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Abstract:
This paper examines different approaches to the measurement of multidimensional inequality and poverty. First, it outlines three aspects preliminary to any multidimensional study: the selection of the relevant dimensions; the indicators used to measure them; and the procedures for their weighting. It then considers the counting approach and the axiomatic treatment in poverty measurement. Finally, it reviews the axiomatic approach to inequality analysis. The paper provides a selective review of a rapidly growing theoretical literature with the twofold aim of highlighting areas for future research and offering some guidance on how to use multidimensional methods in empirical and policy-oriented applications.

Keywords: D3, D63, I30, I32

JEL classification: inequality, poverty, deprivation, multidimensional well-being, capabilities and functionings

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Sammendrag

Artikkelen «Multidimensional Poverty and Inequality» inngår som kapittel 4 i Handbook of Income Distribution, Vol. 2A, redigert av A. B. Atkinson og F. Bourguignon. Artikkelen diskuterer forskjellige metoder for måling av flerdimensjonal ulikhet og fattigdom.

1. Introduction[1]

Few people would question that well-being is the outcome of many different attributes of human life and that the level of income, or expenditure, is only a crude proxy of the quality of living that a person enjoys.[2] Should we then account for the multiple facets of well-being in the social evaluation of inequality and poverty? If so, how can we do it?

Acknowledging the multidimensional nature of well-being does not necessarily imply that the social evaluation has also to be multi-dimensioned. Some could argue that a single variable can still subsume all various dimensions of well-being. This is typically the case of the utilitarian approach, where such a single indicator is represented by "utility", that is the level of well-being as assessed by individuals. Individuals themselves reduce the vector $\boldsymbol{x}$ of the different constituents of well-being to the level of utility $u(\boldsymbol{x})$. The social evaluation may then consider estimated utility levels as revealed by individuals, either directly through their answers to questions on subjective well-being and life satisfaction, as in the happiness literature,[3] or indirectly through their consumption patterns, as suggested by Jorgenson and

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[2] Throughout the paper, we use interchangeably terms such as "well-being", "quality of life" and "standard of living", without adopting any precise definition, except for the recognition of their multidimensional nature. The ensuing ambiguity is not a problem for our presentation, but it might be in a different context. For a discussion of this point, see for instance the exchange between Williams (1987) and Sen (1987), and Sen (1993). Likewise, we use indifferently terms such as "attributes", "dimensions" or "domains" to indicate the components of a multivariate notion of deprivation or well-being, although we acknowledge that in certain areas of the literature on social indicators they may be used to indicate different concepts.

[3] Well before the recent surge of interest for happiness among economists, the "Leyden approach" to the measurement of poverty proposed exploiting the information on people's subjective evaluation of their own economic condition to identify poverty thresholds. See, for

Slesnick (1984a, 1984b). Apart from requiring analytical restrictions (e.g. shape of indirect utility functions, integrability of demand functions), these approaches run into the difficulty that individual utilities must be assumed to be interpersonally comparable. Alternatively, the reduction of multiple dimensions to a single indicator can be considered to be carried out by a social evaluator. This composite indicator would then represent a "utility-like function of all the attributes received", as put by Maasoumi (1986, p. 991) to which standard univariate techniques could be applied. Maasoumi suggests applying Information Theory to find the utility-like function whose distribution is as close as possible to the distributions of the constituent attributes, but other approaches can lead to the definition of analogous individual-level functions. The common practice of adjusting household income for the household size and the age of its members by an equivalence scale is another example of this type of multidimensional analysis, where command over resources (income) and individual needs (varying by age and living arrangements) are the two dimensions reputed to be relevant in assessing well-being. The chosen equivalence scale is assumed to represent the preferences of the social evaluator.

At the opposite extreme are those who argue, on philosophical or practical grounds, that dimensions must be kept distinct in the social evaluation. If well-being domains are characterised by specific criteria and arrangements, some might adhere to Walzer's (1983, p. 19) view of "complex equality" whereby "no citizen's standing in one sphere or with regard to one social good can be undercut by his standing in some other sphere, with regard to some other good". If inequalities in certain domains (e.g. basic life necessities or health) are less acceptable than in others (e.g. luxury goods), it might be justifiable to adopt a piecemeal

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instance, Goedhart et al. (1977), van Praag et al. (1980), Danziger et al. (1984), van Praag et al. (2003) and van Praag and Ferrer-i-Carbonell (2008).

approach informed by the "specific egalitarianism" advocated by Tobin (1970).[4] It may be the intrinsic incommensurability of domains to imply that "no simple ordered indicator of level of living can be constructed, either on an individual or on an aggregate level", as asserted by Erikson (1993, p. 75) in summarising the Swedish approach to welfare research. Or it may be the need to avoid the "*ad hoc* aggregation" and the unexplained tradeoffs between domains, which are implicit in any composite or "mashup" index, that should advise us "... to derive the best measure possible for each of a logically defensible set of grouped dimensions – such as 'income poverty,' 'health poverty' and 'education poverty'" (Ravallion, 2011a, p. 240; see also Ravallion, 2012a). In all these cases, the recognition of the inherent autonomy of each dimension, however motivated, leads to a piecewise social judgement which does not need any unitary measurement of human well-being. The elements of the vector $\boldsymbol{x}$ of the attributes of well-being are examined one by one, without attempting to reduce complexity by a summary index. It is the "dashboard" approach. The straightforwardness of this strategy is appealing, but it is tempered by the difficulty of drawing a synthetic picture, especially in the presence of a rich information set.

There are reasons, however, to take an intermediate position between these two extremes. This may be because the conditions described above for reducing well-being to a single variable may not hold: we might differ in the view about the appropriate equivalence scale or the weights to be placed on different goods, we might not have access to individual well-being measures, or we might reject the individual valuations altogether. Or it may be because we are worried that the inequalities in different spheres cumulate and that the combination of multiple deprivations makes life much harder than just the sum of such deprivations. In these cases, we may need a social evaluation of poverty and inequality that is

[4] Slesnick (1989) assesses how pursuing equalisation in separate domains affects the inequality of overall utility (specific egalitarianism vs. general egalitarianism) by comparing the inequality of main consumption components with the inequality of total expenditure.

multi-dimensioned and accounts for the joint distributions of all the elements of the vector $\boldsymbol{x}$ of well-being attributes.

Our aim in this paper is to explore this intermediate route. We do not argue further whether we should, or should not, have a multi-dimensioned social evaluation. We take it for granted – and we concentrate on how we can carry it out in a sound way. More precisely, we examine the analytical and ethical foundations of methods for the multidimensional measurement of inequality and poverty, whether it be for descriptive, normative or policy-making purposes. All these methods require numerous arbitrary, and hence debatable, assumptions: elucidating their foundations helps unveiling these assumptions and understanding their normative content. Taking this perspective, we pay little attention to the many multivariate techniques that have been developed in statistics and efficiency analysis. They provide valuable information, but their aggregation of multiple attributes is based on empirically observed patterns of association among the variables and lacks any clear ethical interpretation. We may legitimately hesitate to entrust a mathematical algorithm with an essentially normative task such as deriving an index of well-being.

The theoretical literature on the multidimensional measurement of inequality and poverty has been growing very rapidly in the last quarter of a century, and is still far from consolidation. Rather than engaging in a systematic rationalisation of this literature, we provide a selective reading of it with the twofold objective of, first, identifying areas worthy of further investigation and, second, offering some guidance on how to use the rich and sophisticated machinery now available for empirical and policy-oriented applications. As the multidimensional view of well-being has gained momentum in the policy discourse, its practical implementation has turned into an active battlefield where contenders passionately argue for opposing approaches – a good example being the Forum on multidimensional poverty in the 2011 volume of the *Journal of Economic Inequality* (see Lustig, 2011, for an introduction). Our attempt is to give a balanced account of alternative positions as well as of their strengths and weaknesses.

The paper is divided into three parts, plus a closing Section. In the next Section, we briefly review three questions that are preliminary to any multidimensional analysis of well-being: the selection of the relevant dimensions; the indicators used to measure them; and the procedures for their weighting. These questions are theoretically intriguing and of considerable importance in empirical analyses, but we only outline their main features. It should be borne in mind that the choice made with regard to these issues may condition the analytical methods reviewed later. For instance, the fact that many variables used in multidimensional poverty analysis are dichotomous suggests paying particular attention to methods based on counting deprivations. The assumption that inequality does not change after proportionate variations of the variable under examination (scale invariance) may be reasonable for income, but much less so for life expectancy, impinging on the axiomatic measurement of multidimensional inequality. We then move to the core of the paper: the methods for the multivariate analysis of poverty, in Section 3, and of inequality, in Section 4. In the remaining of this introduction, we give a brief account of the historical developments of the research summarised in these two Sections, while providing a brief tour of the main themes discussed in the paper.

1.1. Historical developments and main themes

The multidimensional literature in economics begins with the seminal articles by Kolm (1977) and by Atkinson and Bourguignon (1982) on the dominance conditions for ranking multivariate distributions. Few years later, Atkinson and Bourguignon (1987) develop sequential dominance criteria for the bivariate space of income and household composition. Their aim is to impose weaker assumptions on social preferences than those implicit in the standard method of constructing equivalent incomes. Whereas the standard approach entails specifying how much a family type is needier than another one, sequential dominance criteria only require ranking family types in terms of needs, although at the cost of obtaining an incomplete ordering. This application paves the way to a specific and fertile strand of research which focuses on the possibility that one attribute (e.g. income) can be used to compensate for another non-transferable attribute (e.g. needs, health).

With the partial exception of Maasoumi (1986, 1989), who recasts the multidimensional analysis into the unidimensional space by means of a utility-like function, it is only around the mid-1990s that Tsui (1995, 1999) moves on to the axiomatic approach to inequality indices to achieve complete orderings. The bases of the axiomatic analysis of partial and complete poverty orderings are laid down at about the same time by Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003, 2009) and Tsui (2002). Note that multidimensional indices of inequality and poverty associate real numbers to each multivariate distribution as does the univariate analysis of a composite well-being indicator, but with the important difference that they do not need to go through the aggregation of well-being attributes at the individual level. Thus, multidimensional poverty indices allow for separate thresholds for each attribute, while a utility-like indicator would usually have a single threshold in the space of well-being. The tradeoffs between the attributes that are built-in in the utility-like indicator used in the latter approach follow from the weighting structure of dimensions in the former approach.

At the turn of the 20th century, the literature on multidimensional poverty and inequality is still in its infancy. The first volume of this *Handbook* (Atkinson and Bourguignon, Eds., 2000) does not feature any specific chapter on the topic, and the comprehensive analytical chapter on the measurement of inequality by Cowell (2000) devotes only three pages to “multidimensional approaches”. Ever since, the theoretical literature has grown conspicuously. We can identify two main lines of research.

The first line devotes considerable effort to developing the axiomatic approach to both poverty and inequality measurement. Researchers delve into the different ways to model the patterns of association (correlation) between the variables, which is the single feature that distinguishes multidimensional from unidimensional analysis, and elaborate alternative axioms. They also come to realise that a mechanical transposition of the properties typically adopted in the univariate analysis of income distribution may not be straightforward, and sometimes not even appropriate. A case in point is the extension to life expectancy of the scale invariance property of inequality measures just mentioned. An even more cogent example is that of the Pigou-Dalton principle of transfers, a central tenet of income inequality measurement (Atkinson and Brandolini, forthcoming). This principle states that a mean-preserving transfer of income from a richer person to an (otherwise identical) poorer person decreases inequality. On the one hand, an interpersonal transfer might be unfeasible and even ethically debatable for a dimension such as the health status, despite being acceptable for income. On the other hand, the generalisation of the principle to a multivariate framework is far from univocal, as explained in detail in Section 4.1.

The second line of research focuses on what Atkinson (2003) labels the “counting approach”. This multidimensional approach is at the same time the newest – as regards theoretical elaboration – and the oldest – as regards empirical practice. For example, the main poverty statistic adopted by a parliamentary commission of inquiry over destitution in Italy in the early 1950s was a weighted count of the number of households failing to achieve minimum levels of food consumption, clothing availability, and housing conditions (Cao-Pinna, 1953). Modern applied research on material deprivation owes much to the pioneering work by Townsend (1979) and Mack and Lansley (1985) in Britain.[5] Ever since, it has had a huge impact on the social policy debate in Ireland and the United Kingdom, and later in the European Union.[6] Nevertheless, we lack a fully-fledged theoretical treatment of the normative basis of the counting approach. The recent work by Alkire and Foster (2011a, 2011b) in part fills this gap by providing the axiomatic characterisation of a family of multidimensional counting poverty indices. Yet, the difficulties illustrated by Atkinson (2003) in reconciling the counting approach with a social welfare approach are still unsettled. In our view, part of the problem may derive from defining welfare criteria in terms of the distributions of the underlying continuous variables rather than in terms of the distribution of deprivation scores, which is the key variable considered in the counting approach. The distribution of deprivation scores contains all the relevant information in the counting approach, which by construction implies neglecting levels of achievement in the original

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[5] Interestingly, Townsend's interest for elaborating a deprivation score was largely instrumental, being conceived as a way to reduce the arbitrariness of fixing income thresholds: "We assume that the deprivation index will not be correlated uniformly with total resources at the lower levels and that there will be a 'threshold' of resources below which deprivation will be marked" (Townsend, 1970, p. 29). There is by now an extensive literature. Some examples of studies for rich countries are Mayer and Jencks (1989), Federman et al. (1996), Nolan and Whelan (1996a, 1996b, 2007, 2010, 2011), Whelan et al. (2001), Halleröd et al. (2006), Guio (2005), Cappellari and Jenkins (2007), Fusco and Dickes (2008), Fusco et al. (2010) and Figari (2012).

[6] Since 1997, the official poverty statistic adopted by the Irish government is "consistent poverty", which is the proportion of people who are both income-poor and deprived of two or more items considered essential for a basic standard of living (Social Inclusion Division, 2014). The British Child Poverty Act 2010 sets four policy targets, among which a combined low income and material deprivation target (The Child Poverty Unit, 2014). One of the five European Union headline targets set by the Europe 2020 strategy for a smart, sustainable and inclusive growth concerns the share of people "at risk of poverty or social exclusion" (European Commission 2010). This indicator combines income poverty, household joblessness, and severe material deprivation, where severe material deprivation occurs whenever a person lives in a household that cannot afford at least four out of nine amenities. On the use of indicators of material deprivation and more generally on the multidimensional perspective adopted in the European Union policy evaluation of social progress see Atkinson et al. (2002), Marlier et al. (2007), Maquet and Stanton (2012), and Marlier et al. (2012).

variables. Disagreement on this point, and on the implicit loss of information, might have some part in the recent controversies surrounding the counting approach.

The less developed analytical structure, in the face of the popularity in applied research, is the main reason for devoting a relatively larger space to the counting approach in this paper. However, counting deprivations is also the simplest way to embed the association between dimensions at the individual level into an overall index of deprivation. It is useful to illustrate two aspects of multidimensional measurement which are recurrent throughout the paper. The first is the order of aggregation. In the counting approach the synthesis of the available information begins with aggregating across the single dimensions for each individual, and then across individuals. Inverting the order of aggregation by computing first the proportions of people suffering from deprivation in each dimension, and then aggregating these proportions into a composite index of deprivation would yield the same result only if the dimensions of well-being were “independent”. If this is not the case, this composite index of deprivation would miss the impact of cumulating failures in more than one dimension. The second aspect is the contrast between the “union criterion” and the “intersection criterion”, which plays a fundamental role in the measurement of multidimensional poverty, as stressed by Atkinson (2003). The occurrence of deprivation in some dimensions need not entail a condition of overall poverty: we may define people to be poor when they are deprived in at least one dimension (union criterion) or in all dimensions (intersection criterion), or else in some fraction of the dimensions considered in the analysis. The choice of a critical number of dimensions to identify the poverty status introduces an additional threshold relative to those already set for defining deprivation in each dimension, which is a central feature of the “dual cut-off” approach proposed by Alkire and Foster (2011a, 2011b).

In Sections 3 and 4, we discuss first the counting approach, then the axiomatic treatment of poverty, and finally the axiomatic treatment of inequality. This sequence reflects a growing complexity of data requirements, rather than a chronological order. In this paper we pay no attention to the assessment of data quality and the elaboration of inference tools, although they are admittedly two crucial issues in empirical analyses.

2. Preliminaries: dimensions, indicators and weights

Three questions are preliminary to any discussion of the methods for the multivariate analysis of poverty and inequality: the selection of the relevant dimensions of well-being; the indicators used to measure people's achievements in these dimensions, and the related issue of the choice of deprivation thresholds in poverty analysis; and the weights assigned to each dimension. An in-depth examination of these issues is beyond the scope of this paper, and our primary aim in this Section is to highlight how they can influence the multivariate methods of analysis reviewed below. However, it has to be borne in mind that the actual solutions given to these questions may affect empirical findings and their substantive interpretation. Robustness and sensitivity exercises are advisable.

2.1. Selection of dimensions

An established tradition of research in the study of deprivation postulates that we can better understand hardship focusing on the inability to consume socially perceived necessities because of lack of economic resources, rather than focusing on income. Typically, this approach considers a battery of indicators concerning the ownership of durable goods, the possibility to carry out certain activities, such as going out for a meal with friends, or the ability to cope with the payment of rent, mortgages, or utility bills. Material deprivation indicators have recently gained an official status in the monitoring of the social situation in the European Union as well as in Ireland and the United Kingdom. The aim of the social evaluation may however be broader than assessing material living conditions, and be concerned with "social exclusion".[7] According to Burchardt et al. (1999), social exclusion is associated with failures in achieving a reasonable living standard, a degree of security, an activity valued by others, some decision-making power, and the possibility to draw support from relatives and friends. The variety of dimensions used to define the overall quality of life may be even larger. The "Scandinavian approach to welfare", a long-established research programme in Nordic countries, considers nine domains of human life: health and access to health care; employment and working conditions; economic resources; education and skills; family and social integration; housing; security of life and property; recreation and culture; and political resources (e.g. Erikson and Uusitalo, 1986-87; Erikson, 1993). Within the "capability approach", Nussbaum (2003) proposes a specific list of ten "central human capabilities": life; bodily health; bodily integrity; senses, imagination, and thought; emotions; practical reason; affiliation; other species; play; and control over one's environment. The Commission on the Measurement of Economic Performance and Social Progress, created at the beginning of 2008 on the French government's initiative, identifies eight key dimensions: material living standards; health; education; personal activities including work; political voice and governance; social connections and relationships; environment (present and future conditions); and economic and physical insecurity (Stiglitz et al., 2009).

These examples well illustrate the wide range and diversity of the domains that can be considered in the multidimensional analysis of inequality and poverty. The choice of the dimensions that they include is mainly due to experts, possibly based on existing data,

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[7] On the somewhat elusive concept of social exclusion and its relationship with poverty, see Atkinson (1998). Ruggeri Laderchi et al. (2003) compare empirical findings for the social exclusion and capability approaches. Poggi (2007a, 2007b) and Devicienti and Poggi (2011) study empirically the persistence of social exclusion, while Poggi and Ramos (2011) investigate the inter-dependency of the dimensions of social exclusion using stochastic epidemic models.

conventions and statistical techniques.[8] It could also result from empirical evidence regarding people's values or from a consultative process involving focus groups and representatives of the civil society or the public at large (Alkire, 2007). In all cases, their selection is a fundamental exercise, which has to blend theoretical rigour, political salience, empirical measurability, and data availability.

In this paper, we simply take as given that a predefined list of *r* attributes fully describes the well-being concept used in the analysis of poverty and inequality. We ignore all questions concerning their selection.[9] Notice, however, that the nature of selected attributes may condition the definition of measurement tools. As noted in the introduction, we cannot mechanically export the Pigou-Dalton principle of transfers which is central in income inequality analysis to other well-being dimensions, such as health (Bleichrodt and van Doorslaer, 2006), happiness (Kalmijn and Veenhoven, 2005), and literacy (Denny, 2002). Leaving aside the practical problem of how to transfer one unit of health from one person to another, we might doubt that imposing the principle of transfers in the health domain is ethically justified. We return to this issue in Section 4.1.

2.2. Indicators

The indicators used to measure people's achievements in the various dimensions are numerous and understandably have different measurement units. Incomes, wealth, quantities consumed or purchased are continuous variables, while the number of durable goods owned or the frequency in the use of consumer services are discrete variables. Education can be

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[8] For instance, Fusco and Dickes (2008) assume that poverty is a latent condition that can be identified by selecting the relevant domains from a set of deprivation indicators by applying a psychometric model.

[9] The topic has attracted considerable attention within the literature on the "capability approach". See, among others, Sen (1985, 1992), Alkire (2002, 2007), Nussbaum (1990, 1993, 2003), Kuklys (2005), Robeyns (2005, 2006) and Basu and López-Calva (2011).

measured by a categorical variable such as the highest school attainment of a person. Transforming it into the minimum number of years necessary to achieve each school level provides some objective way to grade the various levels, but we might wonder whether a person who completed fourteen years of school is really twice as well-educated as a person who only completed seven years; moreover, only in a loose sense such a transformed variable can be interpreted as truly continuous. People's competencies and problem-solving ability are increasingly assessed by complex exercises that produce literacy, numeracy or skill scores generally normalised on a scale from 0 to 500: these scores are bounded continuous ordinal variables[10]. Individual health and physical status are measured with a host of indicators: self-reported measures of health conditions are ordinal variables, while the information on the incidence of specific chronic illnesses is dichotomous; anthropometric indicators such as height, weight or the body mass index are continuous variables. Subjective measures of well-being are typically collected by asking interviewees their personal degree of satisfaction on pre-fixed numerical scales or verbal rating scales ranging from, say, "not very happy" to "very happy". In either case, the outcome is an ordinal variable, which ranks the alternative ratings without however providing any information on how much one rating is better, or worse, than another rating.

Cardinal continuous variables, such as income, probably represent a minority of available indicators of well-being. The application of measurement tools that are standard in income distribution analysis may hence need some reconsideration in moving to non-

[10] Well-known examples are the Programme for International Student Assessment (PISA) for 15-year-old students and the Programme for the International Assessment of Adult Competencies (PIAAC), both coordinated by the Organisation for Economic Co-operation and Development (OECD). Micklewright and Schnepf (2007, p. 133) compare the cross-country inequality in learning achievement scores and call for caution in the use of the income inequality measurement toolbox, as "... it is doubtful whether the measurement of the scores is on a ratio scale. Their nature is therefore quite different from that of data on income or height".

monetary domains.[11] This warning applies to, but is clearly not exclusive of, multidimensional analysis. One specific problem that arises in this context, however, concerns the commensurability of the indicators when they are merged into a single index. It is generally tackled by employing procedures of standardization that, for instance, transform the original variable by taking its (normalised) distance from benchmark values (for some examples of these transformations, see Decancq and Lugo, 2013, p. 12). Alternatively, ordinal criteria might be applied also to quantitative variables (e.g. by classifying units according to the quantile to which they belong).[12] Irrespective of the specific procedure adopted, the transformation of the original values substantially affects the outcome.

Many variables are dichotomous, or binary, either by definition or after comparison of the individual achievement with some “social norm”: for instance, we may classify as being deprived in housing conditions all those who live in households with less than one room per person, transforming the variable “room per person” into a binary one. The use of dichotomous variables is at the centre of the “counting approach” examined below.

In poverty assessments, the choice of the indicators is intertwined with the definition of the respective deprivation thresholds. This problem parallels the analogous problem faced in univariate analyses of income or consumption, with absolute, relative, subjective, and legal criteria being the main alternatives (e.g. Callan and Nolan, 1991). In multivariate analyses, these problems may be amplified by the consideration of intangible dimensions for which it

[11] Growing attention is paid to the measurement of inequality when using qualitative ordinal variables, such as self-reported health status (e.g. van Doorslaer and Jones, 2003; Allison and Foster, 2004; Bleichrodt and van Doorslaer, 2006; Abul Naga and Yalcin, 2008) and happiness (e.g. Kalmijn and Veenhoven, 2005; Dutta and Foster, 2013). Cowell and Flachaire (2012) develop axiomatically a class of inequality indices for categorical data, conditional on a reference point, which are based on the individuals’ position in the distribution. Zheng (2008) suggests that, where data are ordinal, stochastic dominance has limited applicability in ranking social welfare and no applicability in ranking inequality.

[12] Qizilbash (2004) discusses the sensitivity of empirical estimates for poverty in South Africa to transforming the indicators from cardinal to ordinal using Borda score as well as to varying the thresholds used to define deprivation.

may even be more contentious to identify minimum thresholds (Thorbecke, 2007). Similarly to the univariate case, however, it could be argued that the binary distinction between a "bad state" and a "good state" is too sharp, since deprivation is likely to occur by degrees. Moving along these lines, Desai and Shah (1988) focus on the distance of the individual achievements from modal values in each dimension, taken to represent the social norm, whereas the extensive literature in the "fuzzy sets approach" formalises a continuum of grades of poverty by means of a "membership" function.[13] Such a "membership" function may assume any value between 0 and 1: the two extreme values indicate that a person is definitely non-deprived (0) or deprived (1), while all other values indicate "partial" membership of the pool of the deprived. The form of the membership function plays a crucial role in the construction of a "fuzzy" deprivation measure. Although largely seen as a distinct approach in the multivariate analysis of deprivation, there is nothing inherently multidimensional in the theory of fuzzy sets.

2.3. Weighting of dimensions

Weights determine the extent to which the selected attributes contribute to well-being and the degree by which we can substitute one attribute for another, interacting with the functional form used to aggregate dimensions. This can be easily seen by defining individual well-being S_β as the weighted mean of order β of the achievements in the r dimensions, as suggested for instance by Maasoumi (1986),

[13] See Cerioli and Zani (1990), Cheli et al. (1994), Cheli (1995), Cheli and Lemmi (1995), Chiappero Martinetti (1994, 2000), Betti et al. (2002), Dagum and Costa (2004), Qizilbash and Clark (2005), Betti and Verma (2008), Betti et al. (2008), Belhadj (2012), and Belhadj and Limam (2012). Deutsch and Silber (2005), Pérez-Mayo (2007), D'Ambrosio et al. (2011) compare empirical results for multidimensional measures of poverty based on the fuzzy sets approach with those derived from applying alternative approaches (axiomatic approach, Information Theory, efficiency analysis, latent class analysis). Kim (2014) studies the statistical behaviour of fuzzy measures of poverty.

$$(2.1) \quad S_\beta = \begin{cases} \left[\sum_{k=1}^{r} w_k x_k^\beta \right]^{1/\beta} & \beta \neq 0 \\ \prod_{k=1}^{r} x_k^{w_k} & \beta = 0 \end{cases}$$

where x_k is non-negative and represents the level of attribute k, $k = 1, 2, ..., r$, and w_k is the corresponding weight. Notice that expression (2.1) turns into an index of deprivation if the r attributes measure hardship. The weights w_k and the parameter β jointly govern the degree of substitution between any pair of cardinal attributes. Indeed, the marginal rate of substitution between attributes b and a, which is the quantity of b that has to be given up in exchange for one more unit of a in order to leave well-being unchanged, is equal to:

$$(2.2) \quad MRS_{b,a} = \frac{dx_{bi}}{dx_{ai}} = -\left(\frac{w_a}{w_b} \right) \left(\frac{x_a}{x_b} \right)^{\beta - 1}.$$

If $\beta = 1$ well-being is simply the (weighted) arithmetic mean of the achievements in all dimensions, which are then perfectly substitutable at a rate equal to the ratio of their respective weights. In all other cases, the marginal rate of substitution depends also on relative achievements: the further away β is from one, the more an unbalanced achievement in the two dimensions matters. When β goes to infinity (minus infinity), the attributes are perfect complements, and the well-being level depends on the highest (lowest) achievement, regardless of the values assigned to the weights.

The pattern of substitution among attributes can be more muddled than in (2.2), when the functional form of the well-being aggregator is more complex than (2.1), but it is bound to depend critically on weights, except in the extreme cases where the attributes are perfect complements. The choice of weights might have a significant effect on the results of multidimensional analyses of inequality and poverty. For instance, Decancq et al. (2013) find that the identification of the worst-off in a sample of Flemish people is considerably influenced by the use of alternative weighting schemes of the attributes. In a comparison of the incidence of income-and-health poverty in selected European countries in 2000-01, Brandolini (2009) finds that the ranking of Italy and Germany reverses as weights are shifted from one dimension to the other, although the ordering of France and the United Kingdom mostly remains unchanged. Here, we outline approaches to weighting by drawing on Brandolini and D'Alessio (1998) and refer to Decancq and Lugo (2013) for a more comprehensive discussion.

A popular way of setting weights is to treat all attributes equally. This is the case of the Human Development Index, which assigns the same weight (one third) to the three basic dimensions considered: a long and healthy life, access to knowledge, and a decent standard of living (e.g. UNDP, 2013). Equal weighting may result either from an "agnostic" attitude and a wish to reduce interference to a minimum, or from the lack of information about some kind of "consensus" view. For instance, Mayer and Jencks (1989, p. 96) opt for equal weighting, after remarking that: "ideally, we would have liked to weight [the] ten hardships according to their relative importance in the eyes of legislators and the general public, but we have no reliable basis for doing this". (In fact, there may be disagreement among the legislators and the public, let alone within the public itself.)

Some departure from equal weighting is envisaged by Atkinson et al. (2002) and Marlier and Atkinson (2010). They propose a set of principles for the design of social indicators for policy purposes, among which is the principle that the weights should be "proportionate", so that dimensions have "…degrees of importance that, while not necessarily exactly equal, are not grossly different" (Marlier and Atkinson, 2010, p. 289). This criterion only sets some reasonable boundaries, without specifying however how to define non-equal weights.

It is possible to elicit the weighting structure directly from consultations with groups of experts or the public at large, or from the importance assigned to dimensions of well-being by survey respondents; indirectly, from estimates of happiness equations.[14] The last procedure is followed by Decancq et al. (2014) who characterise axiomatically a class of multidimensional poverty indices that are consistent with individual preferences in the aggregation of the different dimensions. In addition to standard axioms, they postulate principles for interpersonal poverty comparisons that lead to measure individual poverty as a function of the fraction of the poverty line vector to which the agent is indifferent. The poverty threshold is therefore defined in terms of well-being using person-specific weights. In some exercises, users of statistics are allowed to build their own set of weights. For instance, the OECD Better Life Index allows people to compare well-being across countries by means of eleven indicators of quality of life that can be rated equally or according to individual preferences (see Boarini and Mira D'Ercole, 2013, and the initiative's website http://www.oecdbetterlifeindex.org/). In all these cases, the choice of weights relies on some implicit or explicit normative criterion.

Under certain hypotheses, market prices provide weights that capture a tradeoff between dimensions that is consistent with consumer welfare. Sugden (1993) and Srinivasan (1994) contend that it is the availability of such "operational metric for weighting commodities" that makes traditional real-income comparison in practice superior to Sen's capability approach. Ravallion (2011a, p. 243) argues that the main multidimensional poverty indices aggregate deprivations in a manner that "... essentially ignores all implications for welfare measurement of consumer choice in a market economy. While those implications need not be decisive in welfare measurement, it is clearly worrying if the implicit tradeoff between any two market goods built into a poverty measure differs markedly from the tradeoff facing someone at the poverty line". On the other hand, market prices may be

[14] See Decancq and Lugo (2013, pp. 24-6) for a discussion, and Bellani (2013), Bellani et al. (2013), Cavapozzi et al. (2013), Decancq et al. (2013), and Mitra et al. (2013) for some examples.

distorted by market imperfections and externalities, and they do not exist for many constituents of well-being and their imputation may be arduous, although various approaches estimate the "willingness to pay" in order to add the monetary value of non-income dimensions to income (e.g. Becker et al., 2005; Fleurbaey and Gaulier, 2009). More importantly, they may be conceptually inappropriate for welfare comparisons, a task for which they are not devised (Foster and Sen, 1997; Thorbecke, 2007).

The main alternative and widely applied approach is "to let the data speak for themselves". Methods differ, but we may cluster them into two main categories: frequency-based approaches and multivariate statistical techniques. Since Desai and Shah (1988) and Cerioli and Zani (1990), many researchers assume that the smaller the proportion of people with a certain deprivation, the higher the weight that should be assigned to that deprivation, on the ground that a hardship shared by few is more important than one shared by many. This approach raises two problems. First, it may lead to a questionably unbalanced structure of weights. As observed by Brandolini and D'Alessio (1998), in 1995 the shares of Italians with low achievements in health and in education were 19.5 and 8.6 per cent, respectively. With these proportions, education insufficiency would be valued more than health insufficiency: a tenth more according to Desai and Shah's formula, over a half more according to Cerioli and Zani's formula. Whether education should attain a weight so much higher than health is certainly a matter of disagreement. Second, this criterion makes the weights endogenous to the distributions being studied. Thus, it implies that we should take country-specific weights in an international comparison of multidimensional poverty, unless we impose a common, but arbitrary, set of weights. This observation applies also to the suggestions by Betti et al. (2008) to take weights proportional to the dispersion of the attributes in the population (adjusted for their bilateral correlations to avoid redundancy), and by Vélez and Robles (2008) to select the weights that allow a set of multidimensional poverty measures to better track the dynamics of self-perceived well-being.

Several multivariate statistical techniques are employed to aggregate dimensions.[15] Maasoumi and Nickelsburg (1988), Klasen (2000) and Lelli (2005) use the analysis of principal components, on the ground that this approach "… uncovers empirically the commonalities between the individual components and bases the weights of these on the strength of the empirical relation between the deprivation measure and the individual capabilities" (Klasen, 2000, p. 39, fn. 13). Schokkaert and Van Ootegem (1990), Nolan and Whelan (1996a, 1996b), and Whelan et al. (2001) aggregate by factor analysis elementary indicators into measures of well-being or deprivation. These papers however tend to use this technique to identify few distinct constituents of well-being: as noted by Schokkaert and Van Ootegem (1990, p. 439-40), their application of factor analysis is "a mere data reduction technique", which does not provide any indication about the relative valuation of each attribute. Several authors apply latent variable models or structural equation modelling to collapse multiple indicators into indices of total or domain-specific deprivation (Kuklys, 2005; Pérez-Mayo, 2005, 2007; Di Tommaso, 2007; Krishnakumar, 2008; Krishnakumar and Ballon (2008); Krishnakumar and Nagar (2008); Navarro and Ayala, 2008; Wagle, 2005, 2008a, 2008b; Tomlinson et al., 2008; Ayala et al., 2011). Dewilde (2004) uses a two-step latent class analysis, evaluating deprivation in specific domains in the first step and the latent concept of overall poverty in the second step. Lovell et al. (1994), Deutsch and Silber (2005), Ramos and Silber (2005), Anderson et al. (2008), Ramos (2008), and Jurado and Pérez-Mayo

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[15] On applied multivariate techniques see, for instance, Sharma (1996). Moreover, see Ferro Luzzi et al. (2008), Pisati et al. (2010), Whelan et al. (2010) Lucchini and Assi (2013) and Caruso et al. (2014) for an application of cluster analysis to identify population subgroups homogeneous by well-being or deprivation level, and Hirschberg et al. (1991) for an analogous comparison across countries; Asselin and Anh (2008) and Coromaldi and Zoli (2012) for an application of multiple correspondence analysis and non-linear principal component analysis, respectively.

(2012) apply methods developed in efficiency analysis to aggregate the various attributes of well-being. These methods allow estimating the level of individual achievement relative to the achievement frontier, providing implicit estimates of the values of the weights. In a related approach, Cherchye et al. (2004) construct a synthetic indicator to assess European countries' performance in achieving social inclusion where weights are variable and such as to provide the most favourable evaluation for each country. They contend that this approach preserves the "legitimate diversity" of countries in pursuing their own policy objectives, since a relatively better performance in a particular dimension is seen as revealing a policy priority.

The methods reviewed in the next Sections generally allow for the possibility that weights can differ across dimensions in the social evaluation of poverty and inequality. Our brief overview suggests some ways to define them. Two comments are in order. First, multivariate statistical techniques differ from other approaches in that their aim is to estimate the level of individual achievement; weights are integral part of the aggregation procedure and have no truly independent meaning. We may then wonder whether it is appropriate to use them in conjunction with many of the methods discussed below. Second, as the weighting structure captures the importance assigned to each attribute, it is bound to reflect different views. On one side, this suggests questioning the use of techniques that may be robust from a statistical viewpoint but ignore the intrinsically normative aspect of the choice of weights. On the other side, it hints that one way to account for this plurality of views is to specify "ranges" of weights rather than a single set of weights, although this approach might lead to a partial ordering, as suggested by Sen (1987, p. 30; see also Foster and Sen, 1997, p. 205).[16]

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[16] Cherchye et al (2008) present a methodology that incorporates a range of weighting schemes in the ranking of vectors of attributes.

3. Multidimensional poverty measurement

A long tradition in social sciences has been concerned with measuring material deprivation by looking at a number of indicators of living conditions, such as the ownership of durables or the possibility to carry out certain activities like going out for a meal with friends. The typical way to summarise the information has been to count the number of dimensions in which people fail to achieve a minimum standard, hence the label of "counting approach". It represents the simplest way to embed the association between deprivations at the individual level into an overall index of deprivation.

In the counting approach, the synthesis of the available information begins with aggregating across the single dimensions for each individual, and then across the individuals. However, we could invert the order of aggregation by computing first the proportions of people suffering in each dimension, and then aggregating these proportions into a composite index of deprivation. This different order of aggregation has the great advantage that we can draw these proportions from various sources. This characteristic makes this "composite index" approach easily understandable and very popular, especially in public debates where there is a need to summarise headline messages from sets of indicators. If the dimensions of well-being are "independent" of each other, the order of aggregation does not matter and the two approaches are equivalent. However, if they are dependent and suffering from multiple deprivations has a more than proportionate effect on people's well-being, ignoring the impact of the association among the achievements in the various dimensions, as with the composite index approach, may imply missing an important aspect of hardship. This is not the case for an indicator such as severe material deprivation in the Europe 2020 strategy, as it would rank a society where one person suffers from four deprivations and three persons do not suffer from any differently from a society where four people fail in one dimension each.

The relationship between the two approaches can be better understood by considering the simple situation where there are only two dimensions. Assume that X_i is equal to 1 if an individual suffers from deprivation in dimension i and 0 otherwise, $i=1,2$. Let $p_{ij}=\Pr\left(\left(X_1=i\right)\cap\left(X_2=j\right)\right)$, $p_{i+}=\Pr\left(X_1=i\right)$ and $p_{+j}=\Pr\left(X_2=j\right)$. Then, assign equal weight to the two deprivation indicators and define the deprivation score $X=X_1+X_2$, which can take the values $(0,1,2)$ with associated probabilities (q_0,q_1,q_2). The parameters (q_0,q_1,q_2) of the count distribution X are determined by the parameters of the original two-dimensional simultaneous distribution in the following manner: $q_0=p_{00}$, $q_1=p_{10}+p_{01}$ and $q_2=p_{11}$. The original and derived distributions are summarised in Table 1.

Table 1. The distribution of deprivations in two dimensions and the derived distribution of deprivations scores

	X_2=0	X_2=1	
X_1=0	p_{00}	p_{01}	p_{0+}
X_1=1	p_{10}	p_{11}	p_{1+}
	p_{+0}	p_{+1}	1

	$X=X_1+X_2$
X=0	$q_0=p_{00}$
X=1	$q_1=p_{10}+p_{01}$
X=2	$q_2=p_{11}$
	1

Source: authors' elaboration.

If only the marginal distributions in the left panel of Table 1 were known, an overall poverty indicator P could be expressed as a function g of p_{1+} and p_{+1} only, that is $P=g(p_{1+},p_{+1})$, which is an example of composite poverty index. If the simultaneous distribution was known, we could turn to the distribution of X in the right panel of Table 1 and the overall index could account for the number of deprivations that each individual suffers from. Counting deprivations highlights two possible ways of identifying someone as poor: either he fails in either dimension ($X=1$), or he fails in both ($X=2$). In the first case, we adopt the "union criterion": the poor are those with at least one deprivation and $P=g(1-p_{00})$. In the second case, we favour the "intersection criterion": the poor are those with two failures and $P = g(p_{11})$. The contrast between union and intersection criteria plays a fundamental role in the measurement of multidimensional deprivation (see Atkinson, 2003). It also suggests that the occurrence of deprivation in some domains need not entail a condition of overall poverty: if we adopt the intersection criterion, only those with two failures are regarded as poor individuals, whereas those with only one failure are not. Setting a critical number of dimensions *c*, $1 \le c \le r$, to identify the poverty status introduces an additional threshold over those already set for defining deprivation in each dimension (see Alkire and Foster, 2011a, 2011b). We return to this issue in Section 3.2.6.

The available information may however be richer than the knowledge about the deprived/not deprived status in a number of dimensions. Rather than dichotomous, variables may be continuous, or discrete with at least three categories. We may then want the overall poverty indicator to account not only for the occurrence of deprivation, that is an individual achievement below the given dimension-specific threshold, but also for its intensity, that is the shortfall of this achievement as compared to the threshold.

These observations illustrate that the reach of the informational basis conditions the multidimensional methods that can be used to measure poverty. When individual-level data on multiple attributes are not available, a composite index may be the only measure that can be calculated. When these data exist but are not publicly available, multidimensional poverty analysis may still be possible by using counting measures, if statistical offices release simple tabulations such as those discussed in the examples in Section 3.2. We use the complexity of informational needs as the criterion to organise the discussion of this Section. We begin with the composite multidimensional poverty indices which only require information on the marginal distributions and can be estimated by gathering data from separate sources. All other multidimensional measures need an integrated database where the information for each relevant dimension is available for each individual unit. We first consider counting measures which use minimal information: the distribution of the population by number of deprivations. With *r* dimensions, it is sufficient to know *r* values (the proportions of the population suffering from deprivation in $0,1,...,r$ dimensions). While being the oldest multidimensional approach in social sciences, the counting approach is arguably the least structured from a theoretical point of view, and we devote relatively more space to its examination. Due to its simplicity the counting approach offers transparent illustrations of alternative aggregation methods as well as the role of various normative rearrangement principles, and helps to clarify the distinction between deprivation and poverty. Next, we turn to multidimensional poverty indices requiring the knowledge of individual achievements in each dimension. Lastly, we discuss criteria for partial ordering.

3.1. The composite index approach

We can measure the overall poverty of a society by aggregating over the proportions of individuals suffering from deprivation in the *r* dimensions of well-being, whenever this is the only available information. A prominent example of this composite index approach is the Human Poverty Index (*HPI*), which was published by the United Nations Development Programme from 1997 to 2009 (UNDP 1997). As originally formalised by Anand and Sen (1997), a general version of the index with *r* dimensions, weighted by w_k, is defined by

$$(3.1) \qquad HPI_\beta = \zeta_1(p_1, p_2, ..., p_r) = \left(\sum_{k=1}^{r} w_k p_k^\beta \right)^{\frac{1}{\beta}},$$

where p_k is the proportion suffering from deprivation in dimension *k* (in the two-dimensional case of Table 1 $p_1 = p_{1+}$ and $p_2 = p_{+1}$), $\beta > 0$ and $w_k > 0$ for all *k*; if the *r* dimensions are equally weighted, $w_k = 1/r$. As β rises, greater weight is given to the dimension in which there is the most deprivation. UNDP (1997) paid particular attention to three dimensions related to longevity, knowledge, and a decent standard of living, and later added a fourth dimension, social exclusion, for rich countries. In either case, β was set equal to 3 to give "additional but not overwhelming weight to areas of more acute deprivation" (UNDP, 2005, p. 342).[17]

Bossert et al. (2013) provide an axiomatic characterization of (3.1) for the case where $\beta = 1$, based on the condition of additive decomposability in attributes as well as in individuals (see also Pattanaik et al., 2011). This case is of some interest: it assumes perfect substitutability among the components, and the index HPI_1 equals the weighted arithmetic mean of the headcount indices across all dimensions. This implies that people that suffer from k deprivations, with $0 \leq k \leq r$, are counted k times by the index HPI_1. Although rather crude and ad hoc, this is a simple way of giving heavier weight to people suffering from multiple deprivations. The implicit assumption is, however, that the effect of deprivations is proportionate: suffering from two deprivations is twice as bad as suffering from one. If there are reasons to question this assumption, then the inability of *HPI*-type measures to discriminate between situations where deprivations are concentrated on few people and situations where an identical total amount of deprivations is spread across many people represents a serious shortcoming.

Dutta et al. (2003) prove that composite indices can lead to the same conclusions as those that would be derived from aggregating first across dimensions and then across individuals only under very restrictive conditions on the aggregation functions. Namely, "… the overall deprivation of an individual must be a weighted average of her deprivations [i.e. proportionate shortfalls relative to benchmark values] in terms of the different attributes, and society's overall deprivation must be a simple average of the overall deprivation levels of the different individuals in the society" (Dutta et al., 2003, p. 202). Both conditions may be

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[17] Chakravarty and Majumder (2005) characterise a general family of deprivation indices that includes an index ordinally equivalent to *HPI* as a member.

debatable: the first because it implies that marginal rates of substitution between any pair of attributes are insensitive to the depths of deprivations; the second because it is liable to the same criticism levelled against the poverty gap by Sen (1976). Analogous results hold when the equivalence condition is set with respect to rankings rather than indices. Pattanaik et al. (2011) discuss further weaknesses of *HPI*-type measures.

Although composite indices may not be consistent with an approach which sees society's overall poverty as a function of individual poverty levels, as in standard welfare economics, they might be justified by taking a different set of ethical assumptions.

3.2. The counting approach

In many cases, we know more than the headcount poverty ratio for each dimension and we observe how many people are suffering from deprivation in one dimension, two dimensions, and so forth. Counting the number of failures is well rooted in the analysis of deprivation in social sciences, but the characteristics of the underlying social judgments and the relationship with standard welfare approaches still need clarification. Atkinson (2003), for instance, draws a parallel between the difficulty of deriving dominance conditions in the counting case and the failure of the headcount poverty measure to satisfy the Pigou-Dalton principle of transfers in the one-dimensional case. However, this difficulty stems from defining welfare criteria in terms of the distributions of the underlying continuous variables across people rather than in terms of the distribution of deprivation scores. As the deprivation score counts the number of dimensions in which an individual fails to achieve the minimum standards, it is by definition a discrete variable ranging from 0 to the number of dimensions considered. The distribution of deprivation scores contains all the relevant information in the counting approach, which by construction implies neglecting levels of achievement in the original variables. Dominance conditions in the counting approach can be established following this line of reasoning. In this section, we discuss these conditions and we show how they can yield counting measures that encompass those proposed by Atkinson (2003), Chakravarty and D'Ambrosio (2006) and Alkire and Foster (2011a, 2011b).

As standard in the counting literature, we assume that individuals might suffer from deprivation in r different dimensions and then sum the number of actual deprivations.[18] Let X_i be equal to 1 if an individual suffers from deprivation in the dimension i and 0 otherwise. Moreover, let

$$X = \sum_{i=1}^{r} X_i$$

be a random discrete variable with cumulative distribution function F and mean μ, and let F^{-1} denote the left inverse of F. Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call X the deprivation count and F the deprivation count distribution. Furthermore, let $q_k = \Pr(X = k)$, which yields

(3.2) $$F(k) = \sum_{j=0}^{k} q_j,\ k = 0,1,...,r$$

and

(3.3) $$\mu = \sum_{k=1}^{r} kq_k .$$

For the sake of simplicity, we are assigning equal weight to all dimensions, but this assumption can be relaxed (see Section 3.2.5).

[18] Cappellari and Jenkins (2007) observe that the practice of constructing raw deprivation sum-scores is "ubiquitous" but has weak theoretical foundations. They suggest that a promising alternative way to summarise multiple deprivations can rely on the item response modelling approach used in pychometrics and educational testing, although they find similar results in a comparison of the two approaches for British data.

In order to compare count distributions, we introduce appropriate dominance criteria to obtain partial orderings (Section 3.2.1) and complete orderings (Sections 3.2.2-3.2.4).[19] While the multidimensional approaches discussed in Section 3.3 focus on the distribution of people's achievements, the dominance criteria formulated for the counting approach are defined in terms of the distribution F of the univariate discrete variable X.

3.2.1. Partial orderings

As standard in the income distribution literature, the first criterion regards first-degree dominance.[20]

Definition 3.1. *A deprivation count distribution* F_1 *is said to first-degree dominate a deprivation count distribution* F_2 *if*

$$F_1(k) \geq F_2(k) \text{ for all } k = 0,1,...,r$$

and the inequality holds strictly for some k.

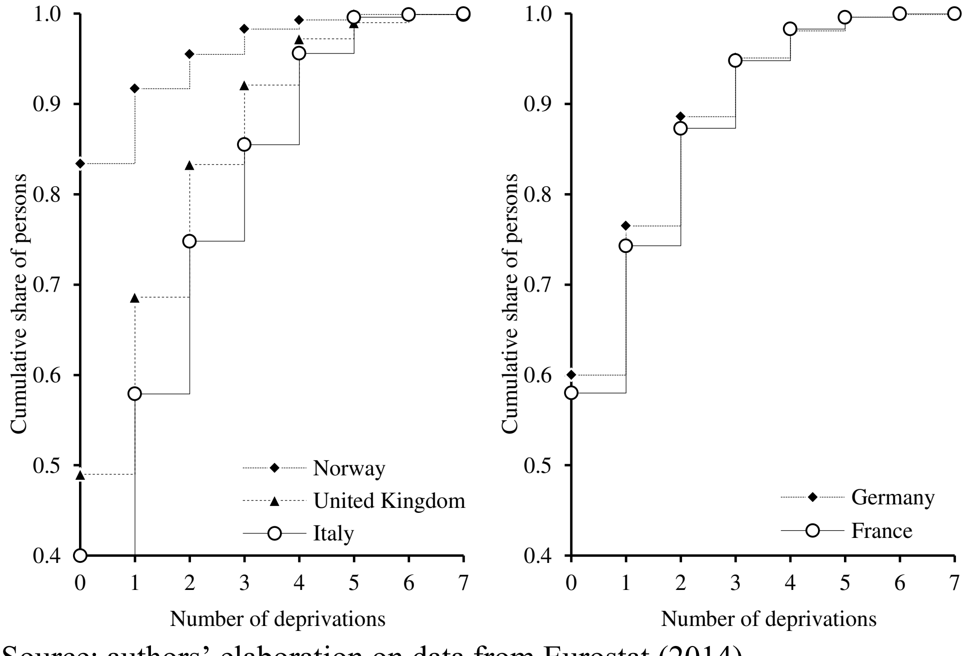
If F_1 first-degree dominates F_2, then F_1 exhibits less deprivation than F_2. An example is given in Figure 1, where we use the material deprivation indicators in five European countries in 2012 drawn from Eurostat (2014) and reported in Table 2. Figure 1 plots on the vertical axis the cumulative proportion of persons that suffer from deprivation in at most the number of dimensions indicated on the horizontal axis. (Figure 1 considers a maximum of seven deprivation items since nobody suffers from more than seven in the countries

[19] Lasso de la Vega (2010) and Yalonetzky (2014) also identify dominance conditions to rank deprivation count distributions.

[20] The first-degree stochastic dominance relations for integer variables representing the counting of people achievements, rather than deprivations, are studied by Chakravarty and Zoli (2012).

considered.) The left panel shows that Norway first-degree dominates both the United Kingdom and Italy, whereas the last two countries cannot be ordered by the criterion of first-degree dominance since their distributions intersect. The United Kingdom clearly lies ahead of Italy for up to five items, but then exhibits a share of people suffering from six or seven deprivations that is more than twice the Italian level (1 vs. 0.4 per cent, see Table 2). The right panel of Figure 1 shows that also the cumulative distributions of deprivations scores for France and Germany intersect, though being much closer. The share of non-deprived is higher in Germany than in France, and the same holds true when we sequentially add those with one, two and three deprivations; however, when we add people suffering from four deprivations the order reverses, and no longer changes when we consider more severe situations.[21]

Figure 1: Cumulative distributions of material deprivation scores in selected European countries in 2012



Source: authors' elaboration on data from Eurostat (2014).

[21] In this example and in all subsequent empirical illustrations, we treat statistics as they were exact and we abstract from the fact that they are subject to sampling and other types of errors. Accounting for these errors would possibly lead us to conclude that neither the observed difference between France and Germany nor the upper tail intersection between France and Norway is statistically significant.

Table 2. Distribution of material deprivations in selected European countries in 2012 (percentage of total population)

Number of deprivations	France	Germany	Italy	Norway	United Kingdom
None	58.0	60.0	39.6	83.4	49.0
1 item	16.3	16.5	18.3	8.3	19.6
2 items	13.0	12.1	16.9	3.8	14.7
3 items	7.5	6.5	10.7	2.8	8.8
4 items	3.5	3.0	10.1	1.0	5.1
5 items	1.3	1.5	4.0	0.6	1.8
6 items	0.4	0.3	0.3	0.0	0.9
7 items	0.0	0.1	0.1	0.1	0.1
8 items	0.0	0.0	0.0	0.0	0.0
9 items	0.0	0.0	0.0	0.0	0.0
All	100.0	100.0	100.0	100.0	100.0

Source: Eurostat (2014).

This example shows that first-degree dominance might be too demanding in practice: where count distributions intersect, they can be ranked only by defining weaker dominance criteria. This implies that we have to impose stricter conditions on the preference ordering of the social evaluator, taking into account that in the study of deprivation we might be leaning towards either the intersection or the union criteria. In the former case, we would start aggregating “from above”, looking first at the proportion of those who are deprived in r dimensions, then adding the proportion of those failing in $r-1$ dimensions, and so forth; in the latter case, we would start “from below”. This distinction leads naturally to the definition of two second-degree dominance criteria as suggested by Aaberge and Peluso (2011):

Definition 3.2A. *A deprivation count distribution F_1 is said to second-degree downward dominate a deprivation count distribution F_2 if*

$$\sum_{k=s}^{r} F_1(k) \geq \sum_{k=s}^{r} F_2(k) \quad \text{for all } s = 0,1,\dots,r$$

and the inequality holds strictly for some s.

Definition 3.2B. *A deprivation count distribution* F_1 *is said to second-degree upward dominate a deprivation count distribution* F_2 *if*

$$\sum_{k=0}^{s} F_1(k) \geq \sum_{k=0}^{s} F_2(k) \quad \textit{for all } s = 0,1,...,r$$

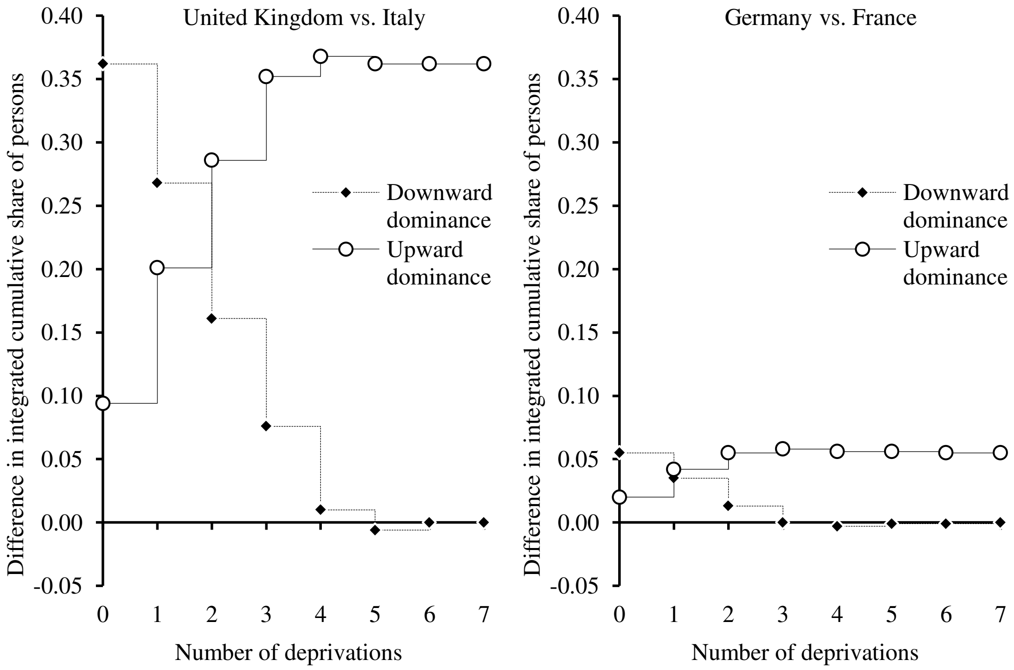
and the inequality holds strictly for some s.

If F_1 second-degree dominates F_2, then F_1 exhibits less deprivation than F_2, as before, but this result is now obtained at the cost of imposing the stricter conditions on the preference ordering that will be shown below by Theorems 3.1A and 3.1B. Moreover, we have to make a choice between being more concerned with the extent to which deprivation is diffused across the population (union criterion) or instead with the occurrence of multiple deprivations (intersection criterion). In the first case, we would adopt second-degree upward dominance. Intuitively, we can see this in Definition 3.2B from the fact that we are making comparisons on (doubly) cumulated population proportions that start by considering the share of people who do not suffer from any deprivation, $F(0)$, and sequentially add the shares of those who suffer from 1 deprivation, then those who suffer from 2 deprivations, and so forth. In calculating the cumulative function we "go up". The opposite happens in the second case, where we aggregate "going down", thus placing more weight on the most deprived. Formally, second-degree upward dominance parallels the dominance criterion used by Atkinson (1970) for ranking income distributions. Second-degree downward dominance has no correspondent in the income inequality literature, as it would be inconsistent with the Pigou-Dalton principle of transfers. It is however analogous to the criterion introduced for Lorenz curves by Aaberge (2009).

Is agreeing on whether "to go up" (union criterion) or "to go down" (intersection criterion) when we aggregate deprivation scores sufficient in empirical applications? Not always. This can be seen by reconsidering the previous comparisons of Italy and the United

Kingdom, and of France and Germany, where neither country in each comparison was found to first-degree dominate the other. In Figure 2 we plot the difference between the integrated cumulative distributions considered by Definitions 3.2A and 3.2B for each pair of countries.

Figure 2: Second-degree dominance for material deprivation scores in selected European countries in 2012



Source: authors' elaboration on data from Eurostat (2014).

If we integrate going up as in Definition 3.2B, the United Kingdom and Germany second-degree (upward) dominate Italy and France, respectively: the lower proportions of people who do not suffer from any deprivation give the first two countries an advantage that is not offset by their worst results for the incidence of people deprived in many dimensions. On the other hand, if we integrate going down as in Definition 3.2A, the difference between the integrated cumulative distributions changes from positive to negative and no country second-degree (downward) dominates the other in either comparison. The distribution of deprivation scores enables social evaluators favouring the union perspective to rank the United Kingdom and Germany ahead of Italy and France, but do not allow social evaluators supporting the intersection perspective to draw unambiguous conclusions. In such a case, higher degree criteria are needed, although they could still provide a partial ordering. The exploration of higher-order dominance criteria is a topic for further research. We turn instead to methods that can lead to a complete ordering.

3.2.2. Complete orderings: the independence axioms

A complete ordering can be achieved by imposing an independence axiom for the preference ordering. This allows us to weight differently certain parts of the distributions and eventually to define a summary measure of deprivation. Formally, let social preferences be represented by the ordering $\succeq$ defined on the family of deprivation count distributions *F*. This preference ordering is assumed to be continuous, transitive and complete and to satisfy the condition of first-degree count distribution dominance. As proved by Debreu (1964), a preference ordering that is continuous, transitive and complete can be represented by a continuous and increasing preference functional. We need, however, further conditions to give social preferences an explicit empirical content. We therefore introduce two alternative independence conditions, which require that the preference ordering is invariant with respect to certain changes in the count distributions being compared:

Axiom (Independence). *Let F_1 and F_2 be members of $\boldsymbol{F}$. Then $F_1 \succeq F_2$ implies $\alpha F_1 + (1-\alpha)F_3 \succeq \alpha F_2 + (1-\alpha)F_3$ for all $F_3 \in \boldsymbol{F}$ and $\alpha \in [0,1]$.*

This axiom focuses on the proportions of people suffering from given numbers of deprivations (the *F*). We could instead focus on the number of deprivations that is associated with a given proportion of people, that is, more technically, the rank in the count distribution (the F^{-1}). This corresponds to an alternative version of the independence axiom, as in the literatures on uncertainty and inequality:

Axiom (Dual Independence). *Let F_1 and F_2 be members of* $\boldsymbol{F}$. *Then* $F_1 \succeq F_2$ *implies*

$$\left(\alpha F_1^{-1}+(1-\alpha)F_3^{-1}\right)^{-1} \succeq \left(\alpha F_2^{-1}+(1-\alpha)F_3^{-1}\right)^{-1} \textit{ for all } F_3 \in \boldsymbol{F} \textit{ and } \alpha \in [0,1].$$

If F_1 is weakly preferred to F_2, then the Independence axiom (similar to the expected utility theory) states that any mixture on F_1 is weakly preferred to the corresponding mixture on F_2: identical mixing interventions on the count distributions do not affect their ranking, which depends solely on how the differences between the mixed count distributions are judged. Thus, if the overall count deprivation is lower in country 1 than in country 2, so that $F_1 \succeq F_2$, the ranking would not change by adding to the population of either country the same group of migrants, whose deprivation distribution is F_3. The ordering relation $\succeq$ is therefore invariant with respect to aggregation of sub-populations across deprivations.

The Dual Independence axiom shifts the attention toward aggregating subsets of deprivation dimensions across proportions of people. Assume that there are only two deprivation indicators, income and health, and that two alternative tax and benefit regimes produce the two count deprivation distributions F_1 and F_2 for income. Next, match F_1 with the count deprivation distribution F_3 for health in such a way that the most deprived person in income is also the most deprived person in health, the second most deprived person in income is the second most deprived person in health, and so on. Match F_2 and F_3 in the same way. If the count deprivation distribution F_1 is preferred to F_2 for income, then the share of income-deprived people under regime 1 is lower than the corresponding share under regime 2. Dual Independence means that, given any distribution F_3 of health deprivation counts, F_1 will continue to be preferred to F_2 after matching either F_1 or F_2 with F_3.[22] The Dual

[22] This argument parallels the rationale offered by Weymark (1981, p. 418) for his "Weak Independence of Income Source" axiom: "if in two income distributions the incomes from all but one type of income are the same in both distributions, then the overall judgement that one

Independence axiom imposes this invariance property regardless of the shape of the count deprivation distribution for health (F_3) and of the weights used for such a matching (α).

The essential difference between the two axioms is that the Independence axiom deals with the relationship between given number of deprivations and weighted averages of the corresponding population proportions, while the Dual Independence axiom deals with the relationship between given population proportions and weighted averages of the corresponding numbers of deprivations. No one has so far provided a convincing justification for preferring one axiom to the other, but the choice of the axiom yields summary measures of deprivation with different decomposition properties. For instance, indices consistent with the Independence axiom can be expressed as weighted averages of the corresponding indices computed for mutually exclusive population subgroups, whereas the indices satisfying the Dual Independence axiom cannot. By contrast, the dual measures offer a convenient decomposition by sources of deprivation, whereas the measures associated with the Independence axiom cannot. Moreover, as measures of income inequality they have the convenient property of being expressed as linear functionals of the Lorenz curve, whereas the primal measures cannot.

The "primal approach", based on the Independence axiom, is analogue to the inequality framework developed by Atkinson (1970) and parallels the discussion of the headcount curves by Aaberge and Atkinson (2013). The "dual approach", based on the Dual Independence axiom, is analogue to the rank-dependent measurement of inequality introduced by Weymark (1981) and Yaari (1988) and to the way to summarise the informational content of Lorenz curves by Aaberge (2001). In what follows we draw on

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distribution is more unequal than a second is completely determined by a comparison of the distributions of income from the variable source". Gajdos and Weymark (2005) call the corresponding multidimensional condition "Weak Comonotonic Additivity".

Aaberge and Peluso (2011) for the dual approach and Aaberge and Brandolini (2014) for the primal approach.

3.2.3. Complete orderings: the dual approach

The Dual Independence Axiom can be used to justify the following family of deprivation measures

$$(3.4)\qquad D_\Gamma(F) = r - \sum_{k=0}^{r-1}\Gamma(\sum_{j=0}^{k} q_j) = \begin{cases} \mu + \Delta_\Gamma(F) & when\, \Gamma\, is\ convex \\ \mu - \Delta_\Gamma(F) & when\, \Gamma\, is\ concave \end{cases},$$

where

$$(3.5)\qquad \Delta_\Gamma(F) = \begin{cases} \sum_{k=0}^{r-1}\left[\sum_{j=0}^{k} q_j - \Gamma(\sum_{j=0}^{k} q_j)\right] & when\, \Gamma\, is\ convex \\ \sum_{k=0}^{r-1}\left[\Gamma(\sum_{j=0}^{k} q_j) - \sum_{j=0}^{k} q_j\right] & when\, \Gamma\, is\ concave \end{cases},$$

and Γ, with $\Gamma(0)=0$ and $\Gamma(1)=1$, is a non-negative, non-decreasing continuous function. Since *F* denotes the distribution of the deprivation count, $D_\Gamma(F)$ can be treated as a summary measure of deprivation exhibited by the distribution *F*. It can be seen as the social evaluation function corresponding to the social preference relation which identifies the most favourable distribution *F* with the one that minimizes $D_\Gamma(F)$. These social preferences are shaped by the specification of the function Γ, which can be considered as a deprivation intensity function. $D_\Gamma(F)$ can be decomposed into the mean number of deprivations, μ, and a term that captures the dispersion of deprivations across the population, Δ_Γ. By definition Δ_Γ is always non-negative and measures left-tail heaviness (left-spread) when Γ is concave and right-tail heaviness (right-spread) when Γ is convex. It follows that $\mu \le D_\Gamma(F) \le r$ when Γ is convex, and $0 \le D_\Gamma(F) \le \mu$ when Γ is concave. If Γ is convex, the minimum value μ of $D_\Gamma(F)$ is attained when $\Delta_\Gamma(F)=0$, that is when each individual suffers from the same number μ of deprivations. If everybody suffers from all *r* deprivations, $\Delta_\Gamma(F)$ still equals 0, but $D_\Gamma(F)$ reaches its maximum value *r*. Conversely, $\Delta_\Gamma(F)$ is maximum when half of the population does not suffer from any deprivation and the remaining half suffer from all, so that $D_\Gamma(F) = r\left[1-\Gamma(0.5)\right]$. The comparison between the last two cases illustrates how the index works: a situation where everybody suffers from *r* deprivations is definitely worse than one where only half of the population suffers from *r* deprivations. But the extent to which the two situations are valued differently depends on the convexity of Γ: the more convex it is, the more weight we give to multiple deprivations, and the closer $D_\Gamma(F)$ is to *r*. A similar reasoning applies, *mutatis mutandis*, for concave Γ.

Expression (3.4) shows that an exclusive concern for the mean number of deprivations implies linear (both convex and concave) social preferences: $\Gamma(t) = t$. There is indifference between a situation where *s* people have one deprivation and a situation where only one person is deprived but in *s* dimensions. It is the same result that we would obtain by applying the composite index approach discussed in Section 3.1. It is another way to appreciate the restrictions imposed on social preferences in that approach. When there is a concern for the distribution of deprivations across the population, the critical judgement is whether this concern should prioritise the intensity or the diffusion of deprivations. In the former case, social preferences pay more attention to one person with *s* deprivations than to *s* people with one deprivation each, and the measure D_Γ should embody a convex Γ. In the latter case, social preferences take the opposite stance, and the measure D_Γ should embody a concave Γ. With Γ concave, for a given μ, D_Γ decreases as Δ_Γ increases because the distribution of deprivations across the population shifts towards people with none or fewer deprivations, i.e. to the left tail of the distribution.

There is then a correspondence between convexity and the intersection criterion, on one side, and concavity and the union criterion, on the other. This can be seen by taking particular specifications of the function Γ. With the union criterion, the focus is on the proportion of people who suffers from deprivation in at least one dimension ($1-q_0$). By specifying Γ as

$$(3.6) \quad \Gamma(t) = \begin{cases} q_0 & \text{if } t = q_0 \\ 1 & \text{if } q_0 < t \le 1 \end{cases},$$

we get $D_\Gamma(F) = 1 - q_0$, which means that the union measure can be considered as a limiting case of the D_Γ-family of deprivation measures in the concave case. With the intersection criterion, the focus is on the proportion of people deprived in all dimensions (q_r). The following alternative specification for Γ,

$$(3.7) \quad \Gamma(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 - q_r \\ 1 - q_r & \text{if } 1 - q_r \le t \le 1 \end{cases},$$

yields $D_\Gamma(F) = r - 1 + q_r$, which means that also the intersection measure represents a limiting case of the D_Γ-family of deprivation measures in the convex case. Although the union and intersection measures do not belong to the D_Γ-family, which is generated by continuous Γ functions, they can be approximated within this class (see Le Breton and Peluso, 2010, for general approximation results).

A Gini version of the measure of deprivation D_Γ obtains by taking $\Gamma(t) = 2t - t^2$ (concave) or $\Gamma(t) = t^2$ (convex), so that Δ_Γ equals the Gini mean difference. A general family of deprivation measures associated with the Lorenz family of inequality measures (Aaberge, 2000) is obtained by using the specification $\Gamma(t) = t^\tau$ where the parameter $\tau > 0$ captures the concern for deprivation inequality, paying more attention towards the lower tail when $0 < \tau < 1$ and to the upper tail when $\tau > 1$ (Aaberge and Peluso, 2011).

3.2.4. Complete orderings: the primal approach

The Independence Axiom provides a justification for the following alternative family of deprivation measures,

$$(3.8) \quad d_\gamma(F) = \sum_{k=0}^{r} \gamma(k) q_k = \begin{cases} \gamma(\mu) + \delta_\gamma(F) & \text{when } \gamma \text{ is convex} \\ \gamma(\mu) - \delta_\gamma(F) & \text{when } \gamma \text{ is concave} \end{cases},$$

where

$$(3.9) \quad \delta_\gamma(F) = \begin{cases} \sum_{k=0}^{r} \left(\gamma(k) - \gamma(\mu)\right) q_k & \text{when } \gamma \text{ is convex} \\ \sum_{k=0}^{r} \left(\gamma(\mu) - \gamma(k)\right) q_k & \text{when } \gamma \text{ is concave} \end{cases},$$

and $\gamma(k)$, with $\gamma(0) = 0$, is a non-negative, non-decreasing continuous function of the number of deprivations *k*. As Γ in the dual case, γ can be considered as a deprivation intensity function, whose curvature determines how much we dislike increasingly severe deprivations in the convex case, or growingly diffused deprivations in the concave case. This family of deprivation measures is analogue to the family of inequality measures introduced by Kolm (1969) and Atkinson (1970). Chakravarty and D'Ambrosio (2006) provide an alternative axiomatic justification of (3.8) with a convex γ for measuring social exclusion.[23]

[23] Unlike the discussion in this Section, Chakravarty and D'Ambrosio (2006) focus on the distribution of deprivation scores across people rather than on the distribution of deprivation scores itself. They also prove that second-degree downward dominance implies a convex γ and is preserved under a "favourable composite change", which is an intervention principle that is closely related to the Pigou-Dalton principle of transfers. This principle differs from the association rearrangement principles motivated by the measurement of multidimensional poverty and discussed in Section 3.2.5. The index characterised by Bossert et al. (2013) is a special (linear) case of Chakravarty and D'Ambrosio's social exclusion measure. In a related paper, Bossert et al. (2007) use the counting approach to derive a further measure of social exclusion. They define axioms such that the degree of deprivation of an individual is proportional to the product of the share of people who suffer from fewer deprivations than he does and the mean difference between his deprivation score and that of all people who are better off: summation of these individual functions across individuals and then over time yield the aggregate deprivation and social exclusion indices, respectively.

As in the dual case, the primal measures $d_\gamma(F)$ can be considered as a social evaluation function where preferences favour the count distribution *F* that minimises $d_\gamma(F)$. The primal measures $d_\gamma(F)$ can be decomposed into a first term which is a transformation of the mean μ and a second term $\delta_\gamma(F)$ which measures the left- or right-tail heaviness when γ is concave or convex. By inserting $\gamma(k) = 2rk - k^2$ (concave) and $\gamma(k) = k^2$ (convex) in (3.9), the term $\delta_\gamma(F)$ equals the variance. When $\gamma(k) = k$ for all *k*, $d_\gamma(F) = \mu$ and only the mean matters: social preferences ignore the dispersion of deprivations.[24] When the dispersion matters, as in the dual case the judgement depends on whether social preferences give more weight to *s* people with one deprivation each or to one person with *s* deprivations, which means choosing a concave function γ in the first case and a convex function in the second. Indeed, the union criterion is a limiting case of the d_γ-family of deprivation measures for concave γ, while the intersection criterion is a limiting case for convex γ.[25] With a concave γ the dispersion term is subtracted from the (transformed) mean and $0 \le d_\gamma(F) \le \gamma(\mu)$, whereas with a convex γ the opposite happens and $\gamma(\mu) \le d_\gamma(F) \le \gamma(r)$.

Unlike the dual measures, the primal measures are exactly decomposable by population subgroups, in the sense that the index computed for the overall population equals the weighted average of the measures calculated for each subgroup, with weights equal to the

[24] As seen, both D_Γ and d_γ can coincide with the mean μ for certain specifications of social preferences ($\Gamma(t) = t$ and $\gamma(k) = k$). From the proof of Theorem 5 in Aaberge (2001), it follows that the mean is the only measure of deprivation that satisfies both the independence and the dual independence axioms. Thus, the independence and the dual independence axioms provide, together with the conditions of transitivity, completeness, continuity and first-degree dominance, a complete axiomatic characterization of the mean μ. In the alternative axiomatic justification for the mean offered by Bossert et al. (2013), two conditions of subgroup decomposability play a similar role as the two independence axioms.
[25] This can be seen by approximating the concave function γ with $\gamma(k) = 1$ for $k = 1, 2, ..., r$ and the convex function γ with $\gamma(k) = 0$ for $k = 1, 2, ..., r-1$ and $\gamma(r) = 1$, which yield $d_\gamma(F) = 1 - q_0$ and $d_\gamma(F) = q_r$, respectively.

respective population shares of the subgroups. Note, however, that the dual measures may admit a different decomposition into within-groups and between-groups components, along the lines suggested by Ebert (2010).

The measure d_γ generalises the counting measure proposed by Atkinson (2003, p. 62) for a bivariate distribution ($r=2$). Atkinson's measure A_θ can be written as

(3.10) $A_\theta = 2^{-\theta}\left[p_{1+}+p_{+1}+2(2^{\theta-1}-1)p_{11}\right] = 2^{-\theta}\left(p_{1+}+p_{+1}\right)+\left(1-2^{1-\theta}\right)p_{11} = 2^{-\theta}q_1+q_2$,

by making use of the notation of Table 1 and after dividing through the original formula by 2^θ. We can obtain (3.10) from (3.8) by inserting $\gamma(k)=\left(k/r\right)^\theta$ and $r=2$. The parameter θ varies from 0 to infinity and is introduced by Atkinson to capture alternative views on the importance of multiple deprivations. (Strictly speaking, both extreme values are inconsistent with the assumed continuity of the function γ, and should be seen as limiting cases.) When $\theta \to 0$, the index counts all people with at least one deprivation, regardless of their number for each individual: $A_0 = p_{1+}+p_{+1}-p_{11} = q_1+q_2$. When $\theta=1$, people with two deprivations are counted twice and A_1 gives the simple mean of the headcount rates in the two dimensions, providing the same result as with a composite index. As θ goes to infinity, the index tends to coincide with the proportion of people deprived on both dimensions: $A_\infty \to p_{11}$. As the original Atkinson's counting deprivation index, its generalisation to more than two dimensions obtained by inserting $\gamma(k)=\left(k/r\right)^\theta$ in (3.8) embodies, as limiting cases, both the union criterion (A_0) and the intersection criterion (A_∞). This index characterises a family of deprivation measures that may be seen as the analogue of the poverty measures proposed by Foster et al. (1984), referred to as the *FGT* measures.

The decomposition of the primal and dual measures of deprivation in terms of mean (or transformation of the mean) and dispersion of the deprivation count distributions parallels the mean-inequality decomposition of the social welfare functions derived from the expected and rank-dependent utility-like theories (see Atkinson, 1970, and Yaari, 1988). Note, however, that differently from the income inequality analysis, the structure of the decomposition of the deprivation measures depends on whether social preferences are associated with the union or the intersection criterion. In the former case the deprivation measures fall and social welfare rises when the dispersion of deprivation across the population goes up, meaning that more people are affected by few or no deprivations. Even though they allow for the decomposition in terms of mean dispersion of deprivation, the primal and dual summary measures are silent about the role played by each dimension. Thus, the information provided by these summary measures should be complemented with estimates of the proportions of people who suffer from deprivation in each of the dimensions. This information reveals whether deprivation is concentrated on few or many dimensions.

Table 3 shows the estimates for some deprivation indices for the five European countries considered earlier. (Some indices are discussed in the next sections.) As regards dual measures, we consider the class of indices associated with the Lorenz family of inequality measures,

$$D_{\tau}^{GG} = r - \sum_{k=0}^{r-1}\left(\sum_{j=0}^{k} q_j\right)^{\tau}$$

for various values of the parameters τ. For $\tau = 2$, the previous expression gives the convex version of the Gini-type measure of deprivation, while the concave version is given by:

$$D_2^{G,concave} = 2\mu - r + \sum_{k=0}^{r-1}\left(\sum_{j=0}^{k} q_j\right)^2 = 2\mu - D_2^{GG} = 2\mu - D_2^{G,convex}.$$

As regards primal measures, we consider the generalised Atkinson-type class of indices

$$d_{\theta}^{GA} = r^{-\theta}\sum_{k=1}^{r} k^{\theta} q_k$$

for various values of the parameters θ. For $\theta = 1$, the previous expression gives the mean headcount ratio, which equals the ratio μ / r. For $\theta = 2$, it coincides with the convex version of the variance-type measure of deprivation $d_2^{V,convex}$ multiplied by r^{-2}, while the concave version ($\gamma(k) = 2rk - k^2$) is given by:

$$d_2^{V,concave} = 2r\mu - \sum_{k=1}^{r} k^2 q_k = 2r\mu - r^2 d_2^{GA} = 2r\mu - d_2^{V,convex}.$$

Norway shows the lowest mean number of deprivations followed by Germany and France, rather close each other, the United Kingdom, and finally Italy. The mean headcount ratio ranges between 3.6 per cent in Norway and 16.3 per cent in Italy. With a concave index, we always find that deprivation is lower in Germany than in France and in the United Kingdom than in Italy, which is not surprising in the light of the results on second-degree upward dominance reported in Section 3.2.1. On the other hand, the lack of second-degree downward dominance in these same comparisons is noticeable in the fact that the rankings reverse as the functions become more convex. For instance, the generalised Atkinson-type deprivation index turns out to be lower in France than in Germany for values of θ higher than 4. The French overall deprivation is below the German level whenever we favour the intersection criterion and weight somebody suffering from $2h$ deprivations at least 16 ($= 2^4$) times somebody suffering from h deprivations (as the index d_4^{GA} assigns each person with h deprivations a weight equal to h^4). Since the United Kingdom fares much better than Italy except than in the occurrence of very severe deprivation (6 or more items), the ranking between the two countries changes only for high values of θ or τ, which correspond to an extreme aversion to the worst conditions of deprivations. Finally, note that the generalised Atkinson-type deprivation index approaches the proportion of people experiencing at least one deprivation (union criterion) as θ tends to 0 and the proportion of people suffering from

Table 3. Indices of material deprivations in selected European countries in 2012

Index	Germany	France	Italy	United Kingdom	Norway	Germany vs. France	United Kingdom vs. Italy
Linear indices							
Mean deprivations	0.822	0.877	1.471	1.109	0.320	-6.3	-24.6
Mean headcount ratio	0.091	0.097	0.163	0.123	0.036	-6.3	-24.6
Concave indices							
D_{τ}^{GG} $\tau = 0.1$	0.096	0.103	0.191	0.136	0.034	-7.3	-28.7
$\tau = 0.5$	0.446	0.479	0.845	0.619	0.165	-6.8	-26.7
$\tau = 0.9$	0.752	0.803	1.360	1.020	0.290	-6.4	-25.0
$D_2^{G,concave}$	0.231	0.262	0.629	0.394	0.037	-11.7	-37.4
d_{θ}^{GA} $\theta \to 0$	0.400	0.420	0.604	0.510	0.166	-4.8	-15.6
$\theta = 0.1$	0.340	0.358	0.523	0.436	0.140	-5.0	-16.6
$\theta = 0.5$	0.184	0.195	0.303	0.241	0.074	-5.7	-20.4
$\theta = 0.9$	0.104	0.111	0.184	0.140	0.041	-6.2	-23.8
$d_2^{V,concave}$	12.550	13.399	21.883	16.747	4.914	-6.3	-23.5
Convex indices							
D_{τ}^{GG} $\tau = 1.1$	0.890	0.948	1.576	1.195	0.350	-6.2	-24.2
$\tau = 5$	2.453	2.537	3.460	2.942	1.280	-3.3	-15.0
$\tau = 19$	3.906	3.910	4.612	4.368	2.799	-0.1	-5.3
$\tau = 21$	4.003	3.998	4.673	4.461	2.917	0.1	-4.5
$\tau = 40$	4.581	4.522	5.020	5.011	3.629	1.3	-0.2
$\tau = 42$	4.622	4.559	5.044	5.050	3.680	1.4	0.1
$\tau = 100$	5.272	5.145	5.414	5.670	4.505	2.5	4.7
$D_2^{G,convex} = D_2^{GG}$	1.413	1.492	2.313	1.824	0.603	-5.3	-21.1
d_{θ}^{GA} $\theta = 1.1$	0.080	0.086	0.146	0.109	0.031	-6.3	-25.3
$\theta = 2$	0.028	0.029	0.057	0.040	0.010	-5.9	-30.0
$\theta = 3$	0.011	0.011	0.023	0.016	0.004	-3.6	-31.6
$\theta = 4$	0.005	0.005	0.010	0.007	0.002	0.4	-30.1
$\theta = 8$	0.001	0.001	0.001	0.001	0.000	20.6	-13.5
$\theta = 9$	0.0003	0.0002	0.0005	0.0005	0.0001	42.8	2.3
$\theta = 20$	7.6×10^{-06}	1.3×10^{-06}	7.8×10^{-06}	9.4×10^{-06}	6.6×10^{-06}	479.9	20.9
$d_2^{V,convex} = r^2 d_2^{GA}$	2.246	2.387	4.595	3.215	0.846	-5.9	-30.0
Other indices							
Eurostat SMD (1)	0.049	0.052	0.145	0.079	0.017	-5.8	-45.5

Source: authors' elaboration on data from Eurostat (2014).
Note: (1) Figures are computed from Table 2 and may differ from published statistics because of rounding.

the maximum number of deprivations (intersection criterion) as θ goes to infinity; as nobody lacks all nine items, in the latter case the index converges to zero in all countries.

3.2.5. Association rearrangements

In many respects, the discussion so far has proceeded as in the case of a single variable, whereas the key feature of the multivariate case is the pattern of association across dimensions. It is then natural to ask how social welfare responds to a change in the distribution of deprivations across the population, though the total number of deprivations remains the same. The standard approach is to consider how social welfare varies after a "marginal-free change" in the association between two variables, which is a change that does not affect the marginal distributions. As in the statistical literature on the measurement of association in multidimensional contingency tables (formed by two or several binary variables), we distinguish association rearrangements for distributions characterised by either positive or negative association. Illustrations of marginal-free association rearrangements are provided by Tables 4 and 5. The right (left) panel of Table 4 is obtained from the left (right) panel by a marginal-free positive association increasing (decreasing) rearrangement, whereas the right (left) panel of Table 5 can be obtained from the left (right) panel by a negative association increasing (decreasing) rearrangement.

Marginal-free rearrangements have been widely used as a basis for evaluating multidimensional measures of poverty and inequality.[26] Bourguignon and Chakravarty (1999, 2003, 2009) and Atkinson (2003) use the principle of marginal-free correlation increasing shifts as a basis for making a normative judgement of poverty measures derived from

[26] For definitions of association increasing rearrangements based on the correlation coefficient we refer to Epstein and Tanny (1980), Atkinson and Bourguignon (1982), Boland and Proschan (1988) Dardanoni (1995), Tsui (1995, 1999, 2002), Bourguignon and Chakravarty (2003), and Duclos et al. (2006a). See also Tchen (1980) who deals with positive association (or concordance) between bivariate probability measures.

continuous variables (attributes) rather than from deprivation scores. They distinguish whether the poverty measure increases or decreases because of a correlation increasing shift, and consider the associated attributes to be substitutes (one attribute can compensate for the lack of the other) in the former case and to be complements in the latter.

Table 4. Illustration of a marginal-free positive association increasing rearrangement

	X_2=0	X_2=1	
X_1=0	0.35	0.20	0.55
X_1=1	0.20	0.25	0.45
	0.55	0.45	1

	X_2=0	X_2=1	
X_1=0	0.36	0.19	0.55
X_1=1	0.19	0.26	0.45
	0.55	0.45	1

Source: authors' elaboration.

Table 5. Illustration of a marginal-free negative association increasing rearrangement

	X_2=0	X_2=1	
X_1=0	0.20	0.25	0.45
X_1=1	0.35	0.20	0.55
	0.55	0.45	1

	X_2=0	X_2=1	
X_1=0	0.19	0.26	0.45
X_1=1	0.36	0.19	0.55
	0.55	0.45	1

Source: authors' elaboration.

Considering marginal-free changes is a neat way to highlight that the multidimensional analysis of poverty and inequality implies making assumptions on the degree to which the different attributes can be substituted one for the other. In the real world, the condition of marginal-free changes may be too restrictive, as policies may reduce deprivation in one dimension at the cost of increasing deprivation in another. We hence adopt a more general approach and we require that only the mean number of deprivations but not the marginal distributions be kept fixed. (The latter implies the former, but not vice versa.) It follows that we need a measure of association that is invariant with regard to changes in the marginal distributions, unlike the correlation coefficient. This is the case of the cross-product κ introduced by Yule (1900). In the 2x2 distribution of Table 1, Yule's measure is defined by

$$(3.11) \quad \kappa = \frac{p_{00}p_{11}}{p_{01}p_{10}},$$

which is invariant to the transformation $p_{ij} \rightarrow a_i b_j p_{ij}$. This association measure, together with the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}), provides complete information on the distribution and does not change if the marginal distributions change.[27] Note that $\kappa \in [0, \infty\rangle$, $\kappa = 1$ if X_1 and X_2 are independent, $\kappa = 0$ if there is perfect negative association ($p_{00} = 0$ and/or $p_{11} = 0$), and $\kappa \rightarrow \infty$ if there is perfect positive association ($p_{01} = 0$ and/or $p_{10} = 0$).

Following Aaberge and Peluso (2011) and Aaberge and Brandolini (2014), we relax the marginal-free condition by introducing an association increasing/decreasing rearrangement principle that relies on the condition of fixed overall mean number of deprivations rather than on the condition of fixed proportions of people suffering from each deprivation. As illustrated by Tables 4 and 5 marginal-free arrangements are special cases of this alternative rearrangement principle.[28]

Definition 3.3. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum_i \sum_j p_{ij} = 1$. *The change* $(p_{00} + \varepsilon, p_{01}, p_{10} - 2\varepsilon, p_{11} + \varepsilon)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\varepsilon > 0$ *(* $\varepsilon < 0$ *) and* $\kappa > 1$*, and a mean preserving negative association increasing (decreasing) rearrangement if* $\varepsilon < 0$ *(* $\varepsilon > 0$ *) and* $\kappa < 1$.

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[27] Yule's measure of association is related to the copula-based measures of association for continuous variables introduced by Spearman and Kendall; see e.g. Nelsen (1998). Decancq (2014) introduces a copula-based generalization of the rearrangement principles for continuous variables and provides an analysis of their links with stochastic dominance. If X_1 and X_2 represent the two social class categories to which an individual can belong at times 1 and 2, the Yule's measure of association also coincides with the "odds ratio" used in mobility studies. See, for instance, Erikson and Goldthorpe (1993, p. 55).

[28] Note that the multinomial distribution defined by the parameters p_{00}, p_{10}, p_{01} and $p_{11} (= 1 - p_{00} - p_{10} - p_{01})$ can alternatively be described by the marginal distributions $(p_{0+}, p_{1+} = 1 - p_{0+})$ and $(p_{+0}, p_{+1} = 1 - p_{+0})$, and the cross-product κ.

It follows from Definition 3.3 that a mean preserving rearrangement reduces the number of people deprived according to indicator X_1 at the cost of increasing the number of people deprived according to indicator X_2 when $\varepsilon > 0$ and vice versa when $\varepsilon < 0$. This is illustrated in Table 6, which shows two distributions where the association is negative ($\kappa < 1$) and the mean is equal to 1. The right (left) panel can be obtained from the left (right) panel by a mean preserving negative association decreasing (increasing) rearrangement where $\varepsilon = 0.01$

Table 6. Illustration of a mean preserving negative association decreasing rearrangement

	X_2=0	X_2=1	
X_1=0	0.20	0.30	0.50
X_1=1	0.30	0.20	0.50
	0.50	0.50	1

	X_2=0	X_2=1	
X_1=0	0.21	0.30	0.51
X_1=1	0.28	0.21	0.49
	0.49	0.51	1

Source: authors' elaboration.

Aaberge and Peluso (2011) show how to extend Definition 3.3 to *r* dimensions. As the standard subscript notation becomes cumbersome for more than two dimensions, they simplify the notation to p_{ijm}, where *i* and *j* represent two arbitrary chosen deprivation dimensions and *m* represents the remaining $r-2$ dimensions. The Yule's measure κ_{ijm} is defined by

$$(3.12) \quad \kappa_{ijm} = \frac{p_{iim} p_{jjm}}{p_{ijm} p_{jim}},$$

where *m* is a $(r-2)$-dimensional vector of any combination of zeroes and ones. In this case, the association is defined by $r(r-1)/2$ cross-products. Aaberge and Peluso (2011) introduce the following generalization of Definition 3.3:

Definition 3.4A. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum_i \sum_j \sum_m p_{ijm} = 1$ *and* $\kappa_{ijm} > 1$. *The following* *change* $(p_{iim}+\varepsilon, p_{ijm}, p_{jim}-2\varepsilon, p_{jjm}+\varepsilon)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\varepsilon > 0$ *(* $\varepsilon < 0$ *).*

Definition 3.4B. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum_i \sum_j \sum_m p_{ijm} = 1$ *and* $\kappa_{ijm} < 1$. *The following change* $(p_{iim}+\varepsilon, p_{ijm}, p_{jim}-2\varepsilon, p_{jjm}+\varepsilon)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\varepsilon < 0$ *(* $\varepsilon > 0$ *).*

Theorems 3.1A demonstrates that social preferences favouring second-degree downward dominance imply that overall deprivation rises after a mean preserving positive association increasing rearrangement as well as a mean preserving negative association decreasing rearrangement, irrespective of whether preferences are consistent with the primal or the dual approach. By contrast, Theorem 3.1B proves that preferences favouring upward second-degree dominance consider such rearrangement as a reduction in the overall deprivation. Moreover, it follows directly from the decompositions (3.4) and (3.8) that the principles of mean preserving association increasing/decreasing rearrangement are equivalent to the mean preserving spread/contraction defined by:

Definition 3.5. *Let* F_1 *and* F_2 *be members of the family* ***F*** *of count distributions based on r deprivations and assume that they have equal means. Then* F_2 *is said to differ from* F_1 *by a mean preserving spread (contraction) if* $\Delta_\Gamma(F_2) > \Delta_\Gamma(F_1)$ *for all convex* Γ *or* $\delta_\gamma(F_2) > \delta_\gamma(F_1)$ *for all convex* γ *(* $\Delta_\Gamma(F_2) < \Delta_\Gamma(F_1)$ *for all concave* Γ *or* $\delta_\gamma(F_2) < \delta_\gamma(F_1)$ *for all concave* γ *).*

Note that Definition 3.5 is equivalent to a sequence of the mean preserving spread introduced by Rothschild and Stiglitz (1970).

Let Ω_1 and Ω_2 be subsets of the Γ-family defined by

$$\Omega_1 = \left\{\Gamma : \Gamma'(t) > 0, \Gamma''(t) > 0 \textit{ for all } t \in \left\langle 0,1\right], \textit{and } \Gamma'(0) = 0\right\}$$

and

$$\Omega_2 = \left\{\Gamma : \Gamma'(t) > 0, \Gamma''(t) < 0 \textit{ for } t \in \left\langle 0,1\right\rangle, \textit{and } \Gamma'(1) = 0\right\},$$

and let ω_1 and ω_2 be subsets of the γ-family defined by

$$\omega_1 = \left\{\gamma : \gamma'(k) > 0, \gamma''(k) > 0 \textit{ for all } k > 0, \textit{and } \gamma'(0) = 0\right\}$$

and

$$\omega_2 = \left\{\gamma : \gamma'(k) > 0, \gamma''(k) < 0 \textit{ for } k > 0, \textit{and } \gamma'(r) = 0\right\}.$$

All members of the sets Ω_1 and ω_1 are increasing convex functions, and all members of Ω_2 and ω_2 are increasing concave functions.

Theorem 3.1A. *Let F_1 and F_2 be members of the family **F** of count distributions based on r deprivations and assume that they have equal means. Then the following statements are equivalent:*

(i) F_1 *second-degree downward dominates* F_2*;*

(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ *for all* $\Gamma \in \Omega_1$*;*

(iii) $d_\gamma(F_1) < d_\gamma(F_2)$ *for all* $\gamma \in \omega_1$*;*

(iv) F_2 *can be obtained from* F_1 *by a sequence of mean preserving positive association increasing rearrangements when* $\kappa > 1$ *for both* F_1 *and* F_2*, a sequence of mean preserving negative association decreasing rearrangements when* $\kappa < 1$ *for both* F_1 *and*

F_2, and a combination of mean preserving positive association increasing and negative association decreasing rearrangements when $\kappa > 1$ for either F_1 or F_2;

(v) F_2 can be obtained from F_1 by a mean preserving spread.

Theorem 3.1B. *Let F_1 and F_2 be members of the family **F** of count distributions based on r deprivations and assume that they have equal means. Then the following statements are equivalent:*

(i) F_1 second-degree upward dominates F_2;

(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ for all $\Gamma \in \Omega_2$;

(iii) $d_\gamma(F_1) < d_\gamma(F_2)$ for all $\gamma \in \omega_2$;

(iv) F_2 can be obtained from F_1 by a sequence of mean preserving positive association decreasing rearrangements when $\kappa > 1$ for both F_1 and F_2, a sequence of mean preserving negative association increasing rearrangements when $\kappa < 1$ for both F_1 and F_2, and a combination of mean preserving positive association decreasing and negative association increasing rearrangements when $\kappa > 1$ for either F_1 or F_2;

(v) F_2 can be obtained from F_1 by a mean preserving contraction.

See Aaberge and Peluso (2011) for a proof of the equivalence between (i), (ii) and (iv) of Theorems 3.1A and 3.1B and Aaberge and Brandolini (2014) for a proof of the equivalence between (i) and (iii). The equivalence between (v) and (ii) and (iii) follows directly from the second terms of equations (3.4) and (3.8).

Following the distinction made by Bourguignon and Chakravarty (2003, 2009) and Atkinson (2003), the results of Theorem 3.1A (3.1B) justify the use of D_Γ and d_γ for convex

Γ and convex γ (concave Γ and concave γ) when the attributes associated with the deprivation indicators can be considered as substitutes (complements). Theorems 3.1A and 3.1B show that D_Γ and d_γ satisfy the mean preserving association rearrangement principles, where a distinction has been made between whether an association rearrangement comes from a distribution characterized by positive or negative association. Consider the specific subfamily of two-dimensional deprivation measures discussed by Atkinson (2003) and defined by (3.10), and assume that there is positive association between the two deprivations ($\kappa > 1$). The d_γ-function associated with the family A_θ is concave for $\theta < 1$ and convex for $\theta > 1$, and approaches the union condition when $\theta \to 0$ and the intersection condition when $\theta \to \infty$. Theorem 3.1B states that a sequence of mean preserving positive association decreasing rearrangements raises the overall deprivation A_θ if $\theta < 1$. Is it reasonable to suppose that the overall deprivation rises as we observe a reduction in the positive association between deprivations in the two attributes? After all, the share of people suffering from deprivation for both attributes falls, while the total number of deprivations does not vary. The answer is positive if we regard the two attributes as complements, which means that we rule out any tradeoff between them, and we dislike the fact that more people are deprived more than the fact that fewer people are hit more.

Until now, we have not considered the cases of unequal weighting of the dimensions. However, all results summarised by Theorems 3.1A and 3.1B remain valid for the distribution of weighted deprivation counts. For the dual approach, Aaberge and Peluso (2011) account for different weights by considering the weighted deprivation counts $X = \sum_{i=1}^{r} w_i X_i$ and the associated distribution *F*, where $w_1 \leq w_2 \leq ... \leq w_r$. For the primal approach, we could apply the procedure suggested by Alkire and Foster (2011a, 2011b) to replace the deprivation count for each person by the sum of the associated weights.

3.2.6. Counting deprivations vs. measuring poverty

So far, we have been concerned with the distribution of deprivation counts, irrespective of how many people are regarded as poor when deprivation and poverty are considered as distinct concepts. In terms of the classical distinction made by Sen (1976), we have focused only on the "aggregation" of the characteristics of deprivation into an overall measure of deprivation, ignoring the first step concerning the "identification" of the poor. The contrast between the union and the intersection criteria emphasised in the previous sections suggests, however, that there is some leeway in defining who is poor. For instance, Brandolini and D'Alessio (1998), Bourguignon and Chakravarty (1999, 2003), Tsui (2002) and Bossert et al. (2013) adopt the more extensive union criterion and define people to be (multidimensional) poor if they suffer from at least one deprivation. In this case deprivation and poverty come to coincide. On the other hand, the European Union regards as severally materially deprived all persons who cannot afford at least four out of nine amenities, moving midway between the union and the (strict) intersection views. Alkire and Foster (2011a, 2011b) formalise what they label the "dual cut-off" identification system, where the dimension-specific thresholds are integrated with a further threshold that identifies the minimum number of deprivations to be classified as poor. If a person is poor when he or she is deprived in at least *c*, $1 \le c \le r$, dimensions, the headcount ratio is uniquely determined by the count distribution *F* and is defined by

(3.13) $$\tilde{H}(c) = 1 - F(c-1) = \sum_{k=c}^{r} q_k .$$

In the case of the European indicator of severe material deprivation, *c* equals 4. As the choice of a specific cut-off *c* is arbitrary, it is useful to check the sensitivity of the ranking of distributions to *c* by treating $\tilde{H}(c)$ as a function of *c*, henceforth labelled headcount curve. As evident from (3.13), the condition of first-degree dominance of headcount curves is equivalent to first-degree dominance of the associated count distributions. If $c > 1$, first-degree dominance for headcount curves is a less demanding condition than that for the overall count distribution, as it ignores what happens to those that suffer from deprivation in fewer than c dimensions. Moreover, the second-degree dominance results of Theorems 3.1A and 3.1B are also valid for the headcount curve, which means that $\tilde{H}(c)$ satisfies the principle of association increasing/decreasing rearrangements when this principle is restricted to be applied among the poor.

To complement the information provided by the headcount ratio when only ordinal data are available, we may employ the measures defined by (3.4) and (3.8) as overall measures of poverty for the conditional count distribution $\tilde{F}$ defined by

$$(3.14)\quad \tilde{F}(k;c) = \Pr(X \le k \mid X \ge c) = \frac{F(k) - F(c-1)}{1 - F(c-1)} = \frac{\sum_{j=c}^{k} q_j}{\sum_{j=c}^{r} q_j}, \quad k = c, c+1, ..., r,$$

with mean given by

$$(3.15)\quad \tilde{\mu}(c) = \frac{\sum_{j=c}^{r} j q_j}{\sum_{j=c}^{r} q_j}.$$

Expressions (3.4) and (3.8) show that the overall measures of poverty for $\tilde{F}$ admits a decomposition into the mean (or a function of the mean) and a measure of dispersion. An analogue to the *FGT* family of poverty measures is obtained by inserting $\gamma(k) = k^{\theta}$ in expression (3.8).

As an alternative, Alkire and Foster (2011a) propose to combine the headcount ratio $\tilde{H}(c)$ and the conditional mean $\tilde{\mu}(c)$ and introduce the adjusted headcount ratio defined by

$$(3.16)\quad \tilde{M}_1(c) = \frac{\tilde{H}(c)\tilde{\mu}(c)}{r} = \frac{1}{r}\sum_{j=c}^{r} j q_j,$$

which is the ratio of the total number of deprivations experienced by the poor to the maximum number of deprivations that could be experienced by the entire population. For $c=1$, the index $\tilde{M}_1(c)$ coincides with the Atkinson-type primal measure of deprivation d_1^{GA}. Expression (3.16) can account for unequal weights for the various deprivations by simply replacing the deprivation count for each person by the sum of the associated weights. Alkire and Foster (2011a, p. 482) underline that both the identification of the poor and the adjusted headcount ratio are invariant to monotonic transformations applied to the deprivation variables and the respective thresholds. Moreover, the index $\tilde{M}_1(c)$ increases if a poor person becomes deprived in an additional dimension (dimensional monotonicity), is decomposable by population subgroups, and can be broken down by indicator as it is the (weighted) average of the deprivations headcount ratios for each dimension computed considering only the poor at the numerator (so-called "censored headcount ratios"). On the other hand, this index is indifferent to changes in the way deprivations are distributed across the poor.[29]

A general family of adjusted poverty measures that take into account not only the average deprivation experienced by the poor, $\tilde{\mu}(c)$, but also the distribution of deprivations across the poor can be derived from the *d*-measure defined by (3.8)

$$(3.17) \quad \tilde{M}_\gamma(c) = \frac{\tilde{H}(c)\tilde{d}_\gamma(c)}{r},$$

where $\tilde{d}_\gamma(c)$ denotes the *d*-index for $\tilde{F}$. Such a measure may weight differently poor persons according to the number of deprivations from which they suffer. Inserting $\gamma(k)=k^\theta$ in $\tilde{d}_\gamma(c)$ in (3.17) yields the general family of adjusted *FGT* measures for count data

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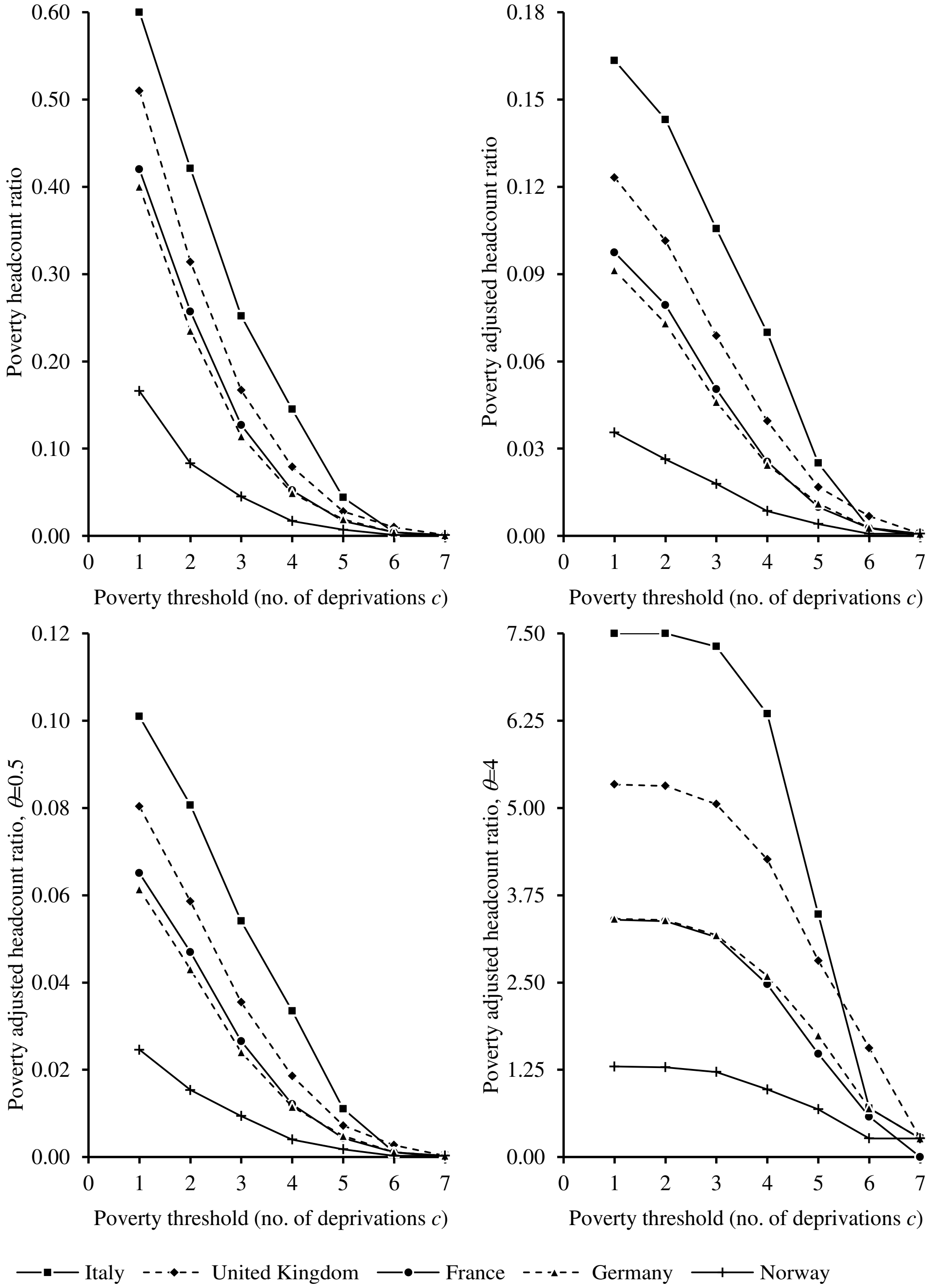
[29] For comments and critiques of the class of multidimensional indices proposed by Alkire and Foster (2011a, 2011b), see, among others, Birdsall (2011), Rippin (2010), Ferreira (2011), Ravallion (2011a, 2012a), Silber (2011), Thorbecke (2011), Ferreira and Lugo (2013), Duclos and Tiberti (forthcoming), and the reply by Alkire et al. (2011).

(3.18) $\tilde{M}_\theta(c) = \frac{1}{r}\sum_{j=c}^{r} j^\theta q_j, \quad \theta > 0$,

which encompasses (3.16) for $\theta = 1$. When $\theta \to 0$, the adjusted *FGT* measure reaches its minimum value $\tilde{H}(c)/r$, which ignores altogether any cumulative effect of multiple deprivations. As θ rises, greater weight is placed on those who suffer from deprivation in several dimensions.

Figure 3 compares how poverty headcount ratios change as we vary the poverty cut-off using the deprivation indicators in the five European countries considered earlier. The proportion of poor people, shown in the top-left panel, fell by three fourths in Italy and around nine tenths in the other countries as the poverty cut-off is raised from one deprivation (union criterion) to four deprivations (the European criterion). Censoring at four deprivations implies excluding from measured poverty a substantial fraction of population suffering from one, two or three deprivations: 15 per cent in Norway and 46 per cent in Italy, accounting for 76 and 57 per cent of all deprivations, respectively. However, the ranking of countries does not change. It changes by setting the cut-off at five deprivations, when Germany and France reverse their order, and again at six deprivations, when the United Kingdom becomes the country with the highest share of poor people. In the top-right panel, the ranking is the same for the adjusted headcount ratio $\tilde{M}_1(c)$, except for a better position granted to France by its lower average intensity of deprivation ($\tilde{\mu}(c)/r$) when the cut-off is set at six deprivations. The bottom panels show results for the adjusted *FGT* measure $\tilde{M}_\theta(c)$: lowering the weights of multiple deprivations ($\theta = 0.5$; left panel) does not modify the sorting produced by the adjusted headcount ratio, whereas significantly raising them ($\theta = 4$; right panel) steadily switches the positions of Germany and France, as seen in Section 3.2.4. This comparison reveals that varying the poverty cut-off has a considerable impact on measured poverty, whereas adjusting the headcount ratio for the deprivations experienced by the poor seems to have minor effects, unless their distribution is taken into account.

Figure 3: Poverty headcount and adjusted headcount ratios for different poverty cut-offs in selected European countries in 2012



Source: authors’ elaboration on data from Eurostat (2014).

The adjusted headcount ratio $\tilde{M}_1(c)$ proposed by Alkire and Foster (2011a, 2011b) provides the theoretical basis for the Multidimensional Poverty Index (*MPI*) developed by Alkire and Santos (2010, 2013, 2014).[30] The *MPI* has replaced the *HPI* in the reports of the United Nations Development Programme since 2010 in order to capture "... how many people experience overlapping deprivations and how many deprivations they face on average" (UNDP, 2010, p. 95). The *MPI* considers ten dichotomous indicators for three dimensions: health, education and living standards. Dimensions, and indicators within each dimension, are equally weighted, and the cut-off *c* for the number of (weighted) deprivations is set at three out of a maximum of ten. Applied research estimating Alkire and Foster's class of indices and the *MPI* is rapidly growing.[31]

3.3. Poverty measurement based on continuous variables

The counting approach focuses on the distribution of deprivation scores that summarise binary variables, defined as having or not goods or performing or not activities that are seen as social necessities. When we have cardinal (continuous or categorical) variables, we can use measures of multidimensional poverty that fully exploit the informational richness of the available data.[32] As in the counting approach, we may aggregate

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[30] Alkire and Foster's method is utilised by Peichl and Pestel (2013a, 2013b) to derive an adjusted headcount ratio for multidimensional richness. This index accounts for the number of individuals who are affluent in a minimum number of dimensions as well as for their average achievements in these dimensions.
[31] See for instance Roelen et al. (2010) for Vietnam, Khan et al. (2011) for Pakistan, Batana (2013) for Sub-Saharan African countries, Battiston et al. (2013) for Latin American countries, Roche (2013) for Bangladesh, Santos (2013) for Buthan, Trani and Cannings (2013) for Western Darfur, Trani et al. (2013) for Afghanistan, Yu (2013) for China, and Cavapozzi et al. (2013) and Whelan et al. (2014) for European countries. See also Mohanty (2011) for a related study on deprivation scores in India. Bennett and Mitra (2013) develop multiple statistical tests for Alkire and Foster's family of poverty measures.
[32] Bosmans et al. (2013b) introduce an approach that deals with joint aggregation of cardinal and ordinal variables. Yalonetzky (2013) derives stochastic dominance conditions for ordinal variables.

attributes first across dimensions and then across individuals. This procedure corresponds to representing each individual's vector of attributes with an interpersonally comparable utility-like function, and then evaluating the distribution of individual well-being using the same tools as in a univariate space. Consumer theory (Slesnick, 1993) or Information Theory (Maasoumi and Lugo, 2008) can provide the analytical framework to derive the utility-like function. This function is then used to aggregate the attribute-specific cut-offs to define an aggregate poverty threshold.[33]

Alternatively, we may employ an axiomatic simultaneous aggregation approach for measuring multidimensional poverty. Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003), and Tsui (2002) consider persons to be poor if they suffer from at least one deprivation (the union approach), whereas Alkire and Foster (2011a) take all those who are deprived in at least c dimensions, where c is comprised between 1 and r. All these papers then aggregate the individual shortfalls relative to dimension-specific cut-offs into a multidimensional poverty measure. The actual functional forms of the poverty indices are determined by the combination of chosen axioms, many of which parallel those considered in the univariate analysis (e.g. Zheng, 1997). In the next Section, we selectively review these indices and illustrate some of their properties. We refer to Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003), Tsui (2002) and Chakravarty and Silber (2008) for proofs and further discussion of the axioms.

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[33] Merz and Rathjen (2014a, 2014b) apply Maasoumi's utility-like approach, estimating a CES function, to study poverty in the bidimensional space of equivalent income and leisure time. Maasoumi and Lugo (2008) shows that the information-theoretic approach can embody attribute-specific thresholds if the utility-like function is replaced by a relative deprivation function whose argument is the relative shortfall of the attribute relative to its threshold. In this case, the aggregation across individuals has to be restricted to those who are deprived in at least one dimension.

3.3.1. Measures of multidimensional poverty

Let $\mathbf{y}_i, i = 1, 2, ..., n$, denotes the vector of the attributes of individual *i*, where $y_{ij} \geq 0$ is the achievement in dimension *j*, and let **z** be a vector of attribute-specific poverty thresholds. Bourguignon and Chakravarty (1999, 2003) introduce the following multidimensional analogue of the *FGT* family of poverty indices

$$(3.19) \quad P_\theta(y;z) = \frac{1}{nr}\sum_{i=1}^{n}\sum_{j=1}^{r} a_{ij}\left(1 - \frac{y_{ij}}{z_j}\right)^{\theta_j}, \quad \theta_j > 1,$$

where a_{ij} is equal to the weight w_j assigned to attribute *j* if $y_{ij} < z_j$ and to 0 otherwise; both a_{ij} and θ_j determine the weight assigned to attribute *j* in the poverty index.[34]

In addition to *Monotonicity*, *Continuity*, and *Scale Invariance*, the members of the family $P_\theta(y;z)$ satisfy three axioms worthy of some comments. *Subgroup Decomposability* requires that overall poverty can be expressed as a weighted sum of the poverty ratios of the subgroups and implies that the poverty index is separable across individuals. Thanks to this property, the poverty index identifies an individual poverty function. The *One-Dimensional Pigou-Dalton Transfer Principle* demands that poverty does not increase in the case of a progressive transfer of one unit of attribute *j* from one poor person to a poorer person. This axiom determines the additivity of (3.19) across attributes. This *Factor Decomposability* may be a useful feature as it allows identifying the contribution of each attribute to the overall poverty level.

The *Focus* axiom highlights the greater complexity of multidimensional analysis. In the univariate income case, it simply entails that the poverty index is independent of the distribution of income among the non-poor. In the multivariate case, it may require the

[34] See Lasso de la Vega and Urrutia (2011) for an axiomatic characterization of a generalised version of $P_\theta(y;z)$.

poverty measure to be invariant with respect to increases in y_{ij} if $y_{ij} > z_j$ for all *i*, poor and non-poor alike (*Strong Focus*), or only with respect to changes in the distribution of attributes among the non-poor (*Weak Focus*). The stronger version of the axiom implies that a better achievement in a dimension where an individual is not deprived cannot compensate for a below-threshold achievement in another dimension. The possibility of trading off one attribute for the other is ruled out. This is the case of (3.19), as a_{ij} equals 0 if $y_{ij} > z_j$.[35]

The family of poverty measures $P_\theta(y;z)$ is a particular specification of the more general class characterised by Chakravarty et al. (1998) and Bourguignon and Chakravarty (1999, 2003) where the power function is replaced by a continuous non-increasing convex function. Assuming that all attributes are positive and specifying such a function to be the negative of the logarithm, Chakravarty and Silber (2008) and Chakravarty et al. (2008) obtain the multidimensional version of the poverty index proposed by Watts (1968) and formalised by Zheng (1993):

(3.20) $$P_\theta(y;z) = \frac{1}{nr}\sum_{i=1}^{n}\sum_{j=1}^{r} a_{ij} \ln\left(\frac{z_j}{y_{ij}}\right),$$

As indicated by Bourguignon and Chakravarty (2003) and Pattanaik et al. (2011), the measures defined by (3.19) are not sensitive to association rearrangement interventions. This is due to the fact that $P_\theta(y;z)$ is uniquely determined by the marginal distributions of the *r* attributes and the associated attribute-specific poverty thresholds. By imposing an additional condition, called the *Poverty Focus* axiom, Alkire and Foster (2011a) provide a justification for the intermediate poverty measure analogue to $P_\theta(y;z)$,

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[35] Permanyer (2014) shows how to modify most of the multidimensional poverty indices commonly considered so that they satisfy the *Weak* rather than the *Strong Focus* axiom. Esposito and Chiappero Martinetti (2010) examine poverty indices that embody a hierarchical ordering of well-being dimensions.

(3.21) $P_\theta(y;z,c)=\frac{1}{nr}\sum_{i=1}^{n}a_i(c)\sum_{j=1}^{r}a_{ij}\left(1-\frac{y_{ij}}{z_j}\right)^\theta, \qquad \theta\geq 0,$

where $a_i(c)=1$ if $\{j: y_{ij}<z_j\}\geq c$ and 0 otherwise. In words, the role of $a_i(c)$ is to select only the poor individuals: these are now all people who suffer from deprivation in at least *c* dimensions, which for $c>1$ is a subgroup of those deprived in any dimension considered with the union criterion. Thus, $P_\theta(y;z,1)=P_\theta(y;z)$. When one or several attributes are dichotomous variables, then $P_\theta(y;z,c)$ is only valid for $\theta=0$. In such a case, we can only use

(3.22) $P_0(y;z,c)=\frac{1}{nr}\sum_{i=1}^{n}a_i(c)\sum_{j=1}^{r}a_{ij},$

which is equal to the average number of deprivations (normalized by the maximum number *r*) experienced by the poor, that is the normalized average among those individuals who suffer from at least *c* deprivations. Note that $P_0(y;z,c)$ is equal to $\tilde{M}_1(c)$, defined by (3.16), when $w_j=1$ for all *j*. For $c=1$, $P_0(y;z,1)$ becomes equal to the average number of deprivations (relative to *r*) for those who suffer from at least one deprivation. As demonstrated by Alkire and Foster (2011a), $P_\theta(y;z,c)$ for $\theta\geq 1$ satisfies a multidimensional transfer principle based on a bistochastic transformation when it is only applied among the poor. Moreover, the measures defined by (3.22) satisfy the association rearrangement principle discussed by Alkire and Foster (2011a), even though these measures are decomposable by subgroups. However, this is only true when $c>1$ and is due to the multidimensional information captured by the counting term $a_j(c)$ of (3.21).[36] By contrast, to account for correlation between attributes when $c=1$ Bourguignon and Chakravarty (1999, 2003) introduce a family

[36] As before, the introduction of the threshold $c>1$ can be criticised because it implies ignoring the condition of those who suffer from deprivation in less than *c* dimensions as well as because of the arbitrariness of the choice of *c*.

of non-additive poverty measures, but limit their discussion to the two-dimensional case.[37] This subfamily of $P_\theta(y;z)$ is defined by

$$(3.23)\quad P^*_{\alpha,\beta}(y;z) = \frac{1}{nr}\sum_{i=1}^{n}\left(\sum_{j=1}^{2} a_{ij}\left(1-\frac{y_{ij}}{z_j}\right)^{\beta}\right)^{\frac{\alpha}{\beta}},$$

where α and β are non-negative parameters, and is used by Bourguignon and Chakravarty (1999, 2003) and Atkinson (2003) as a basis for demonstrating that the effect of an increasing association (correlation) rearrangement depends on whether the attributes are substitutes or complements, and that this corresponds to choosing $\alpha > \beta$ or $\alpha < \beta$.[38] Moreover, Atkinson (2003) demonstrates that the family of counting measures A_θ defined by (3.10) can be seen as a limiting case of (3.23) when α and β tend to zero with $\theta = \alpha / \beta$ and $w_1 = w_2 = 1$.

3.3.2. Partial orderings

Most empirical studies consider few measures of poverty when ranking multidimensional distribution functions. A natural concern is that the conclusions reached in these studies are sensitive to the choice of the specific measures.[39] By drawing on Atkinson and Bourguignon (1982), Bourguignon and Chakravarty (2009) investigate what restrictions

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[37] For alternative families of multidimensional poverty measures and their characterizations we refer to Kolm (1977), Charakvarty et al. (1998), Tsui (2002), Deutsch and Silber (2005), Duclos et al. (2006a, 2007, 2008), Chakravarty and Silber (2008) and Lasso de la Vega and Urrutia (2011). Diez at al. (2008) and Chakravarty and D'Ambrosio (2013) derive subgroup decomposable multidimensional poverty indices that are unit consistent, that is provide poverty rankings that are unaffected by a change in the measurement units of dimensions.
[38] Brandolini (2009) and Madden (2011) use this index to study income and health poverty in selected European countries, and analyse the sensitivity of results to different values of the parameters α and β. See also Bibi and El Lahga (2008).
[39] For similar concern in the single-dimensional case see Atkinson (1987), Zheng (1999), Spencer and Fisher (1992), Jenkins and Lambert (1997) and Aaberge and Atkinson (2013) who introduce poverty dominance criteria as a basis for obtaining more robust results.

two alternative stochastic dominance conditions of first-degree impose on the general family of two-dimensional poverty measures defined by

$$(3.24)\quad \Pi_p(H;z) = \int_0^{z_2}\int_0^{z_1} p(x_1,x_2;z_1,z_2)dH(x_1,x_2)\,,$$

where *H* is the bivariate distribution of the two attributes in question and $p(x_1,x_2;z_1,z_2)$ is the level of poverty associated with attribute levels (x_1,x_2) and poverty thresholds (z_1,z_2) .[40].

Let p_i' denote the derivative of *p* with respect to x_i and let p_{12}'' denote the second derivative of *p* with respect to x_1 and x_2 .The following alternative presentation of a result from Bourguignon and Chakravarty (2009) provides three equivalent statements:

Theorem 3.2A. Let H and H^* *be members of the family* ***H*** *of the bivariate distributions of the attributes* (X_1,X_2) *and let* H_1 *and* H_1^* *and* H_2 *and* H_2^* *be the associated marginal distributions of* X_1 *and* X_2. *Then the following statements are equivalent*

(i) $H_i\,(x_i) \le H_i^*(x_i)$ *for all* $x_i < z_i, i = 1,2$ *and*

$H(x_1,x_2) \le H^*(x_1,x_2)$ *for all* $x_1 < z_1$ *and* $x_2 < z_2$

(ii) $\Pi_p(H;z) < \Pi_p(H^*;z)$ *for all p where* $p_i'(x_1,x_2) \le 0$ *for* $x_i < z_i, i = 1,2$ *and*

$p_{12}''(x_1,x_2) \ge 0,$ *for* $x_1 < z_1$ *and* $x_2 < z_2$

(iii) H^* *can be obtained from H by sequences of Pigou-Dalton regressive transfers and/or a sequence of marginal-free (marginal distribution preserving)correlation increasing rearrangements.*

[40] Poverty orderings of bivariate distributions are studied by Gravel and Moyes (2012), under the hypothesis that one attribute is cardinal and transferable between individuals while the other is ordinal and non-transferable, and Garcia-Diaz (2013), under the hypothesis of asymmetric treatment of the attributes proposed by Muller and Trannoy (2012).

The equivalence between (i) and (ii) is proved by Bourguignon and Chakravarty (2009), whilst the equivalence between (ii) and (iii) is proved by Atkinson and Bourguignon (1982).

Theorem 3.2A shows that poverty measures Π_p that satisfy condition (ii) rank bivariate distributions according to first-degree stochastic dominance for attribute values below the poverty threshold in each dimension and first-degree two-dimensional stochastic dominance below both poverty thresholds. As indicated by Bourguignon and Chakravarty (2003, 2009) and is demonstrated by Theorem 3.2A, the principle of correlation increasing rearrangement (conditional on fixed marginal distributions) is associated with the intersection $H(x_1, x_2)$. Moreover, Theorem 3.2A shows that Π_p increases as a result of correlation increasing rearrangement if the cross-derivative of *p* with respect to x_1 and x_2 is non-negative. This is the reason why Bourguignon and Chakravarty in this case refer to the attributes as substitutes. The case where the cross-derivative is negative corresponds to complements which are associated with non-increasing poverty under a correlation increasing rearrangement. An alternative presentation of this result is given by the following theorem:

*Theorem 3.2B . Let H and H^* be members of the family **H** of the bivariate distributions of the attributes (X_1, X_2) and let H_1 and H_1^* and H_2 and H_2^* be the associated marginal distributions of X_1 and X_2. Then the following statements are equivalent*

(i) $H_1(x_1) + H_2(x_2) - H(x_1, x_2) \leq H_1^*(x_1) + H_2^*(x_2) - H^*(x_1, x_2)$ *for all* $x_1 < z_1$ *and/or* $x_2 < z_2$

(ii) $\Pi_p(H; z) < \Pi_p(H^*; z)$ *for all p where* $p_i'(x_1, x_2) \leq 0$ *for* $x_i < z_i, i = 1,2$ *and* $p_{12}''(x_1, x_2) \leq 0$ *for* $x_1 < z_1$ *and* $x_2 < z_2$

(iii) H^* *can be obtained from* H *by sequences of Pigou-Dalton regressive transfers and/or a sequence of marginal-free (marginal distribution preserving) correlation decreasing rearrangements.*

For the proof of Theorem 3.2B we refer to Atkinson and Bourguignon (1982) and Bourguignon and Chakravarty (2009). Note that Theorems 3.2A and 3.2B are strictly speaking only valid in cases where the association (correlation) between X_1 and X_2 is positive. However, as demonstrated for the counting approach (see Theorems 3.1A and 3.1B), it is straightforward to extend Theorems 3.2A and 3.2B to cover the case where the association (correlation) is negative.

In practice, multivariate distributions might often cross. This is shown, for instance, by Arndt et al. (2012) and Nanivazo (2014) in their multidimensional analyses of first order dominance for child poverty in Vietnam and Mozambique and in the Democratic Republic of Congo, respectively. Thus, it is helpful to introduce weaker criteria than first-degree dominance. Duclos et al. (2006a, 2007, 2008) consider second-degree and higher degree dominance conditions when attributes are regarded as substitutes. A good example is when we consider poverty in relation to both income and wealth, which are perfect or very close substitutes. (Imperfect substitutability may derive from a lower degree of liquidity of an asset.) In these exercises, the asset poverty line is often defined with reference to the income poverty line, as it is taken to be the amount of wealth necessary to maintain the socially defined minimum standard for a certain period for someone who has no other economic sources (e.g. Haveman and Wolff, 2004, and Brandolini et al., 2010). Bourguignon and Chakravarty (2009) remark that this approach gets close to the unidimensional approach since the poverty line of one attribute is assumed to be a function of the poverty line of the other attribute.

4. Multidimensional inequality measurement

The surge of research on the measurement of inequality in multiple dimensions is fairly recent, but the central question is far from new. In "Income Distribution, Value Judgments, and Welfare", Fisher was not interested in money income, a "scalar", but in "real" income, that is "a vector whose components are amounts of commodities" (1956, p. 382). His analysis was however carried out by aggregating commodities either by using constant prices – to which he assigned "no particular significance … as market valuations of the commodities. Any arbitrary set of weights would do as well" (1956, p. 383, fn. 6) – or by means of individual utility functions. Social welfare was thus seen as an aggregation of individual preferences, in the tradition of what Sen (1977) has labelled "welfarism". The modern approach to measuring inequality in multiple dimensions generally departs from this identification by interpreting the individual utility function as "the observer's evaluation of the individual's welfare" (Kolm, 1977, p. 3), so that "the social criterion makes no use of information on individual *i*'s relative valuation of the different elements of [the vector of goods received by person *i*]" (Atkinson and Bourguignon, 1982, p. 184).

As in the study of a single variable pioneered by Kolm (1969) and Atkinson (1970), the analysis proceeds by investigating the conclusions that can be reached on the ranking of multivariate distributions by making alternative assumptions on the order of aggregation and the shape of the social welfare function, or on the desired properties of inequality indices. We first consider the extension of the Pigou-Dalton transfers principle; we then move to partial orderings and sequential dominance criteria, and lastly to inequality indices.[41]

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[41] See Bradburd and Ross (1988) and Fluckiger and Silber (1994) for early proposals of multidimensional inequality index.

4.1. Multidimensional extensions of the Pigou-Dalton transfers principle

Some of the requirements typically specified in the univariate case can be directly transferred to the multivariate context. For instance, the requirement that the social evaluation pays no attention to any other individual characteristics than those included in the vector of attributes – the "anonymity principle" – does not pose any problem, and may in fact result even less restrictive with multiple dimensions. On the other hand, the multidimensional extension of the Pigou-Dalton transfers principle is less straightforward. In its original formulation, it states that inequality should fall as income is transferred from a richer to a poorer person, without modifying their relative ranks. (The last condition is unnecessary if the anonymity principle is assumed.) There is no unique way to reformulate the principle when there are two or more dimensions.[42]

A first possible generalisation is suggested by Fisher's approach discussed above. Suppose that there are *r* attributes and *n* individuals. The distribution is represented by the $n \times r$ matrix $\mathbf{X} = [x_{ij}]$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., r$, where x_{ij} is the quantity of attribute *j* enjoyed by individual i, and $x_i = (x_{i1}, x_{i2}, ..., x_{ir})$ is the vector of the attributes of individual *i*. If attributes are aggregated for each individual by a vector of weights (prices) $\mathbf{p}$, the comparison of two alternative distributions $\mathbf{X}$ and $\mathbf{Y}$ can be reduced to that of the two resulting univariate distributions $\mathbf{Xp}$ and $\mathbf{Yp}$: if $\mathbf{Yp}$ Lorenz-dominates $\mathbf{Xp}$ for any possible $\mathbf{p}$, then $\mathbf{Y}$ is socially preferable to $\mathbf{X}$. This dominance criterion is known as "price majorization", "budget majorization" or "directional majorization". The appropriateness, or the lack, of market prices for attributes such as the health status represents a problem for this

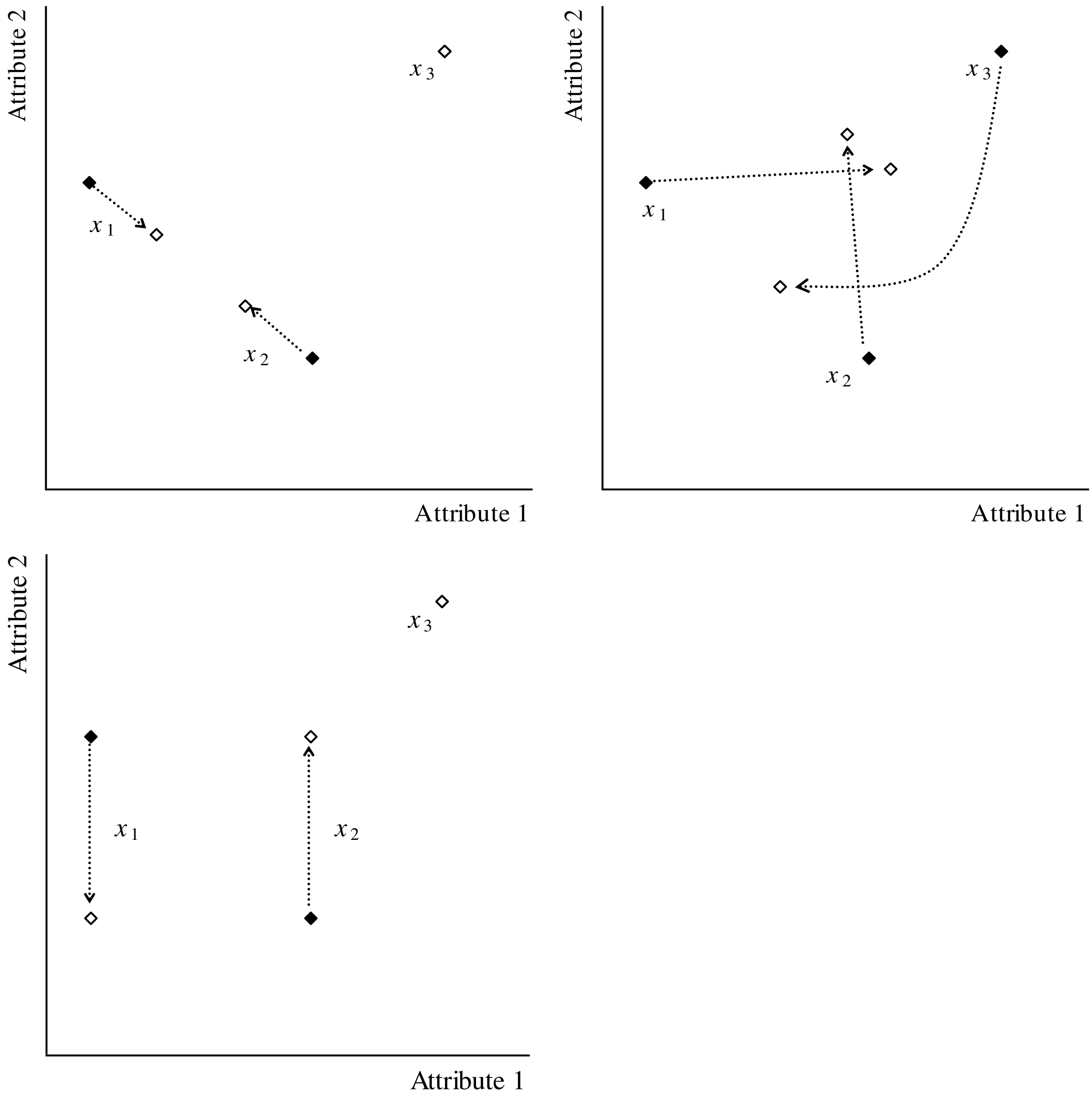
[42] See also Das Gupta and Bhandari (1989), Dardanoni (1995), Fleurbaey and Trannoy (2003), Mosler (2004), Fleurbaey (2006), Savaglio (2006a, 2006b), Diez et al. (2007), Nakamura (2012) and Banerjee (2014a, 2014b).

dominance criterion, but other reasons may lead to question its ethical foundations (Trannoy, 2006).

A second possibility is then to conceive a Pigou-Dalton experiment in the multivariate context as a transfer simultaneously and identically involving all attributes. Suppose that there are two attributes and three individuals. A (strict) Pigou-Dalton (PD) transfer between individuals 1 and 2 can be defined as the transfer to the poorer individual of the fraction λ of the extra quantity of attribute *j* held by the richer individual, or $\lambda\left|x_{2j}-x_{1j}\right|$ with $0<\lambda<1$. Thus, the PD-transfer yields the new vectors $\mathbf{x}_1'=(1-\lambda)\mathbf{x}_1+\lambda\mathbf{x}_2$ and $\mathbf{x}_2'=\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2$, while $\mathbf{x}_3$ is unchanged. An example is shown in the left-top panel of Figure 4, where the empty diamonds correspond to the new distribution. This distribution is socially preferable to the original one, indicated by full diamonds, as it is obtained by a PD-transfer between 1 and 2 (there is no change for individual 3). More generally, a PD-transfer may be described by $\mathbf{Y}=\mathbf{TX}$ where $\mathbf{T}$ is the $n\times n$ matrix $\mathbf{T}=\lambda\mathbf{I}+(1-\lambda)\mathbf{\Pi}_{h,k}$ with $\mathbf{I}$ being the $n\times n$ identity matrix and $\mathbf{\Pi}_{h,k}$ the $n\times n$ permutation matrix interchanging *h* and *k* (e.g. Weymark, 2006, p. 307). A distribution $\mathbf{Y}$ which is obtained from $\mathbf{X}$ by a sequence of PD-transfers is socially preferable to $\mathbf{X}$. This dominance criterion is also known as “chain majorization” in the Marshall and Olkin’s (1979) terminology.

A sequence of PD-transfer matrices $\mathbf{T}$ yields a bistochastic matrix, that is a non-negative square matrix where each row and each column sum up to 1. Though not all bistochastic matrices can be obtained by a sequence of PD-transfers, the multiplication of $\mathbf{X}$ by a bistochastic matrix is a form of averaging that makes the distribution less spread out. An alternative formulation of the dominance criterion is then to require that $\mathbf{Y}$ is socially preferable to $\mathbf{X}$ if there is a bistochastic $n\times n$ matrix $\mathbf{B}$ such that $\mathbf{Y}=\mathbf{BX}$ (“majorization”). An example of a redistribution of this type, which cannot be obtained by a sequence of (strict) PD-transfers, is given in the right-top panel of Figure 4. It is clear at a visual examination that the three individuals are closer each other after the averaging out performed by the **B** matrix; the deterioration suffered by individual 3 is socially acceptable by virtue of the anonymity principle.

Figure 4. Examples of majorization criteria



Source: authors' elaboration.

There are two possible objections to these criteria. The first is that a change in one attribute does not affect the contribution to well-being of other attributes. We could however suppose that the correlation of attributes matter. Tsui (1999) introduces the concept of correlation-increasing transfer, which is an exchange of all attributes between two individuals after which one individual is left with the lowest endowment and the other with the maximum endowment of each attribute. By concentrating attributes, this type of transfer leads to a distribution which is less socially preferable than the original one. An example is shown in the bottom-left panel of Figure 4. Figure 5 summarises majorization criteria.

The second objection is that, unlike income, many constituents of human welfare are not transferable. In general, it does not make much sense to talk of "transferring health" from a healthier individual to a sick one, with the possible exception of organ transplants (e.g. kidney and bone marrow). This has led Bosmans et al. (2009) to study the implications of formulating a version of the Pigou-Dalton principle that applies only to transferable attributes and Muller and Trannoy (2012) to examine dominance conditions when attributes are asymmetric in the sense that one attribute (typically income) can be used to compensate for lower levels of other attribute(s) (e.g. needs, health, etc.).

Figure 5. Majorization criteria

Uniform Pigou-Dalton Majorization (**UPD**): $Y \succ_{UPD} X$ if and only if *Y=TX* for some matrix *T* that is a finite product of PD transfer matrices and is not a permutation matrix.
Uniform Majorization (**UM**): $Y \succ_{UM} X$ whenever *Y=BX*, where *B* is a bistochastic matrix and *Y* cannot be derived by permuting the columns of *X*.
Directional Majorization (**DM**): $Y \succ_{DM} X$ if and only if *Yp* strictly Lorenz dominates *Xp* for any $p \in R^m$.
Positive Directional Majorization (**PDM**): $Y \succ_{PDM} X$ if and only if *Yp* strictly Lorenz dominates *Xp* for any $p \in R^m_{++}$.
Correlation Increasing Majorization (**CIM**): $Y \succ_{CIM} X$ whenever *X*may be derived from *Y* by a permutation of columns and a finite sequence of correlation increasing transfers at least one of which is strict.

Source: adapted from Tsui (1999, pp. 149-152)

4.2. Partial orderings and sequential dominance criteria

As in the univariate case, conclusions based on summary measures of multidimensional inequality might be questioned. Thus, it is helpful to investigate their robustness by using partial orderings like stochastic dominance criteria. The first-degree dominance criterion considered by Atkinson and Bourguignon (1982) was briefly discussed in Section 3.4.2. For a discussion of second-order multidimensional stochastic dominance and the conditions that this criterion imposes on the expected utility type of social welfare functions and associated measures of inequality we refer to Atkinson and Bourguignon (1982). Trannoy (2006) and Duclos et al. (2011) propose extensions of the results provided by Atkinson and Bourguignon (1982). Koshevoy (1995, 1998) and Koshevoy and Mosler (1996, 1997, 2007) introduce an alternative approach based on a multidimensional generalization of the Lorenz curve. Note that the equivalence between second-degree stochastic dominance and first-degree Lorenz dominance for fixed means does not hold in the multidimensional case.

The elaboration of sequential dominance criteria for the bivariate asymmetric space of income and household composition has been an early topic of the research on partial orderings in a multidimensional framework. Following Atkinson and Bourguignon (1987), many authors have seen the advantage of this approach over the standard income equivalisation procedure in the fact that it requires only to rank family types in terms of needs and not to specify how much a family type is needier than another one. Bourguignon (1989), Atkinson (1992), Jenkins and Lambert (1993), Moyes (1994, 2012), Chambaz and Maurin (1998), Ok and Lambert (1999), Ebert (2000), Lambert and Ramos (2002), Duclos and Makdissi (2005), Decoster and Ooghe (2006), and Zoli and Lambert (2012) belong to this strand of research, with a focus either on poverty or on inequality. Sequential dominance analysis can be applied to other bivariate distributions. Brandolini and D'Alessio (1998) present an early application to the joint distribution of equivalent income and health in Italy, whereas Duclos and Échevin (2011) and Madden (2014) carry out a similar exercise to compare Canada and the United States. Duclos et al. (2006b) study the joint distributions of household expenditure and children's heights in Ghana, Madagascar, and Uganda. Bérenger and Bresson (2012) use sequential dominance to test whether growth is "pro-poor" when poverty is measured by income and another discrete well-being attribute. Sequential dominance criteria for more than two attributes are presented by Gravel et al. (2009), Gravel and Mukhopadhyay (2010) and Muller and Trannoy (2011). McCaig and Yatchew (2007) and Batana and Duclos (2011) develop statistical inference techniques to test dominance.

4.3. Measures of multidimensional inequality

As for the measurement of multidimensional deprivation and poverty the informational basis defined by the order of aggregation plays a crucial role in measurement of multidimensional inequality as well. Thus, it is helpful to make a distinction between measures of multidimensional measures of inequality where the order of aggregation either begins with aggregating across individuals for each single attribute or across attributes for each individual. In the former case we obtain measures of overall inequality that aggregate inequality over each of the attributes. If we invert the order of aggregation, we derive an overall measure of inequality that aggregates synthetic functions of the attributes across individuals. The latter approach embeds the association between the achievements in the various dimensions into an overall indicator of individual achievements.

4.3.1. Two-stage approaches: first aggregating across individuals

Two-stage approaches either aggregate, first, individuals' achievements on each dimension and, second, the resulting attribute-specific indicators over the r dimensions or, first, the single attributes into individual-specific well-being indicators and, second, these individual indicators into a summary measure of multidimensional inequality. The former approach forms the basis of the Inequality-adjusted Human Development Index (*IHDI*; e.g.

UNDP, 2013), which belongs to the class of distribution-sensitive composite indices proposed by Foster et al. (2005), as well as of the following family of multidimensional generalized Gini coefficients proposed by Gajdos and Weymark (2005):

$$(4.1) \quad J_{\tau w}(F) = 1 - \frac{W_{\tau w}(F)}{W_{\tau w}(F_{equal})},$$

where $W_{\tau w}(F)$ and $W_{\tau w}(F_{equal})$ are defined by

$$(4.2) \quad W_{\tau w}(F) = \begin{cases} \left[\sum_{j=1}^{r} \tau_j \left(\sum_{i=1}^{n} w_{ij} x_{ij} \right)^{\alpha} \right]^{\frac{1}{\alpha}} & when\ \alpha \neq 0 \\ \prod_{j=1}^{r} \left(\sum_{i=1}^{n} w_{ij} x_{ij} \right)^{\tau_j} & when\ \alpha = 0, \end{cases}$$

and

$$(4.3) \quad W_{\tau w}(F_{equal}) = \begin{cases} \left[\sum_{j=1}^{r} \tau_j \mu_j^{\alpha} \right]^{\frac{1}{\alpha}} & when\ \alpha \neq 0 \\ \prod_{j=1}^{r} \mu_j^{\tau_j} & when\ \alpha = 0, \end{cases}$$

where μ_j is the mean of attribute *j*.

Gajdos and Weymark (2005) demonstrate that the family of social evaluation functions $W_{\tau w}(F)$ is characterized by the following set of distributional associated axioms: *Uniform Pigou-Dalton Majorization Principle* (*UMPM*), *Strong Attribute Separability* (*SAS*), *Weak Comonotonic Additivity* (*WCA*) and *Homotheticity* (*HOM*) and the conventional non-distributional axioms *Ordering*, *Continuity* and *Monotonicity*. *UMPM* is a multidimensional Pigou-Dalton transfer principle, *SAS* requires that any subset of the attributes is independent of the other attributes, *WCA* is a multidimensional extension of the weak independence of income source axiom imposed by Weymark (1981) on the ordering of univariate income distributions, which is equivalent to the *Dual Independence* axiom discussed in Section 3. *HOM* is an extension of the scale invariance axiom for unidimensional inequality measures and requires that a common proportional change in the measurement units of the attributes should not affect the social evaluation ordering.[43]

By specifying $\alpha = 1$ and $\tau_j = 1/r$ in (4.2) and (4.3), $J_{\tau w}(F)$ becomes a weighted average of the attribute-specific generalized-Gini coefficients introduced by Donaldson and Weymark (1980). Alternatively, by choosing $\tau_j = 1/\mu_j$, $J_{\tau w}(F)$ becomes equal to the arithmetic mean of the attribute-specific generalized-Gini coefficients, previously proposed by Koshevoy and Mosler (1997).[44] Replacing *WCA* with a multidimensional extension of the *Independence* axiom gives a normative justification of a multidimensional Atkinson family similar to the generalized-Gini family (4.1).

These types of multidimensional inequality measures ignore the impact of the association between attributes on overall inequality, and therefore do not exploit all information when individual-level data on multiple attributes are available.

4.3.2. Two-stage approaches: first aggregating across attributes

Measures that capture the association between attributes can be derived either from a two-stage aggregation approach or from a direct one-stage approach. The two-stage approach originally proposed by Maasoumi (1986, 1989, 1999) uses a (common) utility-like function (measure of well-being) to aggregate the attributes for each individual in the first stage, and a univariate inequality measure to aggregate the utility-like values across individuals in the

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[43] See Gajdos and Weymark (2005) for a discussion of strengthening the scale invariance axiom to allow for independent proportional changes in the measurement units of the attributes, which is required when one considers both monetary and non-monetary attributes.
[44] Okamoto (2009) provides a decomposition of this class of multivariate Gini indices which satisfies the *Completely Identical Distribution* condition, whereby the between-group inequality is equal to zero if the distribution is the same within all population subgroups.

second stage. Seth (2013) and Bosmans et al. (2013a) give the two-stage approach a normative justification. Let the social evaluation (or welfare) function W associated with the two-stage approach be defined by

(4.4) $$W(F) = v\big(u(x_1), u(x_2), ..., u(x_n)\big),$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{ir})$ is the attribute bundle of individual i, $i = 1,2,...,n$, F is the multidimensional distribution of the r attributes and u is the common utility-like function. Bosmans et al. (2013a) demonstrate that $W(F)$ is characterised by the following axioms:[45] *Monotonicity*, *Continuity*, *Normalization* (provides a cardinalization of the social evaluation function), *Anonymity* (makes the utility function common to all individuals), *Homotheticity* ($W(F)$ is invariant to a common proportional change in each attribute), *Weak Uniform Majorization* (progressive transfers uniformly applied to each attribute do not decrease $W(F)$) and *Individualism* (social evaluation is made in two steps: the first step aggregates across attributes for each individual and the second step aggregates the aggregated attributes across individuals).

Thus, several of the proposed families of multidimensional inequality measures can be ethically justified by drawing on the characterization results of Bosmans et al. (2013a). For example, the common utility-like function can be specified as

(4.5) $$u(x_i) = \sum_j w_j x_{ij},$$

where w_j is the weight associated with attribute j, equal across individuals, and weights are normalised to sum to unity. The hypothesis of additive separability used in (4.5) rules out attributes that are not perfect substitutes. As suggested by Maasoumi (1986), a

[45] Seth (2013) provides an axiomatic characterization of a two-stage approach where generalized means form the basis of the aggregation in each stage. See also Lasso de la Vega et al. (2010) who consider the two-stage generalized mean approach for analysing multidimensional deprivation distributions.

straightforward generalisation of (4.5) is offered by the class of utility functions showing constant elasticity of substitution (CES)

$$(4.6)\quad u(x_i)=\begin{cases}\left[\sum_j w_j x_{ij}^{-\beta}\right]^{-\frac{1}{\beta}} & \beta\neq 0\\ \prod_j x_{ij}^{w_j} & \beta=0,\end{cases}$$

where β is a parameter governing the degree of substitution between the attributes. As β goes to infinity, the attributes are perfect complements, whereas they are perfect substitutes for $\beta=-1$. To aggregate the distribution of $u(x_i)$'s Maasoumi (1986) proposes to use either the entropy family or the Atkinson family of inequality measures.[46] Alternatively, in the second aggregation stage we can rely on the family of rank-dependent measures, which includes the generalised Gini family. List (1999), Banerjee (2010) and Decancq and Lugo (2012) characterise multidimensional Gini indices which aggregate first across attributes and then across individuals.

Tsui (1995, 1999) follows the direct one-stage approach. Tsui (1995) generalises to the multivariate context Kolm's (1969) and Atkinson's (1970) analysis where inequality is identified with the social welfare loss (see Sen, 1978, 1992, for a critique of ethical inequality indices). After restricting the class of social evaluation functions to be continuous, strictly increasing, anonymous, strictly quasi-concave, separable and scale invariant, Tsui (1995) derives the two following multidimensional (relative) inequality indices:[47]

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[46] For instance, in their applications of Maasoumi's approach, Nilsson (2010), Justino (2012) and Rohde and Guest (2013) use the Theil indices.
[47] Abul Naga and Geoffard (2006), Brambilla and Peluso (2010), and Croci Angelini and Michelangeli (2012) provide decompositions of this class of indices into the univariate inequality indices of the attributes and a residual term capturing their joint distributions. See also Kobus (2012) for a stronger definition of decomposition by attributes. Diez et al. (2008) derive unit consistent multidimensional inequality indices. Gigliarano and Mosler (2009) construct multidimensional indices of polarisation. Abul Naga (2010) derives the large sample distribution of a class of multidimensional inequality indices including the Tsui index.

(4.7a) $$I_1 = 1 - \left[\frac{1}{n} \sum_i \prod_j \left(\frac{x_{ij}}{\mu_j} \right)^{r_j} \right]^{\frac{1}{\sum_k r_k}}$$

(4.7b) $$I_2 = 1 - \prod_i \left[\prod_j \left(\frac{x_{ij}}{\mu_j} \right)^{\frac{r_j}{\sum_k r_k}} \right]^{\frac{1}{n}}$$

where μ_j is the mean of attribute j over all persons and parameters r_j's must satisfy certain restrictions. The separability condition implies that the attributes can be aggregated for every person i into an indicator of well-being $u(x_i) = \Pi_j x_{ij}^{w_j}$, where $w_j = r_j / \Sigma_k r_k$ can be seen as a normalised weight on attribute j. By replacing ε for $\Sigma_k r_k$, (4.7a) and (4.7b) can be rewritten as

(4.8) $$I = \begin{cases} 1 - \left[\frac{1}{n} \sum_i \left(\frac{u(x_i)}{u(\mu)} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} & \varepsilon \neq 1 \\ 1 - \prod_i \left(\frac{u(x_i)}{u(\mu)} \right)^{\frac{1}{n}} & \varepsilon = 1 \end{cases}$$

where $u(\mu) = \Pi_j \mu_j^{w_j}$ is the "representative" well-being of the society, that is the well-being of a person showing the mean achievement for each attribute. The restrictions on r_j transfer to w_j and ε; in the bivariate case, it is sufficient that $\varepsilon > 0$ and $0 < w_1 = 1 - w_2 < 1$.

This reformulation has three advantages. Firstly, it demonstrates that the family defined by (4.7a) and (4.7b) could also be justified by the two-stage approach. Secondly, it shows the close link of the Tsui multidimensional inequality measure with the Atkinson univariate index applied to the $u(x_i)$'s, from which it differs only for the replacement of *mean* well-being with *representative* well-being. This is indeed the appropriate normalisation since "… maximizing social welfare under the constraint of fixed total resources of attributes … requires to give each individual the average available quantity of attributes …" (Bourguignon, 1999, p. 478). This observation exposes a conceptual diversity between the direct one-stage approach and the two-stage approach: the first normalises by the representative well-being $u(\mu) = \Pi_j \mu_j^{w_j}$, while the latter would use the mean well-being $(1/n)\Sigma_i \Pi_j x_{ij}^{w_j}$. (Of course, the two indices coincide in the univariate case.) Thirdly, (4.8) brings out the role of ε, i.e. $\Sigma_k r_k$ in the original formulation, as the parameter that governs the degree of concavity, and hence of inequality aversion, of the social evaluation function. In the univariate income space, the range of economically sensible values for ε can be restricted on the basis of considerations on the preference for redistribution. A similar analysis has not been conducted in the multivariate space of well-being, but "… there is not necessarily any reason to change our views about the value of [ε] simply because we have moved to a higher dimensionality" (Atkinson, 2003, p. 59).[48] Fourthly, (4.8) shows that the Tsui index allows for different weightings of the attributes (through the w_j's), but makes no allowance for a variation in the degree of substitution between the attributes: the Cobb-Douglas functional form of the underlying well-being indicator implies that the elasticity of substitution between two attributes is uniformly equal to unity. In the bivariate case, a straightforward generalisation is represented by the index derived by Bourguignon (1999) by assuming a CES functional form for the indicator of well-being, which has the Tsui index as a special case (see Lugo, 2007). Tsui (1999) examines alternative axioms that lead to characterise a class of multidimensional generalized entropy measures.

[48] In the analysis of income inequality, Atkinson and Brandolini (2010) suggest that plausible values for ε are comprised between 0.3 and 3. This range includes the values used by Lugo (2007) and Brandolini (2009) in their empirical analyses. In a cross-national comparison of multidimensional inequality, Aristei and Perugini (2010) use country-specific values of ε, ranging from 1.04 to 1.77, estimated from national tax structures.

4.3.3. Indices for binary variables

Where information is restricted to marginal distributions of zero/one variables, an overall measure of inequality is a function of the proportions of people with attribute values above each of the attribute-specific thresholds, which means that they do not suffer from deprivation in these dimensions.

By contrast, when multiple attributes are observed for the same individuals, let p_j be the proportion of people with *j* attributes that take values above the attribute-specific threshold levels and $G(k) = \sum_{j=0}^{k} p_j$ the cumulative proportion of people with *k* or fewer attributes that take values above the attribute-specific threshold levels. Then, similarly to the discussion for the distribution of deprivation counts in Section 3.1, the social evaluation function

$$(4.9) \quad W_{\Psi}(G) = r - \sum_{k=0}^{r-1} \Psi(\sum_{j=0}^{k} p_j)$$

yields the following measures of dual multidimensional inequality *I*:

$$(4.10) \quad I_{\Psi}(G) = 1 - \frac{W_{\Psi}(G)}{\nu} = 1 - \frac{r - \sum_{k=0}^{r-1} \Psi(\sum_{j=0}^{k} p_j)}{\nu},$$

where ν is the average number of individual achievements above the attribute thresholds, and Ψ, with $\Psi(0) = 0$ and $\Psi(1) = 1$, is a non-negative and non-decreasing concave function capturing the preferences of a social evaluator who supports axioms similar to those underlining the rank-dependent utility theory of Yaari (1987).

Note that $G(k) = 1 - F(r - k - 1)$, where *F* and μ are the count distribution of deprivations and the mean number of deprivations discussed in Section 3.1, which means that $\nu + \mu = r$; that is the sum of the mean number of deprivations and the mean number of achievements is necessarily equal to the number of attributes. By specifying

$\Psi(t) = 1 - \Gamma(1-t)$ it can be demonstrated that this adding up condition also are satisfied by the sum of the deprivation measure *D* and social evaluation function *W*,

$$W_\Psi(G) = W_{1-\Gamma}(1 - F(r-k-1)) = r - \sum_{k=0}^{r-1}\left(1 - \Gamma(1 - \sum_{j=0}^{r-k-1} p_j)\right) = \sum_{k=0}^{r-1}\left(\Gamma(\sum_{j=0}^{k} q_j)\right) = r - D_\Gamma(F).$$

Thus, inequality in the count distribution of achievements, rather than deprivations, can be given the following alternative expression:

$$(4.11)\quad I_\Psi(G) = 1 - \frac{r - D_\Gamma(F)}{r-\mu} = \frac{D_\Gamma(F) - \mu}{r-\mu} = \frac{\Delta_\Gamma(F)}{r-\mu},$$

where Γ is a non-decreasing convex function. Inequality in the distribution of achievements is equivalent to the relative spread of deprivations (divided by the difference between the mean number of deprivations and achievements). Note that the notion of inequality is closely associated with the intersection approach discussed in Section 3, whereas the union approach is in conflict with the notion of inequality.

The primal analogues to $W_\psi(G)$ and $I_\Psi(G)$, and counterpart of $d_\gamma(F)$ defined by (3.8), is given by

$$(4.12)\quad w_\xi(G) = \sum_{k=0}^{r} \xi(k) p_k$$

and

$$(4.13)\quad J_\xi(G) = 1 - \frac{w_\xi(G)}{\xi(\nu)} = 1 - \frac{\sum_{k=0}^{r} \xi(k) p_k}{\xi(\nu)},$$

where ξ is a non-negative and non-decreasing concave function capturing the preferences of a social evaluator who supports the *Independence* axiom for orderings defined on the set of *G*-distributions. By specifying $\xi(k) = \gamma(r) - \gamma(r-k)$ and inserting for $p_k = q_{r-k}$ we get

$$w_{\xi}(G)=\sum_{k=0}^{r}\xi(k)p_k=\sum_{k=0}^{r}\xi(k)q_{r-k}=\sum_{k=0}^{r}\xi(r-k)q_k=\sum_{k=0}^{r}\big(\gamma(r)-\gamma(k)\big)q_k=\gamma(r)-d_{\gamma}(F)$$ and

$\xi(\nu)=\gamma(r)-\gamma(\mu),$ which yield the following alternative expression for J_{ξ}

$$(4.14)\quad J_{\xi}(G)=1-\frac{\gamma(r)-d_{\gamma}(F)}{\xi(\nu)}=\frac{d_{\gamma}(F)-\gamma(\mu)}{\gamma(r)-\gamma(\mu)}=\frac{\delta_{\gamma}(F)}{\gamma(r)-\gamma(\mu)},$$

where $\delta_{\gamma}(F)$ is defined by (3.9) and γ is a non-decreasing convex function such that $\gamma(\mu)\le d_{\gamma}(F)\le\gamma(r)$.

5. Summary and conclusions

Since the 1990s, the measurement of multidimensional inequality and poverty has turned into a thriving research area. Novel analytical results have accompanied a massive production of applied research. The increasing availability of new and rich databases has fuelled the growth, but this process would have not been possible without the spreading of new conceptualisations of well-being, prominently the “capability approach”, and of a policy orientation more inclined to consider the nuances of human well-being. The progress has not always been coherent: applied research has sometimes moved from available data unaware of analytical developments; theoretical research has sometimes ignored the applicability of results to real data. This is common when development is rapid, and can contribute to explain why, while we have enriched our toolbox with so many new instruments, we still disagree on whether and how to use them. Our aim in this paper has been to provide a manual to this toolbox, drawing connections between different strands of the literature, clarifying some ambiguities, and exposing the strict link between analytical tools and the characteristics of the data available for the analysis.

The informational basis of the analysis is indeed crucial: tools thought for cardinal or categorical variables need not be appropriate for dichotomous variables, which often represent the bulk of the available information. This is one reason why we have paid special attention to the counting approach. Another one is expository convenience: the role of marginal distributions and the association between the attributes are particularly transparent for dichotomous variables, especially in the two-dimensional case, although the descriptive and normative issues are similar to those of continuous variables. However, the main motivation for this choice has been the attempt to bridge the gap between a copious empirical literature and a still relatively underdeveloped analytical elaboration. We have derived dominance criteria and measures of deprivation by exploiting the fact that counting deprivations brings us back to a univariate space. Thus, the social evaluation of distributions of deprivation counts is in many respects analogous to the social evaluation of income distributions, although it implicitly accounts for the association among the deprivation indicators. Of course, concave preferences in the income space correspond to convex preferences in the space of deprivations counts, which represent "bads" (loss in welfare) rather than "goods" (gains in welfare). However, despite convex preferences are ruled out in the analysis of income distributions because they would yield a social evaluation function violating the Pigou-Dalton principle of transfers, concave preferences are perfectly legitimate in the analysis of deprivation counts. This happens when we lean towards the union criterion, while convex preferences are associated with the intersection criterion. This example illustrates how the multidimensional case brings in new aspects that are unknown to the univariate case, but also neatly exposes the strict connection between value judgements – where we draw the boundaries of poverty when there are multiple deprivations – and analytical tools – the degree of concavity/convexity of social preferences. There is clearly a need of further work on the analytical foundations of the social evaluation of distributions of deprivations scores.

The opposite situation characterises the axiomatic treatment of poverty and inequality for continuous and categorical variables: a fairly rich theoretical apparatus does not appear to have made yet an impact on empirical investigations, except for sporadic applications. This may be due to the scarcity of suitable variables and databases, but may also reflect the difficulty of discriminating among many equally-sensible alternative tools. In addition to further developing and refining theoretical analysis, in this case empirical work may play an important role in screening the most effective tools. Whatever the approach adopted, the quality and reliability of databases and the elaboration of inference tools, two aspects that we have virtually ignored in this paper, are essential to support the validity of empirical analyses, especially when they are used to inform policy.

Yet, is it really worth devoting so much intellectual effort to develop the multidimensional analysis of poverty and inequality? It is an odd question at the end of such a long paper, but as discussed in the introduction the widely shared view that well-being, and hence poverty, is multidimensional does not necessarily imply that the social evaluation must be itself multi-dimensioned. It may be for philosophical reasons, or more practically because too much is lost in the process of aggregation. Once Sen (1987, p. 33) remarked that "the passion for aggregation makes good sense in many contexts, but it can be futile or pointless in others. ... When we hear of variety, we need not invariably reach for our aggregator". On the other hand, the "eye-catching property" of the Human Development Index was praised by Streeten (1994) as a powerful feature for its affirmation in the public debate, in spite of the theoretical weaknesses pointed out by its critics.[49] Three points may help us to find an answer to the question.

First, there is a pervasive demand by media commentators and policy-makers for multidimensional analyses. This demand must be met, not the least in order to avoid that such analyses are left to practitioners that conceive them as a bunching together of indicators of living standard through some simple averaging or multivariate technique easily available in statistical and econometric packages. Empirical research confirms that broadening the evaluative space to include variables other than income can modify the picture drawn on the basis of income alone. There is a distinct informative value in adopting a multidimensional perspective. The theoretical work surveyed in this paper facilitates the interpretation of empirical findings by bringing to the fore the implicit measurement assumptions and their economic meaning. If we estimate a lower deprivation index in the United Kingdom than in Italy using concave social preferences, as in Section 3.2.4, it is because we favour the union criterion, and hence we tend to be relatively more worried by the spreading of a given number of deprivations across many people than by their concentration on fewer people who are hit more. If, on the contrary, we have convex preferences and are particularly concerned about those suffering from severe deprivations, we cannot unequivocally rank one country ahead of the other.

Second, the difficulties of multidimensional measurement should not be overstated. The choice of the degree of poverty or inequality aversion, or the proper definition of indicators with which we are less familiar than with income also arise in the univariate context. The problems that are new to the multivariate case are the weighting structure of the attributes and their degree of substitutability. Both these aspects are not technical hitches but

[49] For a recent example, see the exchange between Klugman et al. (2011a, 2011b) and Ravallion (2011b, 2012a, 2012b). See also Chakravarty (2011).

the expression of implicit value judgements. Far from being a weakness of multidimensional approaches, the investigation of alternative assumptions is necessary to allow for the different views in the society. This is a sufficient reason for not devolving the resolution of these measurement problems to some statistical algorithm.

Third, the battery of instruments in our toolbox is ample. If we are reluctant to use a summary poverty or inequality index, we may fruitfully use sequential dominance analysis: it may yield a partial ordering, but it may be sometimes sufficient to evaluate, say, the impact on the distribution of well-being of alternative policies. The variety of our toolbox means that there is a "middle ground" between multidimensional summary indices and the dashboard approach, as stressed by Ferreira and Lugo (2013).

These are all good arguments in favour of a multi-dimensioned social evaluation. Are they also compelling enough to push us as far as to accept summary indices? Probably not, but two further comments are in order. The first is a pragmatic suggestion due to Bourguignon (1999, p. 483): when their building assumptions are properly understood, these indices can provide valuable insights if used "… more as a dominance instrument than a strictly cardinal rule of comparison". The second is a somewhat deeper point. In a sense, the uneasiness with such a summary index in sectors of the economics profession may stem from the reluctance to abandon a utility-based conception of well-being. Only individuals are able to assess the tradeoffs between the different constituents of well-being, and prices are the best available way to reveal such tradeoffs, as they derive from the interactions of individuals in a market economy. If externalities, distortions and missing markets prevent us from relying on prices as the aggregator of well-being dimensions, then the dashboard approach may be preferable, since no arbitrary weighting is imposed. The most developed conceptualisation of multidimensional well-being to date, the capability approach, originates exactly from the rejection of a utility-based conception: "*valuing* a life and measuring the *happiness* generated in that life are two different exercises" (Sen, 1985, p. 12). If this is the founding aspect of multidimensional analysis, then the weighting of the different dimensions is an integral part of the evaluation exercise and the reference to market prices loses much of its appeal. The social evaluation may attach more weight to work effort than that revealed by the wage, because jobs are characterized by other attributes that might contribute to reinforce social integration. From this perspective, the practical solutions given to the selection of weights, which often boil down to equal-weighting, may miss a decisive part of the evaluation. If this conjecture is correct, there are little chances to settle ever the controversy between dashboard approach and summary indices.

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