MULTIDIMENSIONAL BALANCED DESIGNS*

BU-513-M by June, 1974

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Abstract

If $v = 4\lambda + 3$ is a prime or prime power, then the $k = 2\lambda + 1$ quadratic residues in $GF(v)$ form a (v,k,λ) difference set. A general construction is given for m-way completely variance balanced designs where each factor has v levels, m is any integer less than or equal to k, and $N = vk$. The construction gives rise to a variety of designs, easily enumerated, with the same parameters pairwise but with differing variance properties. For $m = 3$ there are only two distinct designs possible, and their relative efficiency is shown to be $2\lambda^2/(2\lambda^2 - 1)$.

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1. Introduction and Background. Consider an experiment involving m factors at $s_1, s_2, \cdots, s_m$ levels, and suppose that all interactions between factors can be neglected. A design T for the experiment of size N is the specification of N combinations. Denote by B_{ij} the $s_i \times s_j$ incidence matrix of factors i and j for the design T. That is, the element in B_{ij} corresponding to level x of factor i and level y of factor j is the number of combinations which have factors i and j at levels x and y, respectively. The elements of B_{ij} are integers greater than or equal to zero. In an obvious way we let $B_{ii} = \text{Diag}[r_1, r_2, \cdots, r_{s_i}]$. Then, the information matrix for the design T is the block matrix $\left((B_{ij})\right)$ $i, j = 1, 2, \cdots, m$, where we have already adjusted for the mean. If the design T is connected, $M = \left((B_{ij})\right) + \text{Diag}\left[J_{s_1 s_1}, J_{s_2 s_2}, \cdots, J_{s_m s_m}\right]$ is nonsingular and M^{-1} is a conditional inverse of $\left((B_{ij})\right)$, $\left((B_{ij})\right)^- = M^{-1} = \left((V_{ij})\right)$.

Definition 1.1. The design T is said to be

(i) variance balanced with respect to factor i if $V_{ii} = a_i I_{s_i} + b_i J_{s_i s_i}$, and

(ii) completely variance balanced if it is variance balanced with respect to every $i = 1, 2, \cdots, m$.

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We illustrate the above with the following construction. Suppose for a given n we have a set of t orthogonal Latin squares of order n, $O(n,t)$, which we will assume to be arranged so that each has first row $(0, 1, 2, \cdots, n - 1)$. Cut off the first row from each of the t squares, and consider the $t + 1$ factors corresponding to the column effects and the t treatments of the t Latin squares. It is apparent that for any pair of factors each level x of one occurs exactly one time with each level of the second except x. The combination (x,x) occurs zero times. Thus the incidence matrix for i^{th} and j^{th} factors, B_{ij}, is

$$B_{ij} = J_{nn} - I_n \qquad i \neq j = 1, 2, \cdots, t + 1. \tag{1.1}$$

Since the matrices I_n and $J_{nn} - I_n$ are closed with respect to multiplication, complete variance balance is obvious if the design is connected. It is important to observe that if we have a design with m factors each at n levels in N observations, an obvious necessary condition for connectedness is

$$N \geq 1 + m(n - 1). \tag{1.2}$$

Thus if n is a prime or prime power and $t = n - 1$, we may use any t (not $t + 1$) of the factors to obtain a t-way variance balanced design in $N = n(n - 1)$ runs. If n is a prime power or not and $t < n - 1$, we have $(t + 1)$-way complete variance balanced designs.

The purpose of this paper is to produce families of m-way completely variance balanced designs each with the same parameters pairwise, but which give rise to designs with different efficiencies. The constructions are based on difference sets of the form $(v,k,\lambda) = (4\lambda + 3, 2\lambda + 1, \lambda)$ where $4\lambda + 3$ is a prime or prime power. The complete variance balance follows from the fact (Theorem 2.1) that the matrices I, B, B' form a basis for a linear associative commutative algebra where B is the incidence matrix of the corresponding symmetric balanced incomplete block design.

Hedayat and Raghavarao (1973) have obtained sufficient conditions for the existence of three-way pairwise balanced designs, but not necessarily variance balanced, based on difference sets. Their construction when applied to the (v,k,λ) difference set above is one of the possible constructions in this paper for $m = 3$. Afsarinejad and Hedayat (1972) have given constructions for multistage Youden designs from difference sets in a manner similar to the constructions here; however, the complete variance balance and the differences in variance properties were not observed. Preece (1966) has given several Youden designs for two sets of treatments obtained by cutting rows from Graeco-Latin squares. For the case $v = 4\lambda + 3$, a prime or prime power, his designs are essentially those of the present paper for $m = 3$, and he has noted the different variance properties of the different possible designs. The present paper might be regarded as an extension of these constructions to m-way completely variance balance which easily provide all possible combinatorial configurations of the given type.

Agrawal (1966) has given constructions for three-way designs which are related to those of Hedayat and Raghavarao (1973), and of this paper for $m = 3$. Potthoff (1962a,b, 1963) has given a number of specific designs of three and four dimensions which can be obtained by one of the constructions. Causey (1968) with a related construction produces some designs for the case discussed in this paper of up to five dimensions, but does not obtain a construction for the maximum number of factors.

2. On $(4\lambda + 3, 2\lambda + 1, \lambda)$ Difference Sets. If $v = 4\lambda + 3$ is a prime or prime power and $k = (v - 1)/2 = 2\lambda + 1$, then it is well-known that the quadratic residues in $GF(v)$ form a (v,k,λ) difference set. Let the quadratic residues be arranged in some arbitrary but fixed order (for convenience we take the first element to be the multiplicative identity denoted by 1) as $\underline{Q} = (1, d_2, d_3, \cdots, d_k)$. We observe

that the v - 1 vectors $x\underline{Q} = (x, xd_2, xd_3, \cdots, xd_k)$, $x \in GF(v)$, are $(v - 1)/2$ permutations of the vector $\underline{Q}$ as x ranges over the k quadratic residues of GF(v), and $(v - 1)/2$ permutations of nonquadratic residues as x ranges over the nonquadratic residues. In particular, since if x is a quadratic residue, then (-x) is nonquadratic and $-\underline{Q}$ is a permutation of the nonquadratic residues. Further, $x\underline{Q} - y\underline{Q} = (x - y)\underline{Q}$ is a permutation of quadratic or nonquadratic residues as (x - y) is or is not a quadratic residue. Thus with the ordering of $\underline{Q}$, we have specified (v - 1) ordered vectors $x\underline{Q}$, $x \in GF(v)$, $x \neq 0$, such that the vectors together with $(0, 0, \cdots, 0)$ are closed with respect to vector addition over GF(v).

Corresponding to this (v,k,λ) difference set, we can construct a symmetric balanced incomplete block design $v = b$, $r = k$, λ, whose 0^{th} block is $\underline{Q}$ and whose x^{th} block is $\underline{Q} + (x, x, \cdots, x) = (1 + x, d_2 + x, d_3 + x, \cdots, d_k + x)$, x ranging over the nonzero elements of GF(v). Let B denote the $v \times v$ incidence matrix of this design so that

$$B'B = BB' = (2\lambda + 1)I_v + \lambda(J_{vv} - I_v), \tag{2.1}$$

and

$$BJ = B'J = (2\lambda + 1)J\ . \tag{2.2}$$

We prove the following theorem of central importance in our constructions of multidimensional balanced designs.

Theorem 2.1. *If B is the incidence matrix of the cyclic balanced incomplete block design with initial block $\underline{Q}$, then the three matrices I, B, and B' form a basis for a linear associative and commutative algebra.*

Proof. Since $\underline{Q}$ and $-\underline{Q}$ are permutations of the quadratic and nonquadratic residues, respectively, and 0 is neither in $\underline{Q}$ nor in $-\underline{Q}$, it follows that $I_v + B + B' = J$ and the three matrices are linearly independent. From (2.1)

$$B'B = BB' = (2\lambda + 1)I_v + \lambda B + \lambda B'. \tag{2.3}$$

From (2.2) we have

$$\begin{aligned}(2\lambda + 1)J &= BJ \\ &= B[I + B + B'] \\ &= B + B^2 + BB' \\ &= B + B^2 + (2\lambda + 1)I + \lambda B + \lambda B' \\ &= (2\lambda + 1)I + (\lambda + 1)B + \lambda B' + B^2 ,\end{aligned} \tag{2.4}$$

so that

$$B^2 = \lambda B + (\lambda + 1)B' . \tag{2.5}$$

Similarly,

$$(B')^2 = (\lambda + 1)B + \lambda B' . \tag{2.6}$$

Thus we have shown closure with respect to multiplication, and the commutativity is obvious. The proof is complete.

Corollary 2.1. *If* $C = c_0 I + c_1 B + c_2 B'$, *then*

$$C'C = CC' = aI + b(J - I),$$

is a balanced matrix.

Proof. Since C'C and CC' are obviously symmetric, the coefficients of B and B' in the product must be the same.

Corollary 2.2. $B^{-1} = \dfrac{-1}{(2\lambda + 1)(\lambda + 1)} [\lambda I_v + \lambda B - (\lambda + 1)B']$.

The proof is by direct multiplication of B^{-1} by B.

3. Construction of Multidimensional Balanced Designs from (v,k,λ) Sets. Retaining the notation developed in section 2, consider an experiment with $m \le k$ factors each at v levels. Specify an initial set of treatments by the $m \times k$ matrix whose rows (corresponding to the m factors) are:

$$T_0(x_1, x_2, \cdots, x_m) = \begin{bmatrix} x_1\underline{Q} \\ x_2\underline{Q} \\ \vdots \\ x_m\underline{Q} \end{bmatrix}, \quad x_i \neq x_j \tag{3.1}$$

and each of the k columns corresponds to a treatment combination. Let additional treatments $T_x(x_1, x_2, \cdots, x_m)$ be obtained cyclically from the initial set by adding x to every element of the matrix of (3.1) as x ranges over GF(v). The design depends only on $(x_1, x_2, \cdots, x_m)$, and we may denote it by

$$T(x_1, x_2, \cdots, x_m) = \bigcup_{x \in GF(v)} T_x(x_1, x_2, \cdots, x_m). \tag{3.2}$$

<u>Theorem 3.1.</u> <u>For the design</u> $T(x_1, x_2, \cdots, x_m)$ <u>of</u> (3.2), <u>the incidence matrix for</u> i^{th} <u>and</u> j^{th} <u>factors</u>, $i < j$, B_{ij} <u>is</u> B <u>or</u> B' <u>as</u> $(x_j - x_i)$ <u>is or is not a quadratic residue in</u> GF(v).

<u>Proof.</u> This follows directly from the observation that $x_j\underline{Q} - x_i\underline{Q} = (x_j - x_i)\underline{Q}$ is a permutation of the quadratic or nonquadratic residues as $(x_j - x_i)$ is a quadratic or nonquadratic residue. For example, consider only the i^{th} and j^{th} rows of $T_0(x_1, x_2, \cdots, x_m)$ and subtract (in GF(v)) $x_i\underline{Q}$ from each. Considering the i^{th} factor as blocks and the j^{th} factor as treatments, the cyclic development (3.2) will generate an incidence matrix which is either B or B'.

<u>Theorem 3.2.</u> <u>The design</u> $T(x_1, x_2, \cdots, x_m)$ <u>is</u> m-<u>way balanced</u>.

<u>Proof.</u> The information matrix contains only matrices I, B, and B'. Thus the information matrix of the reduced normal equations is a linear combination of I, B, and B', and since it is necessarily symmetric by Corollary 2.1, it must be of the form aI + bJ.

To the initial set $T_0(x_1, x_2, \cdots, x_m)$ we may adjoin ρ columns of $(0, 0, \cdots, 0)'$ and develop the design as in (3.2). The resultant design has $N = v(k + \rho)$ and incidence matrices either $\rho I + B$ or $\rho I + B'$. The closure of I, B, B' is sufficient to show that the designs are still m-way balanced. We have the following:

<u>Theorem 3.3</u>. *If* $v = 4\lambda + 3$ *is a prime or prime power, there always exists an m-way balanced design if* $m \le (v - 1)/2 = k$ *with* $N = v(k + \rho)$.

<u>Example 3.1</u>. Let $m = 4$, $v = 11$, and $\underline{Q} = (1, 3, 4, 5, 9)$. Take the initial set to be

$$T_0(1, 2, 5, 6) = \begin{bmatrix} 1 & 3 & 4 & 5 & 9 \\ 2 & 6 & 8 & 10 & 7 \\ 5 & 4 & 9 & 3 & 1 \\ 6 & 7 & 2 & 8 & 10 \end{bmatrix}.$$

The full design is

$$T(1, 2, 5, 6) = \left[\begin{array}{ccccc ccccc ccccc ccccc} 1 & 3 & 4 & 5 & 9 & 2 & 4 & 5 & 6 & 10 & 3 & 5 & 6 & 7 & 0 & 4 & 6 & 7 & 8 & 1 \\ 2 & 6 & 8 & 10 & 7 & 3 & 7 & 9 & 0 & 8 & 4 & 8 & 10 & 1 & 9 & 5 & 9 & 0 & 2 & 10 \\ 5 & 4 & 9 & 3 & 1 & 6 & 5 & 10 & 4 & 2 & 7 & 6 & 0 & 5 & 3 & 8 & 7 & 1 & 6 & 4 \\ 6 & 7 & 2 & 8 & 10 & 7 & 8 & 3 & 9 & 0 & 8 & 9 & 4 & 10 & 1 & 9 & 10 & 5 & 0 & 2 \end{array}\right.$$

$$\begin{array}{ccccc ccccc ccccc ccccc} 5 & 7 & 8 & 9 & 2 & 6 & 8 & 9 & 10 & 3 & 7 & 9 & 10 & 0 & 4 & 8 & 10 & 0 & 1 & 5 \\ 6 & 10 & 1 & 3 & 0 & 7 & 0 & 2 & 4 & 1 & 8 & 1 & 3 & 5 & 2 & 9 & 2 & 4 & 6 & 3 \\ 9 & 8 & 2 & 7 & 5 & 10 & 9 & 3 & 8 & 6 & 0 & 10 & 4 & 9 & 7 & 1 & 0 & 5 & 10 & 8 \\ 10 & 0 & 6 & 1 & 3 & 0 & 1 & 7 & 2 & 4 & 1 & 2 & 8 & 3 & 5 & 2 & 3 & 9 & 4 & 6 \end{array}$$

$$\left.\begin{array}{ccccc ccccc ccccc} 9 & 0 & 1 & 2 & 6 & 10 & 1 & 2 & 3 & 7 & 0 & 2 & 3 & 4 & 8 \\ 10 & 3 & 5 & 7 & 4 & 0 & 4 & 6 & 8 & 5 & 1 & 5 & 7 & 9 & 6 \\ 2 & 1 & 6 & 0 & 9 & 3 & 2 & 7 & 1 & 10 & 4 & 3 & 8 & 2 & 0 \\ 3 & 4 & 10 & 5 & 7 & 4 & 5 & 0 & 6 & 8 & 5 & 6 & 1 & 7 & 9 \end{array}\right].$$

Since $(2 - 1)$, $(5 - 1)$, $(6 - 1)$, $(5 - 2)$, $(6 - 2)$, and $(6 - 5)$ are all quadratic residues, the information matrix is

$$\begin{bmatrix} 5I & B & B & B \\ B' & 5I & B & B \\ B' & B' & 5I & B \\ B' & B' & B' & 5I \end{bmatrix} .$$

A five-dimensional design $T(1, 2, 5, 6, 10)$ can be formed by adding the row (10, 8, 7, 6, 2) to the initial set above. The differences (10 - 1), (10 - 5), and (10 - 6) are quadratic residues so the corresponding incidence matrices are each B. The difference 10 - 2 = 8 is a nonquadratic residue so $B_{25} = B'$. It is easy to see the effect of adjoining ρ columns of zeros to the initial sets. The following table gives vectors (x_1, x_2, x_3, x_4) for the various possible designs of dimension 4 when $v = 11$. We have taken the block corresponding to factors one and two to be B. There are five other pairs of factors giving rise to a total of 32 possible designs. Considering all possible vectors (1, 2, x, y) and (1, 2, y, x) it is easy to generate all 32 such designs. They are listed in order of increasing number of B' blocks, and the design obtained by reversing the order of factors three and four is not listed. A similar procedure may be followed to generate the possible five-dimensional designs.

TABLE 3.1

Four-Dimensional Designs for v = 11

(1, 2, 5, 6)	(1, 2, 10, 3)
(1, 2, 3, 6)	(1, 2, 5, 8)
(1, 2, 5, 3)	(1, 2, 10, 4)
(1, 2, 4, 5)	(1, 2, 9, 7)
(1, 2, 6, 4)	(1, 2, 3, 8)
(1, 2, 3, 7)	(1, 2, 9, 10)
(1, 2, 8, 6)	(1, 2, 10, 8)
(1, 2, 3, 4)	(1, 2, 9, 8)

We would note that in a combinatorial sense we have constructed a $(v - 1)$-dimensional pairwise balanced design. The design is obviously not connected, however, since the number of parameters is greater than the number of observations.

4. On the Variance of Linear Contrasts. The construction of section 3 permits considerable variety in the designs as illustrated by example 3.1. Pairwise the designs are all symmetric balanced incomplete block designs with parameters (v,k,λ). However, it is not apparent how the designs $T(x_1, x_2, \cdots, x_m)$ compare in terms of the variance of linear contrasts of treatment effects.

Consider two three-dimensional designs given by $T(1,2,x)$ and $T(1,2,y)$ where $(x - 1)$ and $(x - 2)$ are both quadratic residues and $(y - 1)$ and $(y - 2)$ are non-quadratic and quadratic residues, respectively. The information matrices are, respectively,

$$\begin{array}{ccc} T(1,2,x) & & T(1,2,y) \\ \begin{bmatrix} rI & B & B \\ B' & rI & B \\ B' & B' & rI \end{bmatrix} & \text{and} & \begin{bmatrix} rI & B & B' \\ B' & rI & B \\ B & B' & rI \end{bmatrix} \end{array} . \tag{4.1}$$

After eliminating the first factor, the reduced normal equations for both designs have diagonal blocks of $\lambda v/k\ I_v$ (neglecting constant multiples of J) and for $T(1,2,x)$ and $T(1,2,y)$, respectively, off diagonal blocks of $B - 1/r\ B'B$ and $B - 1/r\ B'B'$. On eliminating the second factor we have, after some simplification, reduced normal equations of $(2\lambda - 1/\lambda)I_v$ for $T(1,2,x)$ and $2\lambda I_v$ for $T(1,2,y)$.

It is easy to show that for each of the designs all three factors have the same reduced normal equations. Thus for each factor the efficiency of design $T(1,2,y)$ to $T(1,2,x)$ is

$$\mathrm{Eff}\Big(T(1,2,y),\ T(1,2,x)\Big) = \frac{2\lambda^2}{2\lambda^2 - 1} . \tag{4.2}$$

When $\lambda = 1$ this is an efficiency of 2; however, the efficiency rapidly approaches 1 as λ increases.

There is thus demonstrated a definite difference between the various possible designs. The problem becomes more complex for four- and higher-dimensional designs as the number of possibilities increases and not all factors have the same variance. A complete study of this phenomenon will be discussed in a later publication.

It was noted in the introduction that three-dimensional designs similar to those constructed in section 3 had been given by Potthoff (1962a), Agrawal (1966), Causey (1968), and Hedayat and Raghavarao (1973). Without exception, the three-dimensional designs given in their constructions correspond to the case here with larger variance, and the fact that two constructions exist with different variance seems not to have been observed. Potthoff (1962b) and Causey (1968) provide some four- and higher-dimensional designs using a similar, but not identical, construction which does not provide for a variety of designs as does the construction of this paper. Causey in fact remarks that no general way of determining the maximum number of factors that can be accommodated in the design was obtained. Here we have shown that this number is always $m = 2\lambda + 1$.

The series of designs given by Anderson (1972) provide three-, four-, and five-dimensional designs with small values of N. These designs have all off-diagonal blocks $B_{ij} = B = B' = B_{ji}$ belonging to a linear associative algebra, but provide only partial balance rather than balance. The multistage designs given by Hedayat, Seiden, and Federer (1972) produce partially balanced designs when considered as m-dimensional designs, though some factors are in fact orthogonal to others.

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