

MULTIDIMENSIONAL CONTACT PROBLEMS IN THERMOELASTICITY*

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Abstract. We consider dynamical and quasi-static thermoelastic contact problems in \mathbb{R}^n modeling the evolution of temperature and displacement in an elastic body that may come into contact with a rigid foundation. The existence of solutions to these dynamical and quasi-static nonlinear problems and the exponential stability are investigated using a penalty method. Interior smoothing effects in the quasi-static case are also discussed.

Key words. quasi-static thermoelasticity, asymptotic behavior

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1. Introduction. We consider dynamical and quasi-static thermoelastic contact problems which model the evolution of temperature and displacement in an elastic body that may come into contact with a rigid foundation. If $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) denotes the reference configuration, we assume that the smooth boundary $\partial\Omega$ consists of three mutually disjoint parts $\Gamma_D, \Gamma_N, \Gamma_C$ such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$ and $\Gamma_D \neq \emptyset$. The body is held fixed on Γ_D , tractions are zero on Γ_N , and Γ_C is the part which may have contact with a rigid foundation. The temperature is held fixed on $\partial\Omega$. Then the dynamical initial boundary value problem for the displacement $u = u(t, x)$ and the temperature difference $\theta = \theta(t, x)$, where $t \geq 0$ and $x \in \bar{\Omega}$, to be considered is the following:

$$(1.1) \quad \rho \partial_t^2 u_i - (C_{ijkl} u_{k,l})_{,j} + (m_{ij} \theta)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(1.2) \quad \rho c \partial_t \theta - (k_{ij} \theta_{,i})_{,j} + m_{ij} \partial_t u_{i,j} = 0,$$

$$(1.3) \quad u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0,$$

$$(1.4) \quad u|_{\Gamma_D} = 0, \quad \sigma_{ij} \nu_j|_{\Gamma_N} = 0, \quad \theta|_{\partial\Omega} = 0,$$

$$(1.5) \quad u_\nu \leq g, \quad \sigma_\nu \leq 0, \quad (u_\nu - g)\sigma_\nu = 0, \quad \sigma_T = 0 \text{ on } \Gamma_C.$$

Here ρ and c denote the density and the heat capacity, respectively, and are assumed in what follows to be equal to one without loss of generality. C_{ijkl} , m_{ij} , and k_{ij} denote the components of the elasticity tensor, those of the thermal expansion tensor, and those of the heat conduction tensor, respectively, and will satisfy

$$(1.6) \quad C_{ijkl} \in L^\infty(\Omega), \quad C_{ijkl} = C_{jikl} = C_{klij},$$

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$$(1.7) \quad \exists \lambda_1 > 0 \quad \forall \eta_{ij} = \eta_{ji}, \quad C_{ijkl} \eta_{ij} \eta_{kl} \geq \lambda_1 |\eta|^2,$$

$$(1.8) \quad m_{ij} \in W^{1,\infty}(\Omega), \quad m_{ij} = m_{ji} \geq 0,$$

$$(1.9) \quad k_{ij} \in W^{1,\infty}(\Omega), \quad k_{ij} = k_{ji},$$

$$(1.10) \quad \exists \lambda_2 > 0 \quad k_{ij} \xi_i \xi_j \geq \lambda_2 |\xi|^2.$$

The comma notation $_{,j}$ denotes the differentiation $\partial_j \equiv \partial/\partial x_j$ with respect to x_j , and a subindex t will denote the differentiation $\partial_t \equiv \partial/\partial t$. The initial values u_0, u_1 , and θ_0 are prescribed with regularity to be made more precise in the final formulation of the system. The stress tensor is given by $\sigma = (\sigma_{ij})$

$$\sigma_{ij} = C_{ijkl} u_{k,l} - m_{ij} \theta.$$

Since on the boundary $\theta = 0$ we have there

$$\sigma_{ij} = C_{ijkl} u_{k,l}.$$

The unit normal vector in $x \in \partial\Omega$ is denoted by $\nu = \nu(x)$ and the normal component of u by u_ν :

$$u_\nu = u \cdot \nu.$$

The normal component σ_ν of the stress tensor is given by

$$\sigma_\nu = \sigma_{ij} \nu_i \nu_j$$

and the tangential part σ_T is

$$\sigma_T = \sigma \nu - \sigma_\nu \nu.$$

The function g describes the initial gap between the part Γ_c of the reference configuration and the rigid foundation and is assumed to satisfy

$$(1.11) \quad g \in H^{1/2}(\Gamma_c), \quad g \geq 0 \quad \text{almost everywhere (a.e.) on } \Gamma_c.$$

Hence the boundary conditions (1.5) describe Signorini's contact condition on Γ_c for a frictionless ($\sigma_T = 0$) contact.

The corresponding quasi-static system arises from (1.1)–(1.5) by omitting $\partial_t^2 u_i$ in (1.1) and prescribing only θ_0 in (1.3).

These thermoelastic contact problems, which are nonlinear because of the contact boundary conditions (1.5), arise in applications such as the manufacturing of castings and pistons; see Shi and Shillor [12] for more details and references. The mathematical treatment of the dynamical problem (1.1)–(1.5) in one space dimension was discussed by Elliott and Tang [4], where more complicated boundary conditions are considered. Existence results in higher dimensions were announced by Figueiredo and Trabucho in [5]. They considered the case of constant contact, which allows them to use only variational methods to solve the problem; this does not apply to the situation of the contact condition of Signorini's type considered here. We shall investigate the existence in $n \geq 1$ space dimensions for radially symmetrical situations following the

approach of Kim [8], who discussed an obstacle problem for a wave equation. Using a penalty method, we obtain an existence result and we also prove the exponential stability, which was proven for a special one-dimensional system by Muñoz Rivera and Lacerda Oliveira [10].

As a motivation for the stability results to be expected and as a tool to be used later, we also discuss the *linear* quasi-static system for classical boundary conditions like

$$u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0,$$

proving existence, exponential stability, and smoothing.

For the quasi-static system, Shi and Shillor [12] proved the existence of a solution, while Ames and Payne [1] proved a uniqueness result; see also these papers for references on the one-dimensional case, where a series of papers has appeared over the last years.

We shall give a new existence proof using a penalty method. This will also allow us to make conclusions on the exponential stability of the system. Finally, we shall prove that the quasi-static system has a smoothing effect in the interior of Ω , where u and θ are shown to be arbitrarily smooth, an effect which cannot be expected up to the boundary because of the mixed and contact boundary conditions.

We remark that we have assumed exterior forces and exterior heat supply to be zero only for simplicity.

The paper is organized as follows. In section 2 we prove an existence and exponential stability result for the dynamical system(1.1)–(1.5) in the case of radial symmetry using a penalty method. Section 3 studies the linear quasi-static system with classical Dirichlet or Neumann type boundary conditions proving existence, uniqueness, and exponential stability. In section 4 the quasi-static contact problem is investigated, and we obtain an existence result and exponential stability using a penalty method. Finally, section 5 will provide the interior smoothing effect for the quasi-static contact problem.

Concerning the notation we remark that we use standard notations for Sobolev spaces and also the Einstein summation convention for repeated indices. By $\langle \cdot, \cdot \rangle$ we denote the inner product in $(L^2(\Omega))^n$, and by $\|\cdot\|$ we denote the corresponding norm.

2. Existence for the dynamical contact problem. A dynamical thermoelastic contact problem was investigated by Elliott and Tang [4] in one space dimension following the approach of Kim [8], who used a penalty method for the treatment of an obstacle problem for a wave equation. We also adopt this approach. The system character of our problem arising in multidimensional elasticity naturally leads to further problems in the estimates that have to be overcome by additional considerations compared to the two papers above. We shall discuss the case of radial symmetry. We look for a solution to (1.1)–(1.5) in the following sense. Let

$$H_{\Gamma_D}^1 := \{u \in (H^1(\Omega))^n \mid u|_{\Gamma_D} = 0\}.$$

DEFINITION 2.1. (u, θ) is a solution to (1.1)–(1.5) for given $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\theta_0 \in L^2(\Omega)$ if, for any $T > 0$,

$$\begin{aligned} u &\in L^\infty((0, T), H_{\Gamma_D}^1(\Omega)), u_t \in L^\infty((0, T), (L^2(\Omega))^n), \\ (2.1) \quad \theta &\in L^\infty((0, T), L^2(\Omega)) \cap L^2((0, T), H_0^1(\Omega)), \end{aligned}$$

$$(2.2) \quad u(t = 0) = u_0, \quad u_t(t = 0) = u_1, \quad \theta(t = 0) = \theta_0,$$

$$(2.3) \quad u_\nu \leq g \text{ on } (0, T) \times \Gamma_C \quad (\text{a.e.});$$

and for all $w \in L^\infty((0, T), H_{\Gamma_D}^1(\Omega)) \cap W^{1,\infty}((0, T), (L^2(\Omega))^n)$ with $w_\nu \leq g$ on Γ_c the following inequality holds:

$$(2.4) \quad \begin{aligned} & \langle u_t(T, \cdot), w(T, \cdot) - u(T, \cdot) \rangle - \langle u_1, w(0, \cdot) - u_0 \rangle \\ & - \int_0^T \langle u_t, w_t \rangle dt + \int_0^T \{ \langle u_t, u_t \rangle - \langle C_{ijkl} u_{k,l}, u_{i,j} \rangle \} dt \\ & + \int_0^T \langle C_{ijkl} u_{k,l}, w_{i,j} \rangle dt - \int_0^T \langle m_{ij} \theta, w_{i,j} - u_{i,j} \rangle dt \geq 0; \end{aligned}$$

and for all $z \in W^{1,2}((0, T), H_0^1(\Omega))$ the following equality holds:

$$(2.5) \quad \begin{aligned} & - \int_0^T \langle \theta, z_t \rangle dt + \langle \theta(T, \cdot), z(T, \cdot) \rangle - \langle \theta_0, z(0) \rangle \\ & + \int_0^T \langle k_{ij} \theta_{,i}, z_{,j} \rangle dt - \int_0^T \langle u_{i,j}, \partial_t(m_{ij} z) \rangle dt \\ & + \langle m_{ij} u(T, \cdot)_{i,j}, z(T, \cdot) \rangle - \langle m_{ij} u_{0i,j}, z(0, \cdot) \rangle = 0. \end{aligned}$$

In what follows we shall write $L^\infty(H_{\Gamma_D}^1)$ instead of $L^\infty((0, T), H_{\Gamma_D}^1(\Omega))$, similarly for the other spaces; moreover we write $u(T)$ instead of $u(T, \cdot)$, and so on.

We remark that a smooth classical solution to (1.1)–(1.5) obviously satisfies (2.5) and also (2.4). On the other hand, it is easy to see that if (u, θ) is a solution in the sense of Definition 2.1 and if (u, θ) is smooth, then (1.1)–(1.5) follows in the classical sense.

The existence of a solution will be proven for the radial symmetrical case by using a penalty method. For this purpose we consider the following approximating problem for a parameter $\varepsilon > 0$. This will be solved—in general, without restriction to radial symmetry—and then a priori estimates are proven that will show the convergence, as $\varepsilon \downarrow 0$, of a subsequence to a solution of (1.1)–(1.5). The penalized problem is the following, first in classical notation. For given $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega))^n \cap (H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ find a solution $(u^\varepsilon, \theta^\varepsilon)$ to the following initial boundary value problem (2.6)–(2.10):

$$(2.6) \quad \partial_t^2 u_i^\varepsilon - (C_{ijkl} u_{k,l}^\varepsilon)_{,j} + (m_{ij} \theta^\varepsilon)_{,j} = 0,$$

$$(2.7) \quad \partial_t \theta^\varepsilon - (k_{ij} \theta_{,i}^\varepsilon)_{,j} + m_{ij} \partial_t u_{i,j}^\varepsilon = 0,$$

$$(2.8) \quad u^\varepsilon(t = 0) = u_0^\varepsilon, \quad u_t^\varepsilon(t = 0) = u_1^\varepsilon, \quad \theta^\varepsilon(t = 0) = \theta_0^\varepsilon,$$

$$(2.9) \quad u_{|\Gamma_D}^\varepsilon = 0, \quad \sigma_{ij}^\varepsilon \nu_j|_{\Gamma_N} = 0, \quad \theta_{|\partial\Omega}^\varepsilon = 0,$$

$$(2.10) \quad \sigma_\nu^\varepsilon = -\frac{1}{\varepsilon} (u_\nu^\varepsilon - g)^+ - \varepsilon \partial_t u_\nu^\varepsilon, \quad \sigma_T^\varepsilon = 0, \quad \text{on } \Gamma_c,$$

where $\sigma^\varepsilon = \sigma(u^\varepsilon)$ is the stress tensor corresponding to u^ε and f^+ denotes the positive part of f . Let $\{w_j\}_{j=1}^\infty \subset H^1_{\Gamma_D}(\Omega)$ be an orthonormal basis in $(L^2(\Omega))^n$, and let $\{z_j\}_{j=1}^\infty \subset H^1_0(\Omega)$ be an orthonormal basis in $L^2(\Omega)$.

DEFINITION 2.2. $(u^\varepsilon, \theta^\varepsilon)$ is a solution to (2.6)–(2.10) for $u^\varepsilon_0, u^\varepsilon_1 \in (H^2(\Omega) \cap H^1_0(\Omega))^n$, $\theta^\varepsilon_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ if, for any $T > 0$,

$$(2.11) \quad u^\varepsilon \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^1_{\Gamma_D}), \quad \theta^\varepsilon \in W^{1,\infty}(L^2) \cap W^{1,2}(H^1_0),$$

$$(2.12) \quad u^\varepsilon(t=0) = u^\varepsilon_0, \quad u^\varepsilon_t(t=0) = u^\varepsilon_1, \quad \theta^\varepsilon(t=0) = \theta^\varepsilon_0,$$

and for all $p \in \mathbb{N}$ and for a.e. $t \in [0, T]$:

$$(2.13) \quad \begin{aligned} & \langle \partial_t^2 u^\varepsilon, w_p \rangle + \langle C_{ijkl} u^\varepsilon_{k,l}, w_{pi,j} \rangle - \langle m_{ij} \theta^\varepsilon, w_{pi,j} \rangle \\ &= -\frac{1}{\varepsilon} \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ w_{p\nu} d\Gamma dt - \varepsilon \int_{\Gamma_c} \partial_t u^\varepsilon_\nu w_{p\nu} d\Gamma, \end{aligned}$$

$$(2.14) \quad \langle \partial_t \theta^\varepsilon, z_p \rangle + \langle k_{ij} \theta^\varepsilon_{,i}, z_{p,j} \rangle + \langle m_{ij} \partial_t u^\varepsilon_{i,j}, z_p \rangle = 0.$$

It is again easy to see that a smooth classical solution to (2.6)–(2.10) is a solution (in the sense of Definition 2.2), and a solution (in the sense of Definition 2.2), which is smooth, is a classical solution; compare the considerations following Definition 2.1.

THEOREM 2.3. For given $u^\varepsilon_0, u^\varepsilon_1 \in (H^2(\Omega) \cap H^1_0(\Omega))^n$, $\theta^\varepsilon_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, there is a solution $(u^\varepsilon, \theta^\varepsilon)$ to the penalized problem (2.6)–(2.10).

Remark. There is no assumption on radial symmetry here.

Proof. A Galerkin method will be the appropriate tool as in [8], [4]. Defining v and ψ by

$$v := u^\varepsilon - u^\varepsilon_0 - tu^\varepsilon_1, \quad \psi := \theta^\varepsilon - \theta^\varepsilon_0,$$

then (v, ψ) should satisfy, for $i = 1, \dots, n$,

$$(2.15) \quad \partial_t^2 v_i - (C_{ijkl} v_{k,l})_{,j} + (m_{ij} \psi)_{,j} = (C_{ijkl} u^\varepsilon_{0k,l} + tu^\varepsilon_{1k,l})_{,j} - (m_{ij} \theta^\varepsilon_0)_{,j} =: f_i,$$

$$(2.16) \quad \partial_t \psi - (k_{ij} \psi_{,i})_{,j} + m_{ij} \partial_t v_{i,j} = (k_{ij} \theta^\varepsilon_{0,i})_{,j} - m_{ij} u^\varepsilon_{1i,j} =: g,$$

$$(2.17) \quad v(t=0) = 0, \quad v_t(t=0) = 0, \quad \psi(t=0) = 0,$$

as well as the boundary conditions (2.9), (2.10) with $(u^\varepsilon, \theta^\varepsilon)$ being replaced by (v, ψ) , all to be understood in the sense of Definition 2.2. We make the ansatz

$$(2.18) \quad v^m(t, x) = \sum_{p=1}^m a_{mp}(t) w_p(x), \quad m \in \mathbb{N},$$

$$(2.19) \quad \psi^m(t, x) = \sum_{p=1}^m b_{mp}(t) z_p(x),$$

where $(a_{mp})_{p=1}^m, (b_{mp})_{p=1}^m$ satisfy

$$(2.20) \quad \begin{aligned} & \langle \partial_t^2 v^m, w_p \rangle + \langle C_{ijkl} v^m_{k,l}, w_{pi,j} \rangle - \langle m_{ij} \psi^m, w_{pi,j} \rangle \\ &= \langle f, w_p \rangle - \frac{1}{\varepsilon} \int_{\Gamma_c} (v^m_\nu - g)^+ w_{p\nu} d\Gamma - \varepsilon \int_{\Gamma_c} \partial_t v^m_\nu w_{p\nu} d\Gamma \end{aligned}$$

for $p = 1, \dots, m$, as well as

$$(2.21) \quad \langle \partial_t \psi^m, z_p \rangle + \langle k_{ij} \psi_i^m, z_{p,j} \rangle + \langle m_{ij} \partial_t v_{i,j}^m, z_p \rangle = \langle g, z_p \rangle,$$

$$(2.22) \quad a_{mp}(0) = 0, \quad \frac{d}{dt} a_{mp}(0) = 0, \quad b_{mp}(0) = 0.$$

Equations (2.20), (2.21) are a system of nonlinear ordinary differential equation for a_{mp} and b_{mp} , $p = 1, \dots, m$, with prescribed initial values given in (2.22). Since the (only) nonlinearity in (2.20) is Lipschitz continuous, there is a smooth solution $\{a_{mp}, b_{mp}\}_{p=1}^m$ in $[0, T]$ for any $T > 0$.

Now we derive a priori estimates for (v^m, ψ^m) . Multiplication of (2.20) by $\frac{d}{dt} a_{mp}(t)$ and (2.21) by $b_{mp}(t)$ and summation over p yields

$$(2.23) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_t v^m\|^2 + \frac{1}{2} \frac{d}{dt} \langle C_{ijkl} v_{k,l}^m, v_{i,j}^m \rangle - \langle m_{ij} \psi^m, \partial_t v_{i,j}^m \rangle \\ & = \langle f, \partial_t v^m \rangle - \frac{1}{\varepsilon} \int_{\Gamma_c} (v_\nu^m - g)^+ \partial_t v_\nu^m d\Gamma - \varepsilon \int_{\Gamma_c} |\partial_t v_\nu^m|^2 d\Gamma, \end{aligned}$$

$$(2.24) \quad \frac{1}{2} \frac{d}{dt} \|\psi^m\|^2 + \langle k_{ij} \psi_{,i}^m, \psi_{,j}^m \rangle + \langle m_{ij} \partial_t v_{i,j}^m, \psi^m \rangle = \langle g, \psi^m \rangle,$$

or

$$(2.25) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|\partial_t v^m\|^2 + \langle C_{ijkl} v_{k,l}^m, v_{i,j}^m \rangle + \|\psi^m\|^2 + \frac{1}{\varepsilon} \int_{\Gamma_c} |(v_\nu^m - g)^+|^2 d\Gamma \right\} \\ & \quad + \langle k_{ij} \psi_{,j}^m, \psi_{,i}^m \rangle \\ & = \langle f, \partial_t v^m \rangle - \varepsilon \int_{\Gamma_c} |\partial_t v_\nu^m|^2 d\Gamma + \langle g, \psi^m \rangle. \end{aligned}$$

Integrating (2.25) with respect to $t \in [0, T]$, we conclude that there is a positive constant

$$(2.26) \quad c = c(T, \|u_0^\varepsilon\|_{H^2}, \|u_1^\varepsilon\|_{H^2}, \|\theta_0^\varepsilon\|_{H^2})$$

not depending on m and depending on ε only in the way indicated in (2.26) such that for any $t \in [0, T]$ the following estimates (2.27)–(2.31) hold:

$$(2.27) \quad \|\partial_t v^m(t)\| \leq c,$$

$$(2.28) \quad \langle C_{ijkl} v_{k,l}^m, v_{i,j}^m \rangle(t) \leq c,$$

$$(2.29) \quad \int_0^t \langle k_{ij} \psi_{,i}^m, \psi_{,j}^m \rangle(s) ds \leq c,$$

$$(2.30) \quad \frac{1}{\varepsilon} \int_{\Gamma_c} |(v_\nu^m - g)^+|^2 d\Gamma \leq c,$$

$$(2.31) \quad \varepsilon \int_0^t \int_{\Gamma_c} |\partial_t v_\nu^m|^2 d\Gamma ds \leq c.$$

Therefore, using Korn’s inequality, we obtain that

$$(2.32) \quad (v^m)_m \text{ is bounded in } W^{1,\infty}(L^2) \cap L^\infty(H_{\Gamma_D}^1),$$

$$(2.33) \quad (\psi^m)_m \text{ is bounded in } L^\infty(L^2) \cap L^2(H_0^1).$$

Next, the boundedness of $\partial_t^2 v^m$ in $L^2(L^2)$ will be proved. The estimate will be uniform only in $m \in \mathbb{N}$, not in $\varepsilon > 0$ (cf. (2.26)).

Differentiation of (2.20), (2.21) with respect to t , then multiplication by $\frac{d^2}{dt^2} a_{mp}(t)$ and $\frac{d}{dt} b_{mp}(t)$, respectively, and summation leads to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ \|\partial_t^2 v^m\|^2 + \langle C_{ijkl} \partial_t v_{k,l}^m, \partial_t v_{i,j}^m \rangle + \|\partial_t \psi^m\|^2 \} + \langle k_{ij} \partial_t \psi_{,i}^m, \partial_t \psi_{,j}^m \rangle \\ &= \langle \partial_t f, \partial_t^2 v^m \rangle - \frac{1}{\varepsilon} \int_{\Gamma_c} (\partial_t(v_\nu^m - g))^+ \partial_t^2 v_\nu^m d\Gamma - \varepsilon \int_{\Gamma_c} |\partial_t^2 v_\nu^m|^2 d\Gamma \\ (2.34) \quad & \leq \|\partial_t f\| \|\partial_t^2 v^m\| + \frac{1}{2\varepsilon^3} \int_{\Gamma_c} |\partial_t(v_\nu^m - g)|^2 d\Gamma - \frac{\varepsilon}{2} \int_{\Gamma_c} |\partial_t^2 v_\nu^m| d\Gamma. \end{aligned}$$

Using (2.31) we conclude, by Gronwall’s inequality,

$$(2.35) \quad \|\partial_t^2 v^m(t)\| \leq c_1(\varepsilon),$$

$$(2.36) \quad \langle C_{ijkl} \partial_t v_{k,l}^m(t), \partial_t v_{i,j}^m(t) \rangle \leq c_1(\varepsilon),$$

$$(2.37) \quad \|\partial_t \psi^m(t)\| \leq c_1(\varepsilon),$$

$$(2.38) \quad \int_0^t \langle k_{ij} \partial_t \psi_{,i}^m, \partial_t \psi_{,j}^m \rangle(s) ds \leq c_1(\varepsilon),$$

where the positive constant $c_1(\varepsilon)$ is independent of m_0 and $t \in [0, T]$, $T > 0$, but may depend on ε . Thus we have that, for fixed $\varepsilon > 0$,

$$(2.39) \quad (\partial_t^2 v^m)_m \text{ is bounded in } L^\infty(L^2),$$

$$(2.40) \quad (\partial_t v^m)_m \text{ is bounded in } L^\infty(H_{\Gamma_D}^1),$$

$$(2.41) \quad (\partial_t \psi^m)_m \text{ is bounded in } L^\infty(L^2) \cap L^2(H_0^1).$$

It follows from (2.32), (2.33), and (2.39)–(2.41) that for fixed ε there is a subsequence, again denoted by $((v^m, \psi^m))_m$, and (v, ψ) such that, as $m \rightarrow \infty$,

$$(2.42) \quad v^m \overset{*}{\rightharpoonup} v \quad \text{in } W^{2,\infty}(L^2) \cap W^{1,\infty}(H_{\Gamma_D}^1),$$

$$(2.43) \quad \psi^m \overset{*}{\rightharpoonup} \psi \quad \text{in } W^{1,\infty}(L^2),$$

$$(2.44) \quad \psi^m \rightharpoonup \psi \quad \text{in } W^{1,2}(H_0^1).$$

With the help of Lemma 1.4 from [8] (essentially Gagliardo–Nirenberg type estimates), the convergence

$$(2.45) \quad (v_\nu^m - g)^+ \rightarrow (v_\nu - g)^+ \text{ in } C^0([0, T], L^2(\Gamma_c)),$$

as well as

$$(2.46) \quad \partial_t v^m \rightarrow \partial_t v \text{ in } C^0([0, T], L^2(\partial\Omega)),$$

follow. Using (2.42)–(2.46) and letting $m \rightarrow \infty$ in (2.20), (2.21), we conclude that

$$u^\varepsilon := v + u_0^\varepsilon + tu_1^\varepsilon, \quad \theta^\varepsilon := \psi + \theta_0^\varepsilon$$

satisfy (2.11)–(2.14). \square

Now we turn to the original problem (1.1)–(1.5) and show that a subsequence of $(u^\varepsilon, \theta^\varepsilon)$, where $(u^\varepsilon, \theta^\varepsilon)$ solves a penalized problem for $\varepsilon > 0$ according to Theorem 2.3, converges to a solution of (1.1)–(1.5).

THEOREM 2.4. *For given $u_0 \in H_0^1(\Omega), u_1 \in (L^2(\Omega))^3, \theta_0 \in L^2(\Omega)$, there is a solution (u, θ) to (1.1)–(1.5) in the case of radial symmetry.*

Radial symmetry means that the domain Ω is radially symmetrical, i.e., invariant under transformations of the special orthogonal group $SO(3)$ if $n = 3$, respectively, $O(2)$ if $n = 2$. The typical examples are balls or annular domains. Radial symmetry of (u, θ) means

$$\forall \mathcal{A} \in SO(2), \text{ respectively, } O(2) \forall x \in \Omega : u(\mathcal{A}x) = \mathcal{A}u(x), \quad \theta(\mathcal{A}x) = \theta(x),$$

or equivalently (see [6]), with $r := |x|$:

There exists a function $w : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ such that $u(x) = xw(r)$, and there exists a function $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ such that $\theta(x) = \psi(r)$.

Of course, for the existence of radially symmetrical solutions to the penalized, and later for the original, problem, the coefficients have to satisfy invariance conditions too. As an example we consider the homogeneous, isotropic case, where, in particular,

$$(2.47) \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}),$$

and the differential equations (1.1), (1.2) turn into

$$(2.48) \quad u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \bar{m} \nabla \theta = 0,$$

$$(2.49) \quad c \theta_t - \lambda_2 \Delta \theta + \bar{m} \operatorname{div} u_t = 0,$$

where μ and λ are the Lamé moduli satisfying $\mu > 0$ and $2\mu + n\lambda > 0, \rho, c, \kappa > 0$, and $m \neq 0$ are constants. We also notice that in this case

$$\sigma_{\nu_i \partial \Omega} = \lambda \operatorname{div} u + 2\mu u_{i,k} \nu_i \nu_k,$$

and it is easy to see that the boundary conditions (2.9), (2.10) allow for radially symmetrical solutions; i.e., if the initial data are radially symmetrical, then the solution according to Theorem 2.3 will have the same symmetry. Namely, if (u, θ) is the solution, then (v, ξ) with $v(x) := \mathcal{A}^{-1}u(\mathcal{A}x), \xi(x) := \theta(\mathcal{A}x)$ will also be a solution to the same initial data and boundary conditions for any (special) orthogonal \mathcal{A} . Hence, by uniqueness, $(u, \theta) = (v, \xi)$.

Proof of Theorem 2.4. Let $\varepsilon > 0$, and let $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega) \cap H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ be radially symmetrical such that, as $\varepsilon \downarrow 0$,

$$(2.50) \quad u_0^\varepsilon \rightarrow u_1^\varepsilon \text{ in } (H_0^1(\Omega))^n,$$

$$(2.51) \quad u_1^\varepsilon \rightarrow u_1 \text{ in } (L^2(\Omega))^n,$$

$$(2.52) \quad \theta_0^\varepsilon \rightarrow \theta_0 \text{ in } L^2(\Omega).$$

Let $(u^\varepsilon, \theta^\varepsilon)$ be the solution to (2.6)–(2.10) according to Theorem 2.3 and corresponding to the initial values $(u_0^\varepsilon, u_1^\varepsilon, \theta_0^\varepsilon)$. By the known regularity of $(u^\varepsilon, \theta^\varepsilon)$ we can substitute w_p by $\partial_t u^\varepsilon$ in (2.13) and z_p by θ^ε in (2.14), respectively, and we obtain (cf. (2.25))

$$(2.53) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|\partial_t u^\varepsilon\|^2 + \langle C_{ijkl} u_{k,l}^\varepsilon, u_{i,j}^\varepsilon \rangle + \|\theta^\varepsilon\|^2 + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma \right\} + \langle k_{ij} \theta_{,i}^\varepsilon, \theta_{,j}^\varepsilon \rangle \\ & = -\varepsilon \int_{\Gamma_c} |\partial_t u_\nu^\varepsilon|^2 d\Gamma. \end{aligned}$$

Hence, there is a positive constant

$$(2.54) \quad c_2 = c_2(T, \|u_0\|_{H^1}, \|u_1\|_{L^2}, \|\theta_0\|_{L^2}),$$

which is independent of ε , such that for any $t \in [0, T]$ we get

$$(2.55) \quad \|\partial_t u^\varepsilon\| \leq c_2,$$

$$(2.56) \quad \langle C_{ijkl} u_{k,l}^\varepsilon, u_{i,j}^\varepsilon \rangle \leq c_2,$$

$$(2.57) \quad \int_0^t \langle k_{ij} \theta_{,i}^\varepsilon, \theta_{,j}^\varepsilon \rangle(s) ds \leq c_2,$$

$$(2.58) \quad \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma \leq c_2,$$

$$(2.59) \quad \varepsilon \int_0^t \int_{\Gamma_c} |\partial_t u_\nu^\varepsilon|^2 d\Gamma ds \leq c_2.$$

The estimates (2.55)–(2.59) imply the existence of a subsequence, again denoted by $(u^\varepsilon, \theta^\varepsilon)$, and of (u, θ) such that, as $\varepsilon \downarrow 0$,

$$(2.60) \quad u^\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(H_{\Gamma_D}^1) \cap W^{1,\infty}(L^2),$$

$$(2.61) \quad \theta^\varepsilon \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(L^2),$$

$$(2.62) \quad \theta^\varepsilon \rightharpoonup \theta \text{ in } L^2(H_0^1).$$

Using Lemma 1.4 from [8] again, we conclude

$$(2.63) \quad u^\varepsilon \rightarrow u \text{ in } C^0(L^2).$$

The equations (2.13), (2.14) imply for $(u^\varepsilon, \theta^\varepsilon)$

$$(2.64) \quad \int_0^T \{ \langle \partial_t^2 u^\varepsilon, w - u^\varepsilon \rangle + \langle C_{ijkl} u_{k,l}^\varepsilon, w_{i,j} - u_{i,j}^\varepsilon \rangle - \langle m_{ij} \theta^\varepsilon, w_{i,j} - u_{i,j}^\varepsilon \rangle \} dt \\ = -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (w_\nu - u_\nu^\varepsilon) d\Gamma dt - \varepsilon \int_0^T \int_{\Gamma_c} \partial_t u_\nu^\varepsilon (w_\nu - u_\nu^\varepsilon) d\Gamma dt$$

for any $w \in L^\infty(H_{\Gamma_D}^1) \cap W^{1,\infty}(L^2)$ such that $w_\nu \leq g$, and

$$(2.65) \quad \int_0^T \{ \langle \partial_t \theta^\varepsilon, z_t \rangle + \langle k_{ij} \theta_{,i}^\varepsilon, z_{,j} \rangle + \langle m_{ij} \partial_t u_{i,j}^\varepsilon, z \rangle \} dt = 0$$

for any $z \in W^{1,2}(H_0^1)$.

Integration by parts in (2.64) yields

$$(2.66) \quad \langle u_t^\varepsilon(T), w(T) - u^\varepsilon(T) \rangle - \langle u_1^\varepsilon, w(0) - u_0^\varepsilon \rangle \\ - \int_0^T \langle u_t^\varepsilon, w_t \rangle dt + \int_0^T \langle u_t^\varepsilon, u_t^\varepsilon \rangle - \langle C_{ijkl} u_{k,l}^\varepsilon, u_{i,j}^\varepsilon \rangle dt \\ + \int_0^T \langle C_{ijkl} u_{k,l}^\varepsilon, w_{i,j} \rangle dt - \int_0^T \langle m_{ij} \theta^\varepsilon, w_{i,j} - u_{i,j}^\varepsilon \rangle dt \\ \geq -\varepsilon \int_0^T \int_{\Gamma_c} \partial_t u_\nu^\varepsilon w_\nu d\Gamma dt + \varepsilon \int_0^T \int_{\Gamma_c} \partial_t u_\nu^\varepsilon u_\nu^\varepsilon d\Gamma dt \\ = -\varepsilon \int_0^T \int_{\Gamma_c} \partial_t u_\nu^\varepsilon w_\nu d\Gamma dt + \frac{\varepsilon}{2} \left\{ \int_{\Gamma_c} |u_\nu^\varepsilon(T)|^2 d\Gamma - \int_{\Gamma_c} |u_{0\nu}^\varepsilon|^2 d\Gamma \right\},$$

where we used

$$-\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (w_\nu - u_\nu^\varepsilon) d\Gamma dt \\ = -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (w_\nu - g) d\Gamma dt + \frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma dt \geq 0.$$

Integration by parts in (2.65) leads to

$$(2.67) \quad - \int_0^T \langle \theta^\varepsilon, z_t \rangle dt + \langle \theta^\varepsilon(T), z(T) \rangle - \langle \theta_0^\varepsilon, z(0) \rangle \\ + \int_0^T \langle k_{ij} \theta_{,i}^\varepsilon, z_{,j} \rangle dt - \int_0^T \langle u_{i,j}^\varepsilon, \partial_t (m_{ij} z) \rangle dt + \langle m_{ij} u_{i,j}^\varepsilon(T), z(T) \rangle - \langle m_{ij} u_{oi,j}^\varepsilon, z(0) \rangle \\ = 0.$$

Using (2.60)–(2.62) we conclude from (2.67), as $\varepsilon \downarrow 0$, that (u, θ) satisfy (2.5).

It remains to justify (2.3) and (2.4). Notice that u satisfies (2.3) because

$$\|(u_\nu - g)^+\|_{L^2(\Gamma_c)} = \lim_{\varepsilon \downarrow 0} \|(u_\nu^\varepsilon - g)^+\|_{L^2(\Gamma_c)} = 0$$

by (2.63), (2.58).

We claim that the following inequality holds:

$$(2.68) \quad \int_0^T \langle u_t, u_t \rangle - \langle C_{ijkl} u_{k,l}, u_{i,j} \rangle dt \geq \limsup_{\varepsilon \downarrow 0} \int_0^T \langle u_t^\varepsilon, u_t^\varepsilon \rangle - \langle C_{ijkl} u_{k,l}^\varepsilon, u_{i,j}^\varepsilon \rangle dt.$$

The convergence

$$(2.69) \quad |u_t^\varepsilon|^2 - C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon \rightarrow |u_t|^2 - C_{ijkl} u_{k,l} u_{i,j}$$

in $\mathcal{D}'((0, T) \times \Omega)$, i.e., in the sense of distributions, follows from the theory of compensated compactness (see Corollary 4.3 of [2]); take

$$v^i := (\partial_t u_i^\varepsilon, -(C_{ijkl} u_{k,l}^\varepsilon)_{j=1, \dots, n}), \quad w^i := (\partial_t u_i^\varepsilon, (u_{i,j}^\varepsilon)_{j=1, \dots, n})$$

and summation over $i = 1, \dots, n$, and observe

$$-(m_{ij} \theta^\varepsilon)_{,j} \in L^2(L^2).$$

The differential equations (2.6), (2.7) for $(u^\varepsilon, \theta^\varepsilon)$ are satisfied in the distributional sense. Since

$$\theta_t^\varepsilon \in L^\infty(L^2) \text{ and } \partial_t u_{i,j}^\varepsilon \in L^\infty(L^2) \quad \Rightarrow \quad (k_{ij} \theta^\varepsilon)_{,j} \in L^\infty(L^2),$$

then

$$(2.70) \quad \theta^\varepsilon \in L^\infty(H^2 \cap H_0^1).$$

Moreover,

$$\partial_t^2 u^\varepsilon \in L^\infty(L^2)$$

and (2.69) imply

$$(C_{ijkl} u_{k,l}^\varepsilon)_{,j} \in L^\infty(L^2),$$

from which we conclude interior regularity in the following sense:

$$(2.71) \quad \forall \delta > 0 : \quad u^\varepsilon \in L^\infty((0, T), H^2(\Omega_\delta)),$$

where

$$\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}, \quad 0 < \delta \leq \delta_0$$

(δ_0 fixed).

Remark. The H^2 -regularity up to $\partial\Omega$ cannot be expected because of the mixed boundary conditions for u^ε .

Let Ω_δ as above, $S_\delta := \partial\Omega_\delta$, and let $h \in (C^1(\overline{\Omega}))^n$ such that

$$h(x) = \begin{cases} \nu(x) = \text{exterior normal in } x \in S_\delta & \text{if } \text{dist}(x, \partial\Omega) = \delta \leq \delta_0/2, \\ 0 & \text{if } \text{dist}(x, \partial\Omega) \geq \delta_0. \end{cases}$$

Writing (u, θ) instead of $(u^\varepsilon, \theta^\varepsilon)$ for simplicity, we multiply the first differential equation (2.6) by $h_k u_{i,k}$ and integrate

$$(2.72) \quad \int_0^T \int_{\Omega_\delta} \partial_t^2 u_i h_k u_{i,k} dx dt = \int_{\Omega_\delta} \partial_t u_i(T) h_k u_{i,k}(T) dx - \int_{\Omega_\delta} u_{1i} h_k u_{0i,k} dx - \frac{1}{2} \int_0^T \int_{S_\delta} |\partial_t u|^2 d\Gamma dt + \frac{1}{2} \int_0^T \int_{\Omega_\delta} h_{k,k} |\partial_t u|^2 dx dt.$$

With

$$(2.73) \quad \tilde{\sigma}_{ij} := \sigma_{ij} + m_{ij}\theta$$

we obtain

$$(2.74) \quad \begin{aligned} - \int_0^T \int_{\Omega_\delta} \sigma_{ij,j} h_k u_{i,k} dx dt &= - \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_k u_{i,k} dx dt - \int_0^T \int_{\Omega_\delta} (m_{ij}\theta)_{,j} h_k u_{i,k} dx dt \\ &= - \int_0^T \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_k u_{i,k} d\Gamma dt + \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{k,j} u_{i,k} dx dt \\ &\quad + \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_k (u_{i,k})_{,j} dx dt - \int_0^T \int_{\Omega_\delta} (m_{ij}\theta)_{,j} h_k u_{i,k} dx dt. \end{aligned}$$

Since

$$\begin{aligned} \tilde{\sigma}_{ij} h_k (u_{i,k})_{,j} &= \tilde{\sigma}_{ij} h_k (u_{i,j})_{,k} = C_{rsij} u_{i,j} h_k (u_{r,s})_{,k} \\ &= (C_{ijrs} u_{r,s})_{,k} h_k u_{i,j} - (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j} \\ &= (\tilde{\sigma}_{ij})_{,k} h_k u_{i,j} - (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j}, \end{aligned}$$

we have

$$\tilde{\sigma}_{ij} h_k (u_{i,k})_{,j} = \frac{1}{2} h_k (\tilde{\sigma}_{ij})_{,k} - \frac{1}{2} (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j}.$$

Inserting this into (2.74), we conclude

$$(2.75) \quad \begin{aligned} - \int_0^T \int_{\Omega_\delta} \sigma_{ij,j} h_k u_{i,k} dx dt &= - \int_0^T \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_k u_{i,k} d\Gamma dt + \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{k,j} u_{i,k} dx dt \\ &\quad + \frac{1}{2} \int_0^T \int_{S_\delta} \tilde{\sigma}_{ij} u_{i,j} d\Gamma dt - \frac{1}{2} \int_0^T \int_{\Omega_\delta} (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j} - \int_0^T \int_{\Omega_\delta} (m_{ij}\theta)_{,j} h_k u_{i,k} dx dt. \end{aligned}$$

The equations (2.6), (2.72), (2.75) imply

$$(2.76) \quad \begin{aligned} &\int_0^T \int_{S_\delta} |u_t|^2 d\Gamma dt - \int_0^T \int_{S_\delta} C_{ijkl} u_{k,l} u_{i,j} d\Gamma dt \\ &= 2 \int_{\Omega_\delta} \{ \partial_t u_i(T) h_k u_{i,k}(T) - u_{1i} h_k u_{0i,k} \} dx \\ &\quad + 2 \int_0^T \int_{\Omega_\delta} \left\{ h_{k,k} |u_t|^2 + \tilde{\sigma}_{ij} h_{k,j} u_{i,j} - \frac{1}{2} (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j} - (m_{ij}\theta)_{,j} h_k u_{i,k} \right\} dx dt \\ &\quad - 2 \int_0^T \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_k u_{i,k} d\Gamma dt \\ &= C_\delta^1 - 2 \underbrace{\int_0^T \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_k u_{i,k} d\Gamma dt}_{=: R_\delta}, \end{aligned}$$

for some C_δ^1 which satisfies

$$(2.77) \quad \exists M = M(\|u_0\|_{H^1}, \|u_1\|_{L^2}, \|\theta_0\|_{L^2}) \quad \forall 0 < \delta \leq \delta_0 : |C_\delta^1| \leq M$$

according to (2.55)–(2.57). In particular, M is independent of ε . In order to get control of the boundary integral R_δ , we multiply (2.6) by $h_m \tilde{\sigma}_{im}$ and obtain (cf. Lemma 2.3 in [6])

$$\begin{aligned}
 \int_{\Omega_\delta} \partial_t^2 u_i h_m \tilde{\sigma}_{im} dx &= \frac{d}{dt} \int_{\Omega_\delta} \partial_t u_i h_m \tilde{\sigma}_{im} dx - \int_{\Omega_\delta} h_m C_{imps} \partial_t u_i \partial_s \partial_t u_p dx \\
 &= \frac{d}{dt} \int_{\Omega_\delta} \partial_t u_i h_m \tilde{\sigma}_{im} dx - \frac{1}{2} \int_{\Omega_\delta} h_m C_{imps} \partial_s (\partial_t u_i \partial_t u_p) dx \\
 (2.78) \quad &+ \frac{1}{2} \int_{\Omega_\delta} h_m C_{imps} [\partial_s \partial_t u_i \partial_t u_p - \partial_t u_i \partial_s \partial_t u_p] dx.
 \end{aligned}$$

Now we shall use—for the first time—the radial symmetry, i.e.,

$$u_i(t, x) = x_i w(t, r), \quad r = |x|,$$

for some function w , in particular, $\text{rot } u = 0$, which implies

$$\begin{aligned}
 [\partial_s \partial_t u_i \partial_t u_p - \partial_t u_i \partial_s \partial_t u_p] &= (\partial_t u_p \partial_i - \partial_t u_i \partial_p) \partial_t u_s \\
 (2.79) \quad &= \partial_t w (x_p \partial_i - x_i \partial_p) x_s \partial_t w \equiv G,
 \end{aligned}$$

and G contains, at most, first-order derivatives (w_t) of u because

$$(x_p \partial_i - x_i \partial_p) \partial_t w = \left(\frac{x_p x_i}{r} - \frac{x_i x_p}{r} \right) (\partial / \partial r) \partial_t w = 0.$$

Combining (2.78) and (2.79) we get

$$\begin{aligned}
 \int_{\Omega_\delta} \partial_t^2 u_i h_m \tilde{\sigma}_{im} dx &= \frac{d}{dt} \int_{\Omega_\delta} \partial_t u_i h_m \tilde{\sigma}_{im} dx - \frac{1}{2} \int_{S_\delta} C_{imps} \nu_m \partial_t u_i \nu_s \partial_t u_p d\Gamma \\
 (2.80) \quad &+ \frac{1}{2} \int_{\Omega_\delta} (h_m C_{imps})_{,s} \partial_t u_i \partial_t u_p dx + \frac{1}{2} \int_{\Omega_\delta} h_m C_{imps} G dx.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 - \int_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_m \tilde{\sigma}_{im} dx &= - \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_m \tilde{\sigma}_{im} d\Gamma + \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j} \tilde{\sigma}_{im} dx + \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_m \tilde{\sigma}_{im,j} dx \\
 &= - \int_{S_\delta} |\tilde{\sigma} \cdot \nu|^2 d\Gamma + \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j} \tilde{\sigma}_{im} dx + \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_j \tilde{\sigma}_{im,m} dx \\
 &+ \int_{\Omega_\delta} \tilde{\sigma}_{ij} \{h_m \partial_j - h_j \partial_m\} \tilde{\sigma}_{im,m} dx.
 \end{aligned}$$

Observing that for the radial symmetrical case

$$h_k = x_k / r,$$

the last integral vanishes, and we obtain

$$(2.81) \quad - \int_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_m \tilde{\sigma}_{im} dx = - \frac{1}{2} \int_{S_\delta} |\tilde{\sigma} \cdot \nu|^2 d\Gamma + \frac{1}{2} \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j} \tilde{\sigma}_{im} dx.$$

Combining (2.6), (2.80), (2.81), and integrating with respect to t , yields

$$(2.82) \quad \int_0^T \int_{S_\delta} |\tilde{\sigma} \cdot \nu|^2 d\Gamma dt + \int_0^T \int_{S_\delta} \underbrace{C_{imps} \nu_m \partial_t u_i \nu_s \partial_t u_p}_{\geq 0 \text{ by (1.7)}} d\Gamma dt = C_\delta^2,$$

where C_δ^2 satisfies the same estimate as C_δ^1 in (2.77).

The boundary integral R_δ can now be estimated by

$$(2.83) \quad \begin{aligned} |R_\delta| &\leq \beta^{-1} \int_0^T \int_{S_\delta} |\tilde{\sigma} \cdot \nu|^2 d\Gamma dt + \beta \int_0^T \int_{S_\delta} |\nabla u|^2 d\Gamma dt \\ &\leq \beta^{-1} |C_\delta^2| + \beta \int_0^T \int_{S_\delta} |\nabla u|^2 d\Gamma dt, \end{aligned}$$

where $\beta > 0$ is still arbitrary.

For $\delta > 0$ let $\varrho_\delta \in C^\infty(\mathbb{R})$, ϱ_δ increasing, $\varrho_\delta(s) = 0$ if $s \leq \delta/2$, $\varrho_\delta(s) = 1$ if $s \geq \delta$, and let

$$\psi_\delta(x) := \varrho_\delta(\text{dist}(x, \partial\Omega)).$$

Then ψ_δ is constant on S_δ for each δ . Integrating (2.76) with respect to $\delta \in [0, \delta^*]$, $0 < \delta^* \leq \delta_0/2$, after multiplication with $(1 - \psi_\delta(x))$, and using (2.83), we obtain

$$\left| \int_0^T \int_{U_{\delta^*}} (1 - \psi_\delta) (|u_t^\varepsilon|^2 - C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon) dx dt \right| \leq \delta^* M + \beta^{-1} \delta^* M + \beta M,$$

where

$$U_{\delta^*} := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \delta^*\} = \Omega \setminus \overline{\Omega}_{\delta^*}.$$

As a consequence, we have for any $\tau > 0$ the existence of a $\bar{\delta} = \bar{\delta}(\tau)$ (independent of ε) such that

$$(2.84) \quad \left| \int_0^T \int_{U_{\delta^*}} (1 - \psi_\delta) (|u_t^\varepsilon|^2 - C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon) dx dt \right| \leq \tau$$

if $\delta^* \leq \bar{\delta}(\tau)$. (Choose $\beta = \beta(\tau) := \tau/(3M)$,

$$\bar{\delta} := \tau/(3M) \min\{1, \beta\} = \tau/(3M) \min\{1, \tau/(3M)\}.)$$

Writing $1 = \psi_\delta + (1 - \psi_\delta)$ we conclude by (2.69) and (2.84) the claimed inequality (2.68).

Now, the desired inequality (2.4) follows as $\varepsilon \downarrow 0$ from (2.66), using (2.60)–(2.63) and (2.68). \square

Finally, we shall prove the exponential decay for the radially symmetrical, homogeneous case. We consider the system (2.48), (2.49), now defined over the set

$$\Omega := B(r_1, 0) \setminus \overline{B(r_0, 0)},$$

where we are denoting by $B(r, 0)$ the ball of radius r with center in 0; i.e., Ω is an annular region. First we treat the penalized problem, and we assume the following boundary conditions:

$$(2.85) \quad u = 0, \quad \frac{\partial \theta}{\partial \nu} = 0 \quad \text{on} \quad \partial B(r_0, 0) = \Gamma_D$$

and the contact conditions on $\partial B(r_1, 0) = \Gamma_c$, which in our case can be written as

$$(2.86) \quad 2\mu \frac{\partial u}{\partial \nu} \cdot \nu + \lambda \operatorname{div} u = -\frac{1}{\varepsilon}(u_\nu - g)^+ \quad \text{on } \Gamma_c,$$

$$(2.87) \quad \theta = 0 \quad \text{on } \Gamma_c.$$

Observe that the boundary condition for θ in (2.85) is different from the Dirichlet boundary condition studied before, but the corresponding existence theorems, both for the penalized and for the original contact problem, hold as well; the proof carries over almost literally, noticing, for example, that the Dirichlet boundary condition is still given on Γ_c .

For simplicity in our notations we will drop the upper index ε . Since u, θ are radially symmetrical solutions, we have

$$\begin{aligned} u_i(t, x) &= x_i w(t, r), \quad r = |x|, \\ \theta(t, x) &= \psi(t, r) \end{aligned}$$

for some w and ψ . Then the system (2.48), (2.49) is equivalent to

$$(2.88) \quad w_{tt} - (2\mu + \lambda)w_{rr} - (2\mu + \lambda)\frac{n+1}{r}w_r + \frac{\bar{m}}{r}\theta_r = 0,$$

$$(2.89) \quad c\theta_t - \lambda_2\theta_{rr} - \lambda_2(n-1)\frac{\theta_r}{r} + \bar{m}nw_t + \bar{m}rw_{rt} = 0,$$

where w and θ satisfy

$$w(r_0) = 0, \quad \theta_r(r_0) = 0$$

among other boundary conditions. Note that there exist positive constants C_0, C_1 (which do not depend on ε) satisfying

$$\begin{aligned} C_0 \int_{\Omega} |\nabla u|^2 dx &\leq \int_{r_0}^1 |w_r|^2 dr \leq C_1 \int_{\Omega} |\nabla u|^2 dx, \\ C_0 \int_{\Omega} |u_t|^2 dx &\leq \int_{r_0}^1 |w_t|^2 dr \leq C_1 \int_{\Omega} |u_t|^2 dx. \end{aligned}$$

To show the exponential decay we shall exploit the following two lemmas.

LEMMA 2.5. *Let q be any $C^2([r_0, 1])$ -function such that $q(r_0) = 0$ and $q(1) = 1$, let $\alpha > 0$, and let f be a function in $W^{1,2}(L^2)$. Then for any solution $\varphi \in L^2(H^1) \cap W^{1,2}(H^1) \cap W^{2,2}(L^2)$ of*

$$(2.90) \quad \varphi_{tt} - \alpha\varphi_{rr} = f,$$

we have that

$$\begin{aligned} -\frac{d}{dt} \left\{ \int_{r_0}^1 q(x)\varphi_t\varphi_r dr \right\} &= -\frac{1}{2} \{ |\varphi_t(1, t)|^2 + \alpha|\varphi_r(1, t)|^2 \} \\ &\quad + \frac{1}{2} \int_{r_0}^1 q'(r) \{ |\varphi_t|^2 + \alpha|\varphi_r|^2 \} dr - \int_{r_0}^1 q(r)\varphi_r f dr. \end{aligned}$$

Proof. To get the above equality, multiply (2.90) by $q\varphi_r$ and make an integration by parts. \square

Let us introduce the function ψ by

$$\psi(t, r) := \int_{r_0}^r \theta(t, \rho) \, d\rho.$$

Then ψ satisfies the following equation:

$$(2.91) \quad c\psi_t - \lambda_2\psi_{rr} - \lambda_2(n-1) \int_{r_0}^r \frac{\theta_r}{r} \, d\rho + \bar{m}n \int_{r_0}^r w_t \, d\rho - \bar{m} \int_{r_0}^r w_t \, d\rho + \bar{m}rw_t = 0.$$

With these notations we have the following.

LEMMA 2.6. *Let q be any $C^2([r_0, 1])$ -function such that $q(r_0) = 0$ and $q(1) = 1$. Then, for $\delta > 0$ small enough, we have*

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{r_0}^1 \psi w_t \, dr - \delta \int_{r_0}^1 q(r) w_t w_r \, dr \right\} \\ & \leq -\frac{\bar{m}}{4} \int_{r_0}^1 |w_t|^2 \, dr - \frac{\delta}{4} \{ |w_t(1, t)|^2 + (2\mu + \lambda) |w_r(1, t)|^2 \} + \delta \int_{r_0}^1 |w_r|^2 \, dr \\ & + C \int_{r_0}^1 |\theta_x|^2 \, dr. \end{aligned}$$

Proof. Multiplying (2.91) by w_t we get

$$\begin{aligned} \frac{d}{dt} \int_{r_0}^1 c\psi w_t \, dr &= \int_{r_0}^1 c\psi_t w_t \, dr + \int_{r_0}^1 c\psi w_{tt} \, dr \\ &= \lambda_2 \int_{r_0}^1 \psi_{rr} w_t \, dr + (n-1)\lambda_2 \int_{r_0}^1 \left(\int_{r_0}^r \frac{\theta_r}{r} \, d\rho \right) w_t \, dr \\ & \quad - \bar{m}(n-1) \int_{r_0}^1 \left(\int_{r_0}^r w_t \, d\rho \right) w_t \, dr - \bar{m} \int_{r_0}^1 r |w_t|^2 \, dr + \int_{r_0}^1 c\psi w_{tt} \, dr \\ &= \lambda_2 \int_{r_0}^1 \theta_r w_t \, dr + (n-1)\lambda_2 \int_{r_0}^1 \int_{r_0}^r \frac{\theta_r}{r} \, d\rho w_t \, dr \\ & \quad - \frac{\bar{m}(n-1)}{2} \left(\int_{r_0}^1 w_t \, d\rho \right)^2 - \bar{m} \int_{r_0}^1 r |w_t|^2 \, dr - (2\mu + \lambda)c \int_{r_0}^1 \theta w_r \, dr \\ & \quad + (2\mu + \lambda)(n+1) \int_{r_0}^1 \psi \frac{w_r}{r} \, d\rho - \bar{m}c \int_{r_0}^1 \psi \frac{\theta_r}{r} \, dr \\ & \quad + c(2\mu + \lambda)\psi(1, t)w_r(1, t). \end{aligned}$$

Using Sobolev’s embedding theorem, we conclude from the above identity

$$\frac{d}{dt} \int_{r_0}^1 \psi w_t \, dr \leq -\frac{\bar{m}}{2} \int_{r_0}^1 r |w_t|^2 \, d\rho + \delta \int_{r_0}^1 |w_r|^2 \, dr + C_\delta \int_{r_0}^1 |\theta_r|^2 \, dr + \frac{\delta}{4} (2\mu + \lambda) |w_r(1, t)|^2.$$

Since $w \in L^2(H^2) \cap W^{1,2}(H^1)$, $\theta \in W^{1,2}(H^1)$ we can use Lemma 2.5 for $\varphi = w$, $\alpha = 2\mu + \lambda$, and $f = -(2\mu + \lambda)n\frac{w_r}{r} + \bar{m}\frac{\theta_r}{r} \in W^{1,2}(L^2)$. Hence, this proves our conclusion. \square

Now we are able to prove the exponential decay for the penalized problem associated to the dynamical contact case.

THEOREM 2.7. *Under the above conditions, and with u_0 such that $u_0 \cdot \nu \leq g$, the energy $\mathcal{E}^\varepsilon = \mathcal{E}^\varepsilon(t)$ to the penalized system (2.48), (2.49) defined by*

$$\mathcal{E}^\varepsilon(t) := \frac{1}{2} \int_{\Omega} |u_t^\varepsilon|^2 + C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 dx + \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma_c$$

decays exponentially as time goes to infinity, i.e.,

$$(2.92) \quad \exists C > 0, \gamma > 0 \forall t \geq 0 : \mathcal{E}^\varepsilon(t) \leq C e^{-\gamma t},$$

where γ and C are independent of ε .

Proof. Multiplying as usual (2.48) by u_t and (2.49) by θ we get, dropping ε for simplicity,

$$\frac{d}{dt} \mathcal{E}(t) = - \int_{\Omega} |\nabla \theta|^2 dx.$$

Multiplying (2.48) by u and integrating by parts over Ω we get

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} uu_t dx \right\} &= \int_{\Omega} |u_t|^2 dx - \int_{\Omega} \mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2 dx \\ &\quad + \bar{m} \int_{\Omega} \nabla \theta u dx - \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_\nu - g)^+|^2 d\Gamma_c, \end{aligned}$$

from where it follows that

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} uu_t dx \right\} &\leq \int_{\Omega} |u_t|^2 dx - \frac{1}{2} \int_{\Omega} C_{ijkl} u_{k,l} u_{i,j} dx \\ &\quad - \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_\nu - g)^+|^2 d\Gamma_c + C_1 \int_{\Omega} |\nabla \theta|^2 dx, \end{aligned}$$

where C_1 , and C_2, C_3 below, are positive constants. Using Lemma 2.6 and choosing η and δ small enough we get

$$\begin{aligned} &\frac{d}{dt} \left\{ \int_{r_0}^1 \psi w_t dr - \delta \int_{r_0}^1 q(x) w_t w_r dr + \eta \int_{\Omega} uu_t dx \right\} \\ &\leq -\kappa_0 \left\{ \int_{\Omega} |u_t|^2 + |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_\nu - g)^+|^2 d\Gamma_c \right\} + C \int_{\Omega} |\nabla \theta|^2 dx. \end{aligned}$$

So taking N large enough we obtain

$$\begin{aligned} &\frac{d}{dt} \left\{ N\mathcal{E}(t) + \int_{r_0}^1 \psi w_t dr - \delta \int_{r_0}^1 q(x) w_t w_r dr + \eta \int_{\Omega} uu_t dx \right\} \\ &\leq -\kappa_0 \left\{ \int_{\Omega} |u_t|^2 + |\nabla u|^2 + |\nabla \theta|^2 dx + \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_\nu - g)^+|^2 d\Gamma_c \right\} \end{aligned}$$

for some $\kappa_0 > 0$ being independent of ε . Since we also have the relation

$$\left| \int_{r_0}^1 \psi w_t dr - \delta \int_{r_0}^1 q(x) w_t w_r dr + \eta \int_{\Omega} uu_t dx \right| \leq C_2 \mathcal{E}(t),$$

we conclude, taking N large enough, that

$$\mathcal{L}(t) := N\mathcal{E}(t) + \int_{r_0}^1 \psi w_t dr - \delta \int_{r_0}^1 q(x) w_t w_r dr + \eta \int_{\Omega} uu_t dx$$

satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leq -\gamma\mathcal{L}(t) \quad \Rightarrow \quad \mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma t}$$

for some positive constant γ which is independent of ε , and hence

$$\mathcal{E}(t) \leq C_3\mathcal{E}(0)e^{-\gamma t}.$$

By our choice of the initial data, the right-hand side of the above inequality is bounded, which proves the theorem. \square

Using the lower semicontinuity of norms, when $\varepsilon \rightarrow 0$ we finally obtain that the energy associated with the original contact problem also decays exponentially; that is, we have proved

THEOREM 2.8. *The energy $\mathcal{E} = \mathcal{E}(t)$ to the system (1.1), (1.2), (1.3), (1.5), (2.85), (2.86),*

$$\mathcal{E}(t) := \frac{1}{2} \int_{\Omega} |u_t|^2 + C_{ijkl}u_{k,l}u_{i,j} + |\theta|^2 \, dx,$$

decays exponentially as time goes to infinity; i.e.,

$$(2.93) \quad \exists C > 0, \gamma > 0, \forall t \geq 0 : \mathcal{E}(t) \leq C\mathcal{E}(0)e^{-\gamma t}.$$

3. Linear quasi-static thermoelasticity. Replacing the boundary conditions for u by Dirichlet type boundary conditions, we consider the following linear elliptic-parabolic initial boundary value problem:

$$(3.1) \quad -(C_{ijkl}u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(3.2) \quad \partial_t\theta - (k_{ij}\theta_{,i})_{,j} + m_{ij}\partial_tu_{i,j} = 0,$$

$$(3.3) \quad \theta(t = 0) = \theta_0,$$

$$(3.4) \quad u_{/\partial\Omega} = 0, \quad \theta_{/\partial\Omega} = 0.$$

According to the paper of Shi and Xu [13], the known results on well-posedness are restricted to one-dimensional situations (see Day [3]) or to homogeneous and isotropic media for Ω being equal to the unit disk on \mathbb{R}^2 ; see [13], where methods from the theory of complex functions are basic ingredients of the proofs.

Here, we discuss the general case of a bounded C^2 -reference configuration Ω in any space dimension, allowing the medium to be anisotropic and nonhomogeneous (the C^2 -assumption on the boundary can be weakened). All coefficients are assumed to be sufficiently smooth functions.

THEOREM 3.1. (i) *Let $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists a unique solution (u, θ) of (3.1)–(3.4) with*

$$u \in C^0([0, \infty), (H^3(\Omega) \cap H_0^1(\Omega))^3),$$

$$\theta \in C^1([0, \infty), L^2(\Omega)) \cap C^0([0, \infty), H^2(\Omega) \cap H_0^1(\Omega)).$$

θ and u decay to zero exponentially as $t \rightarrow \infty$, i.e.,

$$\exists d_1, d_2 > 0 \quad \forall t \geq 0 : \quad \|\theta(t, \cdot)\| + \|u(t)\|_{H^1} \leq d_1 e^{-d_2 t} \|\theta_0\|.$$

(ii) If $\bar{m} := \|(m_{ij})_{ij}\|_{L^\infty}$ is sufficiently small and if $\partial\Omega$ is smooth, and additionally $\theta_0 \in H^{2(m-1)}(\Omega)$, $m \geq 2$, satisfies the usual compatibility conditions, then

$$u \in \bigcap_{j=0}^m C^j([0, \infty), (H^{2(m-j)+1}(\Omega))^3), \quad \theta \in \bigcap_{j=0}^m C^j([0, \infty), H^{2(m-j)}(\Omega)).$$

The higher derivatives decay to zero exponentially; i.e.,

$$\exists d_4 > 0 \quad \forall t \geq 0 : \quad \|\theta(t, \cdot)\|_{H^{2(m-1)}} + \|u(t)\|_{H^{2m-1}} \leq d_4 e^{-d_2 t} \|\theta_0\|_{H^{2(m-1)}}.$$

Proof of Theorem 3.1. The natural decoupling arising from (3.1), (3.2) is obtained using the elasticity operator

$$E = -((\partial_j C_{ijkl} \partial_l))_{ik},$$

which is well defined in $(L^2(\Omega))^n$ with domain

$$D(E) = (H^2(\Omega) \cap H_0^1(\Omega))^n$$

(cf. [9, 11]), and

$$E^{-1} : (L^2(\Omega))^n \rightarrow D(E)$$

is continuous, as well as

$$E^{-1} : H^{-1}(\Omega) \rightarrow (H_0^1(\Omega))^n,$$

where $H^{-1}(\Omega)$ is the dual space to $(H_0^1(\Omega))^n$. Introducing the operator G in $L^2(\Omega)$ by

$$Gv := v - M \nabla E^{-1} \nabla' M' v,$$

where $M = (m_{ij})_{ij}$, and $\mathcal{H} := L^2(\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\langle v, w \rangle_{\mathcal{H}} := \langle Gv, w \rangle,$$

we can rewrite the equation for θ as

$$(3.5) \quad \theta_t + A\theta = 0.$$

Here, $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$, $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$, $Av := -G^{-1}(k_{ij}v_{,i})_{,j}$.

A is self-adjoint and positive definite and has a compact inverse. If $d_1 = d_1(E^{-1}, (m_{ij}))$ denotes the square root of the norm of G in $L^2(\Omega)$ and d_2 denotes the smallest eigenvalue of A , we conclude

$$\|\theta(t, \cdot)\| \leq \|\theta(t, \cdot)\|_{\mathcal{H}} \leq e^{-d_2 t} \|\theta_0\|_{\mathcal{H}} \leq d_1 e^{-d_2 t} \|\theta_0\|.$$

This completes the proof of Theorem 3.1 (i). The proof of (ii) immediately follows from Lemma 3.2 by differentiating the equation for θ with respect to t . \square

LEMMA 3.2. Let $s \in \mathbb{N}_0$.

(i) $G : H^s(\Omega) \rightarrow H^s(\Omega)$ is continuous.

(ii) *A has the usual elliptic regularity property; i.e., $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $Av \in H^s(\Omega)$ imply $v \in H^{s+2}(\Omega)$ and $\|v\|_{H^{s+2}} \leq c_s \|Av\|_{H^s}$.*

(iii) *If \bar{m} ($= \|(m_{ij})_{ij}\|_{L^\infty}$) is small with respect to the norm of E^{-1} , then G is a homeomorphism of $H^s(\Omega)$.*

Proof. (i) Using the boundedness of $E^{-1} : H^{s-1}(\Omega) \rightarrow H^{s+1}(\Omega)$ (cf. [7]), the assertion is obvious.

(ii) is a consequence of (i) and the elliptic regularity of $\partial_j k_{ij} \partial_i$.

(iii) $G = (Id+B)$ with the bounded operator $B := M \nabla E^{-1} \nabla' M'$, and the inverse of G is given by a Neumann series if $\|B\|_{H^s \rightarrow H^s} < 1$; this is fulfilled if $\bar{m}^2 < c_s(E^{-1})$, where $c_s(E^{-1})$ is essentially the norm of $E^{-1} : H^{s-1}(\Omega) \rightarrow H^{s+1}(\Omega)$. \square

4. Quasi-static contact problems—existence and stability. We consider the following problem:

$$(4.1) \quad -(C_{ijkl}u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(4.2) \quad \partial_t \theta - (k_{ij}\theta_{,i})_{,j} + m_{ij} \partial_t u_{i,j} = 0,$$

$$(4.3) \quad \theta(t = 0) = \theta_0,$$

$$(4.4) \quad u|_{\Gamma_D} = 0, \quad \sigma_{ij} \nu_j|_{\Gamma_N} = 0, \quad \theta|_{\partial\Omega} = 0,$$

$$(4.5) \quad u_\nu \leq g, \quad \sigma_\nu \leq 0, \quad (u_\nu - g)\sigma_\nu = 0, \quad \sigma_T = 0 \quad \text{on } \Gamma_c.$$

Shi and Shillor [12] gave an existence proof, provided

$$(4.6) \quad \bar{m} := \sup_{x,i,j} |m_{ij}(x)| \quad \text{is small enough.}$$

Ames and Payne [1] proved a uniqueness and continuous dependence result.

We shall prove an existence result using a penalty method, which will also allow us to prove the exponential stability. The condition (4.6) will also be required.

DEFINITION 4.1. *(u, θ) is a solution to (4.1)–(4.5) for given $\theta_0 \in H_0^1(\Omega)$ if, for any $T > 0$,*

$$(4.7) \quad u \in L^2(H_{\Gamma_D}^1), \quad ((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n} \in L^2(L^2), \quad u_t \in L^2(H^1),$$

$$(4.8) \quad \theta \in L^2(H^2 \cap H_0^1) \cap C^0(H_0^1), \quad \theta_t \in L^2(L^2),$$

$$(4.9) \quad \theta(t = 0) = \theta_0,$$

$$(4.10) \quad u_\nu \leq g \quad \text{on } (0, T) \times \Gamma_c \quad \text{a.e.,}$$

and for all $w \in L^\infty(H_{\Gamma_D}^1)$ with $w_\nu \leq g$ on Γ_c the following inequality holds:

$$(4.11) \quad \int_0^T \langle C_{ijkl}u_{k,l}, w_{i,j} \rangle dt - \int_0^T \langle C_{ijkl}u_{k,l}, u_{i,j} \rangle dt - \int_0^T \langle m_{ij}\theta, w_{i,j} - u_{i,j} \rangle dt \geq 0;$$

and for all $z \in L^2(H_0^1)$ the following equality holds:

$$(4.12) \quad \int_0^T \langle \theta_t, z \rangle + \langle k_{ij} \psi_i, z_{,j} \rangle + \langle m_{ij} \partial_t u_{i,j}, z \rangle dt = 0.$$

A solution will be obtained by studying an associated penalized problem for the parameter $\varepsilon > 0$, then proving a priori estimates, and finally letting ε tend to zero. The advantage will be that finally the exponential stability can be proved. The penalized problem to be solved first is given as follows:

$$(4.13) \quad -(C_{ijkl} u_{k,l}^\varepsilon)_{,j} + (m_{ij} \theta^\varepsilon)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(4.14) \quad \partial_t \theta^\varepsilon - (k_{ij} \theta^\varepsilon)_{,j} + m_{ij} \partial_t u_{i,j}^\varepsilon = 0,$$

$$(4.15) \quad \theta^\varepsilon(t = 0) = \theta_0 \in H_0^1(\Omega),$$

$$(4.16) \quad u^\varepsilon \in L^2(H_{\Gamma_D}^1), \quad ((C_{ijkl} u_{k,l}^\varepsilon)_{,j})_{i=1,\dots,n} \in L^2(L^2), \quad u_t^\varepsilon \in L^2(H^1),$$

$$(4.17) \quad \theta^\varepsilon \in L^2(H^2 \cap H_0^1), \quad \theta_t^\varepsilon \in L^2(L^2),$$

$$(4.18) \quad u^\varepsilon|_{\Gamma_D} = 0, \quad \nu_j C_{ijkl} u_{k,l}^\varepsilon|_{\Gamma_N} = 0, \quad \sigma_{T|_{\Gamma_c}} = 0,$$

$$(4.19) \quad \sigma_\nu(u^\varepsilon)|_{\Gamma_c} = -\frac{1}{\varepsilon}(u_\nu^\varepsilon - g)^+|_{\Gamma_c}.$$

A solution of this penalized problem, for fixed $\varepsilon > 0$, will be obtained by a fixed point argument. For this purpose let $\Gamma > 0$ and

$$W^\Gamma := \{u \in L^2(H^1) \mid ((C_{ijkl} u_{k,l})_{,j})_{i=1,\dots,n} \in L^2(L^2), u_t \in L^2(H^1), \\ \|u_t\|_{L^2(H^1)} \leq \Gamma, \|((C_{ijkl} u_{k,l})_{,j})_{i=1,\dots,n}\|_{L^2(L^2)} \leq \Gamma\}.$$

LEMMA 4.2. W^Γ is a Banach space with norm $\|u\|_{W^\Gamma} := \|u\|_{L^2(H^1)}$.

Proof. This follows from the completeness of $L^2(H^1)$ with respect to the norm $\|\cdot\|_{W^\Gamma}$ and the weak compactness of balls in $L^2(L^2)$, respectively, $L^2(H^1)$. \square

For $v \in W^\Gamma$ we define

$$u = Sv$$

and hence the map S as follows. Let θ be the solution to the parabolic equation

$$(4.20) \quad \theta_t - (k_{ij} \theta)_{,j} = -m_{ij} \partial_t v_{i,j},$$

with initial value

$$(4.21) \quad \theta(t = 0) = \theta_0.$$

Then there exists a solution θ with the regularity

$$(4.22) \quad \theta \in L^2(H^2 \cap H_0^1) \cap C^0(H_0^1), \quad \theta_t \in L^2(L^2).$$

Now let u be the solution to the penalized problem

$$(4.23) \quad -(C_{ijkl}u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(4.24) \quad u|_{\Gamma_D} = 0, \quad \sigma_{ij}\nu_j|_{\Gamma_N} = 0,$$

$$(4.25) \quad \sigma_\nu(u)|_{\Gamma_c} = -\frac{1}{\varepsilon}(u_{\nu|_{\Gamma_c}} - g)^+, \quad \sigma_T = 0 \quad \text{on } \Gamma_c.$$

A solution u to (4.23)–(4.25) is obtained by minimizing

$$J(u) := \langle C_{ijkl}u_{k,l}, u_{i,j} \rangle - 2\langle (m_{ij}\theta)_{,j}, u_i \rangle + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu - g)^+|^2 d\Gamma$$

on

$$\{w \in (H^1(\Omega))^n \mid w = 0 \text{ on } \Gamma_D\},$$

where t is regarded as a parameter.

THEOREM 4.3. *If $\Gamma = \Gamma(\theta_0)$ is chosen large enough and if \bar{m} is small enough, where the required smallness is independent of ε , then*

$$S : W^\Gamma \rightarrow W^\Gamma$$

and S is a contraction mapping.

Proof. Let $v \in W^\Gamma$ and $u := Sv$. By the definition of u we immediately obtain

$$(4.26) \quad \|u\|_{L^2(H^1)} \leq c_1 \bar{m} \|\theta\|_{L^2(L^2)},$$

$$(4.27) \quad \begin{aligned} \|((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n}\|_{L^2(L^2)}^2 &\leq c_2^2 \bar{m}^2 \|\theta\|_{L^2(H^1)}^2 \\ &\leq c_2^2 \bar{m}^2 \|\theta_0\|^2 + c_3^2 \bar{m}^4 \|v_t\|_{L^2(L^2)}^2 \\ &\leq c_2^2 \bar{m}^2 \|\theta_0\|^2 + c_3^2 \bar{m}^4 \Gamma^2, \end{aligned}$$

where, in this proof, c_1, c_2, \dots will denote constants that neither depend on \bar{m} nor on ε . In order to see that $u \in W^\Gamma$, we have to estimate u_t . Let $h \neq 0$ and

$$\begin{aligned} v^h(t, \cdot) &:= (v(t+h, \cdot) - v(t, \cdot))/h, \quad u^h(t, \cdot) := (u(t+h, \cdot) - u(t, \cdot))/h, \\ \theta^h(t, \cdot) &:= (\theta(t+h, \cdot) - \theta(t, \cdot))/h. \end{aligned}$$

From the obvious differential equations for u^h and θ^h we obtain

$$(4.28) \quad \int_{\Omega} C_{ijkl}u_{k,l}^h u_{i,j}^h dx - \int_{\Gamma_c} \nu_j C_{ijkl}u_{k,l}^h u_i^h d\Gamma = \int_{\Omega} \theta^h m_{ij} u_{i,j}^h dx.$$

For the boundary term, we obtain, using the notation

$$u^1(t, \cdot) := u(t+h, \cdot), \quad u^2(t, \cdot) := u(t, \cdot),$$

$$\begin{aligned}
 & \int_{\Gamma_c} \nu_j C_{ijkl} u_{k,l}^h u_i^h d\Gamma = \int_{\Gamma_c} \sigma_\nu^1(u_\nu^1 - g) + \sigma_\nu^2(u_\nu^2 - g) d\Gamma \\
 & \quad - \int_{\Gamma_c} \sigma_\nu^1(u_\nu^2 - g) + \sigma_\nu^2(u_\nu^1 - g) d\Gamma \\
 & = -\frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^1 - g)^+|^2 + |(u_\nu^2 - g)^+|^2 d\Gamma + \frac{1}{\varepsilon} \int_{\Gamma_c} (u_\nu^1 - g)^+ (u_\nu^2 - g) \\
 & \quad + (u_\nu^2 - g)^+ (u_\nu^1 - g) d\Gamma \\
 (4.29) \quad & = \int_{\Gamma_c \cap \Gamma^+} \dots d\Gamma + \int_{\Gamma_c \setminus \Gamma^+} \dots d\Gamma,
 \end{aligned}$$

where

$$\begin{aligned}
 & \Gamma^+ := \{x \in \Gamma_c \mid u_\nu^1 > g \text{ and } u_\nu^2 > g\}. \\
 (4.30) \quad & \int_{\Gamma_c \cap \Gamma^+} \dots d\Gamma = -\frac{1}{\varepsilon} \int_{\Gamma^+} (u_\nu^1 - u_\nu^2)^2 d\Gamma \leq 0,
 \end{aligned}$$

$$\begin{aligned}
 & \int_{\Gamma_c \setminus \Gamma^+} \dots d\Gamma = -\frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} |(u_\nu^1 - g)^+|^2 + |(u_\nu^2 - g)^+|^2 d\Gamma \\
 & \quad + \frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} (u_\nu^1 - g)^+ (u_\nu^2 - g) + (u_\nu^2 - g)^+ (u_\nu^1 - g) d\Gamma \\
 (4.31) \quad & \leq -\frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} |(u_\nu^1 - g)^+|^2 + |(u_\nu^2 - g)^+|^2 d\Gamma \leq 0.
 \end{aligned}$$

The estimates (4.29)–(4.31) imply

$$(4.32) \quad \int_{\Gamma_c} \nu_j C_{ijkl} u_{k,l}^h u_i^h d\Gamma \leq 0.$$

Combining (4.28) and (4.32) we get the estimate

$$\begin{aligned}
 \|u^h\|_{L^2(H^1)}^2 & \leq c_4^2 \bar{m}^2 \|\theta^h\|_{L^2(L^2)}^2 \\
 & \leq c_4^2 \bar{m}^2 \|\theta_t\|_{L^2(L^2)}^2 \\
 (4.33) \quad & \leq c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \|v_t\|_{L^2(H^1)}^2,
 \end{aligned}$$

which implies $u_t \in L^2(H^1)$ and

$$\begin{aligned}
 \|u_t\|_{L^2(H^1)}^2 & \leq c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \|v_t\|_{L^2(H^1)}^2 \\
 (4.34) \quad & \leq c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \Gamma^2.
 \end{aligned}$$

Choosing \bar{m} and Γ such that

$$(4.35) \quad \bar{m}^2 (c_3^2 + c_5^2) \leq \frac{1}{2},$$

$$(4.36) \quad \Gamma^2 = \Gamma^2(\theta_0) = 2\|\theta_0\|^2 \max\{c_2^2, c_4^2\},$$

we conclude from (4.27) and (4.34) that S maps W^Γ into itself.

Now we prove the contraction property. For this purpose let

$$v^j \in W^\Gamma, \quad u^j := Sv^j, \quad j = 1, 2.$$

Let θ^j denote the solution to (4.20), (4.21) with respect to v^j . Replacing u^h by $u^{12} := u^1 - u^2$ and so on, we can derive analogous estimates to those given in (4.28)–(4.33) and conclude that

$$\begin{aligned} \|u^{12}\|_{L^2(H^1)}^2 &\leq c_4^2 \bar{m}^2 \|\theta^{12}\|_{L^2(L^2)}^2 \\ (4.37) \qquad \qquad \qquad &\leq c_5^2 \bar{m}^4 \|v^{12}\|_{L^2(H^1)}^2, \end{aligned}$$

which proves that S is a contraction if

$$(4.38) \qquad \qquad \qquad c_5 \bar{m}^2 < 1. \quad \square$$

The unique fixed point u of S , together with the associated θ , is then the desired solution to the penalized problem; i.e., we have proved the following.

THEOREM 4.4. *For given $\varepsilon > 0$ and $\theta_0 \in H_0^1(\Omega)$ there is a unique solution $(u^\varepsilon, \theta^\varepsilon)$ to the penalized problem (4.13)–(4.19), provided \bar{m} is small enough.*

From the estimates in the proof of Theorem 4.3, we conclude the validity of the following lemma.

LEMMA 4.5. *$(\theta^\varepsilon)_\varepsilon$ is bounded in $W^{1,2}(L^2) \cap L^2(H^2 \cap H_0^1) \cap C^0(H_0^1)$, $(u^\varepsilon)_\varepsilon$ is bounded in $W^{1,2}(H^1)$, and $((C_{ijkl}u_{k,l}^\varepsilon)_{,j})_{i=1,\dots,n}$ is bounded in $L^2(L^2)$.*

Now we can prove the existence of a solution to the quasi-static contact problem.

THEOREM 4.6. *For given $\theta_0 \in H_0^1(\Omega)$ there exists a solution (u, θ) to (4.1)–(4.5) in the sense of Definition 4.1, provided \bar{m} is small enough.*

Remark. The solution is unique, which follows from the result of Ames and Payne [1].

Proof. Let $(u^\varepsilon, \theta^\varepsilon)$ be the solution to the penalized problem according to Theorem 4.4. From Lemma 4.5 we conclude that there exists a subsequence, again denoted by $(u^\varepsilon, \theta^\varepsilon)$, and (u, θ) such that

$$(4.39) \qquad \theta^\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(H_0^1),$$

$$(4.40) \qquad \theta^\varepsilon \rightharpoonup \theta \quad \text{in } W^{1,2}(L^2) \cap L^2(H^2 \cap H_0^1),$$

$$(4.41) \qquad u^\varepsilon \rightharpoonup u \quad \text{in } W^{1,2}(H^1),$$

$$(4.42) \qquad ((C_{ijkl}u_{k,l}^\varepsilon)_{,j})_{i=1,\dots,n} \rightharpoonup ((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n} \quad \text{in } L^2(L^2).$$

It shall be proved that (u, θ) is a solution to (4.1)–(4.5). It remains only to justify the relations (4.11) and (4.12). Since $(u^\varepsilon, \theta^\varepsilon)$ satisfies (4.12), we can use (4.39)–(4.42) to also see that (u, θ) satisfies (4.12).

For $w \in L^\infty(H_{\Gamma_D}^1)$, $w_\nu \leq g$ on Γ_c , we have

$$\begin{aligned} &\int_0^T \langle C_{ijkl}u_{k,l}^\varepsilon, w_{i,j} - u_{i,j}^\varepsilon \rangle dt - \int_0^T \langle m_{ij}\psi_t^\varepsilon, w_{i,j} - u_{i,j}^\varepsilon \rangle dt \\ &= \int_0^T \int_{\Gamma_c} \sigma_\nu^\varepsilon (w_\nu - u_\nu^\varepsilon) d\Gamma dt \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (w_\nu - u_\nu^\varepsilon) d\Gamma dt \\
 &= -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (w_\nu - g) d\Gamma dt + \frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u_\nu^\varepsilon - g)^+ (u_\nu^\varepsilon - g) d\Gamma dt \\
 (4.43) \quad &\geq 0.
 \end{aligned}$$

Using the lower semicontinuity of norms and (4.39)–(4.42), we obtain

$$(4.44) \quad \int_0^T \langle -C_{ijkl} u_{k,l} + m_{ij} \theta, u_{i,j} \rangle dt \geq \limsup_{\varepsilon \downarrow 0} \int_0^T \langle -C_{ijkl} u_{k,l}^\varepsilon + m_{ij} \theta^\varepsilon, u_{i,j}^\varepsilon \rangle dt.$$

From (4.43) and (4.44), we conclude the validity of (4.11). \square

Finally, we want to describe the exponential stability.

Let $u^\varepsilon, \theta^\varepsilon$ denote the solution to the penalized problem as given in Theorem 4.4. Similarly as in [1] we change variables from θ^ε to ψ^ε , where

$$(4.45) \quad \psi^\varepsilon(t, x) := \int_0^t \theta^\varepsilon(s, x) ds + \psi_0^\varepsilon(x).$$

Here, $\psi_0^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ is defined as solution to

$$(4.46) \quad (k_{ij} \psi_{0,i})_{,j} = \theta_0 + m_{ij} u_i^\varepsilon(t=0)_{,j},$$

where $u_i^\varepsilon(t=0)$ is defined by θ_0 via (4.13). Then $(u^\varepsilon, \psi^\varepsilon)$ satisfies

$$(4.47) \quad \psi_t^\varepsilon - (k_{ij} \psi_{,i})_{,j} + m_{ij} u_{i,j}^\varepsilon = 0.$$

Multiplying (4.47) by ψ_t we obtain

$$(4.48) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega k_{ij} \psi_{,i}^\varepsilon \psi_{,j}^\varepsilon dx = - \int_\Omega |\psi_t^\varepsilon|^2 dx - \int_\Omega m_{ij} u_{i,j}^\varepsilon \psi_t^\varepsilon dx.$$

From (4.13) we obtain

$$\begin{aligned}
 &- \int_\Omega m_{ij} u_{i,j}^\varepsilon \psi_t^\varepsilon dx = - \int_\Omega C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx + \int_{\Gamma_c} \nu_j C_{ijkl} u_{k,l}^\varepsilon u_i^\varepsilon d\Gamma \\
 (4.49) \quad &= - \int_\Omega C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma.
 \end{aligned}$$

From (4.48), (4.49) we have

$$(4.50) \quad \frac{1}{2} \frac{d}{dt} \int_\Omega k_{ij} \psi_{,i}^\varepsilon \psi_{,j}^\varepsilon dx \leq - \int_\Omega |\psi_t^\varepsilon|^2 dx - \int_\Omega C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma.$$

On the other hand we obtain from (4.13), (4.14)

$$- \int_\Omega (C_{ijkl} u_{k,l}^\varepsilon)_{,j} \partial_t u_i^\varepsilon dx + \int_\Omega \theta_t^\varepsilon \theta^\varepsilon dx - \int_\Omega (k_{ij} \theta_{,i}^\varepsilon)_{,j} \theta^\varepsilon dx = 0,$$

which implies

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \left\{ \int_\Omega C_{ijkl} u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 dx \right\} = - \int_\Omega k_{,j} \theta_{,i}^\varepsilon \theta_{,j}^\varepsilon dx + \int_{\Gamma_c} \sigma_\nu^\varepsilon \partial_t (u_\nu^\varepsilon - g) d\Gamma \\
 &= - \int_\Omega k_{ij} \theta_{,i}^\varepsilon \theta_{,j}^\varepsilon dx - \frac{1}{\varepsilon} \int_{\Gamma_c} (u_\nu - g)^+ \partial_t (u_\nu - g) d\Gamma,
 \end{aligned}$$

where

$$(4.51) \quad \frac{1}{2} \frac{d}{dt} \left\{ \int_{\Omega} C_{ijkl} u_{k,l}^{\varepsilon} u_{i,j}^{\varepsilon} + |\theta^{\varepsilon}|^2 dx + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_{\nu}^{\varepsilon} - g)^+|^2 d\Gamma \right\} \leq - \int_{\Omega} k_{ij} \theta_{,i}^{\varepsilon} \theta_{,j}^{\varepsilon} dx.$$

Combining (4.50) and (4.51) we obtain

$$(4.52) \quad \begin{aligned} & \frac{d}{dt} \frac{1}{2} \left\{ \int_{\Omega} C_{ijkl} u_{k,l}^{\varepsilon} u_{i,j}^{\varepsilon} + |\theta^{\varepsilon}|^2 + k_{ij} \psi_{,i} \psi_{,j}^{\varepsilon} dx + \frac{1}{\varepsilon} \int_{\Gamma_c} |u_{\nu}^{\varepsilon} - g|^+|^2 d\Gamma \right\} \\ & \leq - \left\{ \int_{\Omega} C_{ijkl} u_{k,l}^{\varepsilon} u_{i,j}^{\varepsilon} + |\theta^{\varepsilon}|^2 + k_{i,j} \theta_{,i}^{\varepsilon} \theta_{,j}^{\varepsilon} dx + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_{\nu} - g)^+|^2 d\Gamma \right\}. \end{aligned}$$

Multiplying (4.14) by ψ^{ε} , we obtain

$$(4.53) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\psi^{\varepsilon}|^2 dx &= - \int_{\Omega} k_{ij} \psi_{,i}^{\varepsilon} \psi_{,j}^{\varepsilon} dx - \int_{\Omega} m_{ij} u_{i,j}^{\varepsilon} \psi^{\varepsilon} dx \\ &\leq - \frac{1}{2} \int_{\Omega} k_{ij} \psi_{,i}^{\varepsilon} \psi_{,j}^{\varepsilon} dx + c_1 \int_{\Omega} C_{ijkl} u_{k,l}^{\varepsilon} u_{i,j}^{\varepsilon} dx, \end{aligned}$$

where the constant $c_1 > 0$ is independent of ε .

Let

$$(4.54) \quad \begin{aligned} E_{\varepsilon}(t) &:= \frac{1}{2} \left\{ \int_{\Omega} C_{ijkl} u_{k,l}^{\varepsilon} u_{i,j}^{\varepsilon} + |\theta^{\varepsilon}|^2 + k_{ij} \psi_{,i}^{\varepsilon} \psi_{,j}^{\varepsilon} dx \right. \\ &\quad \left. + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_{\nu}^{\varepsilon} - g)^+|^2 d\Gamma + \frac{1}{2c_1} \int_{\Omega} |\psi^{\varepsilon}|^2 dx \right\}. \end{aligned}$$

Then the estimates (4.52) and (4.53), together with Poincaré’s estimate applied to ψ^{ε} , imply

$$(4.55) \quad \frac{d}{dt} E_{\varepsilon}(t) \leq -\beta E_{\varepsilon}(t), \quad t \geq 0,$$

for some constant $\beta > 0$ which is independent of ε . Hence we have proved the following.

THEOREM 4.7. *For fixed $\varepsilon > 0$ the “energy” E_{ε} , defined for solutions $(u^{\varepsilon}, \psi^{\varepsilon})$ to the penalized problem (4.13)–(4.19) in (4.54), decays to zero exponentially; i.e.,*

$$(4.56) \quad \exists \beta > 0 \quad \forall t \geq 0 \quad E_{\varepsilon}(t) \leq E_{\varepsilon}(0) e^{-\beta t},$$

and β is independent of ε .

If

$$E(u^{\varepsilon}, \psi^{\varepsilon})(t) := E_{\varepsilon}(t) - \frac{1}{2\varepsilon} \int_{\Gamma_c} |(u_{\nu}^{\varepsilon} - g)^+|^2 d\Gamma,$$

then obviously

$$E(u^{\varepsilon}, \psi^{\varepsilon})(t) \leq E_{\varepsilon}(0) e^{-\beta t}.$$

LEMMA 4.8.

(i)

$$(4.57) \quad \exists \alpha_1 > 0 \forall \varepsilon > 0 : E(u^{\varepsilon}, \psi^{\varepsilon})(0) \leq \alpha_1 \|\theta_0\|^2,$$

(ii)

$$(4.58) \quad \exists \alpha_2 > 0 \quad \forall \varepsilon > 0 \quad \forall t \geq 0 : \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon - g)^+|^2 d\Gamma \leq \alpha_2 \|\theta_0\|^2.$$

Proof.

$$-C_{ijkl} u_{k,l}^\varepsilon(0) = -(m_{,j} \theta_0)_{,j}, \quad i = 1, \dots, n.$$

This implies

$$\begin{aligned} \int_{\Omega} C_{ijkl} u_{k,l}^\varepsilon(0) u_{i,j}^\varepsilon(0) dx + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon(0) - g)^+|^2 d\Gamma &\leq - \int_{\Omega} m_{ij} \theta_0 u_{i,j}(0) \\ &\leq c_1 \|\theta_0\|^2 + \frac{1}{2} \int_{\Omega} C_{ijkl} u_{k,l}^\varepsilon(0) u_{i,j}^\varepsilon(0) dx, \end{aligned}$$

where $c_1 > 0$ is a constant. Therefore

$$(4.59) \quad \int_{\Omega} C_{ijkl} u_{k,l}^\varepsilon(0) u_{i,j}^\varepsilon(0) dx + \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon(0) - g)^+|^2 \leq 2c_1 \|\theta_0\|^2.$$

$$\begin{aligned} \int_{\Omega} |\psi^\varepsilon(0)|^2 dx + \int_{\Omega} k_{ij} \psi_{,i}^\varepsilon(0) \psi_{,j}^\varepsilon(0) dx &\leq c_1 \|\psi_0^\varepsilon\|_{H^1}^2 \\ &\leq c_1 (\|\theta_0\|^2 + \|m_{ij} u_{i,j}(0)\|^2) \quad (\text{by (4.46)}) \end{aligned}$$

$$(4.60) \quad \leq c_1 \|\theta_0\|^2, \quad (\text{by (4.59)}).$$

We use c_1 to denote various constants being independent of ε .

The estimates (4.59), (4.60) prove (4.57). From (4.51) and (4.59) we conclude for $t \geq 0$

$$(4.61) \quad \frac{1}{\varepsilon} \int_{\Gamma_c} |(u_\nu^\varepsilon(t) - g)|^2 d\Gamma \leq \alpha_2 \|\theta_0\|^2$$

for some $\alpha_2 > 0$. Hence (4.58) is proved. \square

COROLLARY 4.9. $\exists \beta_1, \beta_2 > 0 \quad \forall t \geq 0 \quad \forall \varepsilon > 0 : E(u^\varepsilon, \psi^\varepsilon)(t) \leq \beta_1 \|\theta_0\|^2 e^{-\beta_2 t}$.

Now let (u, ψ) be the solution to the quasi-static problem obtained from $(u^\varepsilon, \psi^\varepsilon)_\varepsilon$ as $\varepsilon \downarrow 0$ (Theorem 4.6). Let the “energy” of (u, ψ) be defined by

$$(4.62) \quad E(t) := \int_{\Omega} C_{ijkl} u_{k,l} u_{i,j} + |\psi_t|^2 + k_{ij} \psi_{,i} \psi_{,j} dx.$$

Then the exponential stability is expressed in the next theorem.

THEOREM 4.10.

$$(4.63) \quad \forall t \geq 0 : E(t) \leq \beta_1 \|\theta_0\|^2 e^{-\beta_2 t}$$

(β_1, β_2 from Corollary 4.9).

Proof. Corollary 4.9 and the weak lower semicontinuity of the norm yield the proof. \square

5. Smoothing in the interior for the quasi-static contact problem. The linear, quasi-static problem with Dirichlet boundary conditions has smoothing properties like those known for the solutions to heat equations, in particular, the solution $(u, \theta)(t, x)$ is infinitely smooth in t and x as soon as $t > 0$, no matter how smooth the initial data are (cf. the remarks in section 3). This behavior cannot be expected, in general, for the quasi-static contact problem because of the mixed boundary conditions for u . We shall prove that the solution (u, θ) to the quasi-static thermoelastic contact problem (4.1)–(4.5) is infinitely smooth with respect to x in the interior of Ω if $t > 0$.

Naturally, we assume in this section that all coefficients are C^∞ -smooth. We also assume that (u, θ) is a solution to (1.1)–(1.5) as given in the previous section. Let $\varphi \in C_0^\infty(\Omega)$ such that $\text{supp } \varphi \subset \Omega_{\delta_0/2}$ for some $\delta_0 > 0$, with $\Omega_{\delta_0/2}$ open such that

$$\Omega_{\delta_0/2} = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta_0/2\}.$$

Also let

$$\varphi = 1 \text{ in } \Omega_{3\delta_0/4}.$$

and

$$(\tilde{u}, \tilde{\theta}) := (\varphi u, \varphi \theta).$$

Then $(\tilde{u}, \tilde{\theta})$ satisfy

$$(5.1) \quad -(C_{ijkl}\tilde{u}_{k,l})_{,j} + (m_{ij}\tilde{\theta})_{,j} = R_i(u, \nabla u, \theta), \quad i = 1, \dots, n,$$

$$(5.2) \quad \tilde{\theta}_t - (k_{ij}\tilde{\theta}_{,i})_{,j} + m_{ij}\partial_t\tilde{u}_{i,j} = Q(u_t, \theta, \nabla\theta),$$

$$(5.3) \quad \tilde{\theta}(t = 0) = \tilde{\theta}_0 := \varphi\theta_0,$$

$$(5.4) \quad \text{supp } \tilde{u}, \text{supp } \tilde{\theta} \subset\subset \Omega.$$

Here R_i and Q are given by

$$(5.5) \quad R_i = -\varphi_{,l,j}C_{ijkl}u_k - \varphi_{,l}(C_{ijkl}u_k)_{,j} - \varphi_{,j}C_{ijkl}u_{k,l} + \varphi_{,j}m_{ij}\theta,$$

$$(5.6) \quad Q = -\varphi_{,i,j}k_{ij}\theta - \varphi_{,i}(k_{ij}\theta)_{,j} - \varphi_{,j}\theta_{,i} + \varphi_{,j}m_{ij}\partial_t u_i.$$

In particular,

$$\text{supp } R_i \cup \text{supp } Q \subset \text{supp } \nabla\varphi \subset\subset \Omega.$$

Let $r_1 > 0$ such that $\Omega \subset\subset B := B(0, r_1)$.

Let (w, ϑ) solve in $(0, \infty) \times B$:

$$(5.7) \quad -(C_{ijkl}w_{k,l})_{,j} + (m_{ij}\vartheta)_{,j} = 0, \quad i = 1, \dots, n,$$

$$(5.8) \quad \vartheta_t - (k_{ij}\vartheta_{,i})_{,j} + m_{ij}\partial_t w_{i,j} = 0,$$

$$(5.9) \quad \vartheta(t = 0) = \tilde{\theta}_0,$$

$$(5.10) \quad w|_{\partial B} = 0, \quad \vartheta|_{\partial B} = 0.$$

The linear, quasi-static problem (5.7)–(5.10) can be solved as shown in section 2 assuming

$$(5.11) \quad \theta_0 \in H^2(\Omega) \cap H_0^1(\Omega),$$

and (w, ϑ) is smooth in B as $t > 0$. Let (v, ϱ) be the solution in $(0, \infty) \times B$ to

$$(5.12) \quad -(C_{ijkl}v_{k,l})_{,j} + (m_{ij}\varrho)_{,j} = R_i, \quad i = 1, \dots, n,$$

$$(5.13) \quad \varrho_t - (k_{ij}\varrho_{,i})_{,j} + m_{ij}\partial_t v_{i,j} = Q,$$

$$(5.14) \quad \varrho(t = 0) = 0,$$

$$(5.15) \quad v|_{\partial B} = 0, \quad \varrho|_{\partial B} = 0.$$

The right-hand sides R_i, Q satisfy

$$(5.16) \quad R = (R_1, \dots, R_n) \in W^{1,2}(L^2), \quad Q \in L^2(H^1).$$

The system (5.12)–(5.15) can be transformed into a parabolic equation for ϱ , essentially, as in section 3:

$$(5.17) \quad v = -E^{-1}\nabla' M\theta + E^{-1}R, \quad v|_{\partial B} = 0,$$

where

$$M = (m_{ij})_{ij}, \quad E = (-\partial_j C_{ijkl}\partial_l)_i;$$

hence

$$(5.18) \quad \varrho_t + A\varrho = F,$$

$$(5.19) \quad \varrho(t = 0) = 0,$$

$$(5.20) \quad \varrho|_{\partial B} = 0,$$

where

$$(5.21) \quad A\varrho = -(\text{Id} - M\nabla E^{-1}\nabla' M')^{-1}(k_{ij}\varrho_{,i})_{,j},$$

$$(5.22) \quad F = Q - M\nabla\partial_t E^{-1}R.$$

The regularity of R and Q given in (5.16), as well as the mapping properties of the operator E (cf. section 3), imply

$$(5.23) \quad F \in L^2(H^1).$$

This regularity of F implies for the (existing) solution ϱ to (5.18)–(5.20)

$$(5.24) \quad \varrho \in L^2(H^3(\Omega_{\delta_0/2})), \quad \varrho_t \in L^2(H^1(\Omega_{\delta_0/2}))$$

by the usual regularity for parabolic equations (see, for example, p. 11 in [14]) applied to $\partial_j \varrho$, smoothly cut off in $\Omega_{\delta_0/2}$.

By (5.17) and (5.24) we conclude

$$(5.25) \quad v \in W^{1,2}(H^2(\Omega_{\delta_0/2})).$$

By uniqueness we have in $(0, \infty) \times B$

$$\tilde{u} = w + v, \quad \tilde{\theta} = \vartheta + \varrho.$$

Since (w, ϑ) is smooth for $t > 0$, we conclude from (5.24), (5.25), and

$$(5.26) \quad u = \tilde{u}, \quad \theta = \tilde{\theta} \text{ in } \Omega_{3\delta_0/4}$$

that, for any $\tau > 0$,

$$(5.27) \quad \begin{aligned} u &\in W^{1,2}((\tau, \infty), H^2(\Omega_{3\delta_0/4})), \quad \theta \in L^2((\tau, \infty), H^3(\Omega_{3\delta_0/4})), \\ \theta_t &\in L^2((\tau, \infty), H^1(\Omega_{3\delta_0/4})), \end{aligned}$$

i.e., more regularity for (u, θ) . This implies (cf. (5.16))

$$(5.28) \quad R \in W^{1,2}((\tau, \infty), H^1(\Omega_{3\delta_0/4})), \quad Q \in L^2((0, \infty), H^2(\Omega_{3\delta_0/4})),$$

i.e., higher regularity for F in (5.18).

Proceeding by induction ($\delta_0 > 0$ chosen appropriately) we have proved the following.

THEOREM 5.1. *Let $\theta_0 \in H_0^1(\Omega)$. Then the solution (u, θ) to (4.1)–(4.5) satisfies*

$$(5.29) \quad \forall \tau > 0 \forall \Omega' \subset\subset \Omega \forall m \in \mathbb{N} : u \in W^{1,2}((\tau, \infty), H^m(\Omega')), \theta \in W^{1,2}((\tau, \infty), H^m(\Omega')).$$

This interior smoothing effect in x carries over to smoothing in t , which can be seen by differentiating the equation for $(\tilde{u}, \tilde{\theta})$ with respect to t and applying the same arguments again. As a corollary we have the following.

COROLLARY 5.2. *Let $\theta_0, (u, \theta)$ be as in Theorem 5.1. Then*

$$(5.30) \quad \forall \tau > 0 \forall \Omega' \subset\subset \Omega : u \in C^\infty((\tau, \infty) \times \Omega'), \quad \theta \in C^\infty((\tau, \infty) \times \Omega).$$

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