\alpha-$derivative of a vector valued function, conformable Jacobian matrix are defined; relation between $\alpha-$derivative and usual derivative of a vector valued function is revealed; chain rule for multi-variable conformable derivative is given. In Section 4, conformable partial derivatives of a real valued function with $n-$variables is defined and relation between conformable Jacobian matrix and conformable partial derivatives is given.

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2. Basic Definitions and Theorems

In this section we will give some definitions and properties introduced in [1, 5]

Definition 2.1. *Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. The conformable derivative of the function f of order α is defined by*

$$T_\alpha(f)(x) = \lim_{h\to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h} \tag{1}$$

for all $x > 0$, $\alpha \in (0,1)$. If the conformable derivative of a function f exists for an $\alpha \in (0,1)$, then the function f is called $\alpha-$differentiable.

Theorem 2.2. *[5] If a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is $\alpha-$differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Theorem 2.3. *[1] Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.

(2) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

(3) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{d}{dt}f(t)$.

Theorem 2.4. *Assume $f, g : (0,\infty) \longrightarrow \mathbb{R}$ be two $\alpha-$differentiable functions where $\alpha \in (0,1]$. Then $g \circ f$ is $\alpha-$differentiable and for all t with $t \neq 0$ and $f(t) \neq 0$ we have*

$$T_\alpha(g \circ f)(t) = T_\alpha(g)(f(t))T_\alpha(f)(t)f(t)^{\alpha-1}. \tag{2}$$

3. $\alpha-$Derivative of a Vector Valued Function

Definition 3.1. *Let f be a vector valued function with n real variables such that $f(x_1,...,x_n) = (f_1(x_1,...,x_n),...,f_m(x_1,...,x_n))$. Then we say that f is $\alpha-$differentiable at $a = (a_1,...,a_n) \in \mathbb{R}^n$ where each $a_i > 0$, if there is a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f(a_1,...,a_n) - L(h)\|}{\|h\|} = 0 \tag{3}$$

where $h = (h_1,...,h_n)$ and $\alpha \in (0,1]$. The linear transformation is denoted by $D^\alpha f(a)$ and called the conformable derivative of f of order α at a.

Remark 3.2. *For $m = n = 1$, Definition 3.1 equivalent to Definition 2.1.*

Theorem 3.3. *Let f be a vector valued function with n variables. If f is $\alpha-$differentiable at $a = (a_1,...,a_n) \in \mathbb{R}^n$, each $a_i > 0$, then there is a unique linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f(a_1,...,a_n) - L(h)\|}{\|h\|} = 0.$$

Proof. Suppose $M : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n) - M(h)\|}{\|h\|} = 0.$$

Then,

$$\lim_{h\to 0} \frac{\|L(h) - M(h)\|}{\|h\|}$$
$$\leq \lim_{h\to 0} \frac{\|L(h) - (f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a))\|}{\|h\|}$$
$$+ \lim_{h\to 0} \frac{\|(f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a)) - M(h)\|}{\|h\|} = 0$$

If $x \in \mathbb{R}^n$, then $tx \to 0$ as $t \to 0$. Hence for $x \neq 0$ we have

$$0 = \lim_{h\to 0} \frac{\|L(tx) - M(tx)\|}{\|tx\|} = \frac{\|L(x) - M(x)\|}{\|x\|}.$$

Therefore $L(x) = M(x)$. □

Example 3.4. *Let us consider the function f defined by $f(x, y) = sinx$ and the point $(a, b) \in \mathbb{R}^2$ such that $a, b > 0$, then $D^\alpha f(a, b) = L$ satisfies $L(x, y) = xa^{1-\alpha}\cos a$.*

To prove this, note that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{| f(a + h_1 a^{1-\alpha}, b + h_2 b^{1-\alpha}) - f(a, b) - L(h_1, h_2) |}{\|(h_1, h_2)\|}$$
$$= \lim_{(h_1,h_2)\to(0,0)} \frac{| \sin(a + h_1 a^{1-\alpha}) - \sin a - h_1 a^{1-\alpha}\cos a |}{\sqrt{h_1^2 + h_2^2}}$$
$$\leq \lim_{h_1\to 0} \frac{| \sin(a + h_1 a^{1-\alpha}) - \sin a - h_1 a^{1-\alpha}\cos a |}{| h_1 |}$$
$$= \lim_{h_1\to 0} \left| \frac{\sin(a + h_1 a^{1-\alpha}) - \sin a}{h_1} - a^{1-\alpha}\cos a \right|$$
$$=| a^{1-\alpha}\cos a - a^{1-\alpha}\cos a |= 0$$

Definition 3.5. *Consider the matrix of the linear transformation $D^\alpha f(a) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. This $m \times n$ matrix is called the conformable Jacobian matrix of f at a, and denoted by $f^\alpha(a)$.*

Example 3.6. *If $f(x, y) = \sin x$, then $f^\alpha(a, b) = \begin{bmatrix} a^{1-\alpha}\cos a & 0 \end{bmatrix}$*

Theorem 3.7. *Let f be α−differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$. If f is differentiable at a, then*

$$D^\alpha f(a) = Df(a) \circ L_a^{1-\alpha}$$

where $Df(a)$ is the usual derivative of f and $L_a^{1-\alpha}$ is the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $L_a^{1-\alpha}(x_1, ..., x_n) = (a_1^{1-\alpha}x_1, ..., a_n^{1-\alpha}x_n)$.

Proof. It is suffices to show that

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n) - Df(a) \circ L_a^{1-\alpha}(h)\|}{\|h\|} = 0.$$

Let $\epsilon = (\epsilon_1, ..., \epsilon_n) = (h_1 a_1^{1-\alpha}, ..., h_n a_n^{1-\alpha})$, then $\epsilon \to 0$ as $h \to 0$. On the other hand, if we put $M = \max\{(a_i^{1-\alpha})^2 \mid a_i > 0, i = 1, 2, ..., n\} > 0$, then

$$||\epsilon|| = \sqrt{h_1^2(a_1^{1-\alpha})^2 + ... + h_n^2(a_n^{1-\alpha})^2} \leq \sqrt{h_1^2 M + ... + h_n^2 M} = \sqrt{nM}||h||.$$

Hence we have

$$\frac{1}{\sqrt{nM}}||\epsilon|| \leq ||h||.$$

Finally,

$$\begin{aligned}
&\lim_{h\to 0} \frac{||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n) - Df(a) \circ L_a^{1-\alpha}(h)||}{||h||} \\
&= \lim_{h\to 0} \frac{||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - Df(a)(\epsilon)||}{||h||} \\
&\leq \lim_{\epsilon\to 0} \frac{||f(a+\epsilon) - f(a) - Df(a)(\epsilon)||}{\frac{1}{\sqrt{nM}}||\epsilon||} \\
&= \sqrt{nM} \lim_{\epsilon\to 0} \frac{||f(a+\epsilon) - f(a) - Df(a)(\epsilon)||}{||\epsilon||} = \sqrt{nM}.0 = 0.
\end{aligned}$$

This completes the proof. □

Theorem 3.7 is the generalized case of the part (6) of Theorem 2.3. Also matrix form of the Theorem 3.7 is given by the following:

$$f^\alpha(a) = f'(a).\begin{pmatrix} a_1^{1-\alpha} & 0 & ... & 0 \\ 0 & a_1^{1-\alpha} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_n^{1-\alpha} \end{pmatrix},$$

where $f'(a)$ is the usual Jacobian of f and $\begin{pmatrix} a_1^{1-\alpha} & 0 & ... & 0 \\ 0 & a_1^{1-\alpha} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_n^{1-\alpha} \end{pmatrix}$ is the matrix corresponding to linear transformation $L_a^{1-\alpha}$.

Theorem 3.8. *If a vector valued function f with n variables is α−differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$, then f is continuous at $a \in \mathbb{R}^n$.*

Proof. Since

$$\begin{aligned}
&||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n)|| \\
&= \frac{||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - L(h) + L(h)||}{||h||}||h|| \\
&\leq \frac{||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - L(h)||}{||h||}||h|| + ||L(h)||.
\end{aligned}$$

We have

$$\begin{aligned}
&||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n)|| \\
&\leq \frac{||f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - L(h)||}{||h||}||h|| + ||L(h)||.
\end{aligned}$$

By taking limits of the two sides of the inequality as $h \to 0$, we have

$$\lim_{h\to 0} \|f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a_1, ..., a_n)\|$$
$$\leq \lim_{h\to 0} \frac{\|f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - L(h)\|}{\|h\|} \lim_{h\to 0} \|h\|$$
$$+ \lim_{h\to 0} \|L(h)\| = 0.$$

Let $(\epsilon_1, ..., \epsilon_n) = (h_1 a_1^{1-\alpha}, ..., h_n a_n^{1-\alpha})$, then $\epsilon \to 0$ as $h \to 0$. Since

$$\lim_{\epsilon\to 0} \|f(a + \epsilon) - f(a)\| \leq 0$$

we have

$$\lim_{\epsilon\to 0} \|f(a + \epsilon) - f(a)\| = 0.$$

Hence f is continuous at $a \in \mathbb{R}^n$. □

Theorem 3.9. *(Chain Rule) Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. If $f(x) = (f_1(x), ..., f_m(x))$ is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$ such that $\alpha \in (0, 1]$ and $g(y) = (g_1(y), ..., g_p(y))$ is $\alpha-$differentiable at $f(a) \in \mathbb{R}^m$, all $f_i(a) > 0$ such that $\alpha \in (0, 1]$. Then the composition $g \circ f$ is $\alpha-$differentiable at a and*

$$D^\alpha(g \circ f)(a) = D^\alpha g(f(a)) \circ f(a)^{\alpha-1} \circ D^\alpha f(a) \tag{4}$$

where $f(a)^{\alpha-1}$ is the linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$ defined by
$f(a)^{\alpha-1}(x_1, ..., x_m) = (x_1 f_1(a)^{\alpha-1}, ..., x_m f_m(a)^{\alpha-1})$.

Proof. Let $L = D^\alpha f(a)$, $M = D^\alpha g(f(a))$. If we define

(i) $\varphi(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha})$
$= f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - f(a) - L(h)$,

(ii) $\psi(f_1(a) + k_1 f_1(a)^{1-\alpha}, ..., f_n(a) + k_n f_n(a)^{1-\alpha})$
$= g(f_1(a) + k_1 f_1(a)^{1-\alpha}, ..., f_n(a) + k_n f_n(a)^{1-\alpha}) - g(f(a)) - M(k)$,

(iii) $\rho(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha})$
$= g \circ f(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha}) - g \circ f(a) - M \circ f(a)^{\alpha-1} \circ L(h)$,

then

(iv) $\lim_{h\to 0} \dfrac{\|\varphi(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha})\|}{\|h\|} = 0$,

(v) $\lim_{k\to 0} \dfrac{\|\psi(f_1(a) + k_1 f_1(a)^{1-\alpha}, ..., f_n(a) + k_n f_n(a)^{1-\alpha})\|}{\|k\|} = 0$,

and we must show that

$$lim_{h\to 0} \frac{\|\rho(a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha})\|}{\|h\|} = 0$$

Now,

$$\begin{aligned}
&\rho(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})\\
&=g(f(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))-g(f(a))-M\circ f(a)^{\alpha-1}\circ L(h)\\
&=g(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}),...,f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&-g(f(a))-M\circ f(a)^{\alpha-1}(f(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f(a)\\
&-\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\qquad by(i)\\
&=[g(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}),...,f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&-g(f(a))-M(f(a)^{\alpha-1}(f(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f(a)))]\\
&+M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&=[g(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}),...,f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&-g(f(a))-M(f(a)^{\alpha-1}(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f_1(a),...,\\
&f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f_m(a)))]\\
&+M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&\\
&=[g(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}),...,f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&-g(f(a))-M([f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f_1(a)]f_1(a)^{\alpha-1},...,\\
&[f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f_m(a)]f_m(a)^{\alpha-1})]\\
&+M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))
\end{aligned}$$

If we put $u_i=[f_i(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})-f_i(a)]f_i(a)^{\alpha-1}$, $i=1,...,n$, then we have $f_i(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})=f_i(a)+u_if_i(a)^{1-\alpha}$ and $u\to 0$ as $h\to 0$. Therefore,

$$\begin{aligned}
&\rho(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})\\
&=[g(f_1(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}),...,f_m(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&-g(f(a))-M(u)]+M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\\
&=\psi(f_1(a)+u_1f_1(a)^{1-\alpha},...,f_m(a)+u_mf_m(a)^{1-\alpha})\\
&+M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\qquad by(ii).
\end{aligned}$$

Thus we must show

(vi) $\lim_{u\to 0}\dfrac{\|\psi(f_1(a)+u_1f_1(a)^{1-\alpha},...,f_m(a)+u_mf_m(a)^{1-\alpha})\|}{\|u\|}=0,$

(vii) $\lim_{h\to 0}\dfrac{\|M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\|}{\|h\|}=0.$

For (vi), it is obvious from (v). For (vii), the linear transformation satisfies

$$\begin{aligned}
&\|M\circ f(a)^{\alpha-1}(\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha}))\|\\
&\qquad\le K\|\varphi(a_1+h_1a_1^{1-\alpha},...,a_n+h_na_n^{1-\alpha})\|
\end{aligned}$$

such that $K>0$. Hence, from (iv), (vii) holds. □

Corollary 3.10. *For, $m=n=p=1$, Theorem 3.9 states that*

$$T_\alpha(g\circ f)(a)=T_\alpha g(f(a))T_\alpha f(a)f(a)^{\alpha-1}.$$

Above corollary says that Theorem 3.9 is the generalized case of Theorem 2.4.

Corollary 3.11. *Let all conditions of Theorem 3.9 be satisfied. Then*

$$(g \circ f)^{\alpha}(a) = g^{\alpha}(f(a)) \begin{pmatrix} f_1(a)^{\alpha-1} & 0 & \dots & 0 \\ 0 & f_2(a)^{\alpha-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_m(a)^{\alpha-1} \end{pmatrix} f^{\alpha}(a)$$

where $\begin{pmatrix} f_1(a)^{\alpha-1} & 0 & \dots & 0 \\ 0 & f_2(a)^{\alpha-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_m(a)^{\alpha-1} \end{pmatrix}$ *is the matrix corresponding to the linear transformation* $f(a)^{\alpha-1}$.

Theorem 3.12. *Let f be a vector valued function with n variables such that $f(x_1,...,x_n) = (f_1(x_1,...,x_n),...,f_m(x_1,...,x_n))$. Then f is $\alpha-$differentiable at $a = (a_1,...,a_n) \in \mathbb{R}^n$, each $a_i > 0$ if and only if each f_i is, and*

$$D^{\alpha}f(a) = (D^{\alpha}f_1(a),...,D^{\alpha}f_m(a)).$$

Proof. If each f_i is $\alpha-$differentiable at a and $L = (D^{\alpha}f_1(a),...,D^{\alpha}f_m(a))$, then

$$\begin{aligned} & f(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f(a) - L(h) \\ &= (f_1(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f_1(a) - D^{\alpha}f_1(a)(h),..., \\ & f_m(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f_m(a) - D^{\alpha}f_m(a)(h)). \end{aligned}$$

Therefore,

$$\begin{aligned} & \lim_{h\to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f(a) - L(h)\|}{\|h\|} \\ & \leq \lim_{h\to 0} \sum_{i=1}^{n} \frac{\|f_i(a_1 + h_1a_1^{1-\alpha},...,a_n + h_na_n^{1-\alpha}) - f_i(a) - D^{\alpha}f_i(a)(h)\|}{\|h\|} = 0. \end{aligned}$$

If f is $\alpha-$differentiable at a, then $f_i = \pi_i \circ f$ is $\alpha-$differentiable at a by Theorem 3.9. □

Theorem 3.13. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $a = (a_1,...,a_n) \in \mathbb{R}^n$, each $a_i > 0$. Then*

(i) $D^{\alpha}(\lambda f + \mu g)(a) = \lambda D^{\alpha}f(a) + \mu D^{\alpha}g(a)$ *for all* $\lambda, \mu \in \mathbb{R}$.

(ii) $D^{\alpha}(fg)(a) = f(a)D^{\alpha}g(a) + g(a)D^{\alpha}f(a)$.

Proof. (i) follows from the definition, thus we omitted the proof of (i).

For (ii), let $a_1 + h_1 a_1^{1-\alpha}, ..., a_n + h_n a_n^{1-\alpha} = A$, then

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|(fg)(A) - (fg)(a) - (f(a)D^\alpha g(a) + g(a)D^\alpha f(a))(h)\|}{\|h\|} \\
&\leq \lim_{h\to 0} \frac{\|f(A)g(A) - f(a)g(A) - g(A)D^\alpha f(a)(h)\|}{\|h\|} \\
&+ \lim_{h\to 0} \frac{\|f(a)g(A) - f(a)g(a) - f(a)D^\alpha g(a)(h)\|}{\|h\|} \\
&+ \lim_{h\to 0} \frac{\|g(A)D^\alpha f(a)(h) - g(a)D^\alpha f(a)(h)\|}{\|h\|} \\
&= \lim_{h\to 0} \|g(A)\| \frac{\|f(A) - f(a) - D^\alpha f(a)(h)\|}{\|h\|} \\
&+ \lim_{h\to 0} \|f(a)\| \frac{\|g(A) - g(a) - D^\alpha g(a)(h)\|}{\|h\|} + \lim_{h\to 0} \|D^\alpha f(a)(h)\| \frac{\|g(A) - g(a)\|}{\|h\|} \\
&\leq \lim_{h\to 0} K\|h\| \frac{\|g(A) - g(a)\|}{\|h\|} = 0
\end{aligned}$$

This completes the proof. □

4. Conformable Partial Derivatives

In this section we introduce the definition of conformable partial derivative of a real valued function with n variables by the following.

Definition 4.1. *Let f be a real valued function with n variables and $a = (a_1, ..., a_n) \in \mathbb{R}^n$ be a point whose ith component is positive. Then the limit*

$$\lim_{\epsilon\to 0} \frac{f(a_1, ..., a_i + \epsilon a_i^{1-\alpha}, ..., a_n) - f(a_1, ..., a_n)}{\epsilon}, \tag{5}$$

if it exists, is denoted by $\frac{\partial^\alpha}{\partial x^\alpha} f(a)$, and called the ith conformable partial derivative of f of order $\alpha \in (0, 1]$ at a.

Theorem 4.2. *Let f be a vector valued function with n variables. If f is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$, then $\frac{\partial^\alpha}{\partial x_j^\alpha} f_i(a)$ exists for $1 \leq i \leq m$, $1 \leq j \leq n$ and the conformable Jacobian of f at a is the $m \times n$ matrix $\left(\frac{\partial^\alpha}{\partial x_j^\alpha} f_i(a)\right)$.*

Proof. Let $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$. Suppose first that $m = 1$, so $f(x_1, ..., x_n) \in \mathbb{R}$. Define $h : \mathbb{R} \longrightarrow \mathbb{R}^n$ by $h(y) = (a_1, ..., y, ..., a_n)$ with y in the jth place. Then $\frac{\partial^\alpha}{\partial x_j^\alpha} f_i(a) = D^\alpha(f \circ h)(a_j)$. Hence, by

Corollary 3.11,

$$(f \circ h)^\alpha(a_j) = f^\alpha(h(a_j)) \begin{pmatrix} h_1(a_j)^{\alpha-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \dots & h_j(a_j)^{\alpha-1} \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_n(a_j)^{\alpha-1} \end{pmatrix} h^\alpha(a_j)$$

$$= f^\alpha(a) \begin{pmatrix} a_1^{\alpha-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \dots & a_j^{\alpha-1} \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n^{\alpha-1} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ a_j^{1-\alpha} \\ \vdots \\ 0 \end{pmatrix} = f^\alpha(a) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $(f \circ h)^\alpha(a_j)$ has the single entry $\frac{\partial^\alpha}{\partial x_j^\alpha} f_i(a)$, this shows that $\frac{\partial^\alpha}{\partial x_j^\alpha} f_i(a)$ exists and is the jth entry of the $1 \times n$ matrix $f^\alpha(a)$.

The theorem now follows for arbitrary m since, by Theorem 3.12, each f_i is $\alpha-$differentiable and the ith row of $f^\alpha(a)$ is $(f_i)^\alpha(a)$. □

5. Conclusion

In our study, we have extended the idea given by Khalil at al. in [5] and called it multi-variable conformable fractional calculus. Multi-variable conformable fractional derivative has many interesting properties and is related to classical multi-variable calculus. We have given important theorems and corollaries which reveal this relation and we have obtained useful results.

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References

[1] T. Abdeljawad, On conformable fractional calculus, Journal of Computational and Applied Mathematics 279 (2015) 57–66.
[2] D.R. Anderson, D.J. Ulness, Newly defined conformable derivatives, Advances in Dynamical Systems and Applications 10(2) (2015) 109–137.
[3] N. Benkhettou, S. Hassani, D.F.M. Torres, A conformable fractional calculus on arbitrary time scales, Journal of King Saud University 28(1) (2016) 93–98.
[4] H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, Three-point boundary value problems for conformable fractional differential equations, 2015 (2015) 1–7.
[5] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics 264 (2014) 65–70.