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MULTI-VARIATE STOPPING PROBLEMS WITH A MONOTONE RULE

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Abstract A monotone rule is introduced to sum up individual declarations in a multi-variate stopping problem. The rule is defined by a monotone logical function and is equivalent to the winning class of Kadane. This paper generalizes the previous works on a majority rule. The existence of an equilibrium stopping strategy and the associated gain are discussed for the finite and infinite horizon cases.

1. Formulation

Let X_n , $n \geqq 1$, be p-dimentional random vectors on a probability space $(\Omega, \mathcal{B}, P)$. The process $\{X_n\}$ can be interpreted as a sequence of payoffs to a group of p players. Each of p players observes sequentially the values of $\{X_n\}$. Its distribution is assumed to be known to all of the p players. A player must make a declaration to either "stop" or "continue" on the basis of the observed value at each stage. A group decision whether to stop the process or not is determined by summing up from the individual declarations.

If the group decision is to stop at stage n , then the player i 's net gain is

(1.1) $\quad Y_n^i = X_n^i - n c^i$

where c^i is a constant observation cost. According to the individual declarations, let us define a random variable d_n^i , $n \geqq 1$, $i=1,\dots,p$ by

(1.2) $\quad d_n^i = 1\ (0)$ if player i declares to stop (continue).

We assume, for each n and i ,

(1.3) $\quad d_n^i \in \mathcal{B}(X_n)$

where $\mathcal{B}(X_n)$ denotes the σ-algebra generated by X_n.

Definition 1.1. An individual (stopping) strategy is a sequence of random variables

$$(1.4) \qquad d^i = (d_1^i, d_2^i, \ldots, d_n^i, \ldots)$$

satisfying (1.3). $\mathcal{D}^i$ denotes the set of all individual strategies for player i. A p-dimentional and $\{0,1\}$-valued random vector

$$(1.5) \qquad d_n = (d_n^1, d_n^2, \ldots, d_n^p)$$

denotes the declarations of p players at stage n. A (stopping) strategy is the sequence

$$(1.6) \qquad d = (d_1, d_2, \ldots, d_n, \ldots)$$

and $\mathcal{D}$ denotes the whole set of the strategies.

Now we shall define a stopping rule by which the group decision is determined from the declarations of p players at each stage. A p-variate and $\{0,1\}$-valued logical function

$$(1.7) \qquad \pi = \pi(x^1, \ldots, x^p) \quad : \{0,1\}^p \to \{0,1\}$$

is said to be monotone (cf. Fishburn [2]) if

$$(1.8) \qquad \pi(x^1, \ldots, x^p) \leqq \pi(y^1, \ldots, y^p)$$

whenever $x^i \leqq y^i$ for each i.

Definition 1.2. A monotone rule is a non-constant logical function π, which is

(i) monotone with

(ii) $\pi(1,1,\ldots,1) = 1$.

In this paper a rule does not mean "when to stop the process" but means "how to sum up" the whole players' declarations. The property (ii) is called unanimity in Fishburn [2]. Its dual property $\pi(0,0,\ldots,0) = 0$ is not needed to be assumed here. A constant function makes the problem trivial because the decision is always to stop from (ii).

The monotone rule has a wide variety in choice systems of our real life. Some examples for the monotone rule are given as follows.

Example 1.1. (i) (Equal majority rule) In a group of p players, if no less than r ($\leqq p$) members declare to stop, then the group decision is to stop the process. That is,

$$(1.9) \qquad \pi(d_n^1, \ldots, d_n^p) = 1 \ (0) \qquad \text{if} \quad \textstyle\sum_{i=1}^p d_n^i \geqq (<) \ r.$$

For instance, a simple majority rule for three players, i.e., $(p,r)=(3,2)$, is

$$\pi(d_n^1,d_n^2,d_n^3) = d_n^1\cdot d_n^2 + d_n^2\cdot d_n^3 + d_n^3\cdot d_n^1$$

where + is a logical sum and $\cdot$ is a logical product. The stopping problem of the majority rule is discussed in Kurano, Yasuda and Nakagami [5] .

(ii) (Unequal majority rule) A straightforward extension of (1.9) is

$$\pi(d_n^1,\ldots,d_n^p) = 1\ (0) \quad \text{if} \quad \textstyle\sum_{i=1}^p w^i d_n^i \geqq (<)\ r, \tag{1.10}$$

where $w^i \geqq 0$, $i=1,\ldots,p$, are given weighting constants. See Table 3.1 in Section 3 for several rules with $p=3$.

(iii) (Hierarchical rule) A hierarchical system or Murakami's representative system (cf.Fishburn [2]) is regarded as a composed rule. Since the composition of two monotone logical functions is monotone and satisfies the property (ii) of Def.1.2, the hierarchical rule is also a monotone rule.

Definition 1.3. For a strategy $d=(d_1,d_2,\ldots)\in\mathcal{D}$ with $d_n=(d_n^1,\ldots,d_n^p)$, $n\geqq 1$ and a monotone rule π, a stopping time $t_\pi(d)$ is defined by

$$t_\pi(d) = \begin{cases} \text{first } n \geqq 1 \text{ such that } \pi(d_n^1,\ldots,d_n^p) = 1 \\ \infty \quad \text{if no such } n \text{ exists.} \end{cases} \tag{1.11}$$

For any stopping time $t_\pi(d)$, let

$$Y^i(t_\pi(d)) = \begin{cases} Y_n^i & \text{if } t_\pi(d) = n, \\ \limsup_{n\to\infty} Y_n^i & \text{if } t_\pi(d) = \infty. \end{cases} \tag{1.12}$$

When the group decision is to stop at the time $t_\pi(d)$, player i gets $Y^i(t_\pi(d))$ as a net gain.

Definition 1.4. Let π be a monotone rule. We call $*d=(*d^1,\ldots,*d^p)$ an equilibrium strategy with respect to π if, for each i and any $d^i\in\mathcal{D}^i$,

$$E[Y^i(t_\pi(*d))] \geqq E[Y^i(t_\pi(*d(i)))] \tag{1.13}$$

where $*d(i) = (*d^1,\ldots,*d^{i-1},d^i,*d^{i+1},\ldots,*d^p)$.

In this paper we treat a vector valued expected net gain

$$E[Y(t_\pi(d))], \quad d\in\mathcal{D} \tag{1.14}$$

and our objective is to find an equilibrium strategy $*d\in\mathcal{D}$ for a given monotone rule π. The notion of the equilibrium owes to the non-cooperative game theory by Nash [6].

In order to denote a stopping event of the process for a given rule, we need a set-valued function on $\mathcal{B}^p(X_n)$. For $d=(d_1, d_2,\ldots)$, we shall call

$$D_n^i = \{\omega\in\Omega \mid d_n^i(\omega) = 1\} \in \mathcal{B}(X_n) \tag{1.15}$$

an individual stopping event for player i at stage n. If D_n^i occurs, i.e., $\omega \in D_n^i$, then player i declares to stop. So

(1.16) $$d_n^i = I_{D_n^i}$$

where I_D is the indicator of a set D on Ω. Hence there exists a set-valued function Π on $\mathcal{B}^p(X_n)$ corresponding to a logical function π on $\{0,1\}^p$, such that

(1.17) $$\pi(d_n^1,\ldots,d_n^p) = \pi(I_{D_n^1},\ldots,I_{D_n^p}) = I_{\Pi(D_n^1,\ldots,D_n^p)}$$

Clearly two functions π and Π are related to each other. For example,

$$\pi(d_n^1,d_n^2,d_n^3) = d_n^1 + d_n^2\cdot d_n^3$$

corresponds to

$$\Pi(D_n^1,D_n^2,D_n^3) = D_n^1 \cup (D_n^2 \cap D_n^3).$$

The stopping event of the process at stage n is denoted by

(1.18) $$D_n = \{\omega \in \Omega \mid \pi(d_n^1,\ldots,d_n^p) = 1\} = \Pi(D_n^1,\ldots,D_n^p).$$

We note that, if π is monotone, $A^i \subset B^i$ for each i implies

(1.19) $$\Pi(A^1,\ldots,A^p) \subset \Pi(B^1,\ldots,B^p)$$

from (1.18).

For a given (monotone) rule π, a corresponding set-valued function Π is determined only by the union and the intersection of sets.

Next, a one-stage stopping model is considered to clarify a strategy of our problem. Each player observes a random variable $X = (X^1,\ldots,X^p)$ with $\mathrm{E}|X^i|<\infty$, and player i receives a net gain $X^i - c^i$ if the group decision is to stop, or $v^i - c^i$ if not, where v^i is a given constsnt. For a monotone rule π, the stopping event of the process becomes $\Pi(D^1,\ldots,D^p)$ for $D^i \in \mathcal{D}^i$, $i=1,\ldots,p$. Then the expected net gain for player i is expressed by

(1.20) $$\begin{aligned}&\mathrm{E}[(X^i-c^i)I_{\Pi(D^1,\ldots,D^p)}] + \mathrm{P}(\overline{\Pi(D^1,\ldots,D^p)})(v^i - c^i)\\ &= \mathrm{E}[(X^i - v^i)I_{\Pi(D^1,\ldots,D^p)}] + v^i - c^i.\end{aligned}$$

Since a logical function can be written generally as

$$\pi(x^1,\ldots,x^p) = x^i\cdot \pi(x^1,\ldots,\overset{i}{1},\ldots,x^p) + \bar{x}^i\cdot \pi(x^1,\ldots,\overset{i}{0},\ldots,x^p),$$

it holds

(1.21) $$\Pi(D^1,\ldots,D^p) = \{D^i \cap \Pi(D^1,\ldots,\Omega,\ldots,D^p)\} \cup \{\bar{D}^i \cap \Pi(D^1,\ldots,\phi,\ldots,D^p)\}$$

in terms of the events. A substitution of this for the last expression of (1.20) yields

$$(1.22)\quad \int_{D^i}(X^i - v^i)\{I_{\Pi(D^1,\ldots,\Omega,\ldots,D^p)} - I_{\Pi(D^1,\ldots,\phi,\ldots,D^p)}\}dP + \int_{\Pi(D^1,\ldots,\phi,\ldots,D^p)}(X^i - v^i)dP + v^i - c^i.$$

By (1.19), it is clear that $I_{\Pi(D^1,\ldots,\Omega,\ldots,D^p)} - I_{\Pi(D^1,\ldots,\phi,\ldots,D^p)} \geqq 0$. Therefore we can derive the next proposition.

Proposition 1.1. When $D^1,\ldots, D^{i-1}, D^{i+1},\ldots,D^p$ are fixed, the player i's maximum expected net gain subject to $D^i \in \mathcal{B}(X)$ is attained by

$$(1.23)\quad {}^*D^i = \{X^i \geqq v^i\},$$

and it equals

$$(1.24)\quad \int_\Omega (X^i - v^i)^+ I_{\Pi(D^1,\ldots,\Omega,\ldots,D^p)}dP - \int_\Omega (X^i - v^i)^- I_{\Pi(D^1,\ldots,\phi,\ldots,D^p)}dP + v^i - c^i,$$

where $x^+ = \max(x,0)$ and $x^- = \max(-x,0)$. Especially, when $\Pi(D^1,\ldots,\Omega,\ldots,D^p) = \Pi(D^1,\ldots,\phi,\ldots,D^p)$, player i's expected net gain (1.22) or (1.24) is constant not depending on D^i.

By Prop.1.1, we have solved a one-stage problem where the seeking equilibrium strategy is given as (1.23) and we have shown that the player i's individual strategy depends only on the i-th component X^i of the p-dimensional vector X. Because the larger he perceives his own value, the larger will be his net gain, he is eager to declare to stop when his observed value is high. This situation holds under a monotonicity of the rule, but does not hold under other rules including a negation. It is known that the monotone logical function does not include a negation and vice versa. Another essential point is the "non-cooperative" character in a reward, so other players' net gains do not affect his gain. Therefore, he observes his own value closely.

In the end of this section we shall now refer to the winning class of Kadane [4]. He proved the conjecture of Sakaguchi [8], that is, the reversibility in the juror problem by the choice of many persons. To prove the reversibility affirmative, Kadane used a notion of the winning class as a choice rule.

Definition 1.6. Let p denote a number of players. A family $\mathcal{W}$ of subsets of integers $\{1,2,\dots,p\}$ is called a winning class if

(i) $\{1,2,\dots,p\} \in \mathcal{W}$ and

(ii) $W \in \mathcal{W}$, $W' \supset W$ implies $W' \in \mathcal{W}$.

Assume that r players, e.g., player $i_1, \dots, i_r$ declare to stop. Then the process must be stopped if the set $\{i_1, \dots, i_r\}$ is an element of $\mathcal{W}$, or continued otherwise.

For a non-empty subset $W=\{i_1,\dots,i_r\}$ of $\{1,2,\dots,p\}$ there corresponds a vertex x of the p-dimensional unit cube whose i_1-, i_2-,.. and i_r-th components are equal to 1 and the remaining components 0. Concerning to the two correspondences between W_1, W_2 and x_1, x_2 respectively, a necessary and sufficient condition for $W_1 \subset W_2$ is that $x_1 \leqq x_2$ (component-wise). Let V be a set of vertices corresponding to a winning class $\mathcal{W}$. Define a logical function π by

$$\pi(x^1,\dots,x^p) = 1 \quad \text{if} \quad (x^1,\dots,x^p) \in V,$$
$$= 0 \quad \text{otherwise.}$$

Accordingly the following proposition clearly holds.

Proposition 1.2. The stopping rule defined by a winning class of players, Def.1.6, is equivalent to the one by a monotone logical function, Def.1.2.

2. Finite Horizon Case

Consider a finite horizon case restricted by a prescribed number $N < \infty$. Our objective is to find an equilibrium strategy for a given monotone rule and determine the associated expected net gain under the situation formulated in the previous section.

Assumption 2.1. (a) For any $d = (d_1,\dots,d_n,\dots) \in \mathcal{D}$, $d_N^i = 1$ for $i=1,\dots,p$ with probabilty 1. (b) Random vectors $X_1,\dots,X_N$ are independent and $E|X_n^i| < \infty$ for each n and i. (c) π is a monotone rule and Π is the corresponding rule of events.

Let us consider a sequence of vectors $V_n = (v_n^1,\dots,v_n^p)$ defined by

$$(2.1) \qquad v_{n+1}^i = v_n^i - c^i + E[(X_{N-n}^i - v_n^i)^+ \beta_n^{\Pi\{i\}}(V_n^{\{i\}} | X_{N-n}^i)]$$
$$- E[(X_{N-n}^i - v_n^i)^- \alpha_n^{\Pi\{i\}}(V_n^{\{i\}} | X_{N-n}^i)], \qquad n \geqq 1,$$

(2.2) $v_1^i = E[X_N^i] - c^i,$

where $V_n^{\{i\}} = (v_n^1, \ldots, v_n^{i-1}, v_n^{i+1}, \ldots, v_n^p) \in R^{p-1}, \quad i=1,\ldots,p,$

(2.3) $\beta_n^{\Pi\{i\}}(V_n^{\{i\}} | X_{N-n}^i) = P(\Pi(*D_{N-n}^1, \ldots, *D_{N-n}^{i-1}, \Omega, *D_{N-n}^{i+1}, \ldots, *D_{N-n}^p) | X_{N-n}^i),$

(2.4) $\alpha_n^{\Pi\{i\}}(V_n^{\{i\}} | X_{N-n}^i) = P(\Pi(*D_{N-n}^1, \ldots, *D_{N-n}^{i-1}, \phi, *D_{N-n}^{i+1}, \ldots, *D_{N-n}^p) | X_{N-n}^i),$

and $*D_{N-n}^i = \{X_{N-n}^i \geqq v_n^i\} \in \mathcal{B}(X_{N-n}) \quad i=1,\ldots,p.$

From Assump.2.1 (a) and (c), $P(t_\pi(d) \leqq N) = 1$ holds for all $d \in \mathcal{D}$ even if the corresponding observation cost is negative.

Theorem 2.1. By the sequence $V_n = (v_n^1, \ldots, v_n^p)$, $n \geqq 1$ in (2.1) and (2.2), let us define a strategy $*d \in \mathcal{D}$ as follows: For $n=1,\ldots,N-1$,

(2.5) $$*d_n^i(\omega) = \begin{cases} 1 & \text{if } \omega \in *D_n^i, \text{ i.e., } X_n^i(\omega) \geqq v_{N-n}^i, \\ 0 & \text{otherwise} \end{cases}$$

and

(2.6) $*d_N^i(\omega) = 1, \quad \text{a.e.,} \quad \omega \in \Omega.$

Then, under Assump.2.1, $*d$ is an equilibrium strategy and

(2.7) $E[Y(t_\pi(*d))] = V_N$

holds. That is, v_N^i is an equilibrium expected net gain for player i.

Proof: Define

$$t_n^* = t_n(*d) = \text{first } m \geqq n \text{ such that } \pi(*d_m) = 1$$

for $n=1,\ldots,N$. Clearly $n \leqq t_n^* \leqq N$ and $t_1^* = t(*d)$, where $t(*d) = t_\pi(*d)$ and π is fixed. We will show that

(2.8) $E[Y^i(t_n^*)] = v_{N-n+1}^i - (n-1)c^i, \quad i=1,\ldots,p,$

by backward induction on n.

From $t_N^* = N$ and (2.2), it is trivial for $n=N$. Assume that it is true for $n+1$. From the definition of $*D_n = \Pi(*D_n^1, \ldots, *D_n^p) \in \mathcal{B}(X_n)$,

$$t_n^* = \begin{cases} n & \text{on } *D_n, \\ t_{n+1}^* & \text{on } \overline{*D_n}. \end{cases}$$

Hence

$$E[Y^i(t_n^*)] = E[Y_n^i; *D_n] + E[Y^i(t_{n+1}^*); \overline{*D_n}]$$

where $E[Y;D] = \int Y \cdot I_D dp$. Since $X_{n+1}, X_{n+2}, \ldots$ are independent of X_n, $E[Y^i(t_{n+1}^*); \overline{*D_n}] = P(\overline{*D_n}) E[Y^i(t_{n+1}^*)]$, Therefore we have the recursion:

$$E[Y^i(t_n^*)] = E[Y_n^i;{}^*D_n] + P(\overline{{}^*D_n})E[Y^i(t_{n+1}^*)].$$

By induction, it is equal to

$$E[X_n^i - nc^i;{}^*D_n] + P(\overline{{}^*D_n})(v_{N-n}^i - nc^i) = E[X_n^i - v_{N-n}^i;{}^*D_n] + (v_{N-n}^i - c^i) - (n-1)c^i.$$

The first term of the right hand side in the above is rewritten as

$$E[(X_n^i - v_{N-n}^i)^+;\ \Pi({}^*D_n^1,\ldots,\Omega,\ldots,{}^*D_n^p)] - E[(X_n^i - v_{N-n}^i)^-;\ \Pi({}^*D_n^1,\ldots,\phi,\ldots,{}^*D_n^p)]$$

$$= E[(X_n^i - v_{N-n}^i)^+ \beta_{N-n}^{\Pi\{i\}}(V_{N-n}^{\{i\}}|X_n^i)] - E[(X_n^i - v_{N-n}^i)^- \alpha_{N-n}^{\Pi\{i\}}(V_{N-n}^{\{i\}}|X_n^i)]$$

So, from (2.1),

$$v_{N-n+1}^i = E[X_n^i - v_{N-n}^i;{}^*D_n] + v_{N-n}^i - c^i.$$

This implies (2.8) and we have just proved the latter part of the theorem by letting $n=1$ in (2.8).

Next we must show that, for fixed i,

$$(2.9)\qquad E[Y^i(t({}^*d(i)))] \leqq E[Y^i(t({}^*d))]$$

where ${}^*d(i) = ({}^*d^1,\ldots,d^i,\ldots,{}^*d^p)$ and $d^i = (d_1^i,\ldots,d_N^i)$ is any individual strategy for player i. Define ${}^nd^i$, $n=0,1,\ldots,N$ by

$${}^nd^i = \begin{cases} (d_1^i,\ldots,d_n^i,{}^*d_{n+1}^i,\ldots,{}^*d_N^i) & \text{if } n=1,\ldots,N \\ {}^*d^i & \text{if } n=0 \end{cases}$$

using d^i and ${}^*d^i$. This ${}^nd^i$ is consistent with ${}^*d^i$ after first n periods. Also define a strategy ${}^nd(i)$ by

$${}^nd(i) = ({}^*d^1,\ldots,{}^nd^i,\ldots,{}^*d^p).$$

Clearly ${}^Nd(i) = {}^*d(i)$ and ${}^0d(i) = {}^*d$.

We show

$$(2.10)\qquad E[Y^i(t({}^nd(i)))] \leqq E[Y^i(t({}^{n-1}d(i)))]$$

for $n=1,\ldots,N$ because (2.9) can be proved immediately from (2.10). By the strategy ${}^nd(i)$, it is enough to consider a stopping time t_n instead of t. It is seen that

$$E[Y^i(t_n({}^nd(i)))] = E[Y_n^i;D_n] + P(\overline{D_n})E[Y^i(t_{n+1}({}^nd(i)))],$$

where D_n is a stopping event with respect to ${}^*d^1,\ldots,{}^nd^i,\ldots,{}^*d^p$. Since $t_{n+1}({}^nd(i)) = t_{n+1}({}^*d)$ on $\overline{D_n}$ and $E[Y^i(t_{n+1}({}^*d))] = v_{N-n}^i - nc^i$, it becomes

$$E[X_n^i - c^i;D_n] + P(\overline{D_n})(v_{N-n}^i - c^i) - (n-1)c^i$$

$$\leqq v^{i}_{N-n+1} - (n-1)c^{i} = \mathrm{E}[Y^{i}(t_n({}^{n-1}d(i)))].$$

Q.E.D.

This is an extension of Theorem 3.1 in our previous work [5]. In the result, the player i's region for declaring to stop has the form of $\{ X^{i}_{n} \geqq$ a certain value $\}$. It is intuitively natural, and this rule is called a critical level strategy. In the proof of the theorem we can see the following corollary.

Corollary 2.1. A necessary condition for

$$\{{}^{*}d^{i}_{n} = 1\} = \{X^{i}_{n} \geqq \text{a certain value}\}, \quad n \geqq 1$$

is that π satisfies $\pi({}^{*}d_n, \dots, 0, \dots, {}^{*}d^{p}_{n}) \leqq \pi({}^{*}d_n, \dots, 1, \dots, {}^{*}d^{p}_{n})$, $n \geqq 1$, for the equilibrium strategy ${}^{*}d$.

If we impose further assumptions, then next two corollaries are obtained immediately.

Corollary 2.2. For each n, if components of $(X^{1}_{n}, \dots, X^{p}_{n})$ are mutually independent and identically distributed with X^{0}_{n}, then (2.1) implies

$$v^{i}_{n+1} = v^{i}_{n} - c^{i} + \beta^{\Pi\{i\}}_{n} \mathrm{E}(X^{0}_{N-n} - v^{i}_{n})^{+} - \alpha^{\Pi\{i\}}_{n} \mathrm{E}(X^{0}_{N-n} - v^{i}_{n})^{-} \tag{2.11}$$

where

$$\beta^{\Pi\{i\}}_{n} = \beta^{\Pi\{i\}}_{n}(V^{\{i\}}_{n}) = \mathrm{P}(\Pi({}^{*}D^{1}_{N-n}, \dots, \Omega, \dots, {}^{*}D^{p}_{N-n}))$$

and

$$\alpha^{\Pi\{i\}}_{n} = \alpha^{\Pi\{i\}}_{n}(V^{\{i\}}_{n}) = \mathrm{P}(\Pi({}^{*}D^{1}_{N-n}, \dots, \phi, \dots, {}^{*}D^{p}_{N-n}))$$

Corollary 2.3. In addition, if the monotone rule π is symmetric for i and j, that is,

$$\pi(\dots, d^{i}, \dots, d^{j}, \dots) = \pi(\dots, d^{j}, \dots, d^{i}, \dots) \tag{2.12}$$

and if $c^{i} = c^{j}$, then $v^{i}_{n} = v^{j}_{n}$ for each n. If π is symmetric for any pairs, this leads to the majority case discussed in [5].

Example 2.1. Similarly to Example 4.2 in [5], we shall consider a varient of the secretary problem (cf. [1], [3]) with a monotone rule. Three players want to choose one secretary and we impose the following unequal rule:

$$\pi(x^{1}, x^{2}, x^{3}) = x^{1} + x^{2}x^{3}, \quad x^{i} \in \{0,1\}, \quad i=1,2,3. \tag{2.13}$$

This means that a secretary is accepted only when either player 1 sayes "yes", or both of player 2 and player 3 say "yes".

From Thm.2.1, the equilibrium strategy $^*d$ is determined by the sequence of v_n^i, $n \geqq 1$, in (2.11), where $c^i=0$ and $v_1^i=1/N$. Since the rule π of (2.13) is symmetric for players 2 and 3, $v_n^2 = v_n^3$ from Cor.2.3. Define

$$r^i = \inf\{ r;\ v_{N-n}^i \leqq r/N \},\quad i=1,2.$$

The strategy for player 1 is that he observes until the (r^1-1)th stage and then declares to accept if the relatively best one appears. For player 2 and 3, the strategy is similar. Numerical results are as follows.

N	r^1	v_N^1	r^2	v_N^2
10	3	.3642	1	.1685
30	10	.3649	2	.0801
100	36	.3673	3	.0322
300	110	.3677	4	.0135
1000	367	.3678	5	.0050
10000	3678	.3679	6	.0007

We have applied our result to a secretary problem with an unequal monotone rule and showed the equilibrium strategy is a critical level strategy. But, as a remark, the asymptotic numerical result is non-interesting. Under the rule (2.13), player 1 behaves as if it were a one-person-game and player 2, 3 are neglected. A modified setting of the secretary problem has been discussed by Presman and Sonin [7] and Sakaguchi [9].

3. Infinite Horizon Case

In this section we shall treat an infinite horizon case $N = \infty$. The class of rules is therefore $\{ d \in \mathcal{D};\ P(t_\pi(d) \leqq \infty) = 1 \}$. The problem is worth studying when the observation cost is non-negative. Thm.3.1 discusses the case of $c^i > 0$ for all i, in which case the stopping time is finite. When $c^i = 0$, $i=1,\dots,p$, some trouble occurs in the multi-variate problem. Though we have defined $Y_\infty^i = \limsup_{n\to\infty} Y_n^i$ in (1.12) in the analogy of one-dimentional problem, apparently this definition is not natural for all players under some rules. To avoid this, we assume that the equilibrium stopping time is finite. Then, we can establish the continuity from the finite horizon case and compare the expected gains between rules and between players. From the formulation of our model, this assumption is often satisfied because the process is forced to stop by the conflict among players.

Assumption 3.1. (a) Random vectors $X_1, X_2, \ldots, X = (X^1, \ldots, X^p)$ are independent and identically distributed with $E|X^i| < \infty$ for all i. (b) Each element of cost vector $C = (c^1, \ldots, c^p)$ is strictly positive. (c) π is a monotone rule and let Π be the corresponding rule of events. (d) The following simultaneous equation of $V = (v^1, \ldots, v^p)$:

$$(3.1) \qquad E[(X^i - v^i)^+ \beta^{\Pi\{i\}}(V^{\{i\}}|X^i)] - E[(X^i - v^i)^- \alpha^{\Pi\{i\}}(V^{\{i\}}|X^i)] = c^i$$

$i=1,\ldots,p$ has a solution. Where $V^{\{i\}} = (v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^p) \in R^{p-1}$,

$$\beta^{\Pi\{i\}}(V^{\{i\}}|X^i) = P(\Pi(D^1, \ldots, \Omega, \ldots, D^p)|X^i),$$

$$\alpha^{\Pi\{i\}}(V^{\{i\}}|X^i) = P(\Pi(D^1, \ldots, \phi, \ldots, D^p)|X^i)$$

and $D^i = \{X^i \geqq v^i\}$, $i = 1, 2, \ldots, p$.

Theorem 3.1. Under Assump.3.1, A strategy $*d = (*d^1, \ldots, *d^p)$ determined by

$$(3.2) \qquad *d_n^i(\omega) = 1 \ (0) \quad \text{if} \quad X_n^i(\omega) \geqq (<) \ *v^i,$$

for each n and i, is an equilibrium strategy in the class $\{d \in \mathcal{D}; P(t_\pi(d) \leqq \infty) = 1\}$ and

$$(3.3) \qquad P(t_\pi(*d) < \infty) = 1,$$

$$(3.4) \qquad E[Y^i(t_\pi(*d))] = *v^i, \ i = 1, \ldots, p$$

hold where $*V = (*v^1, \ldots, *v^p)$ is a solution of (3.1).

By (3.4), $*V$ is called an equilibrium extected net gain. Since the proof is similar to that of Theorem 5.3, 5.4 of [5], we omit it here.

For the rest of the section we shall restrict our attention to the case:

(b') $\quad C = 0.$

Under the assumption (b'), it may happen that the equilibrium stopping time is not finite. But if the assumption (e) should be added, the following corollary must then hold.

(e) $\quad P(t_\pi(*d) < \infty) = 1$ where $*d$ is defined by (3.2).

It is seen in Ex.3.2 that there are cases which satisfy (e).

Corollary 3.1. Assume the assumptions (a), (b'), (c), (d), and (e). If X is bounded with prob. 1, then $*d$ is an equilibrium strategy in the restricted class $\{d \in \mathcal{D}; P(t_\pi(d) < \infty) = 1\}$ and (3.4) holds.

Proof: The proof is immediate by Theorem 5.3, 5.4 of [5].

Hereafter we sssume that

(a') (a) and components of ($X^1,\ldots,X^p$) are independent.

Corollary 3.2. Under the assumptions (a'), (b'), (c), (d) and (e), if $P(X^i=y)=0$ where $y=\sup\{y;P(X^i>y)>0\}$, then $^*d$ is an equilibrium strategy in the class $\{d\in\mathcal{D};P(t_\pi(d)\leqq\infty)=1\}$ and (3.4) holds.

Proof: By Assump. (e), $P(\Pi(^*D^1,\ldots,{}^*D^p))>0$ where $^*D^i=\{X^i\geqq{}^*v^i\}$. If we assume that $P(\Pi(^*D^1,\ldots,\phi,\ldots,{}^*D^p))=0$, then $P(\Pi(^*D^1,\ldots,\Omega,\ldots,{}^*D^p))>0$ from the monotonicity of the rule. From (3.1), it follows that (a'), $P(\Pi(^*D^1,\ldots,\Omega,\ldots,{}^*D^p))>0$ and $P(\Pi(^*D^1,\ldots,\phi,\ldots,{}^*D^p))=0$ imply $(X^i-{}^*v^i)^+=0$ a.e., that is, $^*v^i\geqq y$. This means $^*D^i=\phi$ a.e. by the assumption. We have

$$P(\Pi(^*D^1,\ldots,{}^*D^i,\ldots,{}^*D^p))=P(\Pi(^*D^1,\ldots,\phi,\ldots,{}^*D^p)).$$

This is a contradiction because the left hand side is " > 0" but the right hand side equals zero. Hence we obtain $P(\Pi(^*D^1,\ldots,\phi,\ldots,{}^*D^p))>0$. For the strategy $^*d^{\{i\}}=(^*d^1,\ldots,d^i,\ldots,{}^*d^p)$ where d^i is any individual strategy, it is seen that $P(t_\pi(^*d^{\{i\}})<\infty)=1$. Hence the proof is immediately completed from Thm.3.1.

Q.E.D.

For a rule π with $P(t_\pi(^*d)=\infty)=1$, there is player i such that

(3.5) $$^*v^i=\sup\{y;P(X^i>y)>0\}.$$

Clearly (3.4) is satisfied for player i by (1.12). But for another player $j\ (\neq i)$, $^*v^j$ does not necessary satisfy (3.4). Therefore the solution of (3.1) does not always consist with the equilibrium expected gain in this case. In order to discuss the associated gain including this case, we simply call an expected gain (omitting "equilibrium") by the solution $^*V$, which is the limiting value as $N\to\infty$ in the finite horizon case. For this see Figure 4.1 in [5] and Table 3.1.

Now we shall derive a bound of the expected gain by varying π 's. The expected gain $v^i=v^i(\pi)$ associated with a monotone rule π satisfies that

(3.6) $$EX^i\leqq v^i\leqq\sup\{y;(X^i>y)>0\}.$$

This is proved by using a ratio (3.8) as follows. By (a') and (b'), equation (3.1) implies

(3.7) $$E[(X^i-v^i)^+]/E[(X^i-v^i)^-]=\rho_\Pi^{\{i\}}(V^{\{i\}})$$

where

$$(3.8)\qquad \rho_{\Pi}^{\{i\}}(V^{\{i\}}) = \alpha^{\Pi\{i\}}(V^{\{i\}})/\beta^{\Pi\{i\}}(V^{\{i\}})$$
$$= P(\Pi(D^1,\ldots,\phi,\ldots,D^p))/P(\Pi(D^1,\ldots,\Omega,\ldots,D^p)),$$

providing that the denominator is non-zero. Since π is monotone,

$$(3.9)\qquad 0 \leqq \rho_{\Pi}^{\{i\}}(V^{\{i\}}) \leqq 1$$

holds. Therefore (3.9) implies (3.6) immediately.

From the above argument, $\rho_{\Pi}^{\{i\}}(V^{\{i\}}) = 1$ implies $v^i = EX^i$, and $\rho_{\Pi}^{\{i\}}(V^{\{i\}}) = 0$ implies $v^i = \sup\{y; P(X^i > y) > 0\}$. The second assertion corresponds to $P(t_{\pi}({}^*d) = \infty) = 1$ as remarked at (3.5). Here these two extreme cases are interpreted as follows.

First, $\rho_{\Pi}^{\{i\}}(V^{\{i\}}) = 0$ is equivalent to $\Pi(D^1,\ldots,\phi,\ldots,D^p) = \phi$ a.e. and also to $\pi(d^1,\ldots,0,\ldots,d^p) = 0$ with prob. 1. This means that whenever player i declares to continue, the decision process surely continues. But it does not mean that declaring to stop causes to stop the process. Player i is endowed with a veto power. This brings him the maximum expected gain. Secondly $\rho_{\Pi}^{\{i\}}(V^{\{i\}}) = 1$ is equivalent to $\Pi(D^1,\ldots,\phi,\ldots,D^p) = \Pi(D^1,\ldots,\Omega,\ldots,D^p)$ a.e. and also to $\pi(d^1,\ldots,0,\ldots,d^p) = \pi(d^1,\ldots,1,\ldots,d^p)$ with prob. 1. For player i, declaring to stop or to continue does not affect the resulting process. He is ranked as an outsider of the game, and his expected gain EX^i is the least one.

Now we shall make a comparison of gains between players under a fixed monotone rule in Cor.3.3 and also between two different monotone rules in Cor.3.4. The next theorem is immediately proved from (3.7).

Theorem 3.2. Let $V_{\Pi} = (v_{\Pi}^1,\ldots,v_{\Pi}^p)$ and $V_{\tilde{\Pi}} = (v_{\tilde{\Pi}}^1,\ldots,v_{\tilde{\Pi}}^p)$ be the expected gains corresponding to Π and $\tilde{\Pi}$ respectively. For player i and j, let us assume that $\rho_{\Pi}^{\{i\}}$ and $\rho_{\tilde{\Pi}}^{\{j\}}$ are defined by (3.8). If X^i and X^j are identically distributed, we have

$$(3.10)\qquad v_{\Pi}^i \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} v_{\tilde{\Pi}}^j$$

if and only if

$$(3.11)\qquad \rho_{\Pi}^{\{i\}}(V_{\Pi}^{\{i\}}) \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} \rho_{\tilde{\Pi}}^{\{j\}}(V_{\tilde{\Pi}}^{\{j\}})$$

Corollary 3.3. Under a fixed Π, if X^i and X^j are identically distributed and if

$$(3.12)\qquad \rho_{\Pi}^{\{i\}}(V_{\Pi}^{\{i\}}) \leqq \rho_{\Pi}^{\{j\}}(V_{\Pi}^{\{j\}})$$

then $v^i_\Pi \geqq v^j_\Pi$.

Corollary 3.4. If, for player i,

(3.13) $$\Pi(D^1,\ldots,\Omega,\ldots,D^p) \supset \tilde{\Pi}(D^1,\ldots,\Omega,\ldots,D^p)$$

and

$$\Pi(D^1,\ldots,\phi,\ldots,D^p) \subset \tilde{\Pi}(D^1,\ldots,\phi,\ldots,D^p)$$

for every $D^k \in \mathcal{B}(X)$, $k \neq i$, or

(3.14) $$\rho^{\{i\}}_\Pi(U^{\{i\}}) \leqq \rho^{\{i\}}_{\tilde{\Pi}}(U^{\{i\}})$$

for every $U^{\{i\}} = (u^1,\ldots,u^{i-1}, u^{i+1},\ldots, u^p)$ such that $EX^k \leqq u^k \leqq \sup\{y\,;P(X^k > y) > 0\}$, $k \neq i$, then $v^i_\Pi \geqq v^i_{\tilde{\Pi}}$ holds.

Example 3.1. Consider a majority rule $\Pi[r] = (p, r)$ of players, where $r\ (1 \leqq r \leqq p)$ is a majority level. Let X^i, $i=1,\ldots,p$, be independent and identically distributed with X. If $EX < \sup\{y\,;P(X> y) > 0\}$, then the equilibrium expected gains for each rule are

(3.15) $$v_{\Pi[p]} > v_{\Pi[p-1]} > \cdots > v_{\Pi[1]}.$$

In fact, since the rule $\Pi[r]$ is symmetric, we can set the players' gains being equal:

$$v_{\Pi[r]} = v^i_{\Pi[r]},\quad i=1,2,\ldots,p.$$

Hence

$$\rho^{\{i\}}_{\Pi[r]}(V^{\{i\}}) = \rho_{\Pi[r]}(v) = 1 - \eta(r,\bar{v}),$$

where

$$\eta(r,\bar{v}) = \binom{p-1}{r-1}(1-\bar{v})^{r-1}\,\bar{v}^{\,p-r} \Big/ \sum_{k=r-1}^{p-1} \binom{p-1}{k}(1-\bar{v})^k\bar{v}^{\,p-k-1}$$

and $\bar{v}=P(X \leqq v)$. Since $\eta(r,\bar{v})$ is increasing in $\bar{v}$ and $\eta(r,\bar{v}) < \eta(r+1,\bar{v})$, we can see $\rho_{\Pi[r]}(v)$ is decreasing in v and $\rho_{\Pi[r]}(v)>\rho_{\Pi[r+1]}(v)$ for each v. Similarly as Cor.3.4, it implies (3.15).

Figure 4.1 in [5] shows each expected gain of (3.15) for $p = 5$ players. For $r=1,\ldots,p-1$, $\Pi[r]$ has an equilibrium strategy and $v_{\Pi[r]}$ is an equilibrium expected gain from Cor.3.2. But for $r=p$, each player has a veto power and so $v_{\Pi[p]} = \sup\{y\,;P(X>y) > 0\}$. Though its stopping time is such that $P(t_{\Pi[p]}= \infty) = 1$, the associated expected gain is shown to an equilibrium one directly from (1.12), (1.13) and (3.4).

Example 3.2. Let components of random vectors be independent and identically distributed with a common uniform distribution U(0,1). Table 3.1 shows a numerical example with $p = 3$ for non-trivial monotone rules. In the first four rules $P(t_\Pi(^*d) < \infty) = 1$, but this does not hold in other cases.

From (3.5), there exist players who attain the maximum expected gain "unity" in the last four rules. Each expected gain is the limiting value of the finite horizon case. Except for the 5-th, 6-th and 7-th rule, the value is an equilibrium one by Cor.3.2.

Table 3.1 Monotone rules with p = 3.

Monotone rule $\pi(x^1,x^2,x^3)$		$x^1+x^2+x^3$	x^1+x^2	$x^1+x^2x^3$	$x^1x^2+x^2x^3+x^3x^1$	x^1	$x^1x^2+x^1x^3$	x^1x^2	$x^1x^2x^3$
Comments for the rule		majority rule for $(p,r)=(3,1)$	pl.3 is an outsider	asymmetric case	majority rule for $(p,r)=(3,2)$	pl.1 is a dictator	pl.1 has a veto power	pl.3 is an outsider	unanimity $(p,r)=(3,3)$
(Equi-librium) expected gain v^i	v^1	0.5437	$(\sqrt{5}-1)/2$ ≒0.6180	$\sqrt{2}/2$ ≒0.7071	$\sqrt{2}/2$	1	1	1	1
	v^2	0.5437	$(\sqrt{5}-1)/2$	$2-\sqrt{2}$ ≒0.5858	$\sqrt{2}/2$	0.5	$(\sqrt{5}-1)/2$	1	1
	v^3	0.5437	0.5	$2-\sqrt{2}$	$\sqrt{2}/2$	0.5	$(\sqrt{5}-1)/2$	0.5	1

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