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Multicut Benders Decomposition Algorithm for Process Supply Chain Planning under Uncertainty

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Abstract

In this paper, we present a multicut version of the Benders decomposition method for solving two-stage stochastic linear programming problems, including stochastic mixed-integer programs with only continuous recourse (two-stage) variables. The main idea is to add one cut per realization of uncertainty to the master problem in each iteration, that is, as many Benders cuts as the number of scenarios added to the master problem in each iteration. Two examples are presented to illustrate the application of the proposed algorithm. One involves production-transportation planning under demand uncertainty, and the other one involves multiperiod planning of global, multiproduct chemical supply chains under demand and freight rate uncertainty. Computational studies show that while both the standard and the multicut versions of the Benders decomposition method can solve large-scale stochastic programming problems with reasonable computational effort, significant savings in CPU time can be achieved by using the proposed multicut algorithm.

Keywords: Benders decomposition, stochastic programming, planning, supply chain

1. Introduction

Many problems for supply chain planning under uncertainty can be formulated as two-stage stochastic programming problems with fixed recourse (Birge & Louveaux, 1997; Infanger, 1994; Shapiro, 2008). In the two-stage framework, the first-stage decisions are made "here-and-now" prior to the resolution of uncertainty, while the second-stage decisions are postponed in a "wait-and-see" mode after the uncertainties are revealed. The scenario planning approach is used to represent the uncertainties through a number of discrete realizations of the stochastic quantities, constituting distinct scenarios. The objective is to find a solution that performs well on average under all scenarios. This approach provides a straightforward way to account for uncertainty, but the resulting stochastic programming models are often computationally demanding because their model size increases exponentially as the number of scenarios increases.

In order to address the computational challenge, a number of methods have been proposed for the solution of two-stage stochastic programming problems (Ruszczyński, 1997), such as Benders decomposition (Benders, 1962; Van Slyke & Wets, 1969), stochastic decomposition (Higle & Sen, 1991), subgradient decomposition (Sen, 1993; Sen and Huang, 2009), disjunctive decomposition (Ntaimo, 2010), and nested decomposition (Archibald et al., 1999). Among these methods, Benders decomposition (Benders, 1962), also called the L-shaped method, has become the major approach to tackle stochastic programming problems because of its ease of implementation. This method takes advantage of the special decomposable structure of the two-stage stochastic programming model and generates duality cuts based on the subgradient information iteratively. Since the standard Benders decomposition returns only one cut to the master problem in each iteration, its convergence might be slow for some computationally demanding problems (Birge & Louveaux, 1997). To address this issue, numerous researchers have proposed variants to accelerate the algorithm (Bahn et al., 1995; Escudero et al., 2007; Fragniere et al., 2000; Gerd Infanger, 1993; Latorre et al., 2009; Linderoth & Wright, 2003; Mulvey & Ruszczynski, 1995; Ruszczynski, 1993; Saharidis & Ierapetritou, 2010; Saharidis et al., 2010; Contreras, et al., 2010; Miller & Ruszczyński, 2010; Trukhanov et al., 2010).

In this paper, we consider the solution methods for stochastic linear programming problems and stochastic mixed-integer linear programs with only continuous recourse.

We first describe a multicut version of the Benders decomposition, which is a variant of the standard Benders decomposition method but converges faster in general cases. We discuss the theory behind this algorithm and prove its convergence. Two applications of this algorithm are then presented to illustrate the effectiveness of this method. The first application involves production-transportation planning under demand uncertainty. Because of the relatively small problem size, the global optimal solution of this problem can easily be obtained to validate the proposed solution approach and illustrate its effectiveness. The second application involves global chemical supply chain planning under uncertainty, which originates from a real-world application in the Dow Chemical Company. The model was taken from the authors' earlier work (You et al., 2009). Three testing data sets with different sizes are considered. In both applications, the results show that the multicut version of the Benders decomposition method requires fewer iterations and less computational time than does the standard version to obtain a solution with a specified optimality tolerance.

The rest of this paper is organized as follows. Section 2 presents the multicut Benders decomposition algorithm, for problems where the first-stage decision variables can include both discrete and continuous variables, while the second-stage decision variables must all be continuous variables. The problem statements, model formulations, and computational results for the two applications are given in Sections 3 and 4. In Section 5, we summarize our conclusions.

2. Multicut Benders Decomposition Algorithm

Consider the following general form of the two-stage stochastic programming model (**P0**):

(P0)
$$\min_{x, y_s} \quad c^{\mathrm{T}} x + \sum_{s \in S} p_s q_s{}^{\mathrm{T}} y_s \tag{1}$$

$$\text{s.t.} \quad Ax = b, \qquad x \geq 0 \tag{2}$$

$$W y_s = h_s - T_s x, \quad y(w_s) \geq 0, \; s \in S \tag{3}$$

where x is a vector that stands for the first-stage decision variables, which may include 0-1 variables; y_s are the continuous second-stage decisions for each scenario s; A and b are parameter matrices independent of the scenarios; and W, h_s

and T_s are parameter matrices for each scenario $s \in S$.

The expanded version of the general model (**P0**) is given in equation (4). The problem has a special "angular" form, which can be decomposed into a master problem and a number of scenario subproblems.

$$
\begin{array}{llllllll}
\text{Min} & c^T x & + & p_1 q_1^T y_1 + p_2 q_2^T y_2 & \cdot \quad \cdot \quad \cdot & + p_s q_s^T y_s & & \\
\text{s.t.} & Ax & & & & & = b & \rightarrow \text{Master problem} \\
 & T_1 x & + & W_1 y_1 & & & = h_1 & \\
 & T_2 x & + & \quad W_2 y_2 & & & = h_2 & \\
 & \vdots & + & & \cdot & & = \vdots & \text{Scenario subproblems} \\
 & \vdots & + & & \quad \cdot & & = \vdots & \\
 & \vdots & + & & \qquad \cdot & & = \vdots & \\
 & T_s x & + & & & W_s y_s & = h_s & \\
 & & x \geq 0, & y_1 \geq 0, \ y_2 \geq 0, & \cdot \quad \cdot \quad \cdot & y_s \geq 0 & &
\end{array}
\tag{4}
$$

The special decomposable structure of (4) is suitable for Benders decomposition because it takes advantage of subgradient information to construct convex estimates of the recourse function and iteratively generates a Benders cut to be added to the decomposed master problem (Benders, 1962; Van Slyke & Wets, 1969). In the first step, a decomposed subproblem with those constraints that do not include the second-stage variables is solved to obtain the values of the first-stage decisions. Then we fix the first-stage decisions and solve all the scenario subproblems that include second-stage decisions, in order to obtain the optimal values of the second-stage decisions.

Let $Q_s(x)$, the value function, be the objective function value of each scenario subproblem s.

$$
\begin{aligned}
Q_s(x) = \min_{y_s} \quad & q_s^{\mathrm{T}} y_s \\
\text{s.t.} \quad & W y_s = h_s - T_s x, \quad y(w_s) \geq 0
\end{aligned}
\tag{5}
$$

The reformulation of (**P0**) is then as follows.

$$
(\mathbf{P0}) \qquad \min_{x} \quad c^{\mathrm{T}} x + \sum_{s \in S} p_s Q_s(x) \tag{6}
$$

$$
\text{s.t.} \qquad Ax = b, \qquad x \geq 0 \tag{7}
$$

To solve (**P0**), we can take advantage of the dual properties of (6) by introducing a new variable θ for $\sum_{s \in S} p_s Q_s(x)$ and iterating between the master problem (**P1**) and the scenario subproblems (**P2**).

The master problem (**P1**) is given by

$$
\begin{aligned}
(\mathbf{P1}) \qquad & \min_{x,\theta} \quad c^{\mathrm{T}} x + \theta \\
& \text{s.t.} \quad \theta \ge d^{iter} x + e^{iter}, \quad iter = 1..N \qquad (8) \\
& \qquad\quad Ax = b, \quad x \ge 0
\end{aligned}
$$

while the subproblem (**P2**) for scenario s is given by

$$
\begin{aligned}
(\mathbf{P2}) \qquad & \min_{y_s} \quad {q_s}^{\mathrm{T}} y_s \\
& \text{s.t.} \quad Wy_s = h_s - T_s x, \quad y(w_s) \ge 0 \qquad (9)
\end{aligned}
$$

where the inequalities in (**P1**) are the "cuts" that link the master problem and the scenario subproblems. Here, d_l and e_l are coefficients for the Benders cut; they are given by

$$d^{iter} = \sum_{s \in S} p_s \pi^{T}_{iter,s} T_s \qquad (10)$$

$$e^{iter} = \sum_{s \in S} p_s \pi^{T}_{iter,s} h_s \qquad (11)$$

where π_s are the optimal dual vectors of constraint (5) in the subproblem (**P2**) for scenario s.

In this paper, we assume that the problem (**P0**) has complete recourse and that (**P2**) is always feasible. We note that feasibility cuts (Birge and Louveaux, 1997) can be added to the algorithmic framework to deal with problems with infeasible subproblems, although in this work we limit our scope on those that have complete recourse after introducing additional slack variables for shortfalls or back orders in supply chain planning

Under this assumption, feasibility cuts are not present in the master problem (**P1**). Our algorithms and this analysis can be generalized to handle situations in which the aforementioned assumption does not hold; but for the sake of simplifying the analysis, we avoid discussing this more general case here.

The major steps for the standard Benders decomposition algorithm are given in Figure 1. In this algorithm, we first solve the master problem to obtain a lower bound of the objective value. We then fix all the first-stage decisions and solve each scenario subproblem to get an upper bound. If the lower bound and the upper bound are within a tolerance, then the algorithm stops. Otherwise, we use the duals of the scenario subproblems to add a Benders cut and return to the master problem.

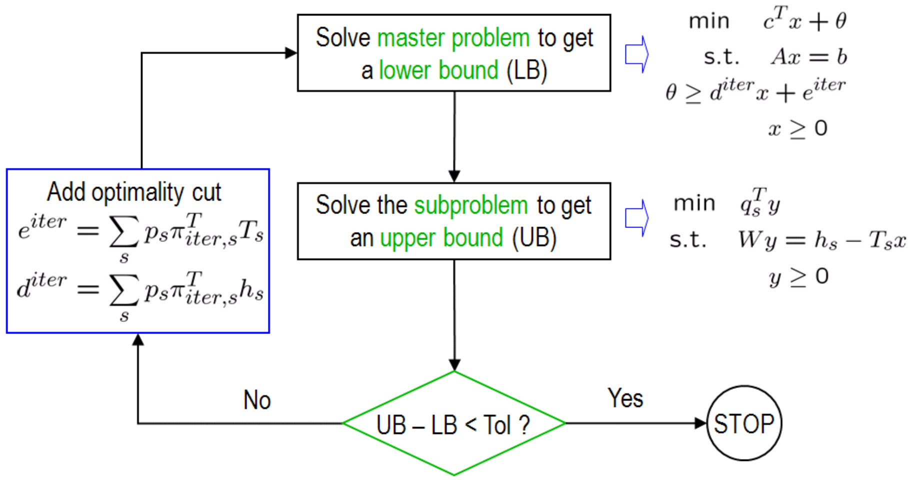

Figure 1 Algorithm for standard Benders decomposition

The standard Benders decomposition algorithm only returns one cut per iteration to the master problem. For large-scale problem, its convergence might be slow and the algorithm might need many iterations to reach a predefined optimality tolerance.

To speed up the algorithm, we can decompose the variable θ by scenarios to return as many cuts as the number of scenarios at each iteration. In this variant, the master problem is then given by (**P3**).

(**P3**)

$$\min_{x,\theta_s} \quad c^{\mathrm{T}} x + \sum_{s \in S} p_s \theta_s$$

$$\text{s.t.} \quad \theta_s \geq d_s^{iter} x + e_s^{iter}, \quad iter = 1..N, \ s = 1..S \tag{12}$$

$$Ax = b, \quad x > 0$$

The coefficients d_{sl} and e_{sl} for the cut (12) are updated as follows

$$d_s^{iter} = \pi_{iter,s}^T T_s \tag{13}$$

$$e_s^{iter} = \pi_{iter,s}^T h_s \tag{14}$$

where π_s are the optimal dual vectors of constraint (5) in the subproblem (**P2**) for scenario s.

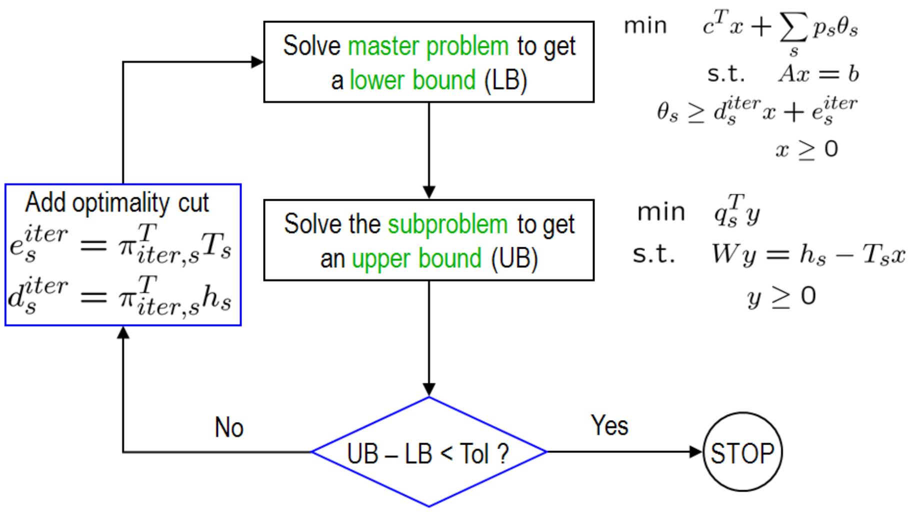

Figure 2 Algorithm for multicut Benders method

The algorithm framework for the multicut Benders algorithm is similar to that for the standard Benders algorithm (see You et al, 2009) and is given as follows (see also Figure 2).

<u>Step 1</u>

Set $iter = 1$, $LB = -\infty$, $UB = +\infty$.

<u>Step 2</u>

At iteration *iter*, solve the master problem (**P3**) with all the optimality cuts generated in the previous iterations. Denote the optimal objective function value as φ^{iter} and the optimal solution of the first-stage decision variables x as X^{iter}. If $\varphi^{iter} > LB$, set $LB = \varphi^{iter}$.

<u>Step 3:</u>

Solve all the scenario subproblems (**P2**) with the values of first-stage decision variables fixed as $x = X^{iter}$. Let the optimal solution of the second-stage decision variables y_s be Y_s^{iter} and the optimal dual vectors of constraint (5) in the subproblem (**P2**) for scenario s be $\pi_{iter,s}$. Compute the value of the objective of the original problem (**P0**); that is, set $\Phi^{iter} = c^{\mathrm{T}} X^{iter} + \sum_{s \in S} p_s q_s^{\mathrm{T}} Y_s^{iter}$. If $\Phi^{iter} < UB$, then update $UB = \Phi^{iter}$.

Step 4

If $UB - LB \le \varepsilon$ (e.g., 10^{-3}), stop and output the optimal solution (X^{iter} , Y_s^{iter}); otherwise, compute the coefficients of the optimality cuts $d_s^{iter} = \pi_{iter,s}^T T_s$ and $e_s^{iter} = \pi_{iter,s}^T h_s$, and add the optimality cuts $\theta_s \ge d_s^{iter} x + e_s^{iter}$, $\forall s$, to the master problem (**P2**). Then, set $iter = iter + 1$, and go to Step 2.

Convergence is guaranteed in this algorithm by the following propositions.

Proposition 1. *The recourse function* $R(x) = \sum_{s\in S} p_s Q_s(x)$ *is a convex piecewise linear function.*

Proof: The proof of this proposition is given in (Birge & Louveaux, 1997). □

Proposition 2. *Each optimality cut* (12) *supports the recourse function* $R(x)$ and $Q_s(x)$ *from below.*

Proof: The proof of this proposition is given by (Birge & Louveaux, 1997). □

Proposition 3. *Given some (* X^{iter} , θ_s^{iter} *) such that* $\theta_s^{iter} = d_s^{iter} X^{iter} + e_s^{iter}$, $\forall s$, *then* X^{iter} *is an optimal solution of the original problem (**P0**).*

Proof:

The original problem (**P0**) is equivalent to the following problem (**P4**).

$$\min_x \quad c^{\mathrm{T}} x + \sum_{s\in S} p_s \theta_s$$

$$\text{s.t.} \quad Ax = b, \qquad x \ge 0$$

$$Q_s(x) \le \theta_s$$

Based on Proposition 2 and the duality property of (**P2**), we have $\theta_s^{iter} = e_s^{iter} X^{iter} + d_s^{iter} = Q_s\left(X^{iter}\right)$, $\forall s$. Therefore, (X^{iter}, θ_s^{iter}) is a feasible solution of (**P4**). Since (X^{iter}, θ_s^{iter}) is the optimal solution of (**P2**), it is also an optimal solution of (**P4**), which is equivalent to the original problem (**P0**). □

Proposition 4. **(Convergence)** *Since the algorithm generates a finite sequence of*

(X^{iter} *,* θ_s^{iter} *) and since (* $\overline{X}^{iter}$ *,* $\overline{\theta}_s^{iter}$ *) is the limit of this sequence, and* $\lim_{iter \to M} d_s^{iter} x + e_s^{iter} - \theta_s^{iter} = 0$*, where M is a sufficiently large integer, then* $\overline{X}^{iter}$ *is an optimal solution of the original problem (**P0**).*

Proof:

From Proposition 2 and $\lim_{iter \to M} d_s^{iter} x + e_s^{iter} - \theta_s^{iter} = 0$, we have $\overline{\theta}_s = Q_s\left(\overline{X}^{iter}\right)$.

Thus, ($\overline{X}^{iter}$, $\overline{\theta}_s^{iter}$) is a feasible solution of problem (**P4**). Because function $R(x) = \sum_{s \in S} p_s Q_s(x)$ is a convex piecewise linear function as shown in Proposition 1, ($\overline{X}^{iter}$, $\overline{\theta}_s^{iter}$) is also an optimal solution of (**P4**), which is equivalent to the original problem (**P0**). □

We note that while the multicut L-shaped method can provide more cuts to support the recourse function from below and most likely reduce the number of iterations, it introduces more variables and constraints in the master problem, which may potentially slow the computation. This algorithm would benefit from solving it with parallel computing, which could significantly reduce the wall-clock times.

3. Production-Transportation Planning under Uncertainty

The first application of the proposed algorithm is about a single product, single-period production-transportation planning under demand uncertainty. This problem can be formally stated as follows.

We are given a set of plants $i \in I$ with production capacity cap_i and a set of demand zones $l \in L$. The selling price at demand zone l is $price_l$, the unit transportation cost from plant i to demand zone l is $ctr_{i,l}$, the unit production cost at plant i is cpd_i, and the unit waste disposal cost in demand zone l is cus_l. Here, $s \in S$ is the set of scenarios, p_s is the scenario probability, and $demand_{l,s}$ is the demand at demand zone l of scenario s. The major decisions include the production level ($prod_i$), transportation amount ($ship_{i,l}$), sales amount ($sale_{l,s}$), and unsold product amount ($unsold_{,s}$). We note that in the two-stage stochastic linear programming framework, the production and transportation decisions are made "here and now" prior to the resolution of demand uncertainty, whereas the sales and waste disposal decisions are postponed in a "wait-and-see" mode after the uncertainties are revealed. Thus, the production and transportation decisions are independent of the scenarios, whereas the sales decisions are made for each scenario. The objective of this problem is to maximize the total expected profit ($E[profit]$) by optimizing the aforementioned decisions.

Based on the problem statement, a two-stage stochastic linear programming model can be formulated as follows.

$$\max \quad E[profit] = \sum_{l \in L}\sum_{s \in S} p_s \cdot price_l \cdot sale_{l,s} - \sum_{i \in I}\sum_{l \in L} ctr_{i,l} \cdot ship_{i,l} - \sum_{i \in I} cpd_i \cdot prod_i - \sum_{l \in L}\sum_{s \in S} p_s \cdot cus_l \cdot unsold_{l,s} \qquad (15)$$

s.t.

$$prod_i \le cap_i, \qquad \forall i \in I \qquad (16)$$

$$prod_i = \sum_{l \in L} ship_{i,l}, \qquad \forall i \in I \qquad (17)$$

$$\sum_{i \in I} ship_{i,l} = sale_{l,s} + unsold_{l,s}, \qquad \forall l \in L,\ s \in S \qquad (18)$$

$$sale_{l,s} \le demand_{l,s}, \qquad \forall l \in L,\ s \in S \qquad (19)$$

$$prod_i \ge 0,\ ship_{i,l} \ge 0,\ sale_{l,s} \ge 0,\ unsold_{l,s} \ge 0$$

Table 1 Probability distribution of demand realizations for the production-transportation planning problem

Demand Zones	Demand Realization (ton)			Probability		
	Low	Medium	High	Low	Medium	High
1	150	160	170	0.25	0.5	0.25
2	100	120	135	0.25	0.5	0.25
3	250	270	300	0.25	0.5	0.25
4	300	325	350	0.3	0.4	0.3
5	600	700	800	0.3	0.4	0.3

Table 2 Unit transportation cost for the production-transportation planning problem ($/ton)

Plants/Demand Zones	1	2	3	4	5
P1	2.49	5.21	3.76	4.85	2.07
P2	1.46	2.54	1.83	1.86	4.76
P3	3.26	3.08	2.6	3.76	4.45

In this case study we consider a production-transportation network with three plants and five demand zones. The probability distribution of the demand realization is given in Table 1. In each demand zone there are three possible demand realizations. We assume these probabilities are independent. By considering the joint probability distribution, we generate a total of 3^5=243 scenarios for this problem. The unit production cost is $14/ton, the sales price is $24/ton, and cost of removal of unsold products is $4/ton. The unit transportation cost between plants and demand zones is given in Table 2.

The deterministic equivalent of the resulting two-stage stochastic linear program includes 2,448 continuous variables and 2,436 constraints. Less than one second was needed to obtain the optimal solution ($10,793) with 0% gap using GAMS 23.4.3/CPLEX 12 (Rosenthal, 2010), on an IBM T400 laptop with an Intel 2.53 GHz CPU and 2 GB RAM.

To illustrate the application of the proposed multicut algorithm and compare its performance with that of the standard Benders decomposition, we solved this problem with both algorithms. The results are shown in Figures 3 and 4. As can be seen from Figure 3, the upper bounds decrease and the lower bounds increase as the number of iterations increase. However, whereas the standard Benders method requires 22 iterations to reach the optimality tolerance, the multicut version requires only 6 iterations. The results in Figure 3 clearly show that the multicut version converges much faster than does the standard Benders method. The reason rests mainly with the improved approximation of the value function in (5), since a larger number of Benders cuts are added to the master problem at each iteration.

The computational times for both algorithms show little difference for this case study (0.15 CPU seconds for the single cut version and 0.13 CPU seconds for the multicut version), although the multicut version requires far fewer iterations than does the standard Benders decomposition. The main reason is that the master problem of the multicut Benders method includes more variables and constraints (Benders cuts) than does the standard version and thus requires longer computational time per iteration. Another reason is that the computational times for this case study are so short that the scaling effect and the advantage of the multicut version cannot be fully illustrated.

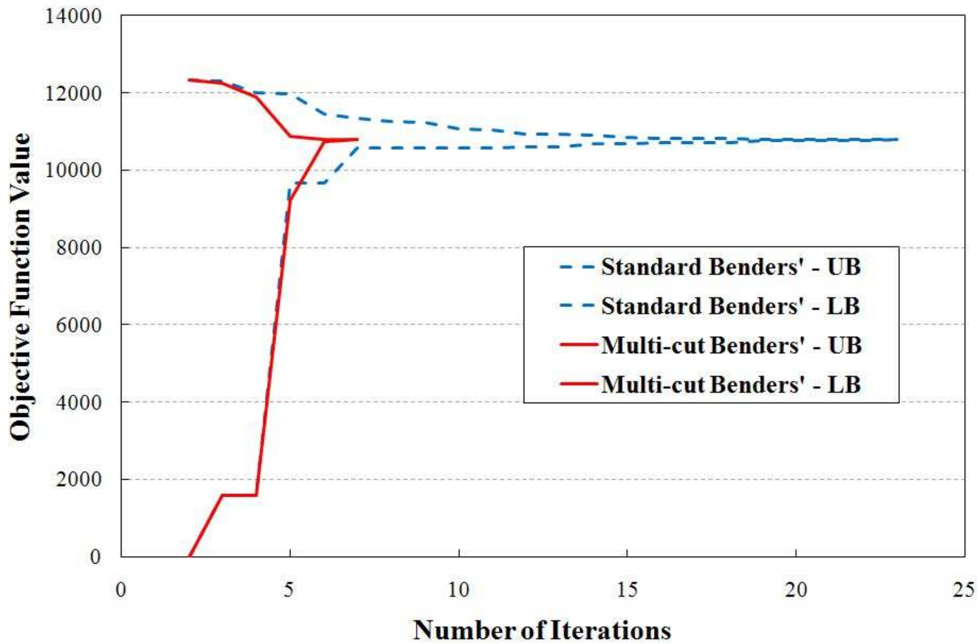

Figure 3 Comparison between the standard Benders method and the multicut version in terms of number of iterations for the production-transportation planning problem

4. Global Chemical Supply Chain Planning under Uncertainty

The second case study considered in this work is based on the problem described by You et al. (2009), which originates from a real-world application in the Dow Chemical Company. Global supply chains in the process industries are usually large-scale systems that can comprise hundreds or even thousands of production facilities, distribution centers, and customers (Wassick, 2009). This case study addresses the midterm planning for a global multiproduct chemical supply chain under demand and freight rate uncertainty. A two-stage stochastic mixed-integer linear programming model is used, incorporating a multiperiod planning model that takes into account the production and inventory levels, transportation modes, times of shipments, and customer service levels. In the two-stage framework, the production, distribution, and inventory decisions for the current time period, which include 0-1 variables, are made "here and now" prior to the resolution of uncertainty, while the decisions for the remaining time periods, which only involve continuous variables, are postponed in a "wait-and-see" mode. The problem includes a large number of uncertain parameters as a result of the multiperiod nature and the large size of the supply chain network. A Monte Carlo sampling approach is used to discretize the continuous probability distribution functions and to generate the scenarios.

To demonstrate the effectiveness of the proposed decomposition algorithms, we solve three instances for small, medium, and large supply chain networks, using both the standard Benders decomposition method and the multicut version. We present the problem statement, model formulation, and computational results in the following subsections.

4.1. Problem statement

This case study can be stated as follows. We are given a midterm planning horizon (for instance, one year), which can be subdivided into a number of time periods (for instance, one month as a time period). A set of products are manufactured and distributed through a given global supply chain that includes a large number of worldwide customers and a number of geographically distributed plants and distribution centers. All the facilities (plants and distribution centers) can hold inventory and are connected to each other by an associated transportation link. Each customer is served by one or more facilities with specified transportation links. A simplified version of the network is shown in Figure 4. The network has multiple echelons whereby material may flow from the manufacturing plant through several distribution centers on its way to the final customer. Freight rates are specific to the transportation link involved and depend on distance and mode of transport. Generally, the transportation links are classified into two types: from one facility to another facility (plant or distribution center) and from a facility to a customer. Some transportation links with certain transportation modes are managed by third-party logistics companies; these require either that no products be shipped through these links with the corresponding transportation mode or that a minimum quantity be shipped in each time period.

Besides the supply chain network topology, we are given the minimum and initial inventory of each facility. The inventory holding costs and the facility throughput costs are already known, together with future monthly demand of each product by each customer. The transportation time of each shipping lane is known and should be taken into account.

The uncertainties arise from the customer demands and freight rates. The values of these uncertain parameters follow some probability distribution (such as normal distribution) with a given mean and variance. Usually, the probability distribution of the uncertain parameters can be obtained by fitting the historical data for different probability distributions or can be based on expert opinions. The mean values of these uncertain parameters typically come from forecasting, and the variances come from historical data. We allow the demands and freight rates to have different levels of uncertainties changing with time. For example, in January the uncertain demand of May has a standard deviation as much as 20% of the mean value, but in April the standard deviation of that demand of May reduces to 5% of the mean value as a result of more accurate forecasting and information. Different levels of uncertainties are important for the operations of industrial supply chains and should be taken into account in the models.

The problem is to determine the monthly or weekly production and inventory levels of each facility, and the monthly shipping quantities between network nodes such that the total expected cost and the total risks of the global supply chain are minimized, while satisfying customer demands over the specified planning horizon.

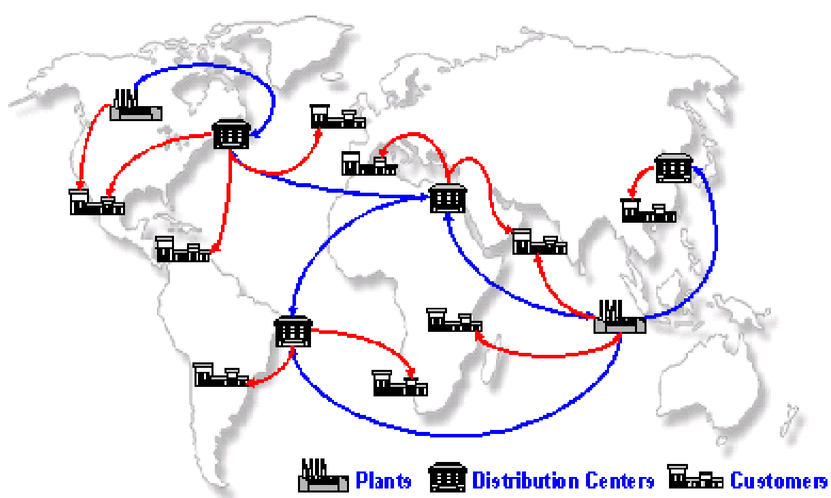

Figure 4 Global chemical supply chain

4.2. Two-stage stochastic programming model

We consider a two-stage stochastic mixed-integer programming approach to deal with different levels of uncertainties. We incorporate this approach into a multiperiod planning model that takes into account the production and inventory levels, transportation modes, times of shipments, and customer service levels. In principle, the problem can be formulated as a multistage stochastic programming model. To reduce the computational effort, we consider only a two-stage approach. In this two-stage framework, the production, distribution, and inventory decisions for the current time period and the transportation mode selection decisions are made "here and now" prior to the resolution of uncertainty, while the decisions for the rest of the time periods are postponed in a "wait-and-see" mode after the uncertainties are revealed. The scenario planning approach is used to represent the uncertainties. A resulting challenge is that a large number of scenarios are required because the problem includes a very large number of uncertain parameters as a result of the multiperiod nature of the model and the large size of the global supply chain network.

To reduce the model size and the number of scenarios, we use a Monte Carlo sampling approach to generate the scenarios (Linderoth et al., 2006; Shapiro, 2000; A. Shapiro & Homem-de-Mello, 1998). Each scenario is then assigned the same probability, with the summation of the probabilities for all the scenarios equal to 1. For example, if we use Monte Carlo sampling to generate 100 scenarios, the probability of each scenario is given as 0.01. The number of scenarios is determined by using a statistical method to obtain solutions within specific confidence intervals for a desired level of accuracy. This method is effective for scenario reduction, particularly for large-scale problems. As an example, for a problem with 5^{1000} scenarios, a sample size of around 400 can find the true optimal solution with probability 95%. The process of generating scenarios by Monte Carlo sampling is illustrated through Figure 5. As the statistical analysis method for determining the required number of scenarios is not the focus of this paper, we do not introduce the details here and the readers can refer to our earlier works for details (You et al., 2009).

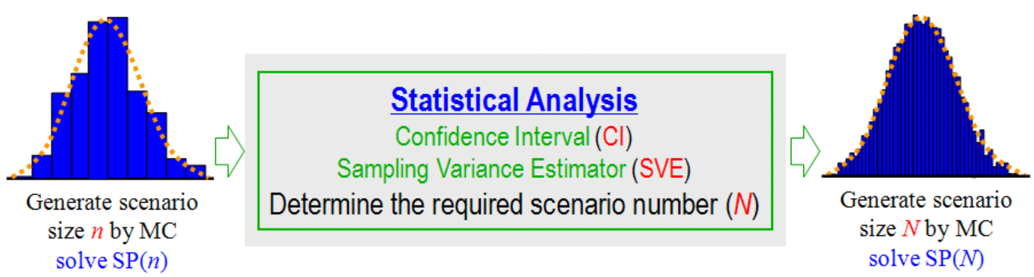

Figure 5 Discretization of the continuous probability distribution by using Monte Carlo sampling for scenario generation

In this work, we use a multiperiod formulation to allow the costs and sourcing decisions to change with time while taking into account the transportation time for each shipment. Sets, variables, and parameters of the model are defined at the end of this paper. The mathematical formulation of the multiperiod mixed-integer linear programming planning model is given below.

$$\min: \quad E[Cost] = Cost1 + \sum_{s \in S} p_s \cdot Cost2_s \tag{20}$$

s.t.

$$\begin{aligned} Cost1 = & \sum_{k \in K} \sum_{j \in J} \sum_{t=1} h_{k,j,t} I_{k,j,t} \\ & + \sum_{k \in K} \sum_{k' \in K} \sum_{j \in J} \sum_{m \in M} \sum_{t=1} \gamma_{k,k',j,m,t} F_{k,k',j,m,t} \\ & + \sum_{k \in K} \sum_{r \in R} \sum_{j \in J} \sum_{m \in M} \sum_{t=1} \gamma_{k,r,j,m,t} S_{k,r,j,m,t} \\ & + \sum_{k \in K} \sum_{k' \in K} \sum_{j \in J} \sum_{m \in M} \sum_{t=1} \delta_{k,j,t} F_{k,k',j,m,t} \\ & + \sum_{k \in K} \sum_{r \in R} \sum_{j \in J} \sum_{m \in M} \sum_{t=1} \delta_{k,j,t} S_{k,r,j,m,t} \end{aligned} \tag{21}$$

$$\begin{aligned} Cost2_s = & \sum_{k \in K} \sum_{j \in J} \sum_{t \geq 2} h_{k,j,t} I_{k,j,t,s} \\ & + \sum_{k \in K} \sum_{k' \in K} \sum_{j \in J} \sum_{m \in M} \sum_{t \geq 2} \gamma_{k,k',j,m,t,s} F_{k,k',j,m,t,s} \\ & + \sum_{k \in K} \sum_{r \in R} \sum_{j \in J} \sum_{m \in M} \sum_{t \geq 2} \gamma_{k,r,j,m,t,s} S_{k,r,j,m,t,s} \\ & + \sum_{k \in K} \sum_{k' \in K} \sum_{j \in J} \sum_{m \in M} \sum_{t \geq 2} \delta_{k,j,t} F_{k,k',j,m,t,s} \\ & + \sum_{k \in K} \sum_{r \in R} \sum_{j \in J} \sum_{m \in M} \sum_{t \geq 2} \delta_{k,j,t} S_{k,r,j,m,t,s} \\ & + \sum_{r \in R} \sum_{j \in J} \sum_{t \in T} \eta_{r,j,t} SF_{r,j,t,s} \end{aligned}, \qquad \forall s \tag{22}$$

$$\sum_{k' \in K} \sum_{m \in M} F_{k,k',j,m,t} + \sum_{r \in R} \sum_{m \in M} S_{k,r,j,m,t} = I^0_{k,j} - I_{k,j,t} + W_{k,j,t} + \sum_{k' \in K} \sum_{m \in M} F_{k',k,j,m,t-\lambda_{k',k,j,m}}, \qquad \forall j,\ k \in K_P,\ t = 1 \tag{23}$$

$$\sum_{k' \in K} \sum_{m \in M} F_{k,k',j,m,t,s} + \sum_{r \in R} \sum_{m \in M} S_{k,r,j,m,t,s} = I_{k,j,t-1,s} - I_{k,j,t,s} + W_{k,j,t,s} + \sum_{k' \in K} \sum_{m \in M} F_{k',k,j,m,t-\lambda_{k',k,j,m},s}, \qquad \forall j,s,k \in K_P,\ t \geq 2 \tag{24}$$

$$\sum_{k' \in K} \sum_{m \in M} F_{k,k',j,m,t} + \sum_{r \in R} \sum_{m \in M} S_{k,r,j,m,t} = I^0_{k,j} - I_{k,j,t} + \sum_{k' \in K} \sum_{m \in M} F_{k',k,j,m,t-\lambda_{k',k,j,m}}, \qquad \forall j,\ k \in K_{DC},\ t = 1 \tag{25}$$

$$\sum_{k' \in K} \sum_{m \in M} F_{k,k',j,m,t,s} + \sum_{r \in R} \sum_{m \in M} S_{k,r,j,m,t,s} = I_{k,j,t-1,s} - I_{k,j,t,s} + \sum_{k' \in K} \sum_{m \in M} F_{k',k,j,m,t-\lambda_{k',k,j,m},s}, \qquad \forall j,s,\ k \in K_{DC},\ t \geq 2 \tag{26}$$

$$\sum_{k \in K} \sum_{m \in M} S_{k,r,j,m,t-\lambda_{k,r,j,m}} + SF_{r,j,t,s} \geq d_{r,j,t,s}, \qquad \forall r,j,s,\ t = 1 \tag{27}$$

$$\sum_{k \in K} \sum_{m \in M} S_{k,r,j,m,t-\lambda_{k,r,j,m},s} + SF_{r,j,t,s} \geq d_{r,j,t,s}, \qquad \forall r,j,s,\ t \geq 2 \qquad (28)$$

$$W_{k,j,t} \leq Q_{k,j}, \qquad \forall j,\ t=1,\ k \in K_P \qquad (29)$$

$$W_{k,j,t,s} \leq Q_{k,j}, \qquad \forall j,s,\ t \geq 2,\ k \in K_P \qquad (30)$$

$$I_{k,j,t} \geq I^m_{k,j,t}, \qquad \forall k,j,\ t=1 \qquad (31)$$

$$I_{k,j,t,s} \geq I^m_{k,j,t}, \qquad \forall k,j,s,\ t \geq 2 \qquad (32)$$

$$F_{k,k',j,m,t} \geq ZF_{k,k',j,m} \cdot F^L_{k,k',j,m} \qquad \forall (k,k',j,m) \in KKJM,\ t=1 \qquad (33)$$

$$F_{k,k',j,m,t,s} \geq ZF_{k,k',j,m} \cdot F^L_{k,k',j,m} \qquad \forall (k,k',j,m) \in KKJM,\ s,\ t \geq 2 \qquad (34)$$

$$S_{k,r,j,m,t} \geq ZS_{k,r,j,m} \cdot S^L_{k,r,j,m} \qquad \forall (k,r,j,m) \in KRJM,\ t=1 \qquad (35)$$

$$S_{k,r,j,m,t,s} \geq ZS_{k,r,j,m} \cdot S^L_{k,r,j,m} \qquad \forall (k,r,j,m) \in KRJM,\ s,\ t \geq 2 \qquad (36)$$

$$ZF_{k,k',j,m} \in \{0,1\},\ ZS_{k,r,j,m} \in \{0,1\}$$

$$Cost1 \geq 0,\ Cost2_s \geq 0,\ F_{k,k',j,m,t} \geq 0,\ F_{k,k',j,m,t,s} \geq 0,\ I_{k,j,t} \geq 0,\ I_{k,j,t,s} \geq 0,\ S_{k,r,j,m,t} \geq 0,$$

$$S_{k,r,j,m,t,s} \geq 0,\ SF_{r,j,t,s} \geq 0,\ W_{k,j,t} \geq 0,\ W_{k,j,t,s} \geq 0$$

The objective function of this stochastic mixed-integer linear programming model is to minimize the total expected cost given in (20), which includes the first-stage cost, $Cost1$, and the expected second-stage cost. Since the scenarios follow discrete distribution, the expected second-stage cost is equal to the product of the scenario probability, p_s, and the associated second-stage scenario cost, $Cost2_s$, summed over all the scenarios s. Both the first-stage cost given in (21) and the second-stage scenario cost given in (22) are equal to the sum of the following items:

- Inventory holding cost for all products at all facilities for all time periods
- Freight cost for interfacility freight shipments in all the shipping lanes of all the products in all time periods
- Freight cost for facility-customer shipments in all the shipping lanes of all the products in all the time periods
- Facility throughput cost for interfacility shipments for all the shipping lanes of all the products in all the time periods
- Facility throughput cost for facility-customer shipments for all the shipping lanes of all the products in all the first-stage time periods

- Penalty costs of all the products for unmet demand of all the customers in all the time periods

Six types of constraints are included in the model. The mass balance relationships for the plants are given in constraints (23) and (24), the mass balance for distribution centers are given in constraints (25) and (26), the demand balance for customers are given in constraints (27) and (28), production capacity constraints are given in (29) and (30), and minimum inventory level constraints are given in (31) and (32); constraints (33)–(36) are minimum transportation level constraints for selected transportation links/modes managed by third-party logistic companies. Constraints (23), (25), (29), (31), (33), and (35) are first-stage constraints that do not include any scenario-dependent (second-stage) variables, while the remaining constraints are second-stage constraints for each scenario. The first-stage constraints are for the production, inventory, and transportation planning of the first time period ($t=1$), except for the demand balance constraint (27) that accounts for the uncertain demand realization. Binary variables $ZF_{k,k',j,m}$ and $ZS_{k,r,j,m}$ are introduced to model the semi-continuous transportation levels for selected transportation links or modes. A slack variable $SF_{r,j,t,s}$ is used to model the shortfalls and avoid infeasibility of the planning problem. An additional feature of this model is that the transportation times are taken into account through the multiperiod formulation, where shipments across multiple time periods are explicitly modeled.

Minimizing the objective function in (20), subject to the constraints in (21) – (36), we can obtain the solution for the two-stage stochastic programming model. Computational results for solving this model with the standard and the multicut versions of the Benders decomposition method are presented in the next section.

4.3. Computational results

The problem is based on the global supply chain of a major commodity chemical producer. We consider a planning horizon of one year, which is subdivided into 12 time periods, one month as a time period. Two products are produced and distributed in the global supply chain. The customer demands and freight rates, which are uncertain, follow normal distributions, with the forecast as the mean value and the variance coming from the historical record. The demand uncertainty has three levels of standard deviations. For the current month the standard deviation of demand is 5% of the mean value; in the coming three months, the standard deviation is 10% of the mean value; and for the remaining eight months, the demand has a standard deviation of 20% of the mean value. Similarly, the freight rate has two levels of uncertainty. For the current month, the variance is 0 (i.e., deterministic case); for the remaining 11 months, the freight rate has a standard deviation of 10% of the mean value. Three makeup instances are considered, representing three supply chain networks. The first instance is for a small network with 2 plants, 4 distribution centers, 2 customers, 1 transportation mode and 9 transportation links. The second instance is for a medium size supply chain network with 5 plants, 17 distribution centers, 46 customers, 4 transportation modes and 75 transportation links. The third instance is for a large network with 14 plants, 70 distribution centers, 126 customers, 14 transportation modes, and 328 transportation links. Although the size of the stochastic programming problem exponentially increases as the number of scenarios increases, we found that at least 1,000 scenarios are required in order to achieve reasonable confidence intervals. Thus, we consider 1000 scenarios for each of the three instances. The problem sizes of the deterministic equivalents for three instances are given in Table 3, and the sizes of the first-stage and second-stage subproblems are listed in Tables 4-5. All the instances are modeled with GAMS 23.4.3 and solved with the CPLEX 12 solver on an IBM T400 laptop with an Intel Core Duo 2.53 GHz CPU and 2 GB RAM. We note that none of these instances can be solved directly because of their large size. Thus, the standard and multicut versions of the Benders decomposition method are used. The optimality tolerances for both methods are set to 0.001%.

Table 3 Problem sizes of the deterministic equivalents of the numerical examples

Problem Size	Instance 1	Instance 2	Instance 3
No. of Binary Variables	7	22	158
No. of Continuous Variables	423,036	3,703,384	75,356,014
No. of Constraints	201,018	1,301,189	52,684,187

Table 4   Problem sizes of the first-stage problem of the numerical examples

Problem Size	Instance 1	Instance 2	Instance 3
No. of Binary Variables	7	22	158
No. of Continuous Variables	36	384	4,014
No. of Constraints	18	189	2,187

Table 5   Problem sizes of the second-stage problem of the numerical examples

Problem Size	Instance 1	Instance 2	Instance 3
No. of Continuous Variables	423	3,703	75,352
No. of Constraints	201	1,301	52,682

The computational performances of the standard and multicut versions of the Benders algorithm are shown in Figures 6 – 11. We can see that how the upper bound decreases and the lower bound increases with the number of iterations, and how the computational time increases for both solution methods in all these figures. For Instance 1, the small-scale problem (results shown in Figures 6 and 7), the standard Benders method requires 21 iterations (around 12 CPU-seconds) to converge, while the multicut versions can reach the same optimality gap in 6 iterations (4 CPU-seconds). Similarly, for Instance 2 with a medium-size supply chain network (results shown in Figures 8 and 9), the multicut method requires only 45 iterations (around 5 CPU-minutes) to converge, while the standard Benders method takes 534 iterations (around 45 CPU-minutes) to reach to the same optimality tolerance. As the problem size becomes larger, the multicut Benders method is computationally much more efficient than the standard method. For Instance 3, the largest problem (results shown in Figures 10 and 11), the multicut version needs only 47 iterations (around 11 CPU-minutes), while the standard Benders method requires 564 iterations (about 3.5 CPU-hours).

The high computational efficiency of the multicut Benders method is because its master problem requires relatively small solution times despite its large size, and the number of iterations is significantly reduced as a result of the "multiple" cuts. In contrast, while the master problem in the standard Benders method is smaller in size and faster to solve, it also requires a significantly larger number of iterations. Note that both algorithms would benefit from solving the scenario subproblems with parallel computing and coordinate through a master-worker computational framework

(Linderoth & Wright, 2003), which could significantly reduce the computational times.

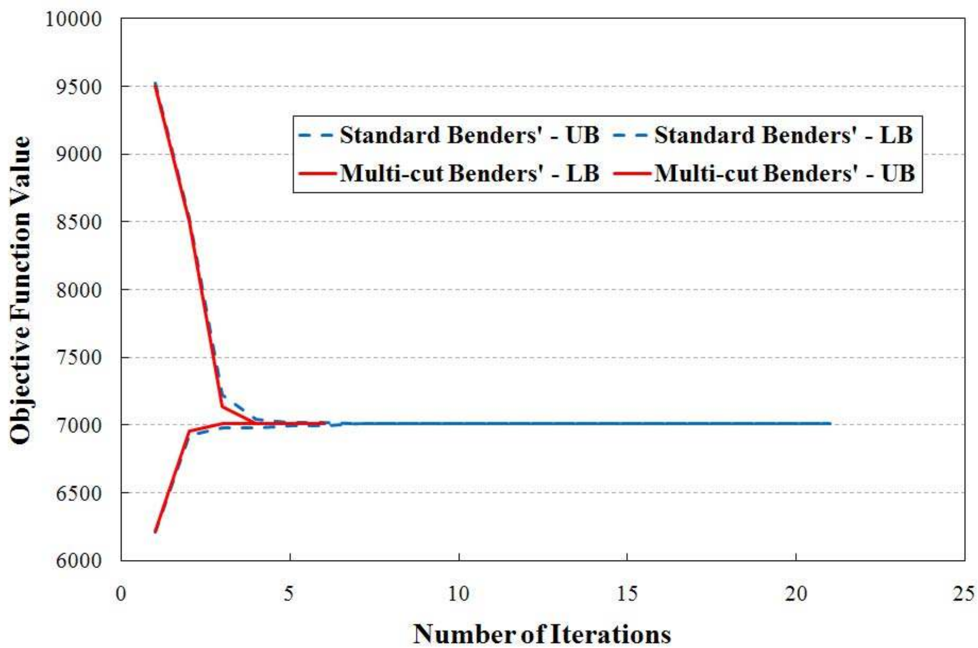

Figure 6 Comparison between the standard Benders method and the multicut version in terms of number of iterations for the first instance of the global supply chain planning problem

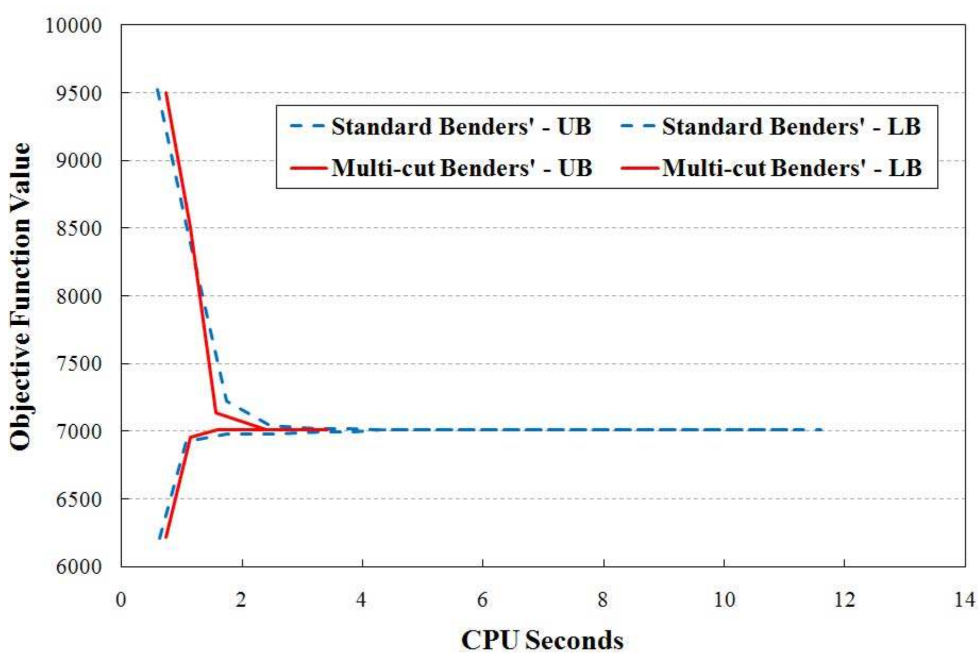

Figure 7 Comparison between the standard Benders method and the multicut version in terms of CPU-seconds for the first instance of the global supply chain

planning problem

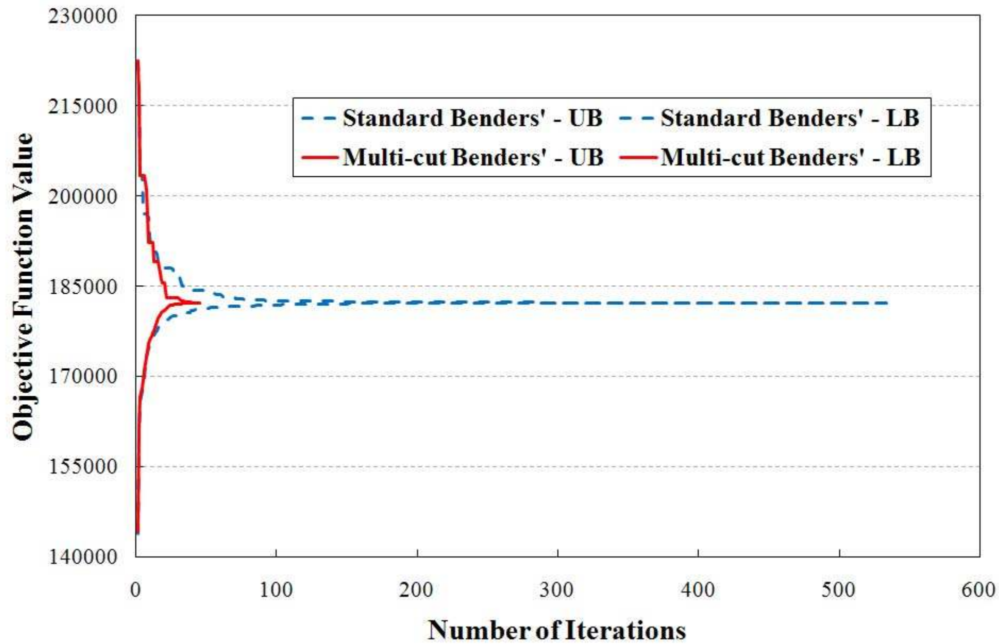

Figure 8 Comparison between the standard Benders method and the multicut version in terms of number of iterations for the second instance of the global supply chain planning problem

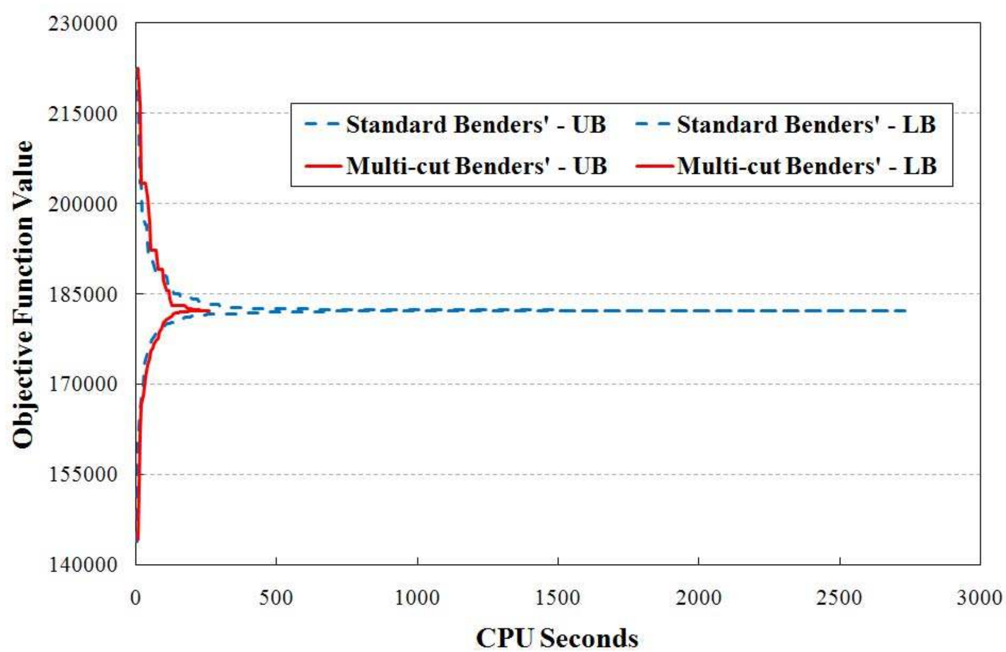

Figure 9 Comparison between the standard Benders method and the multicut version in terms of CPU-seconds for the second instance of the global supply chain planning problem

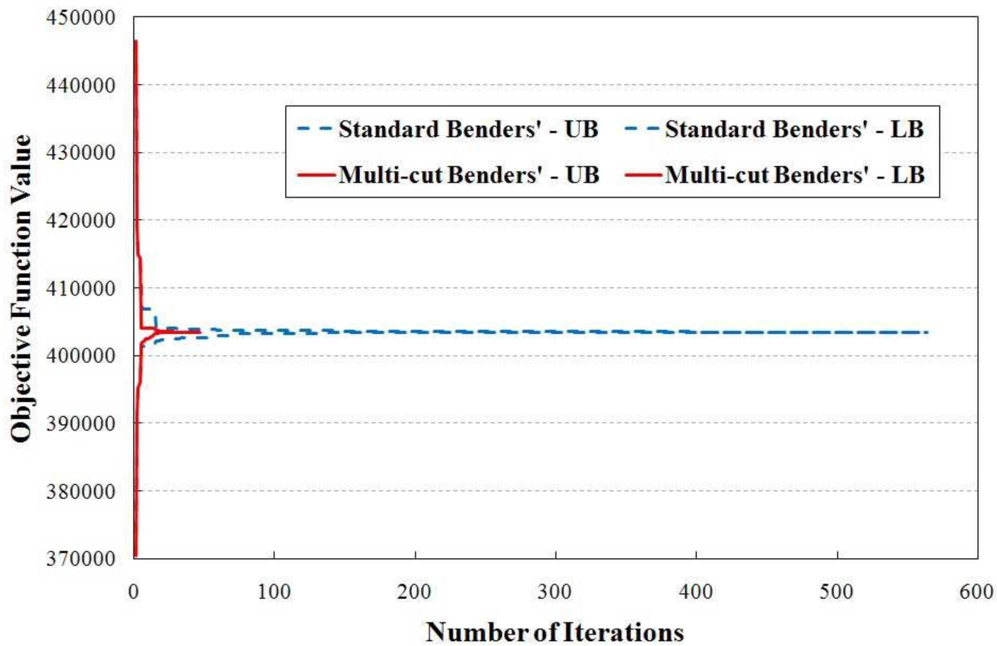

Figure 10 Comparison between the standard Benders method and the multicut version in terms of number of iterations for the third instance of the global supply chain planning problem

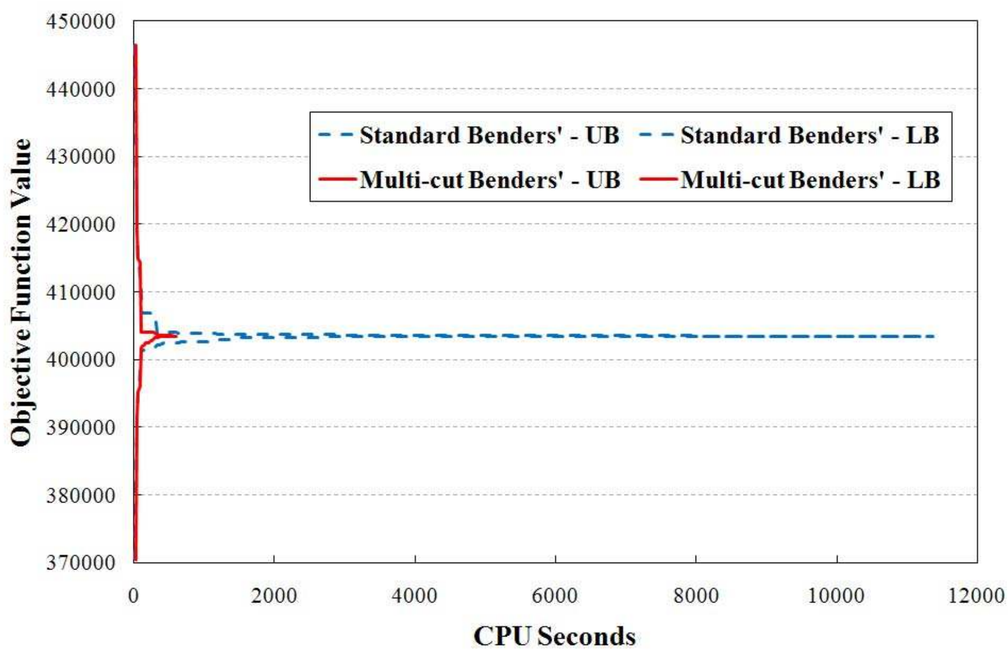

Figure 11 Comparison between the standard Benders method and the multicut version in terms of CPU-seconds for the third instance of the global supply chain planning problem

5. Conclusion

In this work, we described a multicut version of the Benders decomposition method for the solution of two-stage stochastic programming problems. We discussed the theory behind this algorithm and proved its convergence property. Two examples were presented to illustrate the application of the proposed solution method. The first example involves production-transportation planning under demand uncertainty. A small example, for which the global optimal solution can be easily obtained by solving its deterministic equivalent, was solved with both the standard and the multicut versions of the Benders decomposition method. The results illustrated the effectiveness of the multicut method. The second example involved a global chemical supply chain planning under demand and freight rate uncertainty. The decomposition method was tested on three large-scale instances, which cannot be solved directly with a regular personal computer. Computational studies showed that although both versions of the Benders decomposition method can solve large-scale stochastic programming problems with reasonable computational effort, significant savings in CPU time can be achieved by using the proposed multicut algorithm.

Future work will focus on investigating valid inequalities, such as the ones proposed by Georgios et al. (2011) and Santoso et al (2005) that can be used to initialize the decomposed problems and improve the efficiency of the proposed algorithm. Another future research direction is to investigate how to accelerate the Benders decomposition algorithm, such as developing efficient cut bundle generation method (Saharidis, et al. 2010).

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Nomenclature for Section 3

Sets/Indices

I Set of production plants indexed by i

L	Set of demand zones indexed by l
S	Set of scenarios indexed by s

Decision Variables (values: 0 to $+\infty$)

$E[profit]$	Total expected profit
$prod_i$	Production amount at plant i
$sale_{l,s}$	Total amount of the product sold to demand zone l of scenario s
$ship_{i,l}$	Transportation amount from plant i to demand zone l
$unsold_{l,s}$	Unsold amount at demand zone l of scenario s

Parameters

cap_i	Production capacity of plant i
cpd_i	Unit production cost at plant i
$ctr_{i,l}$	Unit transportation cost from plant i to demand zone l
cus_l	Unit unsold product cost in demand zone l
$demand_l$	Demand in demand zone l of scenario s
$demand_{l,s}$	Demand in demand zone l of scenario s
p_s	Probability of scenario s
$price_l$	Sale price at demand zone l

Nomenclature for Section 4

Sets, Subsets, and Indices

J	Set of products indexed by j
K	Set of facilities (including plants and distribution centers) indexed by k
K_{DC}	Set of distribution centers indexed by k
K_P	Set of manufacturing plants indexed by k
M	Set of transportation modes indexed by m
R	Set of customers indexed by r
S	Set of scenarios indexed by s

T	Set of time periods indexed by t
$KKJM$	Subset of the combination of (*k, k', j, m*) that has a minimum transportation level requirement if selected
$KRJM$	Subset of the combination of (*k, r, j, m*) that has a minimum transportation level requirement if selected

Decision Variable (values: 0 or 1)

$ZF_{k,k',j,m}$	Binary variable, equal to 1 if transportation mode *m* for interfacility freight of product *j* from facility *k* to *k'* is selected
$ZS_{k,r,j,m}$	Binary variable, equal to 1 if transportation mode *m* for facility-customer freight of product *j* from facility *k* to customer *r* is selected

Decision Variables (values: 0 to $+\infty$)

$Cost1$	First-stage cost
$Cost2_s$	Second-stage cost of scenario *s*
$E[Cost]$	Total expected cost
$F_{k,k',j,m,t}$	Interfacility freight of product *j* from facility *k* to *k'* with mode *m* at time period *t*
$F_{k,k',j,m,t,s}$	Interfacility freight of product *j* from facility *k* to *k'* with mode *m* at time period *t* of scenario *s*
$I_{k,j,t}$	Inventory level of product *j* at facility *k* at time period *t*
$I_{k,j,t,s}$	Inventory level of product *j* at facility *k* at the end of time period *t* of scenario *s*
$S_{k,r,j,m,t}$	Facility-customer freight of product *j* from facility *k* to customer *r* with mode *m* at time period *t*
$S_{k,r,j,m,t,s}$	Facility-customer freight of product *j* from facility *k* to customer *r* with mode *m* at time period *t* of scenario *s*
$SF_{r,j,t,s}$	Unmet demand of product *j* in customer *r* at time period *t* of scenario *s*
$W_{k,j,t}$	Production amount of product *j* at plant *k* at time period *t*, $k \in K_P$
$W_{k,j,t,s}$	Production amount of product *j* at plant *k* at time period *t* of scenario *s*, $k \in K_P$

Parameters

$d_{r,j,t,s}$	Demand of product *j* in customer *r* at time period *t* of scenario *s*
$h_{k,j,t}$	Unit inventory cost of product *j* in facility *k* at time period *t*
$F^L_{k,k',j,m}$	Minimum transportation amount of product *j* from facility *k* to *k'* with mode *m* at each time period if this transportation link/mode is selected

$I_{k,j}^{0}$	Initial inventory level of product *j* at facility *k*
$I_{k,j,t}^{m}$	Minimum inventory of product *j* at facility *k* at time period *t*
p_s	Probability of scenario *s*
$Q_{k,j}$	Capacity of plant *k* for product *j*, $k \in K_P$
$S_{k,r,j,m}^{L}$	Minimum transportation amount of product *j* from facility *k* to customer *r* with mode *m* at each time period if this transportation link/mode is selected
$\gamma_{k,k',m,j,t}$	Freight rate of product *j* from facility *k* to *k'* with mode *m* at time period *t*
$\gamma_{k,r,j,m,t}$	Freight rate of product *j* from facility *k* to customer *r* with mode *m* at time period *t*
$\gamma_{k,k',j,m,t,s}$	Freight rate of product *j* from facility *k* to *k'* with mode *m* at time *t* of scenario *s*
$\gamma_{k,r,j,m,t,s}$	Freight rate of product *j* from facility *k* to customer *r* with mode *m* at time period *t* of scenario *s*
$\delta_{k,j,t}$	Unit throughput cost of product *j* in facility *k* at time period *t*
$\eta_{r,j,t}$	Unit penalty cost of product *j* for lost unmet demand in customer *r* at time period *t*
$\lambda_{k,k',j,m}$	Shipping time of product *j* from facility *k* to facility *k'* with mode *m*
$\lambda_{k,r,j,m}$	Shipping time of product *j* from facility *k* to customer *r* with mode *m*

References

Archibald, T. W., Buchanan, C. S., McKinnon, K. I. M., & Thomas, L. C. (1999). Nested Benders decomposition and dynamic programming for reservoir optimisation. Journal of the Operational Research Society, 50, 468-479.

Bahn, O., Dumerle, O., Goffin, J. L., & Vial, J. P. (1995). A cutting plane method from analytic centers for stochastic-programming. Mathematical Programming, 69, 45-73.

Benders, J. F. (1962). Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik, 4, 238–252.

Birge, J. R., & Louveaux, F. (1997). Introduction to Stochastic Programming. New York: Springer-Verlag.

Birge, J.R., & Louveaux, F.V. (1988). A multicut algorithm for two-stage stochastic linear programs. European Journal of Operational Research, 34, 384–392.

Contreras, I., Cordeau, J. F., & Laporte, G. (2010) Benders Decomposition for Large-Scale Uncapacitated Hub Location. Operations Research, In press

Escudero, L. F., Garín, A., Merino, M., & Pérez, G. (2007). A two-stage stochastic integer programming approach as a mixture of branch-and-fix coordination and Benders decomposition schemes. Annals of Operations Research, 152, 395-420.

Fragniere, E., Gondzio, J., & Vial, J. P. (2000). Building and solving large-scale stochastic programs on an affordable distributed computing system. Annals of Operations Research, 99, 167-187.

Higle, J. L., & Sen, S. (1991). Stochastic decomposition - an algorithm for 2-stage linear-programs with recourse. Mathematics of Operations Research, 16, 650-669.

Infanger, G. (1993). Monte Carlo (importance) sampling within a Benders decomposition algorithm for stochastic linear programs. Annals of Operations Research, 39, 69-95.

Infanger, G. (1994). Planning under Uncertainty: Solving Large-Scale Stochastic Linear Programs. Danvers: Boyd and Fraser.

Latorre, J. M., Cerisola, S., Ramos, A., & Palacios, R. (2009). Analysis of stochastic problem decomposition algorithms in computational grids. Annals of Operations Research, 166 Issue: 1 Pages: Published: FEB 355-373.

Linderoth, J., Shapiro, A., & Wright, S. (2006). The empirical behavior of sampling methods for stochastic programming Annals of Operations Research, 142, 215-241.

Linderoth, J., & Wright, S. (2003). Decomposition Algorithms for Stochastic Programming on a Computational Grid. Computational Optimization and Applications, 24, 207–250.

Miller, N., Ruszczyński, A. (2010). Risk-Averse Two-Stage Stochastic Linear Programming: Modeling and Decomposition , Operations Research , In press.

Mulvey, J. M., & Ruszczynski, A. J. (1995). A new scenario decomposition method for large-scale stochastic optimization. Operations Research, 43, 477-490.

Ntaimo, L. (2010). Disjunctive Decomposition for Two-Stage Stochastic Mixed-Binary Programs with Random Recourse. Operations Research, 58, 229-243.

Rosenthal, R. E. (2010). GAMS- A User's Manual: GAMS Development Corp.

Ruszczynski, A. (1993). Parallel decomposition of multistage stochastic-programming

problems. Mathematical Programming, 58, 201-228.
Ruszczyński, A. (1997). Decomposition methods in stochastic programming. Mathematical Programming, 79, 333-353.
Saharidis, G. K.D., Boile, M., & Theofanis, S. (2011). Initialization of the Benders master problem using valid inequalities applied to fixed-charge network problems. Expert Systems with Applications, 38, 6627-6636
Saharidis, G. K. D., & Ierapetritou, M. G. (2010). Improving Benders decomposition using maximum feasible subsystem (MFS) cut generation strategy. Computers & Chemical Engineering, 34, 1237.
Saharidis, G. K. D., Minoux, M., & Ierapetritou, M. G. (2010). Accelerating Bender's method using covering cut bundle generation. International Transactions in Operational Research, 17, 221.
Santoso, T., Ahmed, S., Goetschalckx, M., & Shapiro, A. (2005). A stochastic programming approach for supply chain network design under uncertainty. European Journal of Operational Research, 167, 96-115
Sen, S. (1993). Subgradient decomposition and differentiability of the recourse function of a 2-stage stochastic linear program. Operations Research Letters, 13, 143-148.
Sen, S., Zhou, Z., & Huang, K. (2009). Enhancements of two-stage stochastic decomposition. Computers & Operations Research, 36, 2434-2439.
Shapiro, A. (2000). Stochastic programming by Monte Carlo simulation methods. Stochastic Programming E-Prints Series, 03.
Shapiro, A. (2008). Stochastic programming approach to optimization under uncertainty. Mathematical Programming, 112, 183-220.
Shapiro, A., & Homem-de-Mello, T. (1998). A simulation-based approach to two-stage stochastic programming with recourse. Mathematical Programming, 81, 301-325.
Trukhanov, S., Ntaimo, L., & Schaefer, A. (2010). Adaptive multicut aggregation for two-stage stochastic linear programs with recourse. European Journal of Operational Research, 206, 395-406.
Van Slyke, R. M., & Wets, R. (1969). L-Shaped Linear Programs with Applications to Optimal Control and Stochastic Programming. SIAM Journal on Applied Mathematics, 17, 638-663.
Wassick, J. M. (2009). Enterprise-wide optimization in an integrated chemical complex. Computers & Chemical Engineering, 33, 1950-1963.
You, F., Wassick, J. M., & Grossmann, I. E. (2009). Risk management for global supply chain planning under uncertainty: models and algorithms. AIChE Journal, 55, 931-946.