J. Austral. Math. Soc. (*Series A*) 27 (1979), 305–312

MULTIDIMENSIONAL VOLTERRA INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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(Received 16 February; revised 7 August 1978)

Communicated by J. B. Miller

Abstract

In this article, it is shown that the Volterra integral equation of convolution type $w - w \otimes g = f$ has a continuous solution w when f,g are continuous functions on R^n and $\otimes$ denotes a truncated convolution product. A similar result holds when f,g are entire functions of several complex variables. Also simple proofs are given to show when f,g are entire, $f \otimes g$ is entire, and, if $f \otimes g = 0$, then $f = 0$ or $g = 0$. Finally, the set of exponential polynomials and the set of all solutions to linear partial differential equations are considered in relation to this convolution product.

Subject classification (*Amer. Math. Soc.* (*MOS*) *1970*): 32 A 15, 35 E 99; 44 A 35; 45 D 05; 46 E 25.

1. Introduction

In this article, we consider the truncated convolution product

$$f \otimes g(x) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} f(x-\xi)\, g(\xi)\, d\xi, \tag{1}$$

where x, $\xi \in R^n$, $x = (x_1, x_2, \ldots, x_n)$ and f, g are continuous functions on real n dimensional space R^n. We show that for such functions, $f \otimes g$ is also continuous and that the Volterra integral equation

$$w - w \otimes g = f \tag{2}$$

always has a continuous solution w. In addition, corresponding results are obtained for entire functions of several complex variables.

The truncated convolution product as defined above for functions of a single real variable and Titchmarch's convolution theorem for function on a half line are widely known (see, for example, Erdélyi (1962)). This convolution product for continuous functions on R^2 has been considered by Ditkin and Prudnikov (1962). As well, Mikusiński and Ryll-Nardzewski (1953) (see also Mikusiński (1961)) have shown that if $S = \{x \in R^n: x_j \geq 0 \text{ for } j = 1, 2, \ldots, n\}$, and if f, g are continuous functions on S with $f \otimes g = 0$ on S, then $f = 0$ or $g = 0$ on S. This fact has been used by Gutterman (1969) in developing an operational calculus for continuous function on S. Multidimensional Mikusiński-type operators are also considered by Hughes and Struble (1973).

The integral equation (2) in the case of one real variable has been considered by several writers, including Yoshida (1960), Bellman and Cooke (1963), Laird (1974a) and also finds application in probability theory (see, for example, Feller (1966), Chapter XI). For several real variables, integral equations of Volterra type have been considered and we may refer, by way of example, to Parodi (1950), Walter (1970) and Suryanarayana (1972) (for multidimensional Fredholm integral equations: Petrovskii (1957)). Our multidimensional results differ from the results in these references in that the integral equation (2) is simpler with a corresponding ease of treatment.

With regard to complex variables, we set

$$f \otimes g(z) = \int_0^{z_1} \int_0^{z_2} \cdots \int_0^{z_n} f(z-\xi) g(\xi) \, d\xi,$$

where z, $\xi \in C^n$, $z = (z_1, z_2, \ldots, z_n)$ and f, g are entire functions on complex n dimensional space C^n. In the case of one complex variable, it has been observed by Dieudonné (1970), p. 282, Dickson (1973), Laird (1975) and Rubel (1977) that if f, g are entire, then so also is $f \otimes g$. As well, simple proofs have been given by Laird (1975) and Rubel (1977) to show that if f, g are entire, and if $f \otimes g = 0$, then $f = 0$ or $g = 0$ with applications to operational calculus being developed by Rubel (1977). Also, if f, g are entire functions on C, the existence of a unique entire solution to the equation (2) has been shown by Laird (1975). The method of proof will be now shown to extend to multidimensional Volterra integral equations of convolution type.

For much of this article, it is convenient to take the real and the complex variable cases together. Let $C(R^n)$ $(H(C^n))$ denote the set of all continuous complex valued functions on R^n (entire functions on C^n) endowed with the topology of uniform convergence on all compact subsets of R^n (C^n). For $f, g \in C(R^n)$ $(H(C^n))$ and k any positive integer, set

$$p_k(f) = \sup\{|f(x)| : \|x\| \leq k\}$$

and

$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k} \min(1, p_k(f-g))$$

where

$$\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\} \quad \text{and} \quad x = (x_1, x_2, \ldots, x_n) \in R^n\,(C^n).$$

Then p_k is a seminorm and d is a metric for $C(R^n)$ $(H(C^n))$ and each space is complete in this metric (see Treves (1966), p. 469) for these and additional details in the case of $H(C^n)$). Throughout, convergence in either space shall be referred to as locally uniform convergence.

We shall have use for 'multi-indices'. For $p = (p_1, p_2, \ldots, p_n)$ (where p_j is a non-negative integer and $x = (x_1, x_2, \ldots, x_n)$), set

$$p! = p_1! p_2! \ldots p_n! \quad \text{and} \quad u_p(x) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}.$$

From the definition (1) and elementary integration, we have

$$\frac{u_p}{p!} \otimes \frac{u_q}{q!} = \frac{u_{p+q+1}}{(p+q+1)!}, \tag{3}$$

where, p, q and $p+q+1$ are all multi-indices with

$$p+q+1 = (p_1+q_1+1, p_2+q_2+1, \ldots, p_n+q_n+1).$$

Clearly this formula holds for both complex and real variables.

2. Main results

The first two propositions extend results stated by Erdélyi (1962) for continuous functions on a half line. Throughout this section, we let X denote either $C(R^n)$ or $H(C^n)$ as defined above. Thus X is a complex vector space and also a complete metric space.

PROPOSITION 1. (a) *If $f, g \in X$, then $f \otimes g \in X$.*

(b) *If $\{f_m\}$, $\{g_m\}$ are sequences in X that converge, locally uniformly to f, g as $m \to \infty$, then $f_m \otimes g_m \to f \otimes g$ locally uniformly as $m \to \infty$.*

(c) *$X(+, \otimes)$ is an algebra over C.*

PROOF. For $C(R^n)$, we may use the fact that any continuous function on R^n may be uniformly approximated on any compact subset of R^n by polynomials in n variables (extend the result given by Royden (1963), p. 151 from real-valued continuous functions on R^n, say u, v to $f \in C(R^n)$ by $f = u + iv$). From this, if $f, g \in C(R^n)$, f, g may be approximated, locally uniformly by sequences of polynomials, say $\{f_m\}$, $\{g_m\}$. By use of formula (3), $f_m \otimes g_m$ is a polynomial.

With $f_m \to f$ and $g_m \to g$ locally uniformly, an application of elementary estimates to

$$f \otimes g - f_m \otimes g_m = (f - f_m) \otimes g + f_m \otimes (g - g_m)$$

on any compact subset shows that $f_m \otimes g_m \to f \otimes g$ locally uniformly, as $m \to \infty$. Hence, when f, g are continuous, so also is $f \otimes g$. As well, part (b) is established.

https://doi.org/10.1017/S1446788700012416 Published online by Cambridge University Press

For $H(C^n)$, the same proof suffices except we use the facts that any entire function f has a power series expansion on C^n, and, if a sequence of entire functions is locally uniformly convergent, then the limit function is entire (Dieudonné (1960), p. 229).

For part (c), we have noted that X is a vector space over C. The remaining details that establish $C(R^n)(+,\otimes)$ is an algebra over C (and so also a commutative ring) are easily verified (see Ditkin and Prudnikov (1962) for $n = 2$) and shall be omitted. When $X = H(C^n)$, we may use the fact (Dieudonné (1960), p. 204) that when f is any entire function defined on R^n, it may be extended to a unique entire function defined on C^n to conclude $H(C^n)(+,\otimes)$ is an algebra from $C(R^n)(+,\otimes)$ being an algebra.

In the next proposition, when $g \in X$, we set $g^{\otimes 1} = g$ and $g^{\otimes(p+1)} = g^{\otimes p} \otimes g$ for $p = 1, 2, \ldots$.

PROPOSITION 2. *Let $f, g \in X, k, p$ be positive integers and $M = p_k(g)$. Then*

(a) $|f \otimes g(x)| \leqslant p_k(f)\, M k^n$, *and*

(b) $|g^{\otimes(p+1)}(x)| \leqslant M^{p+1} |x_1 x_2 \ldots x_n|^p / (p!)^n \leqslant M^{p+1} k^{pn} / (p!)^n$ *when* $\|x\| \leqslant k$.

PROOF. Both parts are based on the application of elementary estimates of the absolute values of the integrals involved when $\|x\| \leqslant k$. Initially we have

$$|f \otimes g(x)| \leqslant p_k(f)\, M |x_1 x_2 \ldots x_n| \quad \text{for } \|x\| \leqslant k.$$

So part (a) follows. A simple inductive argument gives part (b).

From this, it is immediate that if $g \in X$, then $g^{\otimes p} \to 0$ locally uniformly as $p \to \infty$.

THEOREM 3. *Let $f, g \in X$. Then there exists a unique solution w in X satisfying the integral equation $w - w \otimes g = f$.*

PROOF. When $f, g \in X$, on application of the bounds in Proposition 2(b) shows that if

$$w_p = f + f \otimes g + \ldots + f \otimes g^{\otimes p},$$

then $\{w_p\}$ is a Cauchy sequence in X. Since X is complete, $\{w_p\}$ has a limit, say w, in X. Moreover, w satisfies $w - w \otimes g = f$.

If φ is the difference of two solutions in X, then $\varphi = \varphi \otimes g$. Thus

$$\varphi = \varphi \otimes g = (\varphi \otimes g) \otimes g = \ldots = \varphi \otimes g^{\otimes p} \quad \text{for } p = 1, 2, \ldots.$$

Since $g^{\otimes p} \to 0$ as $p \to \infty$, $\varphi = 0$.

Theorem 3 has an extension to

THEOREM 4. *Let* $f_j, g_{jk} \in X$ *for* $j,k = 1,2,\ldots,m$. *Then the system of equations*

$$w_j - \sum_{k=1}^{m} g_{jk} \otimes w_k = f_j, \quad j = 1,2,\ldots m$$

has a unique solution $w_1, w_2, \ldots, w_m \in X$.

PROOF. The proof is omitted as it requires some detail that is inherent in the proofs of Theorem 3 and in the case of $C(R)$ (Laird (1974b), p. 416).

3. Other results

It is clear that $C(R^n)$ $(+, \otimes)$ has non-zero divisors of zero. However, for entire functions, we have

THEOREM 5. *Let* $f, g \in H(C^n)$ *and* $f \otimes g = 0$. *Then* $f = 0$ *or* $g = 0$.

PROOF. Using p as a multi-index, for $f \in H(C^n)$ and

$$f = \sum_{p=0}^{\infty} f_p u_p / p!,$$

set

$$T(f)(\xi) = \sum_{p=0}^{\infty} f_p \xi^{p+1} = \sum_{p=0}^{\infty} f_p u_{p+1}(\xi)$$

for $\xi \in C^n$. Then T is a linear map from $H(C^n)$ to the ring $U(+, \times)$ of formal power series in n indeterminates. From (3), it is clear that $T(f \otimes g) = T(f) \times T(g)$ when $f = u_p$ and $g = u_q$. Hence this relation holds when f, g are polynomials and so when f, g are entire functions. Moreover, if $T(f) = 0$, then $f = 0$ and so T is an isomorphism between $H(C^n)(+, \otimes)$ and a subring of $U(+, \times)$. Since U has no non-zero divisors of zero (Zariski and Samuel (1958), p. 35), from $f \otimes g = 0$, we have $T(f) = 0$ or $T(g) = 0$ and so $f = 0$ or $g = 0$.

The results so far represent positive extensions from one aspect of functions of one variable to functions of several variables. Some other aspects that may be of interest now follow.

The first concerns the equation $w \otimes g = f$. In the case when f, g are entire functions of a single variable, a necessary and sufficient condition for the existence of an entire function may be readily found to be the existence of a non-negative integer p such that $D^p g(0) \neq 0$ but $D^q f(0) = 0$ for $q = 0, 1, 2, \ldots, p$ (Laird (1975)). For the equation $w \otimes g = f$ in the case of several variables, it would appear that there are no simple conditions on f, g that will guarantee the existence of an entire solution.

https://doi.org/10.1017/S1446788700012416 Published online by Cambridge University Press

The second aspect concerns exponential polynomials. Such a function in n variables is a finite linear combination of terms

$$u_p \exp(a\cdot): z \to z_1^{p_1} z_2^{p_2} \dots z_n^{p_n} \exp(a_1 z_1 + a_2 z_2 + \dots + a_n z_n),$$

where $p_1, p_2, \dots, p_n$ are non-negative integers and $a_1, a_2, \dots, a_n \in C$. If we let MQ denote the set of all exponential polynomials in X, then MQ is a subalgebra of $X(+, \otimes)$ (the fact that MQ is closed under convolution being readily verified by use of elementary calculus).

It is easy to show that if f, g are exponential polynomials in one variable and if w satisfies $w - w\otimes g = f$, then w is also an exponential polynomial (Laird (1974a)). This is not the case for w when f, g are exponential polynomials of several variables as seen by the following example. Let $n = 2$ and $e(x, y) = 1$ for all (x, y). Then the equation $w - w\otimes e = e$ has solution

$$w(x, y) = \sum_{k=0}^{\infty} (xy)^k/(k!)^2$$

which is not an exponential polynomial.

Finally, we consider MC, the set of all solutions in Y to homogeneous linear partial (ordinary when $n = 1$) differential equations with constant coefficients where Y is either $H(C^n)$ or the set of indefinitely differentiable functions in $C(R^n)$. Let E denote the set of all linear differential operators in n variables with constant coefficients so that $MC = \{f \in Y: Pf = 0$ for some $P \in E$ and $P \neq 0\}$. When $n = 1$, $MC = MQ$ and when $n > 1$, MQ is a proper subset of MC. However, by the Malgrange—Ehrenpreis theorem (see, for example, Treves (1966), p. 102), if $f \in MC$ with $Pf = 0$ where $P \in E$ and $P \neq 0$, then f is the locally uniform limit of a sequence of exponential polynomials $\{f_m\}$ where $Pf_m = 0$.

Clearly, MC is a complex vector space. However, for functions of several variables, MC is not closed under convolution. For $n = 2$, let $f(x, y) = 2x \exp x^2$ and $g(x, y) = 2y \exp y^2$. Then $D_2 f = 0 = D_1 g$ where $D_1 = \partial/\partial x$ and $D_2 = \partial/\partial y$. Also $f\otimes g(x, y) = h(x, y) - \exp x^2 - \exp y^2 + 1$ where $h(x, y) = \exp(x^2 + y^2)$. Suppose now that $h \in MC$ with $S \in E$, $S \neq 0$ and $Sh = 0$. Then $S(ph) = 0$ where p is any polynomial of two variables whence $S(\varphi) = 0$ for all $\varphi \in Y$. So $S = 0$ and a contradiction results. Hence $h \notin MC$ and so $f\otimes g \notin MC$ although both $f, g \in MC$.

The next theorem concerns equation (2) and gives a sufficient condition for its solution to belong to MC. Here D_j is the partial derivative with respect to the jth variable and the results quoted as given by Laird (1975), pp. 815–817.

THEOREM 6. *Let $f \in MC$, $g \in MQ$ and w satisfy $w - w\otimes g = f$. Then $w \in MC$.*

PROOF. If g is an exponential polynomial in Y then there are n non-zero linear differential operators $L_j = L_j(D_j) \in E$ where each L_j only involves D_j and is such that $L_j g = 0$ for $j = 1, 2, \ldots, n$. Moreover, $(L_1 L_2 \ldots L_n)(w \otimes g) = Tw$ for all $w \in Y$ where $T \in E$ and $T \neq L_1 L_2 \ldots L_n$. With $f \in MC$, choose $S \in E$, $S \neq 0$ such that $Sf = 0$. Then

$$S(L_1 L_2 \ldots L_n - T)w = S(L_1 L_2 \ldots L_n)(w - w \otimes g) = (L_1 L_2 \ldots L_n) Sf = 0$$

and so $w \in MC$.

It appears at present to be an open problem whether $f, g \in MC$ is sufficient to ensure that $w \in MC$ when $w - w \otimes g = f$.

Acknowledgements

The author would like to thank Professor W. A. Al-Salam of the University of Alberta for the references to work on multidimensional integral equations and express his appreciation of the facilities of the University of Calgary. In addition, he is indebted to the referees for their helpful comments.

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