Multidimensional Solitons

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To my mother Bertha, my wife Marina, my son Ilya, my daughter Olga, and my granddaughters Eleanor, Natalie, Negev, and Nili

ACRONYMS

1D	One-dimensional
(1+1+1)D	Two-dimensional, with one temporal and one transverse spatial coordinates, while the propagation coordinate plays the role of the evolutional variable
2D	Two-dimensional
(2+1)D	Two-dimensional, with two transverse coordinates, while the third (propagation) coordinate plays the role of the evolutional variable
3D	Three-dimensional
(3+1)D	Three-dimensional, with the propagation coordinate playing the role of the evolutional variable and the temporal variable playing the role of an additional coordinate
AL	Ablowitz–Ladik (integrable discretization of the NLS equation)
a.r.	Aspect ratio
b.c.	Boundary condition(s)
BdG	Bogoliubov–de Gennes (linearized equations for perturbations around stationary solutions of GP/NLS equations)
BEC	Bose–Einstein condensate
BG	Bragg grating
BW	Bloch wave
BZ	Brillouin zone
CGL	Complex Ginzburg–Landau (equation)
CQ	Cubic–quintic (nonlinearity)
CSV	Crater-shaped vortex (the usual localized vortex mode)
CW	Continuous wave
DM	Dispersion management
DS	Dissipative soliton
FF	Fundamental frequency
FK	Frenkel–Kontorova (a dynamical lattice model)
FP	Fixed point
FR	Feshbach resonance
FT	Flat-top (profiles of solitons and quantum droplets)
FWHM	Full-width at half-maximum
FWM	Four-wave mixing
GP	Gross–Pitaevskii (equation)
GS	Gap soliton
GVD	Group-velocity dispersion
HO	Harmonic-oscillator (potential)
HV	Hidden vorticity
IS	Inter-site

IST	Inverse-scattering transform (the method for solving integrable nonlinear partial differential equations)
JJ	Josephson junction
KdV	Korteweg–de Vries (equation)
KP	Kadomtsev–Petviashvili (equation)
LB	“Light bullet” (a 3D spatiotemporal optical soliton)
LFE	Local-field effect
LHY	Lee–Huang–Yang (correction to the MF theory)
LI	Lévy index
MF	Mean-field (approximation)
MM	Mixed mode
MW	Microwave
NLS	Nonlinear Schrödinger (equation)
NM	Nonlinearity management
OAM	Orbital angular momentum
OL	Optical lattice
OS	On-site
PhC	Photonic crystal
PhR	Photorefractive (optical material)
$\mathcal{PT}$	Parity-time (symmetry)
QD	Quantum droplet
RL	Rayleigh length (alias diffraction length)
SG	Sine-Gordon (equation)
SH	Second harmonic
SM	Skyrme model
SOC	Spin–orbit coupling
SPM	Self-phase modulation
SSB	Spontaneous symmetry breaking
STOV	Spatiotemporal vortex
SV	Semi-vortex
TB	Three-body (loss)
TF	Thomas–Fermi (approximation)
TOF	Time of flight
TPA	Three-photon absorption
TS	Townes soliton
VA	Variational approximation
VAV	Vortex–antivortex (a two-component bound state in SOC systems)
VK	Vakhitov–Kolokolov (stability criterion)
VR	Vortex ring
XPM	Cross-phase modulation
ZS	Zeeman splitting

PREFACE

The absolute majority of work that has been performed in a huge area of theoretical and experimental studies of solitons, i.e., self-trapped solitary waves found in a great variety of nonlinear systems, dealt with one-dimensional (1D) settings. Extension of the soliton concepts to the multidimensional world is a very promising, but also really challenging, direction of the work for theorists and experimentalists. The expected and, to a certain extent, realized gain is the fascinating possibility to create completely new species of solitary states, as the two- and three-dimensional (2D and 3D) geometries make it possible to build localized modes with intrinsic topological arrangements. The most obvious possibility is to create 2D and 3D vortex solitons, which are described by a complex wave function. This means building solitons with an intrinsic angular momentum, which may be considered as a classical counterpart of spin of quantum particles. Solitons can carry an orbital angular momentum in a different form if they are rotating modes with a non-axially-symmetric shape. Multi-component solitons can be used to build more sophisticated topological structures, such as famous skyrmions, hopfions (alias twisted vortex tori in the 3D space), monopoles coupled to non-Abelian gauge fields, knots, and others, which have no counterparts in 1D realizations. As concerns dynamics of multidimensional solitons, they may demonstrate various scenarios of interactions and formation of bound states (often called “soliton molecules”), as well as many-soliton structures (such as soliton lattices). These and other possibilities are considered in detail in the corresponding chapters of this book.

On the other hand, the work with solitons in the 2D and 3D geometry encounters fundamental difficulties. In terms of the underlying mathematical theory, the fact that the most fundamental models that give rise to 1D solitons, such as the Korteweg–de Vries (KdV), sine-Gordon (SG), and nonlinear Schrödinger (NLS) equations, are *integrable* is limited to one dimension. Therefore, the unprecedentedly powerful methods [the inverse-scattering transform (IST), the Darboux transform, the Hirota method, etc.] that produce highly nontrivial exact solutions, such as those for collisions between solitons, formation of bound states in the form of breathers, prediction of the soliton content of a given input, etc., are not available in the studies of 2D and 3D models. In 2D, there are some exceptional integrable equations [most notably, the celebrated Kadomtsev–Petviashvili (KP) equations], but most important soliton-generating models, such as 2D and 3D NLS equations, lose the integrability that they had in 1D and admit no exact solutions. In 3D settings, no integrable equations, which would be capable to produce physically relevant solutions for solitons, are known.

The lack of the integrability of basic 2D and 3D equations underlying the soliton theory may be considered a technical difficulty because relevant solutions, once they are not available in an exact form, can often be constructed by means of approximate analytical or semi-analytical methods, such as the ubiquitous variational approximation (VA). In any case, the availability of powerful computational facilities suggests a possibility to produce numerical solutions of nearly all theoretical problems, even if simulations of 2D and, especially, 3D nonlinear equations are more difficult than their 1D counterparts.

However, the exit from 1D models to the real 3D world (or intermediate 2D settings) leads to principal problems, related to *stability* of the expected soliton states. First, a well-known result is that stationary solutions of the 3D nonlinear Klein–Gordon equations are always subject to instability, as they cannot represent an energy minimum (Derrick, 1964). Next, the instability problem is very clearly exhibited by the most important model based on the NLS equation with the self-attractive cubic nonlinearity (a commonly known example is the Kerr term in various models originating from optics): while stationary soliton solutions of the 1D NLS equations are completely stable, the 2D and 3D versions of the same equation produce soliton families (which can be found in an approximate form by means of the VA, or easily constructed, with an extremely high accuracy, as numerical stationary solutions) that are *completely unstable*, due to the fact that precisely the same 2D and 3D NLS equations give rise to the *collapse*, alias blowup (catastrophic self-compression leading to the formation of a true singularity after a finite evolution time). The collapse is *critical* in 2D, and *supercritical* in 3D, which means that the 2D collapse sets in if the norm of the input (total power, in terms of optics) exceeds a certain finite critical (threshold) value, while in 3D, the threshold is zero; i.e., an arbitrarily weak input may initiate the supercritical collapse. In 2D, the input whose norm falls below the threshold value does not blow up but rather decays into "radiation" (small-amplitude waves). Small perturbations added to any soliton of the 3D NLS equation trigger its blowup, while in 2D, the addition of small perturbations initiates either the blowup or fast decay. In this connection, it is relevant to mention that the first species of solitons, which was ever considered in optics, is the family of the so-called *Townes solitons* [TSs, predicted by Chiao *et al.* (1964)]. These are stationary solutions of the 2D NLS equation, which predict self-trapped shapes of laser beams propagating in the bulk Kerr medium under the condition of paraxial diffraction. Actually, many other species of solitons, which were predicted later, had been created in the experiment [see the books of Kivshar and Agrawal (2003) and Dauxois and Peyrard (2006)], but the TSs in their pure form have never been created, as they are unstable states, which represent a separatrix between collapsing and decaying solutions of the 2D NLS equation.

As concerns 2D and 3D solitons with embedded vorticity (alias *vortex rings*, VRs), they are subject to still stronger instability, which develops faster than the collapse, viz., spontaneous splitting of the VR into two or several fragments, which are close to the corresponding fundamental (zero-vorticity) solitons. At a later stage of the evolution, the secondary solitons are destroyed by the intrinsic collapse.

The NLS equation with the self-attractive nonlinearity may be a physically correct model for many physical realizations in optics, dynamics of Bose–Einstein condensates (BECs) in ultracold atomic gases, physics of plasma (Langmuir waves), etc., but the occurrence of the collapse implies that these physical settings cannot be used for the creation of multidimensional solitons, once the solitons are the objective of the work. Therefore, a cardinal problem is search for physically realistic multidimensional systems, which include additional ingredients that make it possible to suppress the collapse and help to predict and create stable 2D and 3D solitons. This is possible in various physical setups. For instance (as is discussed in detail in this book), stable 2D and 3D optical solitons can be predicted and, eventually, experimentally created if the optical medium features, in addition to the cubic self-focusing, higher-order quintic self-defocusing, which arrests the blowup and provides the stabilization of 2D and

3D optical solitons (2D solitons stabilized by the quintic self-defocusing have been reported in experimental works, while the creation of 3D solitons remains a challenging problem). Another recently discovered and extremely interesting option is to consider a binary BEC with the collapse driven by the attractive cubic interaction between its two intrinsically self-repulsive components. In this system, the collapse is actually arrested by a higher-order quartic self-repulsive term, which is induced in each component by the so-called Lee–Huang–Yang (LHY) effect, i.e., a correction to the usual cubic mean-field (MF) interaction induced by quantum fluctuations around the MF state (Lee *et al.*, 1957). As a result, the binary BEC creates completely stable 3D and quasi-2D self-trapped "quantum droplets" (QDs), which seem as multidimensional solitons (even if they are not usually called "solitons," as the name of QDs is preferred in the literature). The prediction of QDs, made by Petrov (2015), was quickly realized experimentally by several experimental groups (Cabrera *et al.*, 2018; and Semeghini *et al.*, 2018). These theoretical and experimental findings (as well as many others) are considered in detail in this book.

The collapse-suppressing mechanisms mentioned above, i.e., the cubic–quintic (CQ) focusing–defocusing nonlinearity in optical media, and the attractive cubic MF interaction in the binary BEC with the LHY quartic self-repulsive correction may stabilize not only 3D and 2D fundamental (i.e., structureless) soliton-like states, but also ones with embedded vorticity (3D and 2D VRs), although these results remain, as yet, a theoretical prediction [optical soliton-like modes with embedded vorticity were observed by Eilenberger *et al.* (2013) and Reyna *et al.* (2016), but only as transient states in different experimental settings]. The stabilization of vortex solitons is another topic considered in detail in this book.

The broad topic of multidimensional solitons was a subject of several review articles (Malomed *et al.*, 2005; 2016; Malomed, 2016; 2018; 2019; Mihalache, 2017; and Kartashov *et al.*, 2019). These reviews were focused on particular aspects of the theme; see a detailed discussion in Chap. 1. However, the topic as a whole was not previously summarized in a relatively full form. This is the objective of the present book. Because the stabilization is the critically important issue in the theoretical and experimental work with multidimensional solitons, it is natural to organize a survey of this broad subject *according to particular stabilization mechanisms*. This principle determines the structure of the book, as can be seen in the Table of Contents. However, it is relevant to stress that, although the intention is to include many essential aspects of the subject, the book is not designed as a comprehensive presentation of all aspects relevant to studies of multidimensional solitons. Some topics that are not included are briefly mentioned in Chap. 15 (Conclusion). One such topic, which has become a subject of many recent works, is spatiotemporal propagation of light in nonlinear multimode fibers, and another recently active one, which also belongs to the realm of photonics, is nonlinear self-trapping in exciton–polariton condensates.

It is relevant to stress that settings which give rise to multidimensional solitons and soliton-like states may be essentially conservative (if losses are negligible on temporal and spatial scales relevant to the experiment) or dissipative (if losses are essential, and the system must include gain or external pump, necessary for compensation of the losses). In this book, dissipative multidimensional solitons are considered, in some detail, in Chap. 14, while Chaps. 2–13 address lossless systems or ones including weak losses.

As concerns physical realizations in which multidimensional solitons can be (and have been) created, they are included in the chapters presenting the respective results. In fact, all the realizations considered in the book in detail belong to two broad areas: first, optics and photonics, and second, various realizations of BEC in atomic gases. The Table of Contents presents the material in a sufficiently detailed form to help the reader find particular physical realizations and particular results predicted and/or experimentally reported in the corresponding settings. Another vast realm in which 3D solitons (such as famous skyrmions) are a subject of great significance is the classical field theory, with applications to particles and nuclei, ferromagnetic media, semiconductors, etc. This topic is discussed in the Introduction (Chap. 1) but is not addressed in other chapters of the book.

Of course, the selection of the material for the presentation in the book is biased by personal work and interests of the author. Actually, different topics included in the book are considered with different degrees of detail. Some settings, which play a paradigmatic role, are presented in a sufficiently full form (dropping minor technicalities, in many cases), while some others, which seem as less significant additions to the basic points, are considered schematically or, sometimes, are only briefly mentioned.

Different chapters of the book are linked by relevant cross-references. Nevertheless, some basic principles are briefly repeated in particular chapters to make them relatively independent from each other. In particular, the fact that the zero-vorticity (fundamental) Townes solitons (TSs), i.e., solutions of the 2D NLS equation with the cubic self-attraction, are subject to the instability driven by the critical collapse, and similar solitons with embedded vorticity (vortex rings, VRs) are prone to the still stronger splitting instability, is mentioned in several different chapters, as these facts are basically important for the work with 2D solitons in various settings.

The book is drafted as a monograph, which may be appropriate for reading and using by active researchers, both theorists and experimentalists, working in the realm of solitons and nonlinear waves. It may also be useful for members of research communities in broad areas of nonlinear optics and photonics, matter waves (in BEC), the general theory of nonlinear waves and nonlinear dynamics, and some others. The book may also be used by graduate students who start or continue their work in these areas.

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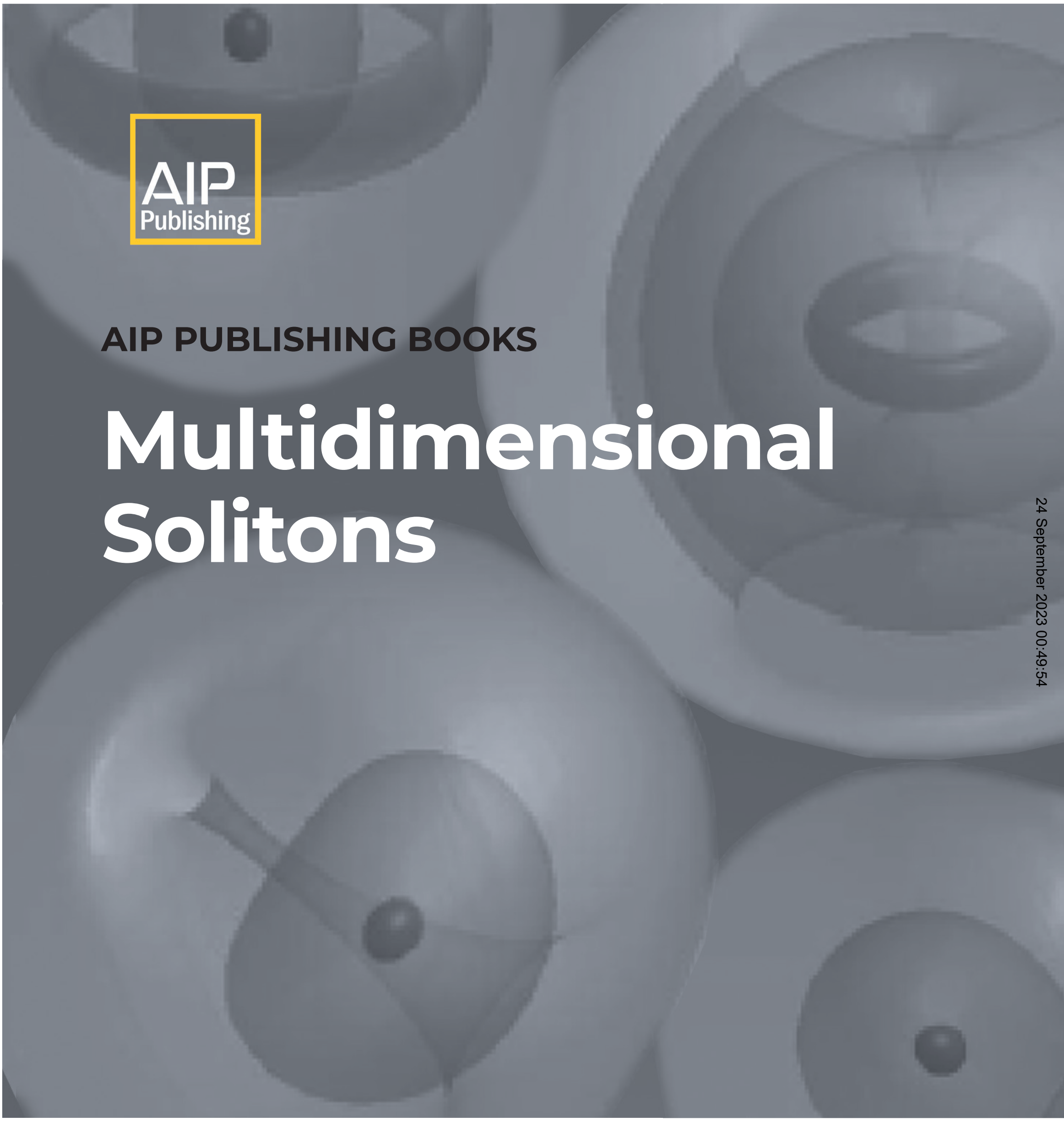
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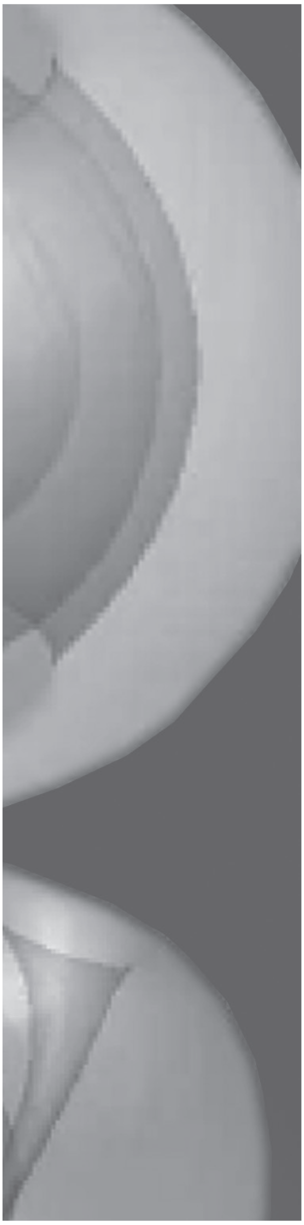

TABLE OF CONTENTS