MULTICHANNEL L-FILTER DESIGN BASED ON MARGINAL DATA ORDERING

C. Kotropoulos *I. Pitas*
Department of Electrical Engineering, University of Thessaloniki, Thessaloniki 540 06, GREECE
E-mail: pitas@vergina.eng.auth.gr

ABSTRACT

In this paper, we address the design of multichannel L-filters based on marginal data ordering using the Mean-Squared-Error as fidelity criterion. Design procedures subject to the constraints of unbiased or location-invariant estimation or without imposing any constraint are discussed. It is shown by simulations that the proposed multichannel L-filters perform better than other multichannel nonlinear filters such as the vector median, the marginal alpha-trimmed mean, the marginal median, the multichannel modified trimmed mean and the multichannel double-window trimmed mean, the multivariate ranked-order estimators as well as their single-channel counterparts.

1. INTRODUCTION

Multichannel one-dimensional and two-dimensional signals appear frequently in practice, for example in the cases involving multiple sources and receivers as well as in the processing of color images and sequences of images. Single-channel nonlinear filtering techniques have exhibited a tremendous growth in the past decade as alternatives of linear filtering in problems that cannot be efficiently solved by using linear techniques, e.g. in the case of non-Gaussian or signal-dependent noise filtering [4]. A class of nonlinear filters that has found extensive applications in digital signal and image processing are the L-filters (sometimes also called order statistic filters) whose output is defined as a linear combination of the order statistics of the input sequence [5]. Recently, increasing attention has been given to nonlinear processing of vector-valued signals [6,7,8,9].

The main contribution of this paper is in the design of multichannel L-filters that are based on marginal data ordering (M-ordering) using the Mean-Squared-Error (MSE) as fidelity criterion. M-ordering implies independent data ordering in each channel. We assume that a multichannel signal is corrupted by additive white multivariate noise which generally exhibits correlation between different channels. The unconstrained minimization of the MSE is treated first. Structural constraints such as unbiasedness and location-invariance are also incorporated in the minimization procedure. The unconstrained minimization is shown that it leads to a global minimum. In order to test the performance of the designed multichannel marginal L-filters, long-tailed multivariate distributions are required. The derivation and design of such a distribution, namely, the Laplacian (bi-exponential) distribution which belongs to Morgenstern's family in the two-channel case is discussed.

The outline of the paper is as follows. The design of multichannel marginal L-filters is described in Section 2. Practical considerations aiming at alleviating the difficulties that are encountered in the design of the proposed multichannel nonlinear filters are discussed in Section 3. Simulation examples are included and conclusions are drawn in Section 4.

2. MULTICHANNEL MARGINAL L-FILTER DESIGN

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be a random sample of a p-dimensional random variable $\mathbf{X}$ where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^T$. The M-ordering scheme orders each of the vector components independently yielding:

$$x_{j(1)} \leq x_{j(2)} \leq \cdots \leq x_{j(N)} \quad j = 1, \ldots, p. \tag{1}$$

Let us suppose that an observed p-dimensional signal $\{\mathbf{x}(k)\}$ can be expressed as the sum of a known p-dimensional signal $\mathbf{s}(k)$ and a noise vector sequence $\{\mathbf{n}(k)\}$ of zero-mean vector having the same dimensionality, i.e., $\mathbf{x}(k) = \mathbf{s}(k) + \mathbf{n}(k)$. The noise vector $\mathbf{n}(k) = (n_1(k), \ldots, n_p(k))^T$ is a p-dimensional vector of random variables characterized by the joint pdf of its components which are assumed to be correlated in the general case. In addition, we assume that the noise vectors at different time-instants are independent, identically distributed (i.i.d.) and that at every time-instant the signal $\mathbf{s}(k)$ and the noise vector $\mathbf{n}(k)$ are uncorrelated. The output of a p-channel L-filter of length N operating on the sequence of p-dimensional vectors $\{\mathbf{x}(k)\}$, for N odd, can be expressed as

$$\mathbf{y}(k) = \sum_{j=1}^{p} \mathbf{A}_j \tilde{\mathbf{x}}_j(k) \tag{2}$$

where $\mathbf{A}_j$ are appropriate $(p \times N)$ coefficient matrices and $\tilde{\mathbf{x}}_j(k) = (x_{j(1)}(k), \ldots, x_{j(N)}(k))^T$ are the $(N \times 1)$ vectors of the order statistics along each channel. We shall design the p-channel marginal L-filter which operates on the p-dimensional observed signal $\{\mathbf{x}(k)\}$ and is the optimal estimator of $\mathbf{s}(k)$ by using the MSE between $\mathbf{s}(k)$ and the output of the p-channel marginal L-filter as fidelity criterion.

Let $\mathbf{a}_{jl}^T$, $l = 1, \ldots, p$ be $(1 \times N)$ row vectors corresponding to the rows of matrix $\mathbf{A}_j$. Let $\mathbf{R}_{ji} = \mathrm{E}[\tilde{\mathbf{x}}_j \tilde{\mathbf{x}}_i^T]$ denote the correlation matrix of the ordered input samples in channels j and i. For $j = i$, $\mathbf{R}_{ii}$ $i = 1, \ldots, p$ consists of moments of the order statistics from a univariate population and well-known formulae [1,5] can be used to devise a discrete algorithm for their computation by vector quantizing the p-dimensional observations. For $j \neq i$, the elements of $\mathbf{R}_{ji}$ are moments of the order statistics from a bivariate population. The results reported in [7,8] can be exploited to their computation. Let also $\underline{\mu}_j = (\mathrm{E}[x_{j(1)}], \mathrm{E}[x_{j(2)}], \ldots, \mathrm{E}[x_{j(N)}])^T$ denote the mean vector of the order statistics in channel j.

The MSE between $s(k)$ and $y(k)$ is given by:

$$\varepsilon = \sum_{i=1}^{p} \mathbf{a}_{(i)}^T \hat{\mathbf{R}}_p \mathbf{a}_{(i)} - 2\mathbf{s}(k)^T \begin{bmatrix} \mathbf{a}_{(1)}^T \\ \vdots \\ \mathbf{a}_{(p)}^T \end{bmatrix} \hat{\underline{\mu}}_p + \mathbf{s}(k)^T\mathbf{s}(k) \quad (3)$$

where

$$\mathbf{a}_{(i)} = \begin{pmatrix} \mathbf{a}_{1i}^T & \mathbf{a}_{2i}^T & \cdots & \mathbf{a}_{pi}^T \end{pmatrix}^T \quad (4)$$

$$\hat{\mathbf{R}}_p = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \cdots & \mathbf{R}_{1p} \\ \mathbf{R}_{12}^T & \mathbf{R}_{22} & \cdots & \mathbf{R}_{2p} \\ \vdots & & \ddots & \vdots \\ \mathbf{R}_{1p}^T & \mathbf{R}_{2p}^T & \cdots & \mathbf{R}_{pp} \end{bmatrix} \quad (5)$$

$$\hat{\underline{\mu}}_p = \begin{pmatrix} \underline{\mu}_1^T & \underline{\mu}_2^T & \cdots & \underline{\mu}_p^T \end{pmatrix}^T \quad (6)$$

In the following, the time-index k will be suppressed without any lack of generality. We shall treat first the unconstrained minimization of the MSE and then we shall impose constraints on the output of the multichannel marginal L-filter.

Minimizing (3) over $\mathbf{a}_{ji}$, $j, i = 1, \ldots, p$ is a quadratic optimization problem which has a unique solution provided that the symmetric matrix $\hat{\mathbf{R}}_p$ is positive definite. The components of the vectors of the order statistics along each channel $\tilde{\mathbf{x}}_j$, $j = 1, \ldots, p$ are linearly independent variables with probability 1 [2, pp. 179,180] due to the independence of the observations at different time instants. Thus, the diagonal submatrices of $\hat{\mathbf{R}}_p$, $\mathbf{R}_{jj}$, are positive definite with probability 1. The $2N$ random variables that form the vector of the order statistics from two different channels $(\tilde{\mathbf{x}}_i^T \ \tilde{\mathbf{x}}_j^T)^T$ with $i \neq j$ are linearly dependent in the general case, due to the correlation that exists between the ordered samples that correspond to the same time instant. Consequently, $\hat{\mathbf{R}}_p$ is positive semi-definite in general. We shall assume that $\hat{\mathbf{R}}_p$ is not singular, i.e., that $\hat{\mathbf{R}}_p$ is indeed positive definite. Equating the derivatives of ε with respect to $\mathbf{a}_{ji}$ with zero, i.e., $\frac{\partial \varepsilon}{\partial \mathbf{a}_{ji}} = \mathbf{0}$, and solving for the p-channel marginal L-filter coefficients, we obtain:

$$\begin{aligned} \mathbf{a}_{(1)}^* &= s_1 \hat{\mathbf{R}}_p^{-1} \hat{\underline{\mu}}_p \\ \mathbf{a}_{(m)}^* &= \frac{s_m}{s_1} \mathbf{a}_{(1)}^* \qquad m = 2, \ldots, p \end{aligned} \quad (7)$$

The minimum MSE (MMSE) associated with the optimal coefficients (7) is:

$$\varepsilon_{\min} = (1 - \Delta)\mathbf{s}^T\mathbf{s} \; ; \; \Delta = \hat{\underline{\mu}}_p^T \hat{\mathbf{R}}_p^{-1} \hat{\underline{\mu}}_p \quad (8)$$

The fact that ε is always nonnegative implies that $\varepsilon_{\min} \geq 0$. Therefore, $0 \leq \Delta \leq 1$.

In (7), the optimal coefficients for the unconstrained multichannel marginal L-filter depend on the knowledge of the signal s (to be estimated). In addition, the distributional model (i.e., the marginal and joint probability/cumulative density functions) of the components of the input vector-valued signal $\mathbf{x}(k)$ must be known in order to calculate $\hat{\mathbf{R}}_p$ and $\hat{\underline{\mu}}_p$. In many practical applications, the signal s to be estimated is unknown, unless the detection of a known signal in noise is investigated. Furthermore, in general, the distributional model of input vector data is unknown. Efficient procedures for estimating s and for the calculation of $\hat{\mathbf{R}}_p$ and $\hat{\underline{\mu}}_p$ based on estimates of the marginal and joint probability density functions of the input vector data are developed in Section 3.

In the univariate case, L-filters are designed by imposing local structural constraints on the output of the L-filter. Two types of constraints have been incorporated in the design of single-channel L-filters [5]: unbiasedness and location-invariance. A multichannel marginal L-filter is said to be unbiased multichannel estimator of location, if $E[\mathbf{y}(k)] = \mathbf{s}$ holds, or equivalently:

$$\mathbf{a}_{(i)}^T \hat{\underline{\mu}}_p = s_i \qquad i = 1, \ldots, p \quad (9)$$

Under the set of constraints (9), it can easily be shown that the optimal coefficients of the unbiased p-channel marginal L-filter are given by:

$$\begin{aligned} \mathbf{a}_{(1)}^* &= \frac{s_1}{\hat{\underline{\mu}}_p^T \hat{\mathbf{R}}_p^{-1} \hat{\underline{\mu}}_p} \hat{\mathbf{R}}_p^{-1} \hat{\underline{\mu}}_p \\ \mathbf{a}_{(l)}^* &= \frac{s_l}{s_1} \mathbf{a}_{(1)}^* \qquad l = 2, \ldots, p \end{aligned} \quad (10)$$

and the MMSE associated with the optimal coefficients (10) is:

$$(\varepsilon_{\text{unb}})_{\min} = \frac{1 - \Delta}{\Delta} \mathbf{s}^T\mathbf{s}. \quad (11)$$

Since $\Delta \leq 1$, the MMSE associated with the optimal unbiased p-channel marginal L-filter is always greater than the MMSE (8) produced by the optimal unconstrained p-channel marginal L-filter. The need for an estimate $\hat{\mathbf{s}}(k)$ of $\mathbf{s}(k)$ as well as for the design of unbiased multichannel L-filters based on estimates of the marginal and joint probability density functions of input vector-valued observations is recognized in this case, too. A multichannel marginal L-filter is said to be location-invariant, if its output is able to track small perturbations of its input, i.e., $\mathbf{x}'(k) = \mathbf{x}(k) + \mathbf{b}$ implies that

$$\mathbf{y}'(k) = \mathbf{T}[\mathbf{x}'(k)] = \mathbf{y}(k) + \mathbf{b} \quad (12)$$

where $\mathbf{y}(k) = \mathbf{T}[\mathbf{x}(k)]$. The definition of location-invariant multichannel marginal L-filter (12) yields the following set of constraints imposed on the filter coefficients:

$$\begin{aligned} \mathbf{e}^T\mathbf{a}_{jj} &= 1 \;\; \forall j, \;\; j = 1, \ldots, p \\ \mathbf{e}^T\mathbf{a}_{ji} &= 0 \;\; \forall i, \;\; i \neq j, \;\; j = 1, \ldots, p \end{aligned} \quad (13)$$

where $\mathbf{e}$ denotes the $(N \times 1)$ unitary vector, i.e., $\mathbf{e} = (1, 1, \ldots, 1)^T$. By incorporating (13) into (3) we obtain:

$$\varepsilon_{\text{loc}} = \sum_{i=1}^{p} \mathbf{a}_{(i)}^T \check{\mathbf{R}}_p \mathbf{a}_{(i)} \quad (14)$$

where $\check{\mathbf{R}}_p = \{\tilde{\mathbf{R}}_{ji}\}$ with $\tilde{\mathbf{R}}_{ji} = E[\tilde{\mathbf{n}}_j \tilde{\mathbf{n}}_i^T]$, $j, i = 1, \ldots, p$ and $\tilde{\mathbf{n}}_l = (n_{l(1)}, \ldots, n_{l(N)})^T$. By using Lagrange multipliers, the minimization of the MSE subject to (13) yields the following optimal coefficients for the location-invariant p-channel marginal L-filter:

$$\mathbf{a}_{(i)}^* = \frac{1}{\det(\mathbf{G})} \check{\mathbf{R}}_p^{-1} \begin{bmatrix} c_{i1}(\mathbf{G})\mathbf{e} \\ c_{i2}(\mathbf{G})\mathbf{e} \\ \vdots \\ c_{ip}(\mathbf{G})\mathbf{e} \end{bmatrix} \quad i = 1, \ldots, p \quad (15)$$

where we assume that $\check{\mathbf{R}}_p^{-1}$ exists and it has been decomposed in terms of the $(N \times N)$ square matrices $\mathbf{P}_{ij}$, $i, j = 1, \ldots, p$ as in (5). $\mathbf{G}$ is the $(p \times p)$ square matrix $\mathbf{G} = \{\mathbf{e}^T\mathbf{P}_{ij}\mathbf{e}\}$, $i, j = 1, \ldots, p$. In Eq. (15), det( ) denotes the determinant of a matrix and $c_{ij}(\mathbf{G})$ stands for the cofactor of the ij-element of $\mathbf{G}$. The associated with L-filter coefficients (15) MMSE is given by:

$$(\epsilon_{\rm loc})_{\rm min} = \frac{1}{\det(\mathbf{G})} \sum_{i=1}^{p} c_{ii}(\mathbf{G}) \tag{16}$$

It is seen that the optimal coefficients (15) are independent of the two-channel constant signal to be estimated. Unfortunately, it has been found by experiments that the location-invariant two-channel marginal L-filter leads only to a slightly higher noise suppression than its single-channel counterparts.

3. PRACTICAL CONSIDERATIONS

As can be seen in the preceding analysis, the following difficulties are met in the design of the unconstrained and the unbiased multichannel L-filters: (i) The marginal L-filter coefficients depend explicitly on the signal s (to be estimated). (ii) The marginal and joint probability density functions of the input vector-valued observations must be known in order to calculate $\hat{\mathbf{R}}_p$ and $\underline{\hat{\mu}}_p$.

Let us consider first the case of a constant signal s. We restrict the discussion in one-dimensional signals without any loss of generality. In this case, an initial estimate $\hat{\mathbf{s}}(k)$ for $\mathbf{s}(k)$ can be obtained from the past L-filter outputs $\mathbf{y}(l)$, $l = k-1, k-2, \ldots$ as follows:

$$\hat{\mathbf{s}}(k) = \frac{1}{N_e} \sum_{l=k-N_e}^{k-1} \mathbf{y}(l) \tag{17}$$

where N_e is chosen to be sufficiently large. The initial estimate (17) can be used in the place of s in (7) or (10) to determine the multichannel marginal L-filter coefficients at each time instant k, provided that $\hat{\mathbf{R}}_p^{-1}$ and $\underline{\hat{\mu}}_p$ have already been computed based on the knowledge of the distributional model of input vector-valued observations. For $k < N_e$, the marginal median can be used to initialize filtering. Another suitable estimator of s is the vector median of the past L-filter outputs which is based on the L_2 norm (VM_{L_2}) [6]. An even better estimate, $\hat{\mathbf{s}}(k)$, can be obtained by employing the arithmetic mean or the marginal median of the N_e past input vector-valued observations (i.e., noisy observations).

Next, we proceed to the treatment of non-constant signals $\mathbf{s}(k)$ (e.g. a multichannel edge). A segmentation of a multichannel non-constant signal to homogeneous regions where the signal $\mathbf{s}(k)$ is locally constant (i.e., edges do not occur) and to transition regions where an edge occurs in a certain input channel by using an edge detection algorithm is proposed. Such an edge detection algorithm may be the one described in [10]. Having segmented the input vector-valued observations to homogeneous regions where a locally constant multichannel signal is corrupted by additive white multivariate noise and to transition regions, we may use one of the previously described techniques to estimate the constant multichannel signal within each homogeneous region by restricting either the past L-filter outputs or the past noisy input vector-valued observations to lie within the homogeneous region.

In the sequel, the design of multichannel marginal L-filters based on estimates of the marginal and joint pdf of input vector-valued observations is considered. Both in the case of a multichannel constant signal corrupted by additive white multivariate noise as well as within the homogeneous regions in the case of a noisy nonconstant multichannel signal, the proposed filters will operate on identically distributed observations. Therefore, we can estimate the marginal statistics of input vector data from the empiric pdf. i.e., by uniformly quantizing the range of input observations in each channel to a number, say M, of discrete values and constructing their histogram. Moreover, the input joint statistics can be estimated from the empiric joint pdf, i.e., by exploiting the uniform quantization of any couple of input vector data components to a set of pairs of discrete values and by estimating their cooccurrence matrix.

In the transition regions, input vector-valued observations may be considered to a first approximation as independent non-identically distributed random variables. The formulae for calculating $\hat{\mathbf{R}}_p$ and $\underline{\hat{\mu}}_p$ for independent i.i.d. input variates do not hold anymore. for the design of (unconstrained/unbiased) two-channel marginal L-filters close to the edge. Since the framework for calculating $\hat{\mathbf{R}}_p$ and $\underline{\hat{\mu}}_p$ for independent non-identically distributed input variates is very complicated [11,12], we shall employ the marginal median to filter the input vector data that belong to transition regions.

4. SIMULATION EXAMPLES

We shall discuss the treatment of two-channel one-dimensional signals. The previous attempts to use nonlinear filters based on order statistics for vector-valued signal processing either have been derived from a natural generalization of univariate exponential distributions [6] or have been tested on a contaminated multinormal distribution which has been used to model long-tailed multivariate distributions [7]–[9]. In the following, the design of the bivariate Laplacian distribution is examined.

It is well-known [3] that a joint distribution $F_{x_1,x_2}(x_1,x_2)$ given by:

$$F_{x_1,x_2}(x_1,x_2) = F_{x_1}(x_1)\, F_{x_2}(x_2)\, [1+\alpha \times (1-F_{x_1}(x_1))\,(1-F_{x_2}(x_2))] \tag{18}$$

where $\alpha \in [-1,+1]$, has as marginal cdf's $F_{x_i}(x_i)$ $i = 1,2$. The family of joint distributions (18) is the so-called Morgenstern's family. We are interested in the case:

$$F_{x_i}(x_i) = \begin{cases} \frac{1}{2}\,\exp[\sqrt{2}\,\frac{x_i}{\sigma_i}] & \text{if } x_i < 0 \\ 1 - \frac{1}{2}\,\exp[-\sqrt{2}\,\frac{x_i}{\sigma_i}] & \text{if } x_i \geq 0 \end{cases} \tag{19}$$

for $i = 1,2$. This approach yields the bivariate Laplacian distribution.

The performance of the proposed multichannel marginal L-filters as estimators of location (or equivalently, in multichannel noise filtering) has been compared to the performance of other multichannel nonlinear filters as well as of their single channel counterparts. The following nonlinear filters have been considered: the vector median [6], the marginal median [7,8], the marginal α-trimmed mean [7,8], the multichannel modified trimmed mean (MTM) [8], the multichannel double window modified trimmed mean (DW-MTM) [8] and the ranked-order estimator $\mathcal{R}_{\rm E}$ [9]. We have also included the arithmetic mean in the comparative study, because it is a straightforward choice for noise filtering in many practical applications. The performance of the single-channel L-filter counterparts, i.e., the unbiased and location-invariant single-channel L-filter [5] used to filter the noise in each channel independently has also been taken into consideration. The quantitative criterion we used was the noise reduction index (NR) defined as the ratio of the output noise power to the input noise power, i.e.,:

$$\mathrm{NR} = 10 \log \frac{\sum_k (\mathbf{y}(k)-\mathbf{s})^T(\mathbf{y}(k)-\mathbf{s})}{\sum_k (\mathbf{x}(k)-\mathbf{s})^T(\mathbf{x}(k)-\mathbf{s})}. \tag{20}$$

First, the performance of the nonlinear filters under study when a vector-valued constant signal $s = (1.0, 2.0)^T$ is corrupted by additive white bivariate noise $n(k)$ whose components are distributed according to the Laplacian-Morgenstern distribution has been studied. The case of a bivariate noise with zero-mean vector, $\sigma_1 = \sigma_2 = \sqrt{2}$ and $\alpha = 1.0$ is considered. The NR index is shown in Table 1 for filter length $N = 5$. It can clearly be seen that the unconstrained two-channel L-filter attains the highest noise reduction. The noise reduction capability of the unbiased two-channel L-filter approaches the one of the unconstrained two-channel L-filter. Furthermore, the unbiased two-channel L-filter attains again almost 2 dB higher noise suppression than its single-channel counterpart.

Next, we consider the case where the distributional model of input vector-valued observations is unknown. In such a case, the moments of the order statistics that form the matrix $\hat{\mathbf{R}}_2$ and the mean vector $\hat{\underline{\mu}}_2$ are calculated by using estimates of the marginal and joint probability density function of input vector data. The design of an unbiased two-channel marginal L-filter of length $N = 5$ is treated for the Laplacian-Morgenstern noise model. As can be seen in Table 2, the deterioration in noise suppression is almost negligible (0.056 dB). When s is estimated from the marginal median of the N_e past input data vectors and $\hat{\mathbf{R}}_2$ and $\hat{\underline{\mu}}_2$ are calculated based on estimates of the marginal and joint pdfs of input data vectors, the total deterioration varies between 1.12 dB for $N_e = 49$ to 2 dB for $N_e = 25$. The same procedure has also been applied with similar success to the design of two unbiased single-channel L-filters that have been used to filter the two input channels independently. Again, it is verified experimentally that the unbiased two-channel marginal L-filter is superior to its single-channel counterparts yielding an almost 2 dB higher NR index.

Finally, the treatment of a two-channel one-dimensional edge corrupted by additive white Laplacian-Morgenstern bivariate noise is discussed. The case of a bivariate noise with zero-mean vector, $\sigma_1 = \sigma_2 = \sqrt{2}$ and $\alpha = 1.0$ is considered. In the first channel, a transition from level 1 to level 10 occurs at time instance k_1. In the second channel, a transition from level 2 to level 15 occurs at time instant $k_2 \neq k_1$. The edge-detection algorithm described in [10] has been used to segment the signal in homogeneous regions and in transition ones. The results for noise reduction are summarized in Table 1. The unbiased single-channel L-filters that have been included in Table 1 have also been designed based on estimates for $\hat{\mathbf{R}}_2$, $\hat{\underline{\mu}}_2$ and s. Once more it is seen that the unbiased two-channel L-filter is the best yielding an almost 2 dB higher noise suppression.

In general, the multichannel marginal L-filters have better performance than all the other multichannel estimators included in the comparative study. The price which is paid for the superior performance is the complicated design procedure.

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Table 1. NOISE REDUCTION (in dB) FOR THE LAPLACIAN-MORGENSTERN DISTRIBUTION NOISE MODEL (FILTER LENGTH $N = 5$).

Filter	NR	
	Constant signal	Edge
multichannel DW-MTM	-4.931	-4.8945
$\mathcal{R}_E$	-5.631	-5.7939
vector median L_1	-6.174	-6.2295
arithmetic mean	-6.966	-6.9302
multichannel MTM	-7.138	-7.1135
marginal median	-7.642	-7.6085
α-trimmed mean	-7.953	-7.8996
unbiased single-channel L-filter	-10.494	-8.2951
unbiased two-channel marginal L-filter	-12.617	-10.2365
unconstrained two-channel marginal L-filter	-12.774	–

Table 2. NOISE REDUCTION (in dB) FOR AN UNBIASED TWO-CHANNEL L-FILTER OF LENGTH $N = 5$, WHEN $\hat{\mathbf{R}}_2$ AND $\hat{\underline{\mu}}_2$ AND/OR s ARE ESTIMATED.

Method	NR		
	two-channel	single-channel	N_e
$\hat{\mathbf{R}}_2$ and $\hat{\underline{\mu}}_2$ are estimated, s is known	-12.5610	-10.4117	–
$\hat{\mathbf{R}}_2$, $\hat{\underline{\mu}}_2$ and s are estimated	-10.5719	-8.9349	25
	-11.4929	-9.0498	49
optimal unbiased filter	-12.617	-10.494	–