Multi-Dimensional Backward Stochastic Riccati Equations, and Applications*

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Abstract

Multi-dimensional backward stochastic Riccati differential equations (BSRDEs in short) are studied. A closed property for solutions of BSRDEs with respect to their coefficients is stated and is proved for general BSRDEs, which is used to obtain the existence of a global adapted solution to some BSRDEs. The global existence and uniqueness results are obtained for two classes of BSRDEs, whose generators contain a quadratic term of L (the second unknown component). More specifically, the two classes of BSRDEs are (for the regular case $N > 0$)

$$\left\{\begin{array}{rcl} dK & = & -[A^*K + KA + Q - LD(N + D^*KD)^{-1}D^*L]\,dt + L\,dw, \\ K(T) & = & M \end{array}\right.$$

and (for the singular case)

$$\left\{\begin{array}{rcl} dK & = & -[A^*K + KA + C^*KC + Q + C^*L + LC \\ & & -(KB + C^*KD + LD)(D^*KD)^{-1}(KB + C^*KD + LD)^*]\,dt + L\,dw, \\ K(T) & = & M. \end{array}\right.$$

This partially solves Bismut-Peng's problem which was initially proposed by Bismut (1978) in the Springer yellow book LNM 649. The arguments given in this paper are completely new, and they consist of some simple techniques of algebraic transformations and direct applications of the closed property mentioned above. We make full use of the special structure (the nonnegativity of the quadratic term, for example) of the underlying Riccati equation. Applications in optimal stochastic control are exposed.

Key words: backward stochastic Riccati equation, stochastic linear-quadratic control problem, algebraic transformation, Feynman-Kac formula

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Abbreviated title: Multi-dimensional backward stochastic Riccati equation

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1 Introduction

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a fixed complete probability space on which is defined a standard d-dimensional $\mathcal{F}_t$-adapted Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))^*$. Assume that $\mathcal{F}_t$ is the completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w. Denote by $\{\mathcal{F}_t^2, 0 \leq t \leq T\}$ the P-augmented natural filtration generated by the $(d-d_0)$-dimensional Brownian motion $(w_{d_0+1}, \dots, w_d)$. Assume that all the coefficients A, B, C_i, D_i are $\mathcal{F}_t$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0, T]$, of dimensions $n \times n, n \times m, n \times n, n \times m$ respectively. Also assume that M is an $\mathcal{F}_T$-measurable nonnegative bounded $n \times n$ random matrix, and Q and N are $\mathcal{F}_t$-progressively measurable, bounded, nonnegative and uniformly positive, $n \times n$ and $m \times m$ matrix processes, respectively.

Consider the following **backward stochastic Riccati differential equation** (BSRDE in short):

$$\left\{\begin{array}{rcl} dK & = & -[A^*K + KA + \sum_{i=1}^{d} C_i^* K C_i + Q + \sum_{i=1}^{d}(C_i^* L_i + L_i C_i) \\ & & -(KB + \sum_{i=1}^{d} C_i^* K D_i + \sum_{i=1}^{d} L_i D_i)(N + \sum_{i=1}^{d} D_i^* K D_i)^{-1} \\ & & \times(KB + \sum_{i=1}^{d} C_i^* K D_i + \sum_{i=1}^{d} L_i D_i)^*]\, dt + \sum_{i=1}^{d} L_i\, dw_i, \qquad 0 \leq t < T, \\ K(T) & = & M. \end{array}\right. \tag{1}$$

It will be called the BSRDE $(A, B; C_i, D_i, i = 1, \dots, d; Q, N, M)$ in the following for convenience of indicating the associated coefficients. When the coefficients A, B, C_i, D_i, Q, N, M are all deterministic, then $L_1 = \cdots = L_d = 0$ and the BSRDE (1) reduces to the following nonlinear matrix ordinary differential equation:

$$\left\{\begin{array}{rcl} dK & = & -[A^*K + KA + \sum_{i=1}^{d} C_i^* K C_i + Q - (KB + \sum_{i=1}^{d} C_i^* K D_i) \\ & & \times(N + \sum_{i=1}^{d} D_i^* K D_i)^{-1}(KB + \sum_{i=1}^{d} C_i^* K D_i)^*]\, dt, \\ & & \quad 0 \leq t < T, \\ K(T) & = & M, \end{array}\right. \tag{2}$$

which was completely solved by Wonham [28] by applying Bellman's principle of quasilinearization and a monotone convergence approach. Bismut [2, 3] initially studied the case of random coefficients, but he could solve only some special simple cases. He always assumed that the randomness of the coefficients only comes from a smaller filtration $\{\mathcal{F}_t^2\}$, which leads to $L_1 = \cdots = L_{d_0} = 0$. He further assumed in his paper [2] that

$$C_{d_0+1} = \cdots = C_d = 0, \quad D_{d_0+1} = \cdots = D_d = 0, \tag{3}$$

under which the BSRDE (1) becomes the following one:

$$\left\{\begin{array}{rcl} dK &=& -[A^*K+KA+\sum_{i=1}^{d_0}C_i^*KC_i+Q \\ && -(KB+\sum_{i=1}^{d_0}C_i^*KD_i)(N+\sum_{i=1}^{d_0}D_i^*KD_i)^{-1}(KB+\sum_{i=1}^{d_0}C_i^*KD_i)^*]\,dt \\ && +\sum_{i=d_0+1}^{d}L_i\,dw_i, \qquad 0\le t<T, \\ K(T) &=& M, \end{array}\right. \tag{4}$$

and the generator does not involve L at all. In his work [3] he assumed that

$$D_{d_0+1}=\cdots=D_d=0, \tag{5}$$

under which the BSRDE (1) becomes the following one

$$\left\{\begin{array}{rcl} dK &=& -[A^*K+KA+\sum_{i=1}^{d}C_i^*KC_i+Q+\sum_{i=d_0+1}^{d}(C_i^*L_i+L_iC_i) \\ && -(KB+\sum_{i=1}^{d_0}C_i^*KD_i)(N+\sum_{i=1}^{d_0}D_i^*KD_i)^{-1}(KB+\sum_{i=1}^{d_0}C_i^*KD_i)^*]\,dt \\ && +\sum_{i=d_0+1}^{d}L_i\,dw_i, \qquad 0\le t<T, \\ K(T) &=& M, \end{array}\right. \tag{6}$$

and the generator depends on the second unknown variable $(L_{d_0+1},\ldots,L_d)^*$ in a linear way. Moreover his method was rather complicated. Later, Peng [18] gave a nice treatment on the proof of existence and uniqueness for the BSRDE (6), by using Bellman's principle of quasi-linearization and a method of monotone convergence—a generalization of Wonham's approach to the random situation.

As early as in 1978, Bismut [3] commented on page 220 that:"Nous ne pourrons pas démontrer l'existence de solution pour l'équation (2.49) dans le cas général." (We could not prove the existence of solution for equation (2.49) for the general case.) On page 238, he pointed out that the essential difficulty for solution of the general BSRDE (1) lies in the integrand of the martingale term which appears in the generator in a quadratic way. Two decades later in 1998, Peng [19] included the above problem in his list of open problems on BSDEs. Recently, Kohlmann and Tang [13] solved the one dimensional case of the above Bismut-Peng's problem.

In this paper, we prove the global existence and uniqueness result for BSRDE (1) for the following special multi-dimensional case:

$$d=1, \quad B=C=0.$$

That is, we solve the following BSRDE

$$\left\{\begin{array}{rcl} dK &=& -[A^*K+KA+Q-LD(N+D^*KD)^{-1}D^*L]\,dt+L\,dw, \\ && \quad 0\le t<T, \\ K(T) &=& M. \end{array}\right. \tag{7}$$

This BSRDE is special but typical, for the generator contains a quadratic term on L. This result is stated as Theorem 2.3.

Consider then the case where the control weight matrix N reduces to zero. Kohlmann and Zhou [14] discussed such a case. However, their context is rather restricted, as they make the following assumptions: (a) all the coefficients involved are deterministic; (b) $C_1 = \cdots = C_d = 0, D_1 = \cdots = D_d = I_{m\times m}$, and $M = I$;(c) $A + A^* \geq BB^*$. Their arguments are based on applying a result of Chen, Li and Zhou [4]. Kohlmann and Tang [12] considered a general framework along those analogues of Bismut [3] and Peng [18], which has the following features: (a) the coefficients A, B, C, D, N, Q, M are allowed to be random, but are only $\mathcal{F}_t^2$-progressively measurable processes or $\mathcal{F}_T^2$-measurable random variable; (b) the assumptions in Kohlmann and Zhou [14] are dispensed with or generalised; (c) the condition (5) is assumed to be satisfied. Kohlmann and Tang [12] obtained a general result and generalised Bismut's previous result on existence and uniqueness of a solution of BSRDE (6) to the singular case under the following additional two assumptions:

$$M \geq \varepsilon I_{n\times n},\ \sum_{i=1}^d D_i^* D_i(t) \geq \varepsilon I_{m\times m} \text{ for some deterministic constant } \varepsilon > 0. \tag{8}$$

Kohlmann and Tang [13] proved the existence and uniqueness result for the one-dimensional singular case $N = 0$ under the assumption (8), but for a more general framework of the following features: the coefficients A, B, C, D, N, Q, M are allowed to be $\mathcal{F}_t$-progressively measurable processes or $\mathcal{F}_T$-measurable random variable, and the coefficient D is not necessarily zero. In this paper we obtain the global existence and uniqueness for the following multi-dimensional singular case:

$$d = 1, \quad D^* D \geq \varepsilon I_{m\times m},\ M \geq \varepsilon I_{n\times n} \text{ for some deterministic constant } \varepsilon > 0.$$

That is, we solve the following BSRDE:

$$\begin{cases} dK &= -[A^*K + KA + C^*KC + Q + C^*L + LC \\ &\quad -(KB + C^*KD + LD)(D^*KD)^{-1}(KB + C^*KD + LD)^*]\, dt + L\, dw, \\ &\qquad 0 \leq t < T, \\ K(T) &= M. \end{cases} \tag{9}$$

This result is stated as Theorem 2.2.

The BSRDE (1) arises from solution of the optimal control problem

$$\inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u;0,x) \tag{10}$$

where for $t \in [0,T]$ and $x \in R^n$,

$$J(u;t,x) := E^{\mathcal{F}_t}[\int_t^T [(Nu,u) + (QX^{t,x;u}, X^{t,x;u})]\, ds + (MX^{t,x;u}(T), X^{t,x;u}(T))] \tag{11}$$

and $X^{t,x;u}(\cdot)$ solves the following stochastic differential equation

$$\begin{cases} dX &= (AX + Bu)\, ds + \sum_{i=1}^d (C_iX + D_iu)\, dw_i, \quad t \leq s \leq T, \\ X(t) &= x. \end{cases} \tag{12}$$

The following connection is well known: if the BSRDE (1) has a solution (K, L), the solution for the above **linear-quadratic optimal control problem** (LQ problem in short) has the following closed form (also called the feedback form):

$$u(t) = -(N + \sum_{i=1}^{d} D_i^* K D_i)^{-1} [B^* K + \sum_{i=1}^{d} D_i^* K C_i + \sum_{i=1}^{d} D_i^* L_i] X(t) \tag{13}$$

and the associated value function V is the following quadratic form

$$V(t,x) := \inf_{u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x) = (K(t)x, x), \qquad 0 \le t \le T, x \in R^n. \tag{14}$$

In this way, on one hand, solution of the above LQ problem is reduced to solving the BSRDE (1). On the other hand, the formula (14) actually provides a representation—of Feynman-Kac type— for the solution of BSRDE (1). The reader will see that this kind of representation plays an important role in the proofs given here for Theorems 2.1, 2.2 and 2.3.

The arguments given in this paper are completely new. They results from two observations. The first one is that in the following simple case

$$\begin{array}{l} A = B = C = 0, d = 1, m = n, \\ D \text{ is nonsingular, and } D \text{ and } N \text{ are constant matrices,} \end{array} \tag{15}$$

the difficult quadratic term of L can be removed by doing some simple algebraic transformation, and the resulting BSRDE is globally solvable in view of the result of Bismut [3] and Peng [18]. As a consequence, the above simple case is globally solvable. However, this case is too restricted. Then comes out the second observation: by using some other tricks and by applying the closedness theorem 2.1, some more general cases can be attacked. Specifically, the following restrictions

$$A = 0, m = n, \text{ and } D \text{ is nonsingular} \tag{16}$$

are all removed, and the restricted assumption

$$D \text{ and } N \text{ are constant matrices} \tag{17}$$

is improved. For the singular case, we only have the one restriction $d = 1$ remained. Theorem 2.1 provides a way to obtain the solvability of more general BSRDEs from that of simple ones. We hope that Bismut-Peng's problem will be completely solved in the near future, by using the above-mentioned methodology.

The rest of the paper is organized as follows. Section 2 contains a list of notation and two preliminary propositions, and the statement of the main results which consist of Theorems 2.1-2.3. The proofs of these three theorems are given in Sections 3-5, respectively. Finally, in Section 6, application of Theorems 2.2 and 2.3 is given to the regular and singular stochastic LQ problems, both with and without constraints.

2 Preliminaries and the Main Results

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a fixed complete probability space on which is defined a standard d-dimensional $\mathcal{F}_t$-adapted Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))^*$. Assume that $\mathcal{F}_t$ is the completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w. Denote by $\{\mathcal{F}_t^2, 0 \leq t \leq T\}$ the P-augmented natural filtration generated by the $(d-d_0)$-dimensional Brownian motion $(w_{d_0+1}, \ldots, w_d)$. Assume that all the coefficients A, B, C_i, D_i are $\mathcal{F}_t$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m$ respectively. Also assume that M is an $\mathcal{F}_T$-measurable, nonnegative, and bounded $n \times n$ random matrix. Assume that Q and N are $\mathcal{F}_t$-progressively measurable, bounded, nonnegative and uniformly positive, $n\times n$ and $m \times m$ matrix processes, respectively.

Notation. Throughout this paper, the following additional notation will be used:

M^*	:	the transpose of any vector or matrix M;
$\lvert M\rvert$	:	$=\sqrt{\sum_{ij} m_{ij}^2}$ for any vector or matrix $M=(m_{ij})$;
(M_1, M_2)	:	the inner product of the two vectors M_1 and M_2;
R^n	:	the n-dimensional Euclidean space;
R_+	:	the set of all nonnegative real numbers;
$\mathcal{S}^n$	:	the Euclidean space of all $n \times n$ symmetric matrices;
$\mathcal{S}^n_+$	:	the set of all $n\times n$ nonnegative definite matrices;
$C([0,T];H)$	:	the Banach space of H-valued continuous functions on $[0,T]$, endowed with the maximum norm for a given Hilbert space H;
$\mathcal{L}^2_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued $\mathcal{F}_t$-adapted square-integrable stochastic processes f on $[0,T]$, endowed with the norm $(E\int_0^T \lvert f(t)\rvert^2\, dt)^{1/2}$ for a given Euclidean space H;
$\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued, $\mathcal{F}_t$-adapted, essentially bounded stochastic processes f on $[0,T]$, endowed with the norm $\operatorname{ess\,sup}_{t,\omega} \lvert f(t)\rvert$ for a given Euclidean space H;
$L^2(\Omega, \mathcal{F}, P; H)$	:	the Banach space of H-valued norm-square-integrable random variables on the probability space $(\Omega, \mathcal{F}, P)$ for a given Banach space H;

and $L^\infty(\Omega, \mathcal{F}, P; C([0,T]; R^n))$ is the Banach space of $C([0,T];R^n)$-valued, essentially maximum-norm-bounded random variables f on the probability space $(\Omega, \mathcal{F}, P)$, endowed with the norm $\operatorname{ess\,sup}_{\omega\in\Omega} \max_{0\leq t\leq T} \lvert f(t,\omega)\rvert$.

Proposition 2.1. *Assume that all the coefficients A, B, C_i, D_i are $\mathcal{F}_t^2$-progressively measurable bounded matrix-valued processes, defined on $\Omega\times[0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m$ respectively. Also assume that M is an $\mathcal{F}_T^2$-measurable, nonnegative, and bounded $n\times n$ random matrix. Assume that Q and N are $\mathcal{F}_t^2$-progressively measurable, bounded, nonnegative and uniformly positive, $n \times n$ and $m\times m$ matrix processes, respectively. Then, the* BSRDE (6) *has a unique $\mathcal{F}_t^2$-adapted global solution (K, L) with*

$$K \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+)), \quad L \in \mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n).$$

Proposition 2.1 is due to Bismut [3] and Peng [18], and the reader is referred to

Bismut [3] and Peng [18] for the proof.

Consider the optimal control problem

$$\inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u;0,x) \tag{18}$$

where for $t\in[0,T]$ and $x\in R^n$,

$$J(u;t,x) := E^{\mathcal{F}_t}[\int_t^T [(Nu,u)+(QX^{t,x;u},X^{t,x;u})]\,ds + (MX^{t,x;u}(T),X^{t,x;u}(T))] \tag{19}$$

and $X^{t,x;u}(\cdot)$ solves the following stochastic differential equation

$$\begin{cases} dX &= (AX+Bu)\,ds + \sum_{i=1}^d (C_iX+D_iu)\,dw_i, \quad t\le s\le T,\\ X(t) &= x. \end{cases} \tag{20}$$

Proposition 2.2. *Let (K,L) be an $\mathcal{F}_t$-adapted solution of the BSRDE (1) with*

$$K\in\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathcal{S}^n)\cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n)), \quad L\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n),$$

and $N(t)+\sum_{i=1}^d D_i^*KD_i(t)$ being uniformly positive. Then,

$$(K(t)x,x) = V(t,x) := \inf_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \quad \forall x\in R^n.$$

This proposition is a special case of Theorem 6.1, and the reader is referred to Section 6 for the proof.

The main results of this paper are stated by the following three theorems.

Theorem 2.1. *Assume that $\forall\gamma\ge 0$ the coefficients $A^\gamma, B^\gamma, C_i^\gamma, D_i^\gamma, Q^\gamma$, and N^γ are $\mathcal{F}_t$-progressively measurable matrix-valued processes, defined on $\Omega\times[0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m, n\times n$, and $m\times m$, respectively. Assume that M^γ is an $\mathcal{F}_T$-measurable and nonnegative $n\times n$ random matrix. Assume that Q^γ is a.s.a.e. nonnegative. Assume that there are two deterministic positive constants ε_1 and ε_2 which are independent of the parameter γ, such that*

$$|A^\gamma(t)|, |B^\gamma(t)|, |C_i^\gamma(t)|, |D_i^\gamma(t)|, |Q^\gamma(t)|, |N^\gamma(t)|, |M^\gamma| \le \varepsilon_1$$

and

$$N^\gamma \ge \varepsilon_2 I_{m\times m}.$$

Assume that as $\gamma\to 0$, $A^\gamma(t), B^\gamma(t), C_i^\gamma(t), D_i^\gamma(t), Q^\gamma(t)$, and $N^\gamma(t)$ converge uniformly in (t,ω) to $A^0(t), B^0(t), C_i^0(t), D_i^0(t), Q^0(t)$ and $N^0(t)$, respectively. Assume that M^γ uniformly in ω converges to M^0 as $\gamma\to 0$. Assume that $\forall\gamma>0$ the BSRDE $(A^\gamma, B^\gamma; C_i^\gamma, D_i^\gamma, i = 1,\dots,d; Q^\gamma, N^\gamma, M^\gamma)$ has a unique $\mathcal{F}_t$-adapted global solution (K^γ, L^γ) with

$$K^\gamma\in\mathcal{L}^\infty_{\mathcal{F}}(0,T;\mathcal{S}^n_+)\cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n_+)), \quad L^\gamma\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n).$$

Then, there is a pair of processes (K, L) *with*

$$K \in \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n_+)), \quad L \in \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n),$$

such that

$$\begin{aligned} &\lim_{\gamma\to 0} K^\gamma = K \quad \textit{strongly in } \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n_+)),\\ &\lim_{\gamma\to 0} L^\gamma = L \quad \textit{strongly in } \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n), \end{aligned} \tag{21}$$

and such that (K, L) *is a unique* $\mathcal{F}_t$*-adapted solution of the BSRDE* $(A^0, B^0, C^0, D^0, Q^0, N^0, M^0)$.

If the above assumption of uniform convergence of $(A^\gamma, C^\gamma, Q^\gamma, M^\gamma)$ *is replaced with the following one:*

$$\lim_{\gamma\to 0} \operatorname*{esssup}_{\omega\in\Omega} \int_0^T (|A^\gamma - A^0| + |C^\gamma - C^0|^2 + |Q^\gamma - Q^0|)\, ds + |M^\gamma - M^0| \to 0. \tag{22}$$

then the above assertions still hold.

Remark 2.1. *When the assumption of uniform positivity on the control weight matrix* N *is relaxed to nonnegativity, Theorem 2.1 still holds with the additional assumption that there is a deterministic positive constant* ε_3 *such that*

$$\sum_{i=1}^d (D_i^\gamma)^* D_i^\gamma \geq \varepsilon_3 I_{m\times m}, \quad M^\gamma \geq \varepsilon_3 I_{n\times n}.$$

Theorem 2.2. (the singular case) *Assume that* $d = 1$ *and* $Q(t) \geq 0$. *Also assume that there is a deterministic positive constant* ε_3 *such that*

$$M \geq \varepsilon_3 I_{n\times n} \tag{23}$$

and

$$D^* D(t) \geq \varepsilon_3 I_{m\times m}. \tag{24}$$

Then, the BSRDE (9) *has a unique* $\mathcal{F}_t$*-adapted global solution* (K, L) *with*

$$K \in \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n_+)), \quad L \in \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n),$$

and $K(t,\omega)$ *being uniformly positive w.r.t.* (t,ω).

Theorem 2.3. (the regular case) *Assume that* $d = 1, M \geq 0, Q(t) \geq 0$ *and* $N(t) \geq \varepsilon_3 I_{m\times m}$ *for some positive constant* $\varepsilon_3 > 0$. *Further assume that* $B = C = 0$, *and* D *and* N *satisfy the following*

$$\begin{aligned} \lim_{h\to 0+} \operatorname*{esssup}_{\omega\in\Omega} \max_{t_1,t_2\in[0,T];\,|t_1-t_2|\leq h} |D(t_1) - D(t_2)| &= 0,\\ \lim_{h\to 0+} \operatorname*{esssup}_{\omega\in\Omega} \max_{t_1,t_2\in[0,T];\,|t_1-t_2|\leq h} |N(t_1) - N(t_2)| &= 0. \end{aligned} \tag{25}$$

Then, the BSRDE (7) *has a unique* $\mathcal{F}_t$*-adapted global solution* (K, L) *with*

$$K \in \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n_+)), \quad L \in \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n).$$

The proofs of the above three theorems are given in Sections 3, 4, and 5, respectively.

3 The Proof of Theorem 2.1.

For $\forall(t,K,L)\in[0,T]\times\mathcal{S}^n_+\times(\mathcal{S}^n)^d$, write

$$\begin{aligned} F^\gamma(t,K,L) :=\ & -[KB^\gamma(t)+\sum_{i=1}^d C_i^\gamma(t)^*KD_i^\gamma(t)+\sum_{i=1}^d L_iD_i^\gamma(t)] \\ & \times[N^\gamma(t)+\sum_{i=1}^d D_i^\gamma(t)^*KD_i^\gamma(t)]^{-1} \\ & \times[KB^\gamma(t)+\sum_{i=1}^d C_i^\gamma(t)^*KD_i^\gamma(t)+\sum_{i=1}^d L_iD_i^\gamma(t)]^*. \end{aligned} \tag{26}$$

The generator of the BSRDE $(A^\gamma,B^\gamma;C_i^\gamma,D_i^\gamma,i=1,\ldots,d;Q^\gamma,N^\gamma,M^\gamma)$ is

$$\begin{aligned} G^\gamma(t,K,L)\ :=\ & (A^\gamma)^*K+KA^\gamma+\sum_{i=1}^d(C_i^\gamma)^*KC_i^\gamma+Q^\gamma \\ & +\sum_{i=1}^d((C_i^\gamma)^*L_i+L_iC_i^\gamma)+F^\gamma(t,K,L). \end{aligned} \tag{27}$$

We have the following *a priori* estimates.

Lemma 3.1. *Let the set of coefficients $(A^\gamma,B^\gamma;C_i^\gamma,D_i^\gamma,i=1,\ldots,d;Q^\gamma,N^\gamma,M^\gamma)$ satisfy the assumptions made in Theorem 2.1, and let (K^γ,L^γ) be a global adapted solution to the BSRDE $(A^\gamma,B^\gamma;C_i^\gamma,D_i^\gamma,i=1,\ldots,d;Q^\gamma,N^\gamma,M^\gamma)$ with*

$$K^\gamma\in\mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n)\cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];\mathcal{S}^n)),\quad L^\gamma\in\mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n),$$

and $N(t)+\sum_{i=1}^d D_i^*KD_i(t)$ being uniformly positive. Then, there is a deterministic positive constant ε_0 which is independent of γ, such that $\forall\gamma\ge0$, the following estimates hold:

$$0\le K^\gamma(t)\le\varepsilon_0I_{n\times n},\quad E^{\mathcal{F}_t}\left(\int_t^T|L^\gamma|^2\,ds\right)^p\le\varepsilon_0,\quad\forall p\ge1. \tag{28}$$

Proof of Lemma 3.1. From Proposition 2.2, we see that $K^\gamma\ge0$. Note that (K^γ,L^γ) satisfies the BSRDE:

$$\left\{\begin{aligned} dK^\gamma\ =\ & -\Big[(A^\gamma)^*K^\gamma+K^\gamma A^\gamma+\sum_{i=1}^d(C_i^\gamma)^*K^\gamma C_i^\gamma+Q^\gamma+\sum_{i=1}^d((C_i^\gamma)^*L_i^\gamma+L_i^\gamma C_i^\gamma) \\ & +F^\gamma(t,K^\gamma,L^\gamma)\Big]\,dt+\sum_{i=1}^d L_i^\gamma\,dw_i,\qquad 0\le t<T, \\ K^\gamma(T)\ =\ & M^\gamma. \end{aligned}\right. \tag{29}$$

Using Itô's formula, we get

$$\left\{\begin{aligned} d|K^\gamma|^2\ =\ & -\Big[4\,\mathrm{tr}\ \big[(K^\gamma)^2A^\gamma\big]+\sum_{i=1}^d 2\,\mathrm{tr}\ [K^\gamma(C_i^\gamma)^*K^\gamma C_i^\gamma]+2\,\mathrm{tr}\ (K^\gamma Q^\gamma) \\ & +\sum_{i=1}^d 4\,\mathrm{tr}\ (K^\gamma L_i^\gamma C_i^\gamma)+2\,\mathrm{tr}\ [K^\gamma F^\gamma(t,K^\gamma,L^\gamma)]-|L^\gamma|^2\Big]\,dt \\ & +\sum_{i=1}^d 2\,\mathrm{tr}\ (K^\gamma L_i^\gamma)\,dw_i,\qquad 0\le t<T, \\ |K^\gamma|^2(T)\ =\ & |M^\gamma|^2. \end{aligned}\right. \tag{30}$$

We observe that since

$$F^\gamma(t, K^\gamma, L^\gamma) \le 0, \quad K^\gamma \ge 0,$$

we have

$$2\,\mathrm{tr}\ [K^\gamma F^\gamma(t, K^\gamma, L^\gamma)] = 2\,\mathrm{tr}\ \left[(K^\gamma)^{\frac{1}{2}}\, F^\gamma(t, K^\gamma, L^\gamma)\, (K^\gamma)^{\frac{1}{2}}\right] \le 0. \tag{31}$$

Hence,

$$\begin{aligned} |K^\gamma|^2(t) + \int_t^T |L^\gamma|^2\, ds \ \le\ & |M^\gamma|^2 + \int_t^T \Big[4\,\mathrm{tr}\ \left[(K^\gamma)^2 A^\gamma\right] + \sum_{i=1}^d 2\,\mathrm{tr}\ [K^\gamma (C_i^\gamma)^* K^\gamma C_i^\gamma] \\ & + 2\,\mathrm{tr}\ (K^\gamma Q^\gamma) + \sum_{i=1}^d 4\,\mathrm{tr}\ (K^\gamma L_i^\gamma C_i^\gamma)\Big]\, ds \\ & - \int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (K^\gamma L_i^\gamma)\, dw_i, \qquad 0 \le t < T. \end{aligned} \tag{32}$$

Using the elementary inequality

$$2ab \le a^2 + b^2$$

and taking the expectation on both sides with respect to $\mathcal{F}_r$ for $r \ge t$, we obtain that

$$E^{\mathcal{F}_r}|K^\gamma|^2(t) + \frac{1}{2}E^{\mathcal{F}_r}\int_t^T |L^\gamma|^2\, ds \le \varepsilon_4 + \varepsilon_4 \int_t^T E^{\mathcal{F}_r}|K^\gamma|^2(s)\, ds, \quad 0 \le r \le t < T. \tag{33}$$

Using Gronwall's inequality, We derive from the last inequality the first one of the estimates (28). In return, we derive from the second last inequality that

$$\int_t^T |L^\gamma|^2\, ds \le \varepsilon_5 + \varepsilon_5 \int_0^T |L^\gamma|\, ds - \int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (K^\gamma L_i^\gamma)\, dw_i. \tag{34}$$

Therefore,

$$E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|^2\, ds\right)^p \le 3^p\left[\varepsilon_5^p + \varepsilon_5^p E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|\, ds\right)^p + E^{\mathcal{F}_t}\left|\int_t^T \sum_{i=1}^d 2\mathrm{tr} K^\gamma L_i^\gamma\, dw_i\right|^p\right]. \tag{35}$$

We have from the Burkholder-Davis-Gundy inequality the following

$$E^{\mathcal{F}_t}\left|\int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (K^\gamma L_I^\gamma)\, dw_i\right|^p \le 2^p E^{\mathcal{F}_t}\left|\int_t^T |K^\gamma|^2 |L^\gamma|^2\, ds\right|^{p/2},$$

while from the Cauchy-Schwarz inequality, we have

$$E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|\, ds\right)^p \le T^{p/2} E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|^2\, ds\right)^{p/2}.$$

Finally, we get

$$E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|^2\, ds\right)^p \le 3^p \varepsilon_5^p + [3^p T^{p/2}\varepsilon_5^p + 6^p n^{p/2}\varepsilon_0^p] E^{\mathcal{F}_t}\left(\int_t^T |L^\gamma|^2\, ds\right)^{p/2}, \tag{36}$$

which implies the last estimate of the lemma.

Now, consider the optimal control problem

$$\text{Problem } \mathcal{P}_\gamma \qquad \inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J^\gamma(u;0,x) \tag{37}$$

where for $t \in [0,T]$ and $x \in R^n$,

$$J^\gamma(u;t,x) := E^{\mathcal{F}_t}[\int_t^T [(N^\gamma u,u) + (Q^\gamma X_\gamma^{t,x;u}, X_\gamma^{t,x;u})]\,ds + (M^\gamma X_\gamma^{t,x;u}(T), X_\gamma^{t,x;u}(T))] \tag{38}$$

and $X_\gamma^{t,x;u}(\cdot)$ solves the following stochastic differential equation

$$\begin{cases} dX &= (A^\gamma X + B^\gamma u)\,ds + \sum_{i=1}^d (C_i^\gamma X + D_i^\gamma u)\,dw_i, \quad t \le s \le T, \\ X(t) &= x. \end{cases} \tag{39}$$

The associated value function V^γ is defined as

$$V^\gamma(t,x) := \inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J^\gamma(u;t,x). \tag{40}$$

Then, from Proposition 2.2, we have

$$(K^\gamma(t)x,x) = V^\gamma(t,x), \quad \forall (t,x) \in [0,T]\times R^n.$$

From the *a priori* estimates result Lemma 3.1, we have

$$V^\gamma(t,x) \le \varepsilon_0 |x|^2, \quad \forall (t,x) \in [0,T]\times R^n.$$

So, the optimal control $\widehat{u}^\gamma$ for the problem $\mathcal{P}_\gamma$ satisfies

$$\varepsilon_2 E^{\mathcal{F}_t}\int_t^T |\widehat{u}^\gamma|^2\,ds = E^{\mathcal{F}_t}\int_t^T (N^\gamma \widehat{u}^\gamma, \widehat{u}^\gamma)\,ds \le \varepsilon_0 |x|^2.$$

Set

$$\mathcal{U}_{ad}^x(t,T) := \left\{ u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m) : \varepsilon_2 E^{\mathcal{F}_t}\int_t^T |u|^2\,ds \le \varepsilon_0|x|^2 \right\}, \quad \forall x \in R^n. \tag{41}$$

Then, we have

$$V^\gamma(t,x) := \inf_{u(\cdot)\in\mathcal{U}_{ad}^x(t,T)} J^\gamma(u;t,x). \tag{42}$$

Define

$$\begin{array}{l} K^{\gamma\tau} := K^\gamma - K^\tau, \quad L_i^{\gamma\tau} := L_i^\gamma - L_i^\tau, \quad X_{\gamma\tau}^{t,x;u} := X_\gamma^{t,x;u} - X_\tau^{t,x;u}, \\ A^{\gamma\tau} := A^\gamma - A^\tau, \quad B^{\gamma\tau} := B^\gamma - B^\tau, \quad C_i^{\gamma\tau} := C_i^\gamma - C_i^\tau, \\ D^{\gamma\tau} := D^\gamma - D^\tau, \quad Q^{\gamma\tau} := Q^\gamma - Q^\tau, \quad N^{\gamma\tau} := N^\gamma - N^\tau, \\ M^{\gamma\tau} := M^\gamma - M^\tau. \end{array} \tag{43}$$

Lemma 3.2. *Let the assumptions of Theorem 2.1 be satisfied. Then, there are three deterministic positive constants* $\varepsilon_6, \varepsilon_7$, *and* ε_8, *which are independent of the parameters* γ *and* τ *such that the following three estimates hold.* (*i*) *For each* $x \in R^n$,

$$E^{\mathcal{F}_t} \max_{t\le s\le T} |X_\gamma^{t,x;u}(s)|^2 \le \varepsilon_6|x|^2 + \varepsilon_6 E^{\mathcal{F}_t}\int_t^T |u|^2\, ds. \tag{44}$$

(*ii*) *For each* $(t,x) \in [0,T]\times R^n$,

$$\begin{aligned} E^{\mathcal{F}_t} \max_{t\le s\le T} |X_{\gamma\tau}^{t,x;u}(s)|^2 \ \le\ & \varepsilon_7 E^{\mathcal{F}_t}\int_t^T (|A^{\gamma\tau}| + |C^{\gamma\tau}|^2)|X_\gamma^{t,x;u}(s)|^2\, ds \\ & +\varepsilon_7 E^{\mathcal{F}_t}\int_t^T (|B^{\gamma\tau}| + |D^{\gamma\tau}|^2)|u|^2(s)\, ds. \end{aligned} \tag{45}$$

(*iii*) *For each* $(t,x) \in [0,T]\times R^n$,

$$\begin{aligned} & |J^\gamma(u;t,x) - J^\tau(u;t,x)| \\ \le\ & \varepsilon_8 E^{\mathcal{F}_t}[|M^{\gamma\tau}||X_\gamma^{t,x;u}(T)|^2 + |X_{\gamma\tau}^{t,x;u}(T)|(|X_\gamma^{t,x;u}(T)| + |X_\tau^{t,x;u}(T)|)] \\ & +\varepsilon_8 E^{\mathcal{F}_t}\int_t^T |X_{\gamma\tau}^{t,x;u}(s)|[|X_\gamma^{t,x;u}(s)| + |X_\tau^{t,x;u}(s)|]\, ds \\ & +\varepsilon_8 E^{\mathcal{F}_t}\int_t^T |Q^{\gamma\tau}||X_\gamma^{t,x;u}(s)|^2\, ds + \varepsilon_8 E^{\mathcal{F}_t}\int_t^T |N^{\gamma\tau}||u|^2(s)\, ds. \end{aligned} \tag{46}$$

Proof of Lemma 3.2. Note that $X_{\gamma\tau}^{t,x;u}$ satisfies the following stochastic differential equation:

$$\begin{cases} dX_{\gamma\tau} &= (A^\tau X_{\gamma\tau} + A^{\gamma\tau}X_\gamma + B^{\gamma\tau}u)\, ds + \sum_{i=1}^d (C_i^\tau X_{\gamma\tau} + C_i^{\gamma\tau}X_\gamma + D_i^{\gamma\tau}u)\, dw_i, \\ X_{\gamma\tau}(t) &= 0. \end{cases}$$

So, in view of the assumptions of Theorem 2.1, the first two estimates are actually a consequence of the continuous dependence upon the parameters of the solution of a stochastic differential equation, and the proof is standard. The last estimate results from an immediate application of the mean-value formula for a differential function.

Lemma 3.3. *Let the assumptions of Theorem 2.1 be satisfied. Then, we have the following three inequalities.* (*i*) *For each* $x \in R^n, \forall u \in \mathcal{U}_{ad}^x(t,T)$,

$$E^{\mathcal{F}_t} \max_{t\le s\le T} |X_\gamma^{t,x;u}(s)|^2 \le \varepsilon_6(1 + \varepsilon_2^{-1}\varepsilon_0)|x|^2. \tag{47}$$

(*ii*) *For each* $(t,x) \in [0,T]\times R^n, \forall u \in \mathcal{U}_{ad}^x(t,T)$,

$$\begin{aligned} E^{\mathcal{F}_t} \max_{t\le s\le T} |X_{\gamma\tau}^{t,x;u}(s)|^2 \ \le\ & \varepsilon_7\varepsilon_6(1 + \varepsilon_2^{-1}\varepsilon_0)|x|^2 \operatorname*{esssup}_\omega \int_0^T (|A^{\gamma\tau}| + |C^{\gamma\tau}|^2)\, ds \\ & +\varepsilon_7\varepsilon_2^{-1}\varepsilon_0|x|^2 \operatorname*{esssup}_{s,\omega}(|B^{\gamma\tau}| + |D^{\gamma\tau}|^2)(s). \end{aligned} \tag{48}$$

(iii) For each $(t,x)\in[0,T]\times R^n, \forall u\in\mathcal{U}_{ad}^x(t,T)$,

$$
\begin{aligned}
& |J^\gamma(u;t,x)-J^\tau(u;t,x)| \\
\le\ & \varepsilon_8 \operatorname*{esssup}_\omega |M^{\gamma\tau}|\ E^{\mathcal{F}_t}|X_\gamma^{t,x;u}(T)|^2 \\
& +\varepsilon_8\left[E^{\mathcal{F}_t}|X_{\gamma\tau}^{t,x;u}(T)|^2\right]^{1/2}\left[E^{\mathcal{F}_t}(2|X_\gamma^{t,x;u}(T)|^2+2|X_\tau^{t,x;u}(T)|^2)\right]^{1/2} \\
& +\varepsilon_8 T\left[E^{\mathcal{F}_t}\sup_{t\le s\le T}|X_{\gamma\tau}^{t,x;u}(s)|^2\right]^{1/2}\left[E^{\mathcal{F}_t}\sup_{t\le s\le T}[2|X_\gamma^{t,x;u}(s)|^2+2|X_\tau^{t,x;u}(s)|^2]\right]^{1/2} \\
& +\varepsilon_8 \operatorname*{esssup}_\omega\int_0^T|Q^{\gamma\tau}|\,ds\ E^{\mathcal{F}_t}\sup_{t\le s\le T}|X_\gamma^{t,x;u}(s)|^2+\varepsilon_8\varepsilon_2^{-1}\varepsilon_0|x|^2\operatorname*{esssup}_{s,\omega}|N^{\gamma\tau}|(s).
\end{aligned}
\tag{49}
$$

Proof of Lemma 3.3. Since $u\in\mathcal{U}_{ad}^x(t,T)$, we have

$$
E^{\mathcal{F}_t}\int_t^T|u|^2\,ds\le\varepsilon_2^{-1}\varepsilon_0|x|^2. \tag{50}
$$

Putting (50) into the first estimate of Lemma 3.2, we get the first inequality of Lemma 3.3. Putting (50) and the first inequality of Lemma 3.3 into the second estimate of Lemma 3.2, we get the second one. The last one is a combination of (50) and applying the Cauchy-Schwarz inequality in the third estimate of Lemma 3.2.

Combining the first and the last inequalities of Lemma 3.3, we conclude that for each $(t,x)\in[0,T]\times R^n, \forall u\in\mathcal{U}_{ad}^x(t,T)$,

$$
\begin{aligned}
& |J^\gamma(u;t,x)-J^\tau(u;t,x)| \\
\le\ & \varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2\operatorname*{esssup}_\omega|M^{\gamma\tau}| \\
& +2|x|\varepsilon_8(T+1)\sqrt{\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)}\left[E^{\mathcal{F}_t}\sup_{t\le s\le T}|X_{\gamma\tau}^{t,x;u}(s)|^2\right]^{1/2} \\
& +\varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2\operatorname*{esssup}_\omega\int_0^T|Q^{\gamma\tau}|\,ds+\varepsilon_8\varepsilon_2^{-1}\varepsilon_0|x|^2\operatorname*{esssup}_{s,\omega}|N^{\gamma\tau}|(s).
\end{aligned}
\tag{51}
$$

Putting the second inequality of Lemma 3.3 into this, we have that

$$
\begin{aligned}
& |J^\gamma(u;t,x)-J^\tau(u;t,x)| \\
\le\ & \varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2\operatorname*{esssup}_\omega|M^{\gamma\tau}|+2|x|\varepsilon_8(T+1)\sqrt{\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)} \\
& \times\Big[\varepsilon_7\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2\operatorname*{esssup}_\omega\int_0^T(|A^{\gamma\tau}|+|C^{\gamma\tau}|^2)\,ds \\
& +\varepsilon_7\varepsilon_2^{-1}\varepsilon_0|x|^2\operatorname*{esssup}_{s,\omega}(|B^{\gamma\tau}|+|D^{\gamma\tau}|^2)(s)\Big]^{1/2} \\
& +\varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2\operatorname*{esssup}_\omega\int_0^T|Q^{\gamma\tau}|\,ds+\varepsilon_8\varepsilon_2^{-1}\varepsilon_0|x|^2\operatorname*{esssup}_{s,\omega}|N^{\gamma\tau}|(s)
\end{aligned}
\tag{52}
$$

hold for each $(t,x) \in [0,T] \times R^n, \forall u \in \mathcal{U}_{ad}^x(t,T)$. Therefore, we have

$$
\begin{aligned}
& |V^\gamma(t,x) - V^\tau(t,x)| \\
\leq\ & \varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2 \operatorname*{esssup}_{\omega} |M^{\gamma\tau}| + 2|x|\varepsilon_8(T+1)\sqrt{\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)} \\
& \times\Big[\varepsilon_7\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2 \operatorname*{esssup}_{\omega}\int_0^T (|A^{\gamma\tau}| + |C^{\gamma\tau}|^2)\,ds \\
& +\varepsilon_7\varepsilon_2^{-1}\varepsilon_0|x|^2 \operatorname*{esssup}_{s,\omega}(|B^{\gamma\tau}| + |D^{\gamma\tau}|^2)(s)\Big]^{1/2} \\
& +\varepsilon_8\varepsilon_6(1+\varepsilon_2^{-1}\varepsilon_0)|x|^2 \operatorname*{esssup}_{\omega}\int_0^T |Q^{\gamma\tau}|\,ds + \varepsilon_8\varepsilon_2^{-1}\varepsilon_0|x|^2 \operatorname*{esssup}_{s,\omega}|N^{\gamma\tau}|(s)
\end{aligned} \tag{53}
$$

hold for each $(t,x) \in [0,T] \times R^n$.

In view of the assumptions of Theorem 2.1, (53) implies that for each $(t,x) \in [0,T]\times R^n$, $V^\gamma(t,x)$ converges to $V^0(t,x)$ as $\gamma \to 0$. Moreover, this convergence is uniform in (t,ω). Hence, K^γ converges to some K^0 in the Banach space

$$\mathcal{L}_{\mathcal{F}}^\infty(0,T;\mathcal{S}_+^n) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}_+^n)).$$

In the following, we show the strong convergence of L^γ. Note that $(K^{\gamma\tau}, L^{\gamma\tau})$ satisfies the BSDE

$$
\left\{
\begin{aligned}
dK^{\gamma\tau}(t) &= -\left[G^\gamma(t,K^\gamma,L^\gamma) - G^\tau(t,K^\tau,L^\tau)\right]\,dt + \sum_{i=1}^d L_i^{\gamma\tau}\,dw_i, \\
K^{\gamma\tau}(T) &= M^{\gamma\tau}.
\end{aligned}
\right. \tag{54}
$$

Using Itô's formula, we have

$$
\begin{aligned}
& E|K^{\gamma\tau}|^2(t) + E\int_t^T |L^{\gamma\tau}|^2(s)\,ds \\
=\ & E|M^{\gamma\tau}|^2 + E\int_t^T K^{\gamma\tau}[G^\gamma(s,K^\gamma,L^\gamma) - G^\tau(t,K^\tau,L^\tau)]\,ds.
\end{aligned} \tag{55}
$$

Since

$$|G^\gamma(s,K^\gamma,L^\gamma) - G^\tau(t,K^\tau,L^\tau)| \leq \varepsilon(1+|L^\gamma|^2 + |L^\tau|^2) \tag{56}$$

for some deterministic constant ε which is independent of γ and τ, we have

$$E\int_t^T |L^{\gamma\tau}|^2(s)\,ds \leq E|M^{\gamma\tau}|^2 + \varepsilon \operatorname*{esssup}_{s,\omega}|K^{\gamma\tau}(s)| E\int_t^T (1+|L^\gamma|^2+|L^\tau|^2)\,ds. \tag{57}$$

From the second a priori estimate of Lemma 2.1, we conclude that L^γ converges to some L^0 strongly in $\mathcal{L}_{\mathcal{F}}^2(0,T;\mathcal{S}^n)$. By passing to the limit in the BSRDE $(A^\gamma, B^\gamma; C_i^\gamma, D_i^\gamma, i = 1,\ldots,d; Q^\gamma, N^\gamma, M^\gamma)$, we show that (K^0, L^0) solves the BSRDE $(A^0, B^0; C_i^0, D_i^0, i = 1,\ldots,d; Q^0, N^0, M^0)$.

4 The Proof of Theorem 2.2.

This section gives the proof of Theorem 2.2. The main idea is to do the inverse transformation:

$$\widetilde{K} := K^{-1}, \tag{58}$$

which turns out to satisfy a Riccati equation whose generator depends on the martingale term in a linear way.

First, since D is inversable, we can rewrite the BSRDE (9) as

$$\left\{\begin{array}{rcl} dK & = & -[-\widetilde{A}^*K - K\widetilde{A} + Q - K\widetilde{B}K^{-1}\widetilde{B}^*K - LK^{-1}L \\ & & +K\widetilde{B}K^{-1}L + LK^{-1}\widetilde{B}^*K]\,dt + L\,dw, \\ K(T) & = & M, \end{array}\right. \tag{59}$$

where

$$\widetilde{A} := -A + BD^{-1}C, \quad \widetilde{B} := -BD^{-1}.$$

Note that we have the following rule for the first and the second differentials of the inverse of a positive matrix as a matrix-valued function:

$$d\left(K^{-1}\right) = -K^{-1}(dK)K^{-1}, \quad d^2\left(K^{-1}\right) = 2K^{-1}(dK)K^{-1}(dK)K^{-1}. \tag{60}$$

Using Itô's formula, we can write the equation for the inverse $\widetilde{K}$ of K:

$$\left\{\begin{array}{rcl} d\widetilde{K} & = & -[\widetilde{K}\widetilde{A}^* + \widetilde{A}\widetilde{K} - \widetilde{K}Q\widetilde{K} + \widetilde{B}\widetilde{K}\widetilde{B}^* + \widetilde{B}\widetilde{L} + \widetilde{L}\widetilde{B}^*]\,dt + \widetilde{L}\,dw, \\ \widetilde{K}(T) & = & M^{-1}, \end{array}\right. \tag{61}$$

where

$$\widetilde{L} := -K^{-1}LK^{-1}.$$

From Proposition 2.1, the above BSRDE $\left(\widetilde{A}, Q^{1/2}; \widetilde{B}, 0; 0, I_{m\times m}, M^{-1}\right)$ has a unique global adapted solution $(\widetilde{K}, \widetilde{L})$ with

$$\widetilde{K} \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathcal{S}^n_+) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}^n_+)), \quad \widetilde{L} \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n),$$

which implies that $\widetilde{K}^{-1}(t)$ is uniformly positive in (t,ω). Moreover, from the fact that $\widetilde{K}(T) = M^{-1} \geq \varepsilon_1^{-1} I_{n\times n}$, we derive that $\widetilde{K}$ is uniformly positive. This shows that $\widetilde{K}^{-1}(t)$ is uniformly bounded. Therefore (K, L) is a global adapted solution to the BSRDE (9) with

$$\begin{array}{l} K := \widetilde{K}^{-1} \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathcal{S}^n_+) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}^n_+)), \\ L := -\widetilde{K}^{-1}\widetilde{L}\widetilde{K}^{-1} \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n). \end{array}$$

The uniqueness results from the Feynman-Kac representation result Proposition 2.2. In fact, assume that $(\widehat{K}, \widehat{L})$ also solves the BSRDE (9). Then, from Proposition 2.2, we see that

$$(K(t)x, x) = V(t,x) = (\widehat{K}(t)x, x), \quad a.s., \quad \forall (t,x) \in [0,T] \times R^n.$$

So, we have $K(t) = \widehat{K}(t)$ almost surely for $\forall (t,x) \in [0,T] \times R^n$. Set

$$\delta K := K - \widehat{K}, \quad \delta L_i := L_i - \widehat{L}_i, \quad \delta G := G(t,K,L) - G(t,\widehat{K},\widehat{L}).$$

Then, we have $\delta K = 0$. Note that $(\delta K, \delta L)$ satisfies the following BSDE:

$$\begin{cases} d\delta K(t) &= -\delta G\, dt + \sum_{i=1}^{d} \delta L_i(t)\, dw_i(t), \quad 0 \le t < T, \\ \delta K(T) &= 0. \end{cases} \tag{62}$$

From this, proceeding identically as in the last paragraph of Section 3, we have

$$E\int_t^T |\delta L|^2(s)\, ds \le E|\delta K(T)|^2 + \varepsilon \operatorname*{esssup}_{s,\omega} |\delta K(s)| E\int_t^T (1 + |L|^2 + |\widehat{L}|^2)\, ds = 0. \tag{63}$$

Hence, $\delta L = L - \widehat{L} = 0$.

5 The Proof of Theorem 2.3

For the regular case, the situation is a little complex: we easily see that the above inverse transformation on the first unknown variable can not eliminate the quadratic term of the second unknown variable. However, we can still solve some classes of BSRDEs with the help of doing some appropriate transformation.

Proposition 5.1. *Assume that $Q \ge A^*(D^{-1})^*ND^{-1} + (D^{-1})^*ND^{-1}A, m = n$, and D and N are positive constant matrices. Then, Theorem 2.3 holds.*

Proof of Proposition 5.1. Write

$$\widehat{N} := (D^{-1})^*ND^{-1}. \tag{64}$$

Then, the BSRDE (7) reads

$$\begin{cases} dK &= -[A^*K + KA + Q - L(\widehat{N} + K)^{-1}L]\, dt + L\, dw, \\ & \qquad 0 \le t < T, \\ K(T) &= M. \end{cases} \tag{65}$$

The equation for $\widehat{K} := \widehat{N} + K$ is

$$\begin{cases} d\widehat{K} &= -[A^*\widehat{K} + \widehat{K}A + Q - A^*\widehat{N} - \widehat{N}A - \widehat{L}\widehat{K}^{-1}\widehat{L}]\, dt + \widehat{L}\, dw, \\ & \qquad 0 \le t < T, \\ \widehat{K}(T) &= \widehat{N} + M. \end{cases} \tag{66}$$

Note that $\widehat{N} + M$ is uniformly positive. From Theorem 2.2, we see that the BSRDE (66) has a unique global adapted solution $(\widehat{K}, \widehat{L})$. Therefore $(\widehat{K} - \widehat{N}, \widehat{L})$ is a global adapted solution to the BSRDE (7).

Proposition 5.2. *Assume that $A = 0$ and D and N are constant matrices. Then, Theorem 2.3 holds.*

Proof of Proposition 5.2. First assume $m = n$. Consider the following approximating BSRDEs:

$$\left\{\begin{array}{rcl} dK & = & -[Q - LD_\alpha(N + D_\alpha^* K D_\alpha)^{-1} D_\alpha^* L]\, dt + L\, dw, \\ K(T) & = & M \end{array}\right. \tag{67}$$

where

$$D_\alpha := D + \alpha I_{m\times m} > 0, \alpha > 0.$$

From Proposition 5.1, we see that the BSRDE (67) has a unique global adapted solution (K_α, L_α) for every $\alpha > 0$. From Proposition 2.2, K_α can be represented as

$$(K_\alpha(t)x, x) = V_\alpha(t, x), \quad \forall (t, x) \in [0, T] \times R^n. \tag{68}$$

From Theorem 2.1, we see that K_α uniformly converges to some $K \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0, T]; \mathcal{S}^n_+))$ and L_α strongly converges to some $L \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathcal{S}^n)$, and that (K, L) is an adapted solution of the BSRDE (7) when $A = 0$.

Consider the case $n > m$. Then consider the $n\times n$ matrices $\widetilde{D}$ whose first m columns are D and whose last $(n - m)$ columns are zero column vectors, and $\widetilde{N}$ which is defined as

$$\widetilde{N} := \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

The BSRDE (7) when $A = 0$ is rewritten as

$$\left\{\begin{array}{rcl} dK & = & -[Q - L\widetilde{D}(\widetilde{N} + \widetilde{D}^* K \widetilde{D})^{-1} \widetilde{D}^* L]\, dt + L\, dw, \\ K(T) & = & M \end{array}\right.$$

From the preceding result, we obtain the desired existence result.

Consider the case $n < m$. Then, there is a $m \times m$ orthogonal transformation matrix T such that

$$D = [\widehat{D}, 0]T, \quad \widehat{D} \in R^{n\times n} \text{ and is non-singular.}$$

Write

$$\widetilde{N} := (T^{-1})^* N T^{-1} := \begin{pmatrix} \widehat{N}_{11} & \widehat{N}_{12} \\ \widehat{N}_{12}^* & \widehat{N}_{22} \end{pmatrix} > 0.$$

Then, $\widehat{N}_{11} > 0$. The BSRDE (7) when $A = 0$ is rewritten as

$$\left\{\begin{array}{rcl} dK & = & -[Q - L\widehat{D}(\widetilde{N}_{11} + \widehat{D}^* K \widehat{D})^{-1} \widehat{D}^* L]\, dt + L\, dw, \\ K(T) & = & M \end{array}\right.$$

From the preceding result, we obtain the desired existence result.

Proposition 5.3. *Assume that $A = 0$, and D and N are piece-wisely constant $\mathcal{F}_t$-adapted bounded matrix processes. Then, Theorem 2.3 holds.*

Proof of Proposition 5.3. Since D and N are piece-wisely constant $\mathcal{F}_t$-adapted bounded matrix processes, there is a finite partion:

$$0 =: t_0 < t_1 < \cdots < t_J := T$$

such that on each interval $[t_i, t_{i+1}] \subset [0,T]$, D and N are constant $\mathcal{F}_{t_i}$-measurable bounded random matrices. From Proposition 5.2, the BSRDE

$$\begin{cases} dK &= -[Q - LD(N + D^*KD)^{-1}D^*L]\,dt + L\,dw, \\ & \qquad t_{J-1} \le t < T, \\ K(T) &= M \end{cases} \tag{69}$$

has a unique $\mathcal{F}_t$-adapted solution (K^J, L^J) with

$$K^J \in \mathcal{L}^\infty_{\mathcal{F}}(t_{J-1}, T; \mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([t_{J-1}, T]; \mathcal{S}^n_+)), \quad L^J \in \mathcal{L}^2_{\mathcal{F}}(t_{J-1}, T; \mathcal{S}^n).$$

Assume that for some $i = 2, \ldots, J$, the BSRDE

$$\begin{cases} dK &= -[Q - LD(N + D^*KD)^{-1}D^*L]\,dt + L\,dw, \\ & \qquad t_{i-1} \le t < t_i, \\ K(t_i) &= K^{i+1}(t_i) \end{cases} \tag{70}$$

has a unique $\mathcal{F}_t$-adapted solution (K^i, L^i) with

$$K^i \in \mathcal{L}^\infty_{\mathcal{F}}(t_{i-1}, t_i; \mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_{t_i}, P; C([t_{i-1}, t_i]; \mathcal{S}^n_+)), \quad L^i \in \mathcal{L}^2_{\mathcal{F}}(t_{i-1}, t_i; \mathcal{S}^n).$$

Note that when $i = J$, we use the convention $K^{J+1}(t_J) := M$. Then, we conclude from Proposition 5.2 that the BSRDE

$$\begin{cases} dK &= -[Q - LD(N + D^*KD)^{-1}D^*L]\,dt + L\,dw, \\ & \qquad t_{i-2} \le t < t_{i-1}, \\ K(t_{i-1}) &= K^i(t_{i-1}) \end{cases} \tag{71}$$

has a unique $\mathcal{F}_t$-adapted solution (K^{i-1}, L^{i-1}) with

$$K^{i-1} \in \mathcal{L}^\infty_{\mathcal{F}}(t_{i-2}, t_{i-1}; \mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_{t_{i-1}}, P; C([t_{i-2}, t_{i-1}]; \mathcal{S}^n_+)), \quad L^{i-1} \in \mathcal{L}^2_{\mathcal{F}}(t_{i-2}, t_{i-1}; \mathcal{S}^n).$$

In this both inductive and backward way, we may define J paires of processes $\{(K^i, L^i)\}_{i=1}^J$. Define on the whole time interval $[0,T]$ the pair of $\mathcal{F}_t$-adapted processes (K, L) as follows:

$$K(t) := \sum_{i=1}^J K^i(t)\chi_{[t_{i-1}, t_i)}(t), \quad L(t) := \sum_{i=1}^J L^i(t)\chi_{[t_{i-1}, t_i)}(t).$$

We see that (K, L) satisfies the BSRDE (7). We then obtain the desired existence result.

Proposition 5.4. *Assume that* $A = 0$. *Then, Theorem* 2.3 *holds.*

Proof of Proposition 5.4. For an arbitrary positive integer k, consider the 2^k-partion of the time interval. Define

$$D^k(t) = D\left(\frac{i-1}{2^k}T\right), \quad \forall t \in \left[\frac{i-1}{2^k}T, \frac{i}{2^k}T\right), i = 1, 2, \ldots, 2^k;$$

and

$$N^k(t) = N\left(\frac{i-1}{2^k}T\right), \quad \forall t \in \left[\frac{i-1}{2^k}T, \frac{i}{2^k}T\right), i = 1, 2, \ldots, 2^k.$$

For each k, D^k and N^k are are piece-wisely constant, $\mathcal{F}_t$-adapted, bounded matrix processes. Further, in view of (25), $D^k(t)$ and $N^k(t)$ converge respectively to D and N, uniformly in (t,ω). That is, we have

$$\lim_{k\to\infty} \operatorname*{esssup}_{\omega\in\Omega} \max_{t\in[0,T]} |D^k(t)-D(t)| = 0, \quad \lim_{k\to\infty} \operatorname*{esssup}_{\omega\in\Omega} \max_{t\in[0,T]} |N^k(t)-N(t)| = 0.$$

From Proposition 5.3, we see that the BSRDE $(0,0,0,D^k;Q,N^k;M)$ has a global adapted solution (K^k,L^k), and then from Theorem 2.1, we see that Theorem 2.3 holds.

Proof of Theorem 2.3. The case $A=0$ is solved by Proposition 5.4. For the case $A\neq 0$, consider the following transformation

$$\widetilde{K} := \Phi^* K\Phi, \quad \widetilde{L} := \Phi^* L\Phi$$

where Φ solves the differential equation

$$\begin{cases} \dfrac{d\Phi}{dt}(t) &= A(t)\Phi(t), \quad t\in(0,T], \\ \Phi(0) &= I_{n\times n}. \end{cases}$$

Using Itô's formula, we get the BSDE for $(\widetilde{K},\widetilde{L})$:

$$\begin{cases} d\widetilde{K}(t) &= -[\widetilde{Q} - \widetilde{L}\widetilde{D}(N+\widetilde{D}^*\widetilde{K}\widetilde{D})^{-1}\widetilde{D}\widetilde{L}]\,dt + \widetilde{L}\,dw(t), \quad t\in(0,T], \\ \widetilde{K}(T) &= \widetilde{M} \end{cases}$$

where $\widetilde{Q} := \Phi^* Q\Phi, \widetilde{M} := \Phi(T)^* M\Phi(T), \widetilde{D} := \Phi^{-1}D$. Note that the trajectories of $\widetilde{D}$ are still uniformly continuous like D. From Proposition 5.4, we see that the BSRDE $(0,0,0,\widetilde{D};\widetilde{Q},N,\widetilde{M})$ has a global adapted solution $(\widetilde{K},\widetilde{L})$, and thus the pair

$$((\Phi^*)^{-1}\widetilde{K}\Phi^{-1}, (\Phi^*)^{-1}\widetilde{L}\Phi^{-1})$$

solves the original BSRDE $(A,0,0,D;Q,N,M)$.

The uniqueness can be proved in the same way as in the proof of Theorem 2.2.

6 Application to Stochastic LQ Problems

6.1 The unconstrainted case

Assume that

$$\xi\in L^2(\Omega,\mathcal{F}_T,P;R^n), \quad q,f,g_i\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^n). \tag{72}$$

Consider the following optimal control problem (denoted by $\mathcal{P}_0$):

$$\min_{u\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u;0,x) \tag{73}$$

with

$$J(u;t,x) = \begin{array}{l} E^{\mathcal{F}_t}(M(X^{t,x;u}(T)-\xi), X^{t,x;u}(T)-\xi) \\ +E^{\mathcal{F}_t}\int_t^T [(Q(X^{t,x;u}-q), X^{t,x;u}-q)+(Nu,u)]\,ds \end{array} \tag{74}$$

and $X^{t,x;u}$ solving the equation

$$\begin{cases} dX = (AX+Bu+f)\,ds + \sum_{i=1}^d (C_iX+D_iu+g_i)\,dw_i, \quad t < s \le T, \\ X(t) = x, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{75}$$

The value function V is defined as

$$V(t,x) := \min_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \quad (t,x)\in[0,T]\times R^n. \tag{76}$$

Define $\Gamma : [0,T]\times\mathcal{S}^n_+\times R^{n\times d}\to R^{m\times n}$ by

$$\Gamma(\cdot,S,L) = -(N+\sum_{i=1}^d D_i^*SD_i)^{-1}(B^*S+\sum_{i=1}^d D_i^*SC_i+\sum_{i=1}^d D_i^*L_i). \tag{77}$$

and

$$\widehat{A} := A + B\Gamma(\cdot,K,L), \widehat{C}_i := C_i + D_i\Gamma(\cdot,K,L), \quad i=1,\ldots,d. \tag{78}$$

Let (ψ,ϕ) be the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$\begin{cases} d\psi(t) = -[\widehat{A}^*\psi + \sum_{i=1}^d \widehat{C}_i^*(\phi_i - Kg_i) - Kf - \sum_{i=1}^d L_ig_i + Qq]\,dt + \sum_{i=1}^d \phi_i\,dw_i, \\ \psi(T) = M\xi \end{cases} \tag{79}$$

where (K,L) is the unique $\mathcal{F}_t$-adapted solution of the BSRDE (1). The following can be verified by a pure completion of squares.

Theorem 6.1 *Suppose that the assumptions of Theorem* 2.2 *or Theorem* 2.3 *are satisfied. Let (K,L) be the unique $\mathcal{F}_t$-adapted solution of* BSRDE (1). *Then, the optimal control $\widehat{u}$ for the non-homogeneous stochastic LQ problem $\mathcal{P}_0$ exists uniquely and has the following feedback law*

$$\begin{array}{rcl} \widehat{u} &=& -(N+\sum_{i=1}^d D_i^*KD_i)^{-1}[(B^*K+\sum_{i=1}^d D_i^*KC_i+\sum_{i=1}^d D_i^*L_i)\widehat{X} \\ && -B^*\psi + \sum_{i=1}^d D_i^*(Kg_i-\phi_i)]. \end{array} \tag{80}$$

The value function $V(t,x), (t,x)\in[0,T]\times R^n$ has the following explicit formula

$$V(t,x) = (K(t)x,x) - 2(\psi(t),x) + V^0(t), \quad (t,x)\in[0,T]\times R^n \tag{81}$$

with

$$V^0(t) := E^{\mathcal{F}_t}(M\xi,\xi) + E^{\mathcal{F}_t}\int_t^T (Qq,q)\,ds - 2E^{\mathcal{F}_t}\int_t^T (\psi,f)\,ds$$
$$+E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d [(Kg_i,g_i) - 2(\phi_i g_i)]\,ds \tag{82}$$
$$-E^{\mathcal{F}_t}\int_t^T ((N+\sum_{i=1}^d D_i^*KD_i)u^0,u^0)\,ds$$

and

$$u^0 := (N+\sum_{i=1}^d D_i^*KD_i)^{-1}[B^*\psi + \sum_{i=1}^d D_i^*(\phi_i - Kg_i)], \quad t \le s \le T. \tag{83}$$

Proof Set

$$\tilde{u} = u - \Gamma(\cdot,K,L)X. \tag{84}$$

Then the system (75) reads

$$\begin{cases} dX &= (\widehat{A}X + B\tilde{u} + f)\,ds + \sum_{i=1}^d (\widehat{C}_iX + D_i\tilde{u} + g_i)\,dw_i, \quad t < s \le T, \\ X(t) &= x, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{85}$$

Applying Itô's formula, we have the equation for $\mathcal{X} =: XX^*$:

$$\begin{cases} d\mathcal{X} = & [\widehat{A}\mathcal{X} + \mathcal{X}\widehat{A}^* + X(B\tilde{u}+f)^* + (B\tilde{u}+f)X^*]\,ds \\ & +\sum_{i=1}^d [\widehat{C}_iX\widehat{C}_i^* + \widehat{C}_iX(D_i\tilde{u}+g_i)^* + \widehat{C}_iX(D_i\tilde{u}+g_i)X^*\widehat{C}_i^* \\ & +(D_i\tilde{u}+g_i)(D_i\tilde{u}+g_i)^*]\,ds \\ & +\sum_{i=1}^d [\widehat{C}_i\mathcal{X} + \mathcal{X}\widehat{C}_i^* + X(D_i\tilde{u}+g_i)^* + (D_i\tilde{u}+g_i)X^*]\,dw_i, \quad t < s \le T, \\ \mathcal{X}(t) = & xx^*, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{86}$$

Note that the BSRDE (1) can be rewritten as

$$\begin{cases} -dK &= \Big[\widehat{A}^*K + K\widehat{A} + \sum_{i=1}^d \widehat{C}_i^*K\widehat{C}_i + \sum_{i=1}^d (\widehat{C}_i^*L_i + L_i\widehat{C}_i) + Q \\ & \quad +\Gamma(t,K,L)^*N\Gamma(t,K,L)\Big]\,dt - \sum_{i=1}^d L_i\,dw_i, \\ K(T) &= M. \end{cases} \tag{87}$$

So, application of Itô's formula gives

$$
\begin{aligned}
& E^{\mathcal{F}_t}(MX(T), X(T)) + E^{\mathcal{F}_t}\int_t^T([Q+\Gamma(s,K,L)^* N\Gamma(s,K,L)]X, X)\,ds \\
= \;& (K(t)X(t), X(t)) + 2E^{\mathcal{F}_t}\int_t^T (K(B\widetilde{u}+f), X)\,ds \\
& + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d 2(K(D_i\widetilde{u}+g_i), \widehat{C}_i X)\,ds \\
& + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d (K(D_i\widetilde{u}+g_i), D_i\widetilde{u}+g_i)\,ds \\
& + 2E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d (L_i(D_i\widetilde{u}+g_i), X)\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
& E^{\mathcal{F}_t}\left[(M\xi, X(T)) + \int_t^T (Qq, X)\,ds\right] \\
= \;& E^{\mathcal{F}_t}\left[\psi(T)X(T) + \int_t^T QqX\,ds\right] \\
= \;& (\psi(t), X(t)) + E^{\mathcal{F}_t}\int_t^T (\psi, B\widetilde{u}+f)\,ds \\
& + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d (\phi_i, D_i\widetilde{u}+g_i)\,ds \\
& + E^{\mathcal{F}_t}\int_t^T (\sum_{i=1}^d \widehat{C}_i^* K g_i + Kf + \sum_{i=1}^d L_i g_i, X)\,ds.
\end{aligned}
$$

Combining the last two equations, we get

$$
\begin{aligned}
& J(u;t,x) \\
= \;& E^{\mathcal{F}_t}\left[(M(X(T)-\xi),X(T)-\xi)+\int_t^T(Q(X-q),X-q)\,ds+\int_t^T(Nu,u)\,ds\right] \\
= \;& E^{\mathcal{F}_t}\left[(MX(T),X(T))+\int_t^T([Q+\Gamma(s,K,L)^*N\Gamma(s,K,L)]X,X)\,ds\right] \\
& -2E^{\mathcal{F}_t}\left[(M\xi,X(T))+\int_t^T(Qq,X)\,ds\right]+E^{\mathcal{F}_t}\left[(M\xi,\xi)+\int_t^T(Qq,q)\,ds\right] \\
& +E^{\mathcal{F}_t}\int_t^T[(N\tilde u,\tilde u)+2(N\Gamma(s,K,L)X,\tilde u)]\,ds \\
= \;& (KX(t),X(t))-2(\psi(t),X(t))+E^{\mathcal{F}_t}\left[(M\xi,\xi)+\int_t^T(Qq,q)\,ds\right] \\
& +E^{\mathcal{F}_t}\int_t^T\sum_{i=1}^d(K(D_i\tilde u+g_i),D_i\tilde u+g_i)\,ds+E^{\mathcal{F}_t}\int_t^T(N\tilde u,\tilde u)\,ds \\
& -2E^{\mathcal{F}_t}\int_t^T(\psi,B\tilde u+f)\,ds-2E^{\mathcal{F}_t}\int_t^T\sum_{i=1}^d(\phi_i,D_i\tilde u+g_i)\,ds \\
= \;& (K(t)x,x)-2(\psi(t),x)+E^{\mathcal{F}_t}\left[(M\xi,\xi)+\int_t^T(Qq,q)\,ds\right] \\
& -2E^{\mathcal{F}_t}\int_t^T(\psi,f)\,ds+E^{\mathcal{F}_t}\int_t^T\sum_{i=1}^d[(Kg_i,g_i)-2(\phi_i,g_i)]\,ds \\
& +E^{\mathcal{F}_t}\int_t^T((N+\sum_{i=1}^d D_i^*KD_i)(\tilde u-u^0),\tilde u-u^0)\,ds \\
& -E^{\mathcal{F}_t}\int_t^T((N+\sum_{i=1}^d D_i^*KD_i)u^0,u^0)\,ds.
\end{aligned}
$$

This completes the proof.

6.2 The constrainted case

Fix $x_T\in R^n$. Define

$$U_{\rm ad}(t,x):=\{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m):EX^{t,x;u}(T)=x_T\},\quad \forall(t,x)\in[0,T]\times R^n, \tag{88}$$

where $X^{t,x;u}$ solving the equation (75). Then, consider the following constrainted LQ problem (denoted by $\mathcal{P}_c^{t,x}$):

$$\inf_{u\in U_{\rm ad}(0,x)} J(u;0,x) \tag{89}$$

where the cost functional $J(u;t,x)$ is defined by (74). Note that the set of admissible controls $U_{\rm ad}(t,x)$ contains the terminal expected constraint.

Let $\Psi(\cdot,t)$ be the unique solution of the SDE:

$$\left\{\begin{aligned} dY_s &= A(s)Y_s\,ds+\sum_{i=1}^d C_i(s)Y_s\,dw_i(s),\quad t\le s\le T,\\ Y_t &= I_{n\times n}.\end{aligned}\right. \tag{90}$$

To guarantee that $U_{\rm ad}(t,x)$ is not empty, assume that the matrix

$$\Delta := E\int_0^T E^{\mathcal{F}_s}\Psi(T,s)B(s)B^*(s)E^{\mathcal{F}_s}\Psi^*(T,s)\,ds \tag{91}$$

is nonsingular. Then, $\forall x\in R^n$, the following control

$$u(s) := B^*(s)E^{\mathcal{F}_s}\Psi^*(T,s)\Delta^{-1}[x_T - E\int_t^T \Psi(T,s)f(s)\,ds], \quad s\in(t,T], \tag{92}$$

belongs to $U_{\rm ad}(t,x)$.

We have the following existence result.

Theorem 6.2. *Let the assumptions of Theorem 2.2 or Theorem 2.3 be satisfied. Assume that $U_{\rm ad}(0,x)$ is not empty. Then, the problem $\mathcal{P}_c^{0,x}$ has a unique optimal contol.*

Proof of Theorem 6.2. The proof is similar to that of Kohlmann and Tang [12]. The main idea is to choose a sequence $\{u^k; k=1,2,\ldots\}$ such that

$$u^k\in U_{\rm ad}(0,x), \quad \lim_{k\to\infty} J(u^k;0,x) = \inf_{u\in U_{\rm ad}(0,x)} J(u;0,x).$$

Then, we prove that this sequence is a Cauchy sequence by using the uniform convexity of the cost functional $J(u;0,x)$ in the control u. This uniform convexity is obvious for the regular case, and has been proved for the singular case by Kohlmann and Tang [12]. The details are left to the reader.

Due to the limitation of space, we will in what follows just sketch how to solve the unique optimal control of Theorem 6.2 in terms of the solution of the associated BSRDE.

Using the stochastic maximum principle (see Peng [20], and Tang and Li [27], for example), we have the following. Let $\widetilde{u}$ be the optimal control, and $\widetilde{X} := X^{0,x;\widetilde{u}}$. Then, there is some $\lambda\in R^n$, and a pair of processes $(\widetilde{p},\widetilde{q})$, such that

$$\left\{\begin{array}{rcl} d\widetilde{p} &=& -[A^*\widetilde{p} + Q(\widetilde{X}-q) + \sum_{i=1}^d C_i^*\widetilde{q}_i]\,ds + \sum_{i=1}^d \widetilde{q}_i\,dw_i, \quad 0<s\le T,\\ \widetilde{p}(T) &=& M(\widetilde{X}(T)-\xi)-\lambda \end{array}\right. \tag{93}$$

and

$$B^*\widetilde{p} + \sum_{i=1}^d D_i^*\widetilde{q}_i + N\widetilde{u} = 0. \tag{94}$$

Using Itô's formula and the equality (94), we get the equation for $\widetilde{\psi} := K\widetilde{X} - \widetilde{p}$:

$$\left\{\begin{array}{rcl} d\widetilde{\psi}(t) &=& -[\widehat{A}^*\widetilde{\psi} + \sum_{i=1}^d \widehat{C}_i^*(\widetilde{\phi}_i - Kg_i) - Kf - \sum_{i=1}^d L_i g_i + Qq]\,dt + \sum_{i=1}^d \widetilde{\phi}_i\,dw_i,\\ \widetilde{\psi}(T) &=& M\xi+\lambda \end{array}\right. \tag{95}$$

where (K, L) is the unique $\mathcal{F}_t$-adapted solution of the BSRDE (1), and the explicit formula of the optimal control:

$$\begin{aligned}\widetilde{u} &= -(N + \sum_{i=1}^{d} D_i^* K D_i)^{-1}[(B^* K + \sum_{i=1}^{d} D_i^* K C_i + \sum_{i=1}^{d} D_i^* L_i)\widehat{X} \\ &\quad -B^*\widetilde{\psi} + \sum_{i=1}^{d} D_i^*(K g_i - \widetilde{\phi}_i)]\end{aligned} \tag{96}$$

where the Lagrange multiple λ is determined such that the terminal constraint $E\widetilde{X}(T) = x_T$ is satisfied.

6.3 A comment on application of the LQ theory in mathematical finance

One-dimensional singular LQ problems arise from mathematical finance. The mean-variance hedging problem and the dynamic version of Markowitz's mean-variance portfolio selection problem, are one-dimensional singular LQ problems.

The mean-variance hedging problem was initially introduced by Föllmer and Sondermann [7], and later was widely studied among others by Duffie and Richardson [5], Föllmer and Schweizer [8], Schweizer [23, 24, 25], Hipp [11], Monat and Stricker [16], Pham, Rheinländer and Schweizer [21], Gourieroux, Laurent and Pham [10], and Laurent and Pham [15]. All of these works are based on a projection argument. Recently, Kohlmann and Zhou [14] used a natural LQ theory approach to solve the case of deterministic market conditions. Kohlmann and Tang [12, 13] used a natural LQ theory approach to solve the case of stochastic market conditions, and the optimal hedging portfolio and the variance-optimal martingale measure are characterized in terms of the solution of the associated BSRDE.

The continuous time mean-variance portfolio selection problem was initially considered by Richardson [22]. The reader is referred to Zhou and Li [29] for recent developments on this problem.

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References

[1] Bismut, J. M., *Conjugate convex functions in optimal stochastic control,* J. Math. Anal. Appl., 44 (1973), 384–404

[2] Bismut, J. M., *Linear quadratic optimal stochastic control with random coefficients,* SIAM J. Control Optim., 14 (1976), 419-444

[3] Bismut, J. M., *Controle des systems lineares quadratiques: applications de l'integrale stochastique,* Séminaire de Probabilités XII, eds : C. Dellacherie, P. A. Meyer et M. Weil, LNM 649, Springer-Verlag, Berlin 1978

[4] Chen, S., Li, X. and Zhou, X., *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36 (1998)**, 1685-1702

[5] Duffie, D. and Richardson, H. R., *Mean-variance hedging in continuous time,* Ann. Appl. Prob., 1 (1991), 1-15

[6] Föllmer, H. and Leukert, P., *Efficient hedging: cost versus shortfall risk,* Finance & Stochastics, 4 (2000), 117–146

[7] Föllmer, H. and Sonderman, D., *Hedging of non-redundant contingent claims.* In: Mas-Colell, A., Hildebrand, W. (eds.) Contributions to Mathematical Economics. Amsterdam: North Holland 1986, pp. 205-223

[8] Föllmer, H. and Schweizer, M., *Hedging of contingent claims under incomplete information.* In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. (Stochastics Monographs vol. 5) London-New York: Gordon & Breach, 1991, pp. 389-414

[9] Gal'chuk, L. I., *Existence and uniqueness of a solution for stochastic equations with respect to semimartingales,* Theory of Probability and Its Applications, 23 (1978), 751–763

[10] Gourieroux, C., Laurent, J. P. and Pham, H., *Mean-variance hedging and numéraire,* Mathematical Finance, 8 (1998), 179-200.

[11] Hipp, C., *Hedging general claims,* Proceedings of the 3rd AFIR colloqium, Rome, Vol. 2, pp. 603-613 (1993)

[12] Kohlmann, M. and Tang, S., *Optimal control of linear stochastic systems with singular costs, and the mean-variance hedging problem with stochastic market conditions.* submitted

[13] Kohlmann, M. and Tang, S., *Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging.* submitted

[14] Kohlmann, M. and Zhou, X., *Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach,* SIAM J. Control & Optim., 38 (2000), 1392-1407

[15] Laurent, J. P. and Pham, H., *Dynamic programming and mean-variance hedging,* Finance and Stochastics, 3 (1999), 83-110

[16] Monat, P. and Stricker, C., *Föllmer-Schweizer decomposition and mean-variance hedging of general claims,* Ann. Probab. 23, 605–628

[17] Pardoux, E. and Peng, S., *Adapted solution of backward stochastic equation,* Systems Control Lett., 14 (1990), 55–61

[18] Peng, S., *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control and Optimization **30 (1992)** , 284-304

[19] Peng, S., *Some open problems on backward stochastic differential equations,* Control of distributed parameter and stochastic systems, proceedings of the IFIP WG 7.2 international conference, June 19-22, 1998, Hangzhou, China

[20] Peng, S., *A general stochastic maximum principle for optimal control problems,* SIAM J. Control Optim., 28 (1990), 966-979

[21] Pham, H., Rheinländer, T., and Schweizer, M., *Mean-variance hedging for continuous processes: New proofs and examples,* Finance and Stochastics, 2 (1998), 173-198

[22] Richardson, H, *A minimum result in continuous trading portfolio optimization.* Management Science, 35 (1989), 1045-1055

[23] Schweizer, M., *Mean-variance hedging for general claims,* Ann. Appl. Prob., 2 (1992), 171-179

[24] Schweizer, M., *Approaching random variables by stochastic integrals,* Ann. Probab., 22 (1994), 1536–1575

[25] Schweizer, M., *Approximation pricing and the variance-optimal martingale measure,* Ann. Probab., 24 (1996), 206–236

[26] Stein, E. M. and Stein, J. C., *Stock price distributions with stochastic volatility: an analytic approach,* Rev. Financial Studies, 4 (1991), 727-752

[27] Tang, S. and Li, X., *Necessary conditions for optimal control of stochastic systems with random jumps,* SIAM J. Control Optim., 32 (1994), 1447–1475

[28] Wonham, W. M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control Optim., 6 (1968), 312-326

[29] Zhou, X. and Li, D., *Continuous time mean-variance portfolio selection: a stochastic LQ framework,* Applied Math. and Optim., 42 (2000), 19-33