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Multidimensional Filter Banks and Multiscale Geometric Representations

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Abstract

Thanks to the explosive growth of sensing devices and capabilities, multidimensional (MD) signals — such as images, videos, multispectral images, light fields, and biomedical data volumes — have become ubiquitous. Multidimensional filter banks and the associated constructions provide a unified framework and an efficient computational tool in the formation, representation, and processing of these multidimensional data sets. In this survey we aim to provide a systematic development of the theory and constructions of multidimensional filter banks. We thoroughly review several tools that have been shown to be particularly effective in the design and analysis of multidimensional filter banks, including sampling lattices, multidimensional bases and frames, polyphase representations, Gröbner bases, mapping methods, frequency domain constructions, ladder structures and lifting schemes. We then focus on the construction of filter banks and signal representations that can capture directional and geometric features, which are unique and key properties of many multidimensional signals. Next, we study the connection between iterated multidimensional filter banks in the discrete domain and the associated multiscale signal representations in the continuous domain through a directional multiresolution analysis framework. Finally, we show several examples to demonstrate the power of multidimensional filter banks and geometric signal representations in applications.

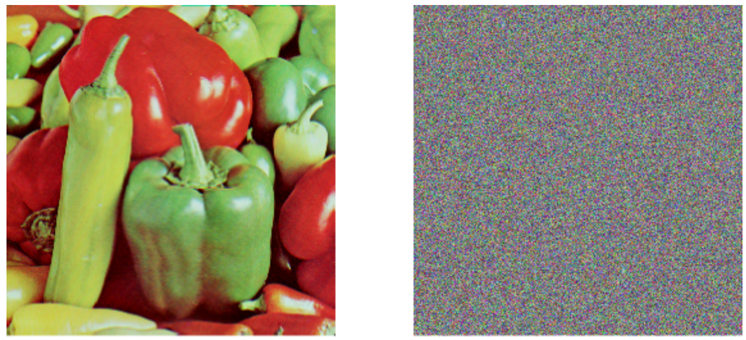
Contents

1

Introduction

Multidimensional (MD) signals are information-carrying physical quantities that depend on several variables, each representing a unique dimension. For example, a video is a three-dimensional (3D) signal with two spatial dimensions (horizontal and vertical) and one temporal dimension. A particularly important and common class of MD signals contains *visual information*, ranging from general images and videos on the Web to special medical images (such as MRI and CT scans) for diagnostics, and from very small scales (molecular images) to very large scales (astronomical images).

Efficient representation of visual information lies at the heart of many image processing tasks such as reconstruction, denoising, compression, and feature extraction. For example, a 512 by 512 color image can be considered as a vector in a $512 \times 512 \times 3$ dimensional space (each pixel is represented by a triplet of color components). However, as we can see in Figure 1.1, a randomly sampled image from this space is far from being a *natural image*. In other words, natural images occupy a very small faction of the huge space of all possible images. Effectively exploring this fact allows us to efficiently compress an image or to separate a clean image from noise.



(a) A natural image **(b) An arbitrary image**

Fig. 1.1 Example of a natural image (a) compared with an arbitrary image (b) that is sampled from the same image space.

As can be seen from Figure 1.1, a key distinguishing feature of natural images is that they have *intrinsic geometric structures*. In particular, visual information is mainly contained in the geometry of object boundaries or edges. For this reason, *wavelets and filter banks* [21, 65, 89, 95, 99] — a breakthrough resulting from the convergence of ideas from several fields — have been found to be particularly well-suited for representing images. In particular, wavelets are good at isolating the discontinuities at *edge points*. However, as a result of their construction by separable extension from 1D bases, wavelets in 2D cannot "see" the *smoothness along the contours*. In addition, separable wavelets can capture only limited *directional* information, which is an important and unique feature of MD signals.

To see how one can improve the 2D separable wavelet transform in representing images with smooth contours, consider the following scenario. Imagine that there are two painters, one with a wavelet-style and the other with a new style, both wishing to paint a natural scene. Both painters apply a refinement technique to increase the resolution from coarse to fine. We consider efficiency as measured by how quickly, that is with how few brush strokes, each painter can faithfully reproduce the scene. In other words, an *efficient* painting style is associated with a *sparse* image representation scheme.

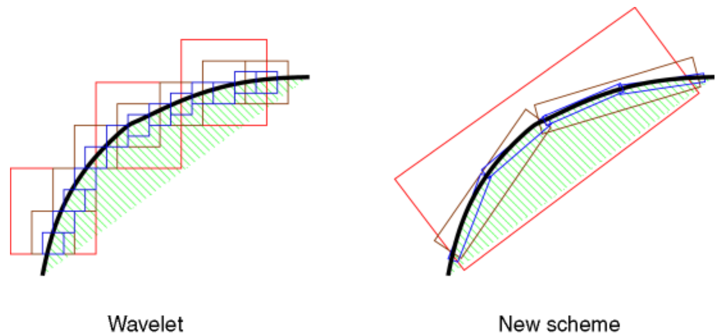


Fig. 1.2 Wavelet versus the new scheme: illustrations of different successive refinement styles by the two systems near a smooth contour, which is shown as a thick curve separating two smooth regions.

Consider the situation when a smooth contour is being painted, as shown in Figure 1.2. Because 2D wavelets are constructed from tensor products of 1D wavelets, the wavelet-style painter is limited to using square-shaped brush strokes along the contour, using different sizes corresponding to the multiresolution structure of wavelets. As the resolution becomes finer, we can clearly see the limitation of the wavelet-style painter who needs to use many fine "dots" to capture the contour.[1] The new style painter, on the other hand, effectively exploits the smoothness of the contour by making brush strokes with different elongated shapes and in a variety of directions following the contour. This intuition was first formalized by Candès and Donoho in the *curvelet* construction [7, 9]. We will also see later an actual realization of the new scheme with the *contourlet* transform in Figure 8.2.

For the human visual system, it is well-known [44] that the receptive fields in the visual cortex are characterized as being *localized*, *oriented*, and *bandpass*. Furthermore, computational experiments in searching for the sparse components of (both still and time-varying) natural images produced basis images that closely resemble the aforementioned characteristics of the visual cortex [72, 73]. These results support the hypothesis that the human visual system has been tuned so as to capture the

[1] Or we could consider the wavelet-style painter as a *pointillist*!

essential information of a natural scene using a least number of active visual cells. More importantly, the results suggest that, for a computational image representation to be efficient, it should be based on a *local*, *directional*, and *multiresolution* expansion.

Over the past decade, a number of concurrent studies in applied mathematics, computer vision, and statistical learning theory have independently developed theories and tools to explore and make use of the geometric structures in multidimensional data. In signal processing, the challenges as well as great research opportunities come from the discrete nature of the data, together with the issues of robustness, efficiency, and speed. For example, directions other than horizontal and vertical can look very different on a rectangular grid typically used to sample images. Because of pixelation, the notion of smooth contours on sampled images is not obvious. Moreover, for practical applications, efficient representation has to be obtained by structured transforms and fast algorithms.

Thus, we are particularly interested in a *discrete-space* framework for the construction of multiscale geometric transforms that can be applied to sampled images and MD signals. Following the success of wavelets and filter banks in 1D, we will focus on the constructions using multidimensional filter banks. However, as mentioned above, the commonly used wavelets and filter banks in MD are simply constructed from separable extensions of their 1D counterparts. Here, we want to exploit the full flexibility of true (non-separable) MD constructions in order to achieve the desired multiscale directional and geometric transforms and representations.

Toward this goal, we first provide a thorough review of the theory and design of multidimensional filter banks in this survey. While there are already several excellent papers and reviews on MD filter banks (see, for example, [14, 49, 57, 100]), our review emphasizes MD filter banks as basis and frame expansions for signal representations, in addition to the traditional view of achieving good frequency partitions. Moreover, we will highlight some modern and effective tools for designing MD filter banks such as Gröbner bases, mapping methods, frequency domain constructions, and ladder structures and lifting schemes. We believe that this MD filter bank review will be useful in its own right. Building upon this background, we then present constructions of iterated and directional filter banks leading to multiscale geometric representations for MD signals, both in discrete and continuous domains. The effectiveness of these constructions will be demonstrated through applications and numerical results.

The outline of this survey is as follows. In Section 2, we define our notation and study the first building block of multidimensional filter bank, namely, MD filtering. In Section 3, we study the other building block: MD sampling. The generalization of sampling from 1D to MD using lattices provides a rich set of new possibilities that will be exploited in later constructions of directional and geometric representations. Section 4 combines these two building blocks into a systematic study of MD filter banks. In particular, we focus on those filter banks that satisfy the perfect reconstruction condition, which lead to bases or frames for MD signal representations. Section 5 presents some of the most effective tools for characterizing and designing MD filter banks. In Section 6, we study the iterated and directional filter banks that are obtained by well-designed combinations of the building blocks for MD filter banks. Based on this directional construction, we present multiscale geometric transforms in Section 7. Moreover, we establish a precise connection between iterated MD filter banks in the discrete domain and the associated multiscale signal representations in the continuous domain through a directional multiresolution analysis framework. Finally, Section 8 illustrates some applications in image and MD signal processing, demonstrating the power of the constructed MD filter banks and signal representations.

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