Multidimensional Systems and Signal Processing, 9, 391–398 (1998)

Multicomponent Multidimensional Signals

JOSEPH P. HAVLICEK* joebob@tobasco.ecn.ou.edu
School of Electrical and Computer Engineering, The University of Oklahoma, Norman, OK 73019-1023

DAVID S. HARDING dave@vision.ece.utexas.edu
Center for Vision and Image Sciences, The University of Texas, Austin, TX 78712-1084

ALAN C. BOVIK bovik@orion.ece.utexas.edu
Center for Vision and Image Sciences, The University of Texas, Austin, TX 78712-1084

Received August 12, 1997; Accepted January 6, 1998

Abstract. In this brief paper, we extend the notion of multicomponent signal into multiple dimensions. A definition for multidimensional instantaneous bandwidth is presented and used to develop criteria for determining the multicomponent nature of a signal. We demonstrate application of the criteria by testing the validity of a multicomponent interpretation for a complicated nonstationary texture image.

Key Words: Multicomponent signals, instantaneous frequency, instantaneous bandwidth, AM-FM models

1. Introduction

Signal descriptions that are inherently capable of capturing nonstationary structure are of great practical interest in an increasing variety of signal processing applications. For many signals, representation in terms of instantaneously varying quantities such as amplitude and frequency are fundamental as well as intuitively appealing. For example, a pure FM chirp is most naturally described as a constant-modulus exponential with linearly increasing frequency. More generally, a nonstationary signal $t : \mathbb{R} \to \mathbb{C}$ may be modeled by the joint amplitude-frequency modulated *AM-FM function*

$$t(x) = a(x)e^{j\varphi(x)}, \tag{1}$$

where $a(x)$ and $\varphi(x)$ are unique; $a(x)$ is referred to as the *instantaneous amplitude*, or *amplitude modulation function* of $t(x)$, whereas $\varphi'(x)$ is known as the *instantaneous frequency*, or *frequency modulation function*. A real signal $s : \mathbb{R} \to \mathbb{R}$ may be analyzed against the model (1) using the unique complex extension $t(x) = s(x) + j\mathcal{H}[s(x)]$, known as the *analytic signal* [6,11], where $\mathcal{H}$ indicates the Hilbert transform. With the analytic signal, the amplitude and frequency of a real-valued signal are unambiguously defined in a way that establishes attractive fundamental relationships between the instantaneous frequency and Fourier spectrum of the signal [1,3,5,6,10,11].

The model (1) does not deliver an intuitively satisfying interpretation for *all* signals, however. Consider the signal $t(x) = a_1 e^{j\omega_1 x} + a_2 e^{j\omega_2 x}$ [3,10]. Intuitively, $t(x)$ is the

* This research was supported in part by the Army Research Office under contract DAAH 049510494 and by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under grant number F49620-93-1-0307.

sum of two components each having constant amplitude and frequency. The interpretation delivered by (1) is

$$a(x) = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos[(w_2 - w_1)x]} \tag{2}$$

and

$$\varphi'(x) = \tfrac{1}{2}(w_2 + w_1) + \tfrac{1}{2}(w_2 - w_1)\frac{a_2^2 - a_1^2}{a^2(x)}, \tag{3}$$

both of which oscillate for all nontrivial choices of the parameters. Indeed, certain signals are *inherently multipartite* in character and are better interpreted as a *sum* of components that each take the form (1).

Cohen and Lee have developed the notion of *multicomponent signal* in 1D [2–5]. They introduced the *instantaneous bandwidth*, which for $t(x)$ in (1) is defined by $B(x) = |a'(x)/a(x)|$. Within the context of certain quadratic time-frequency distributions, $B^2(x)$ admits a rigorous interpretation as the conditional instantaneous spread of frequency about $\varphi'(x)$. Cohen and Lee consider a signal to be multicomponent if there exists a decomposition into components of the form (1) such that the instantaneous bandwidth of each component is smaller than the instantaneous bandwidth of the composite signal and such that the frequency separation between components is large compared to their instantaneous bandwidths. In this brief paper, we discuss the extension of this notion of multicomponent signal into multiple dimensions.

2. Multicomponent Multidimensional Signals

For a multidimensional signal $t : \mathbb{R}^n \to \mathbb{C}$ modeled by the multicomponent AM-FM function

$$t(\mathbf{x}) = \sum_{i=1}^{K} a_i(\mathbf{x}) \exp[j\varphi_i(\mathbf{x})] = \sum_{i=1}^{K} t_i(\mathbf{x}), \tag{4}$$

we define the instantaneous bandwidth of component $t_i(\mathbf{x})$ by [7,8]

$$B_i(\mathbf{x}) = \left|\frac{\nabla a_i(\mathbf{x})}{a_i(\mathbf{x})}\right| = \left|\mathrm{Im}\left[\frac{\nabla t_i(\mathbf{x})}{j t_i(\mathbf{x})}\right]\right|. \tag{5}$$

The magnitudes of the individual components of the vector $\nabla a_i(\mathbf{x})/a_i(\mathbf{x})$ are analogous to the 1D instantaneous bandwidth, and describe the local spread of frequencies in each dimension. $B_i(\mathbf{x})$ in (5) quantifies the spread simultaneously in *all* dimensions. The instantaneous bandwidth for the composite signal $t(\mathbf{x})$ is obtained by taking $K = 1$ in (4) and applying (5). A real-valued signal may be analyzed against the model (4) by applying the directional multidimensional Hilbert transform described in [9].

We consider that $t(\mathbf{x})$ is multicomponent on a region $\mathcal{S} \subset \mathbb{R}^n$ if a decomposition of the form (4) exists over $\mathcal{S}$ with $K > 1$ such that two conditions are satisfied. First,

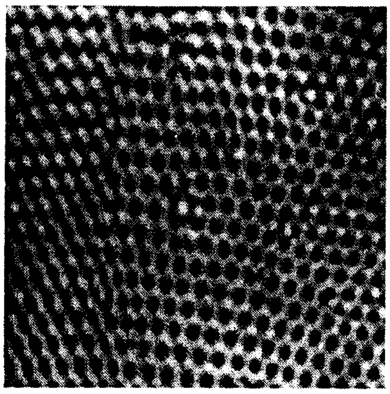

Figure 1. Reptile texture image.

the instantaneous bandwidth of each component must be appreciably smaller than the instantaneous bandwidth of $t(\mathbf{x})$. Second, the frequency separation between any pair of components in the multicomponent interpretation must be large compared to the component instantaneous bandwidths on a pointwise basis. Thus, for each i and each j in $[1, K]$ we require that

$$B_i(\mathbf{x}),\ B_j(\mathbf{x}) \ll \left|\nabla\varphi_i(\mathbf{x}) - \nabla\varphi_j(\mathbf{x})\right|. \tag{6}$$

Note that, as in 1D, this notion of multicomponent signal implies that $t(\mathbf{x})$ may generally be multicomponent in certain regions and not in others.

3. Example

The nonstationary, multipartite texture image *Reptile* is shown in Fig. 1. A six-component interpretation of the image is given in Fig. 2, where components one through six appear in parts (a)–(f) respectively. These components were extracted using the multicomponent AM-FM demodulation techniques described in [8]. In Fig. 2, each component has been independently scaled for display. Note that all of the components exhibit significant nonstationary structure manifest as spatially varying amplitude and frequency modulations.

The amplitude of the composite image computed using the multidimensional Hilbert transform is given in Fig. 3(a). Fig. 3(b) gives the instantaneous bandwidth for the composite image, which has a mean value of approximately 0.35 and lies between zero and 24.0. For comparison, the amplitude and instantaneous bandwidth of component two are shown in Fig. 4(a) and (b), respectively. The instantaneous bandwidth of component two lies between zero and 15×10^{-3}. Its mean value is approximately 5×10^{-3}.

A histogram of $B(\mathbf{x})$ for the composite *Reptile* image appears in Fig. 5(a), while instantaneous bandwidth histograms for components one through six are given in Fig. 5(b)–(g).

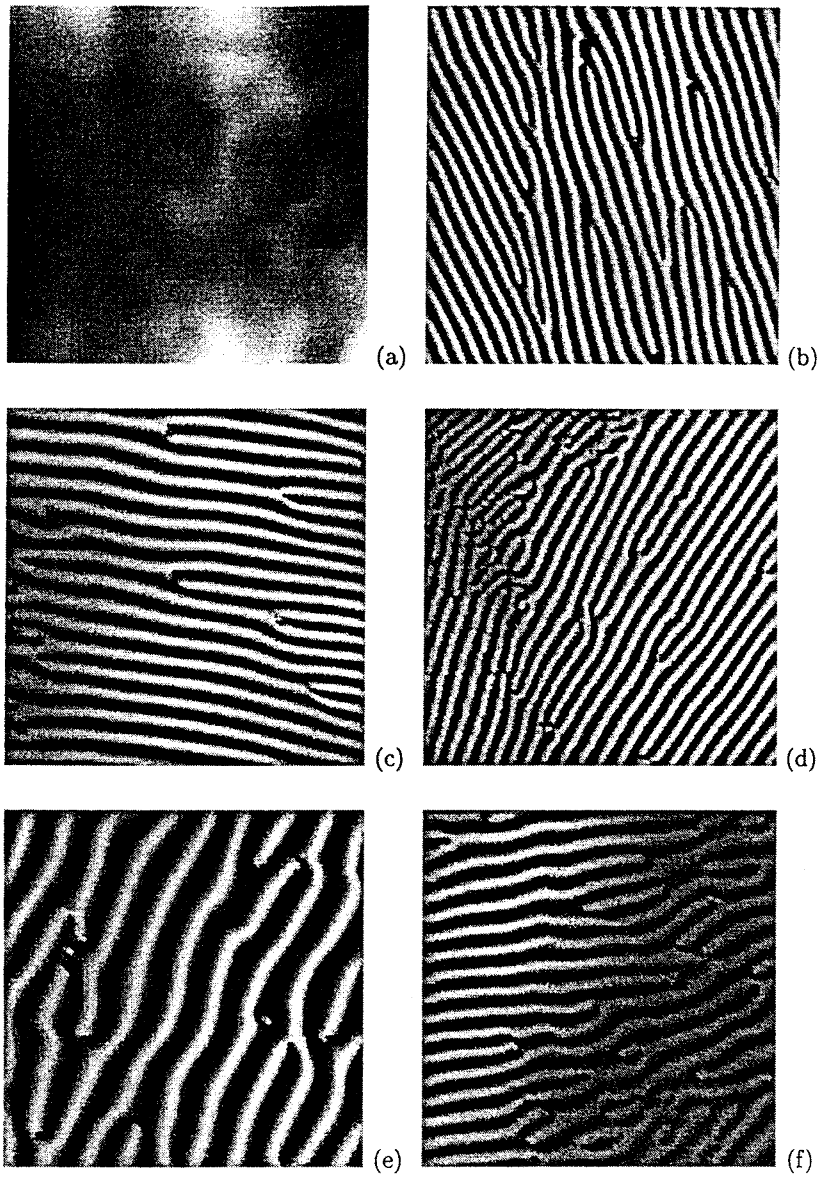

Figure 2. Six-component interpretation of *Reptile* image. (a) Component one. (b) Component two. (c) Component three. (d) Component four. (e) Component five. (f) Component six.

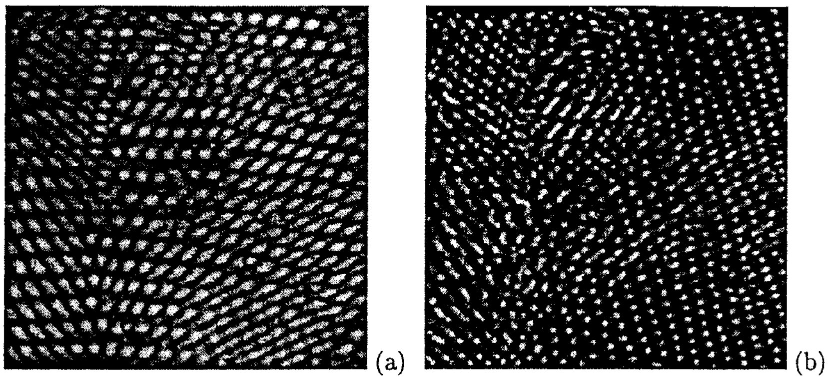

Figure 3. Amplitude and instantaneous bandwidth of composite *Reptile* image. (a) Computed amplitude modulation function $a(\mathbf{x})$. (b) Instantaneous bandwidth $B(\mathbf{x})$.

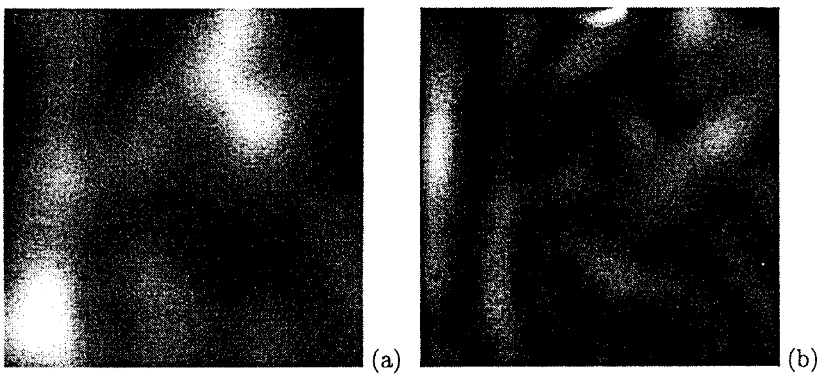

Figure 4. Amplitude and instantaneous bandwidth of component two. (a) Computed amplitude modulation function $a_2(\mathbf{x})$. (b) Instantaneous bandwidth $B_2(\mathbf{x})$.

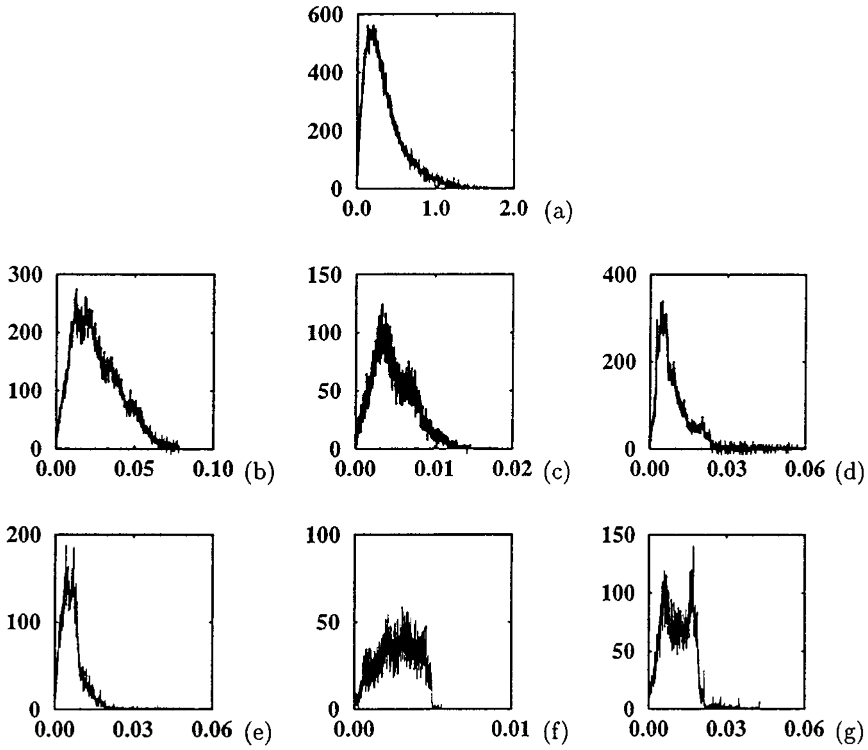

Figure 5. Histograms of the instantaneous bandwidth $B(\mathbf{x})$ for (a) composite image. (b) Component one. (c) Component two. (d) Component three. (e) Component four. (f) Component five. (g) Component six.

Each histogram in Fig. 5(a)–(g) depicts the same number of data points. The areas under the various curves appear to be different because different bin sizes were used for each histogram in order to accurately reflect the spread of values assumed by the instantaneous bandwidth. Note that, on average, the decomposition of this image into components has reduced the instantaneous bandwidth by more than two orders of magnitude.

The ratio of $B_2(\mathbf{x})$ to the quantity on the right side of (6) is histogrammed in Fig. 6 for $i = 2$ and $j = 1, 3, \ldots, 6$. Thus, small abscissa values in these histograms indicate points where the frequency separation between components is large compared to the instantaneous bandwidth of component two. Collectively, the histograms in Fig. 5 and Fig. 6 strongly indicate that the *Reptile* image is indeed multicomponent and that the multicomponent interpretation depicted in Fig. 2 is a valid one.

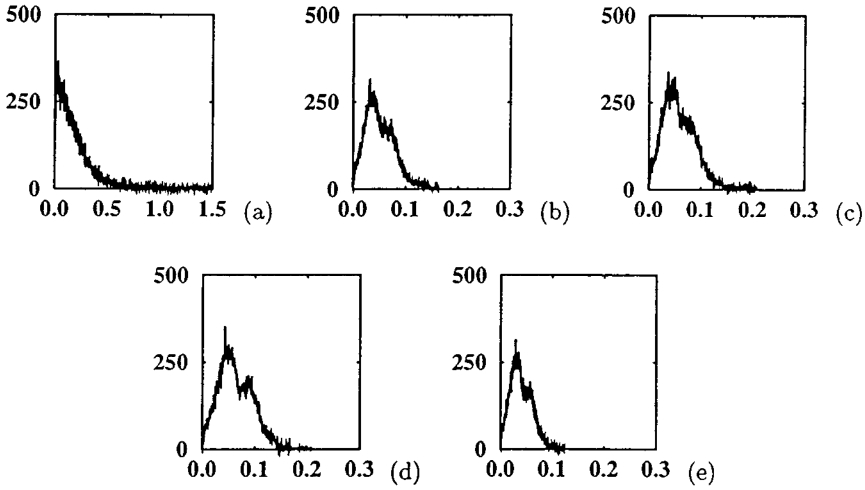

Figure 6. Histograms of (a) the ratio of frequency separation between components one and two to instantaneous bandwidth of component two. (b) the ratio of frequency separation between components two and three to instantaneous bandwidth of component two. (c) the ratio of frequency separation between components two and four to instantaneous bandwidth of component two. (d) the ratio of frequency separation between components two and five to instantaneous bandwidth of component two. (e) the ratio of frequency separation between components two and six to instantaneous bandwidth of component two.

4. Discussion

The two conditions discussed in Section 2 imply that a multidimensional signal is multicomponent if it can be decomposed into a sum of components that are well delineated in instantaneous frequency and that are tightly concentrated on a local basis in the time-frequency or space/spatial frequency hyperplanes. Decompositions that satisfy these conditions generally tend to be physically meaningful and intuitively satisfying. This notion of multicomponent signal does not, however, suggest a procedure for decomposing a multipartite multidimensional signal into components. The computation of valid multicomponent interpretations for complicated natural images and video is extremely difficult in general and remains an active area of research.

References

1. L. Cohen, "Distributions Concentrated Along the Instantaneous Frequency," *SPIE Adv. Signal Proc. Alg., Architectures, Impl.*, vol. 1348, 1990, pp. 149–157.
2. L. Cohen, "What is a Multicomponent Signal?" *Proc. IEEE Int'l. Conf. Acoust., Speech, Signal Proc.*, vol. V, pp. 113–116, San Francisco, CA, March 1992.
3. L. Cohen, *Time-Frequency Analysis*, Englewood Cliffs, NJ: Prentice Hall, 1995.

4. L. Cohen and C. Lee, "Instantaneous Frequency, Its Standard Deviation and Multicomponent Signals," *SPIE Adv. Alg. Architectures Signal Proc. III*, vol. 975, 1988, pp. 186–208.
5. L. Cohen and C. Lee, "Instantaneous Bandwidth," B. Boashash, editor, *Time-Frequency Signal Analysis*, pp. 98–117. Melbourne: Longman Cheshire, 1992.
6. D. Gabor, "Theory of Communication," *J. Inst. Elect. Eng. London*, vol. 93, no. III, 1946, pp. 429–457.
7. J. P. Havlicek, A. C. Bovik, and P. Maragos, "Modulation Models for Image Processing and Wavelet-Based Image Demodulation," *Proc. 26th IEEE Asilomar Conf. Signals, Syst., Comput.*, pp. 805–810, Pacific Grove, CA, October 26–28, 1992.
8. J. P. Havlicek, D. S. Harding, and A. C. Bovik, "The Multi-Component AM-FM Image Representation," *IEEE Trans. Image Proc.*, vol. 5, no. 6, 1996, pp. 1094–1100.
9. J. P. Havlicek, J. W. Havlicek, and A. C. Bovik, "The Analytic Image," *Proc. IEEE Int'l. Conf. Image Proc.*, Santa Barbara, CA, October 26–29, 1997.
10. L. Mandel, "Interpretation of Instantaneous Frequencies," *Am. J. Phys.*, vol. 42, 1974, pp. 840–846.
11. J. Ville, "Théorie et Applications de la Notation de Signal Analytique," *Cables et Transmission*, vol. 2A, 1948, pp. 61–74. Translated from the French in I. Selin, "Theory and applications of the notion of complex signal," Tech. Rept. T-92, The RAND Corporation, Santa Monica, CA, August, 1958.