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Multidimensional Yamada-Watanabe theorem and its applications to particle systems

Piotr Graczyk[1,a)] and Jacek Małecki[2,b)]
[1]*LAREMA, Université d'Angers, 2 Bd Lavoisier, 49045 Angers Cedex 1, France*
[2]*Institute of Mathematics and Computer Science, Wrocław University of Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland*

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We prove a multidimensional version of the Yamada-Watanabe theorem, i.e., a theorem giving conditions on coefficients of a stochastic differential equation for existence and pathwise uniqueness of strong solutions. It implies an existence and uniqueness theorem for the eigenvalue and eigenvector processes of matrix-valued stochastic processes, called a "spectral" matrix Yamada-Watanabe theorem. The multidimensional Yamada-Watanabe theorem is also applied to particle systems of squared Bessel processes, corresponding to matrix analogues of squared Bessel processes, Wishart and Jacobi matrix processes. The β-versions of these particle systems are also considered.
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I. INTRODUCTION

In this paper, we prove a multidimensional analogue of the celebrated Yamada-Watanabe theorem, ensuring the existence and uniqueness of strong solutions of one-dimensional stochastic differential equations (SDEs) with a Hölder coefficient in the Itô integral part. It is proved in Sec. II, Theorem 2.

Consider the space $\mathcal{S}_p$ of symmetric $p \times p$ real matrices and a function $g : \mathbf{R} \to \mathbf{R}$. Recall that the function g acts spectrally on a matrix $X \in \mathcal{S}_p$ in the following way:

$$g(X) = H \operatorname{diag}[g(\lambda_1), \ldots, g(\lambda_p)]\, H^T, \tag{1.1}$$

where $X = H\Lambda H^T$ is a diagonalization of X, with an orthonormal matrix H and an eigenvalue matrix $\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_p]$. Consequently, $g(\Lambda) = \operatorname{diag}[g(\lambda_1), \ldots, g(\lambda_p)]$ and $g(X) = Hg(\Lambda)H^T$.

In Sec. III, we derive a system of SDEs for the eigenvalues and the eigenvectors for a solution of a matrix SDE of the form

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt,$$

where B_t is a Brownian matrix of dimension $p \times p$, the matrix stochastic process X_t takes values in the space of symmetric $p \times p$ matrices and the functions $g, h, b : \mathbf{R} \to \mathbf{R}$ act on the spectrum of X_t. Under some mild conditions on the functions g, h, b it is shown in Theorem 5 that the eigenvalues never collide. The β-versions and complex versions of the eigenvalue system are also considered for the collision time problem (Corollaries 1 and 2).

If the functions g, h, b are such that gh is 1/2-Hölder continuous, and the symmetrized functions $g^2 \otimes h^2$ and b are Lipschitz continuous, then we establish in Theorem 6 the existence and uniqueness of a strong solution of the system of SDEs for the eigenvalues and the eigenvectors of X_t. We call such a result "a spectral matrix Yamada-Watanabe theorem".

[a)]E-mail: piotr.graczyk@univ-angers.fr.
[b)]E-mail: jacek.malecki@pwr.wroc.pl.

Section IV contains interesting applications. We apply Theorems 2, 5, and 6 to

(i) noncolliding particle systems of squared Bessel processes which are intensely studied in recent years in statistical and mathematical physics (Katori and Tanemura[14–17]).

(ii) The systems of SDEs for the eigenvalues of Wishart and Jacobi matrix processes, as well as of the β-Wishart and β-Jacobi processes. We note the importance of the β-Wishart eigenvalues systems in statistical physics: they are statistical mechanics models of "log -gases," see the recent book of Forrester.[11]

Surprisingly, the existence of strong solutions of SDEs for such Hölder-type noncolliding particle systems was not established in general; only some cases of (ii) were treated by Demni[6–8] and Lépingle.[20] In Secs. IV A and IV D, we prove the existence and uniqueness of a strong solution to these systems of SDEs, for the whole range of the drift parameter and $\beta \geq 1$.

The spectral matrix Yamada-Watanabe theorem is applied in Secs. IV B and IV C to matrix valued squared Bessel type processes, Wishart and Jacobi matrix processes. We improve the known results of Bru,[1–3] Mayerhofer *et al.*,[22] and Doumerc,[10] showing the existence and uniqueness of strong solutions of the SDEs system for the eigenvalues and eigenvectors of X_t, for the whole range of the drift parameter.

In the Wishart case, we contribute in this way to realization of a programme started by Donati-Martin, Doumerc, Matsumoto and Yor,[9] claiming that Wishart processes have similar properties as classical one-dimensional squared Bessel processes.

II. A MULTIDIMENSIONAL YAMADA-WATANABE THEOREM

Let us recall the classical Yamada-Watanabe theorem see, e.g., Ref. 13, p. 168 and Ref. 28.

Theorem 1: *Let $B(t)$ be a Brownian motion on* $\mathbf{R}$. *Consider the SDE,*

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt.$$

If $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$ for a strictly increasing function ρ on $\mathbf{R}^+$ with $\rho(0) = 0$ and $\int_{0^+} \rho^{-1}(x)dx = \infty$, and b is Lipschitz continuous, then the pathwise uniqueness of solutions holds; consequently the equation has a unique strong solution.

No multidimensional versions of the Yamada-Watanabe theorem seem to be known, even if their need is great (cf. Ref. 3, p. 738). We propose a useful generalization; however, we stress the fact that the Hölder continuous functions σ_i appearing in the following system of SDEs are one-dimensional. The proof is based on the approach presented in Revuz, Yor,[25] in particular on the results of Le Gall.[19] By $\|\cdot\|$ we mean the Euclidean norm $\|\cdot\|_2$ on $\mathbf{R}^d$.

Theorem 2: *Let $p, q, r \in \mathbb{N}$ and the functions $b_i : \mathbf{R}^p \to \mathbf{R}$, $i = 1, \ldots, p$ and $c_k, d_j : \mathbf{R}^{p+r} \to \mathbf{R}$, $k = p + 1, \ldots, p + q$, $j = p + 1, \ldots, p + r$, be bounded real-valued and continuous, satisfying the following Lipschitz conditions: for a constant $A > 0$,*

$$|b_i(y_1) - b_i(y_2)| \leq A||y_1 - y_2||, \quad i = 1, \ldots, p,$$

$$|c_k(y_1, z_1) - c_k(y_2, z_2)| \leq A||(y_1, z_1) - (y_2, z_2)||, \quad k = p + 1, \ldots, p + q,$$

$$|d_j(y_1, z_1) - d_j(y_2, z_2)| \leq A||(y_1, z_1) - (y_2, z_2)||, \quad j = p + 1, \ldots, p + r,$$

for every $y_1, y_2 \in \mathbf{R}^p$ and $z_1, z_2 \in \mathbf{R}^r$. Moreover, let $\sigma_i : \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, p$, be a set of bounded Borel functions such that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho_i(|x - y|), \quad x, y \in \mathbf{R},$$

where ρ_i: $(0, \infty) \to (0, \infty)$ are Borel functions such that $\int_{0^+} \rho_i^{-1}(x)dx = \infty$. Then the pathwise uniqueness holds for the following system of stochastic differential equations:

$$dY_i = \sigma_i(Y_i)dB_i + b_i(Y)dt, \quad i = 1, \ldots, p, \tag{2.1}$$

$$dZ_j = \sum_{k=p+1}^{p+q} c_k(Y, Z)dB_k + d_j(Y, Z)dt, \quad j = p+1, \ldots, p+r, \tag{2.2}$$

where $B_1, \ldots, B_{p+q}$ are independent Brownian motions.

Proof: Let (Y, Z) and $(\tilde{Y}, \tilde{Z})$ be two solutions with respect to the same Brownian motion $B = (B_i)_{i \le p+q}$ such that $Y(0) = \tilde{Y}(0)$ and $Z(0) = \tilde{Z}(0)$ a.s. For every $i = 1, \ldots, p$, we have

$$Y_i(t) - \tilde{Y}_i(t) = \int_0^t (\sigma_i(Y_i(s)) - \sigma_i(\tilde{Y}_i(s)))dB_i(s) + \int_0^t (b_i(Y(s)) - b_i(\tilde{Y}(s)))ds. \tag{2.3}$$

Then we get

$$\int_0^t \frac{\mathbf{1}_{\{Y_i(s) > \tilde{Y}_i(s)\}}}{\rho_i(Y_i(s) - \tilde{Y}_i(s))} d\left\langle Y_i - \tilde{Y}_i, Y_i - \tilde{Y}_i \right\rangle = \int_0^t \frac{(\sigma_i(Y_i(s)) - \sigma_i(\tilde{Y}_i(s)))^2}{\rho_i(Y_i(s) - \tilde{Y}_i(s))} \mathbf{1}_{\{Y_i(s) > \tilde{Y}_i(s)\}} ds \le t.$$

Thus, applying Lemma 3.3 from Ref. 25, p. 389 , we get that the local time of $Y_i - \tilde{Y}_i$ at 0 vanishes identically. Consequently, by Tanaka's formula we get

$$\begin{aligned} |Y_i(t) - \tilde{Y}_i(t)| &= \int_0^t \operatorname{sgn}(Y_i(s) - \tilde{Y}_i(s))d(Y_i(s) - \tilde{Y}_i(s)) + L_t^0(Y_i - \tilde{Y}_i) \\ &= \int_0^t \operatorname{sgn}(Y_i(s) - \tilde{Y}_i(s))d(Y_i(s) - \tilde{Y}_i(s)). \end{aligned}$$

Since σ_i is bounded, using (2.3), we state that

$$|Y_i(t) - \tilde{Y}_i(t)| - \int_0^t \operatorname{sgn}(Y_i(s) - \tilde{Y}_i(s))(b_i(Y(s)) - b_i(\tilde{Y}(s)))ds$$

is a martingale vanishing at 0. This together with the Lipschitz conditions satisfied by b_i give

$$\mathbf{E}|Y_i(t) - \tilde{Y}_i(t)| \le A \int_0^t \mathbf{E}||Y(s) - \tilde{Y}(s)||ds.$$

Summing up the above-given inequalities, we arrive at

$$\mathbf{E}||Y(t) - \tilde{Y}(t)|| \le C \int_0^t \mathbf{E}||Y(s) - \tilde{Y}(s)||ds$$

and Gronwall's lemma shows that $Y(t) = \tilde{Y}(t)$ for every $t > 0$ a.s.

Using in a standard way the properties of the Itô integral and the Schwarz inequality, similarly as in Ref. 13, p. 165, we get that for every $t \in [0, T]$,

$$\begin{aligned} \mathbf{E}|Z_j(t) - \tilde{Z}_j(t)|^2 &\le C \sum_{k=p+1}^{p+q} \mathbf{E}(\int_0^t (c_k(Y(s), Z(s)) - c_k(\tilde{Y}(s), \tilde{Z}(s))dB_k(s))^2 \\ &+ C\mathbf{E}(\int_0^t (d_j(Y(s), Z(s)) - d_j(\tilde{Y}(s), \tilde{Z}(s))ds)^2 \\ &\le C \sum_{k=p+1}^{p+q} \mathbf{E} \int_0^t (c_k(Y(s), Z(s)) - c_k(\tilde{Y}(s), \tilde{Z}(s)))^2 ds \\ &+ CT\mathbf{E} \int_0^t (d_j(Y(s), Z(s)) - d_j(\tilde{Y}(s), \tilde{Z}(s)))^2 ds \\ &\le CA^2(q+T)\mathbf{E} \int_0^t (||Y(s) - \tilde{Y}(s)||^2 + ||Z(s) - \tilde{Z}(s)||^2)ds. \end{aligned}$$

Thus, using the previously proved fact that $Y = \tilde{Y}$ a.s. we get that

$$\mathbf{E}||Z(t) - \tilde{Z}(t)||^2 \leq CA^2(q+T)r \int_0^t \mathbf{E}||Z(s) - \tilde{Z}(s)||^2 ds.$$

One more application of the Gronwall's lemma ends the proof. □

III. EIGENVALUES AND EIGENVECTORS OF MATRIX STOCHASTIC PROCESSES

A. Real case

Consider the space $\mathcal{S}_p$ of symmetric $p \times p$ real matrices. Denote by B_t a Brownian $p \times p$ matrix. Let X_t be a stochastic process with values in $\mathcal{S}_p$ satisfying the matrix SDE,

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt, \tag{3.1}$$

where $g, h, b : \mathbf{R} \to \mathbf{R}$, and $X_0 \in \tilde{\mathcal{S}}_p$, the set of symmetric matrices with p different eigenvalues. The spectral action of the functions $g, h, b : \mathbf{R} \to \mathbf{R}$ on a symmetric matrix X was explained in (1.1) in the Introduction.

Let $\Lambda_t = \mathrm{diag}[\lambda_i(t)]$ be the diagonal matrix of eigenvalues of X_t ordered increasingly: $\lambda_1(t) \leq \lambda_2(t) \leq \ldots \leq \lambda_p(t)$ and H_t an orthonormal matrix of eigenvectors of X_t. Matrices Λ and H may be chosen (Ref. 24) as smooth functions of X until the first collision time

$$\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}.$$

We want to consider the SDEs satisfied by the processes of eigenvalues and eigenvectors of X_t. In the sequel, we use the notation $dYdZ = d\langle Y, Z\rangle$ for the quadratic variation process. Note that if Y, Z are matrix valued processes, then $dYdZ$ is a matrix process (see, e.g., Ref. 12).

Theorem 3: *Suppose that an $\mathcal{S}_p$-valued stochastic process X_t satisfies the following matrix stochastic differential equation:*

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt,$$

where $g, h, b : \mathbf{R} \to \mathbf{R}$, and $X_0 \in \tilde{\mathcal{S}}_p$.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. Then, for $t < \tau$ the eigenvalues process Λ_t and the eigenvectors process H_t verify the following stochastic differential equations:

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \left(b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) dt, \tag{3.2}$$

$$dh_{ij} = \sum_{k \neq j} h_{ik} \frac{\sqrt{G(\lambda_j, \lambda_k)}}{\lambda_j - \lambda_k} d\beta_{kj} - \frac{1}{2} h_{ij} \sum_{k \neq j} \frac{G(\lambda_j, \lambda_k)}{(\lambda_k - \lambda_j)^2} dt, \tag{3.3}$$

where $(\nu_i)_i$ and $(\beta_{kj})_{k<j}$ are independent Brownian motions and $\beta_{jk} = \beta_{kj}$.

Proof: The proof generalizes ideas of Bru[1] in the case of Wishart processes. See also Ref. 15 for the SDEs for the eigenvalue processes of X_t. Following Ref. 10 in the case of matrix Jacobi processes, it is handy to use the Stratonovich differential notation $X \circ dY = XdY + \frac{1}{2}dXdY$. We then write the Itô product formula

$$d(XY) = dX \circ Y + X \circ dY.$$

We also have $dX \circ (YZ) = (dX \circ Y) \circ Z$ and $(X \circ dY)^T = dY^T \circ X^T$.

Define A, a stochastic logarithm of H, by

$$dA = H^{-1} \circ dH = H^T \circ dH.$$

Observe that by Itô formula applied to $H^T H = I$, the matrix A is skew-symmetric. By Itô formula applied to $\Lambda = H^T X H$, setting $dN = H^T \circ dX \circ H$, we get

$$d\Lambda = dN + \Lambda \circ dA - dA \circ \Lambda.$$

The process $\Lambda \circ dA - dA \circ \Lambda$ is zero on the diagonal. Consequently, $d\lambda_i = dN_{ii}$ and $0 = dN_{ij} + (\lambda_i - \lambda_j) \circ dA_{ij}$, when $i \neq j$. Thus

$$dA_{ij} = \frac{1}{\lambda_j - \lambda_i} \circ dN_{ij}, \quad i \neq j. \tag{3.4}$$

For further computations, we need the quadratic variation $dX_{ij} dX_{km}$ which is easily computed from (3.1).

$$dX_{ij} dX_{km} = \left(g^2(X)_{ik} h^2(X)_{jm} + g^2(X)_{im} h^2(X)_{jk} + g^2(X)_{jk} h^2(X)_{im} + g^2(X)_{jm} h^2(X)_{ik}\right) dt.$$

The martingale part of dN equals the martingale part of $H^T dX\, H$ and by the last formula

$$dN_{ij} dN_{km} = \left(g^2(\Lambda)_{ik} h^2(\Lambda)_{jm} + g^2(\Lambda)_{im} h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk} h^2(\Lambda)_{im} + g^2(\Lambda)_{jm} h^2(\Lambda)_{ik}\right) dt. \tag{3.5}$$

From (3.5) it follows that

$$dN_{ii} dN_{jj} = 4\delta_{ij} g^2(\lambda_i) h^2(\lambda_i) dt. \tag{3.6}$$

Now we compute the finite variation part dF of dN,

$$\begin{aligned} dF &= H^T b(X) H dt + \frac{1}{2}(dH^T dX\, H + H^T dX dH) \\ &= b(\Lambda) dt + \frac{1}{2}\left((dH^T\, H)(H^T dX\, H) + (H^T dX\, H)(H^T dH)\right) \\ &= b(\Lambda) dt + \frac{1}{2}(dN dA + (dN dA)^T). \end{aligned}$$

Using (3.4) and (3.5) we find, writing $G(x, y) = g^2(x) h^2(y) + g^2(y) h^2(x)$,

$$(dN dA)_{ij} = \sum_{k \neq j} dN_{ik} dA_{kj} = \delta_{ij} \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} dt.$$

It follows that the matrix $dNdA$ is diagonal, so also dF is diagonal and

$$dF_{ii} = b(\lambda_i) dt + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} dt.$$

Finally, using (3.6) and the last formula, there exist independent Brownian motions ν_i, $i = 1, \ldots, m$, such that (3.2) holds.

In order to find SDEs for H_t, we deduce from the definition of dA that

$$dH = H \circ dA = H dA + \frac{1}{2} dH dA = H dA + \frac{1}{2} H dA dA.$$

By (3.5), we find $dN_{ij} dN_{ij} = g^2(\lambda_i) h^2(\lambda_j) + g^2(\lambda_j) h^2(\lambda_i)$ and $dN_{ij} dN_{km} = 0$ when the ordered pairs $i < j$ and $k < m$ are different. We infer from (3.4) that

$$dA_{ij} = \frac{\sqrt{G(\lambda_i, \lambda_j)}}{\lambda_j - \lambda_i} d\beta_{ij}, \tag{3.7}$$

where the Brownian motions $(\beta_{ij})_{i<j}$ are independent and $\beta_{ji} = \beta_{ij}$. Moreover, when $k < m$, we have $d\lambda_i dA_{km} = dN_{ii} dN_{km} / (\lambda_m - \lambda_k) = 0$ by (3.5), so the Brownian motions $(\beta_{ij})_{i<j}$ and $(\nu_i)_i$ are

independent. From (3.7), we deduce that the matrix $dAdA$ is diagonal and

$$(dAdA)_{ii} = -\sum_{k\neq i} dA_{ik}dA_{ik} = -\sum_{k\neq i} \frac{G(\lambda_i,\lambda_k)}{(\lambda_k-\lambda_i)^2}dt.$$

Now we can compute $dH = HdA + \frac{1}{2}HdAdA$ and prove (3.3). □

B. Complex case

In this subsection, we study the eigenvalues process for a process X_t with values in the space $\mathcal{H}_p$ of Hermitian $p \times p$ matrices.

Theorem 4: *Let W_t be a complex $p \times p$ Brownian matrix (i.e., $W_t = B_t^1 + iB_t^2$ where B_t^1 and B_t^2 are two independent real Brownian $p \times p$ matrices).*

Suppose that an $\mathcal{H}_p$-valued stochastic process X_t satisfies the following matrix stochastic differential equation:

$$dX_t = g(X_t)dW_t h(X_t) + h(X_t)dW_t^* g(X_t) + b(X_t)dt, \tag{3.8}$$

where $g, h, b : \mathbf{R} \to \mathbf{R}$, and $X_0 \in \tilde{\mathcal{H}}_p$.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. Then, for $t < \tau$ the eigenvalues process Λ_t verifies the following system of stochastic differential equations:

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \left(b(\lambda_i) + 2\sum_{k\neq i}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}\right)dt, \tag{3.9}$$

where $(\nu_i)_i$ are independent Brownian motions.

Proof: We will need the following formula for the quadratic variation $dX_{ij}dX_{kl}$ which is computed from (3.8), using the fact that for a complex Brownian motion w_t, the quadratic variation processes satisfy $dwdw = 0$ and $dwd\bar{w} = 2dt$,

$$dX_{ij}dX_{kl} = 2\left(g^2(X)_{il}h^2(X)_{jk} + g^2(X)_{jk}h^2(X)_{il}\right)dt. \tag{3.10}$$

Define A, a stochastic logarithm of H, by

$$dA = H^{-1}\circ dH = H^*\circ dH.$$

By Itô formula applied to $H^*H = I$, the matrix A is skew-Hermitian. In particular, the terms of $diag(A)$ are purely imaginary (recall that in the real case they were 0). By Itô formula applied to $\Lambda = H^*XH$, we get, setting $dN = H^*\circ dX\circ H$,

$$d\Lambda = dN + \Lambda\circ dA - dA\circ\Lambda.$$

We have

$$dN = H^*dXH + \frac{1}{2}(dH^*dX\,H + H^*dXdH),$$

so the process N takes values in $\mathcal{H}_p$. In particular, its diagonal entries are real. The process $\Lambda\circ dA - dA\circ\Lambda$ is zero on the diagonal, so $d\lambda_i = dN_{ii}$. Moreover, when $i \neq j$, we have $0 = dN_{ij} + (\lambda_i - \lambda_j)\circ dA_{ij}$ and

$$dA_{ij} = \frac{1}{\lambda_j-\lambda_i}\circ dN_{ij}, \quad i\neq j. \tag{3.11}$$

The martingale part of dN equals the martingale part of H^*dXH and by formula (3.10), we obtain

$$dN_{ij}dN_{km} = 2\left(g^2(\Lambda)_{im}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{im}\right)dt. \tag{3.12}$$

From (3.12) it follows that

$$dN_{ii}dN_{jj} = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_i)dt. \tag{3.13}$$

Now we compute the finite variation part dF of dN,

$$\begin{aligned} dF &= H^* b(X) H dt + \frac{1}{2}(dH^* dX\, H + H^* dX dH) \\ &= b(\Lambda)dt + \frac{1}{2}\left((dH^*\, H)(H^* dX\, H) + (H^* dX\, H)(H^* dH)\right) \\ &= b(\Lambda)dt + \frac{1}{2}(dNdA + (dNdA)^*). \end{aligned}$$

Recall that $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. We get

$$(dNdA)_{ij} = \sum_k dN_{ik}dA_{kj} = 2\delta_{ij}\sum_{k\neq i}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}dt + dN_{ij}dA_{jj}.$$

When $i = j$, the term dN_{ii} is real and $dA_{ii} \in i\mathbf{R}$. It follows that

$$dF_{ii} = b(\lambda_i)dt + 2\sum_{k\neq i}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}dt.$$

Finally, using (3.13) and the last formula, there exist independent Brownian motions ν_i, $i = 1, \ldots, m$, such that (3.9) holds. □

Theorem 4 may be applied in a special case $g(x) = \sqrt{x}$, $h(x) = 1$, and $b(x) = 2\delta > 0$, when Eq. (3.8) is the SDE for the complex Wishart process, called also a Laguerre process. This process and its eigenvalues were studied by König-O'Connell[18] and Demni.[5]

Remark 1: The SDEs for the eigenvectors matrix H_t remain an open problem in the complex Hermitian case and a complex analogue of equations (3.3) will be treated in a forthcoming paper. Also, similar problems for more general functions $G, H, B : \mathbf{R}^p \to \mathbf{R}$, acting spectrally on $\mathcal{S}_p$ by $G(X) = HG(\Lambda)H^T$, should be investigated. Note that in this paper, we consider the case when $G = g^{\otimes p}$ is a pth tensor power of a function $g : \mathbf{R} \to \mathbf{R}$.

C. Collision time

In this subsection, we show that under some mild conditions on the functions g, h, and b in the matrix SDE (3.1), the eigenvalues of the process X_t never collide.

Theorem 5: *Let $\Lambda = (\lambda_i)_{i=1\ldots p}$ be a process starting from $\lambda_1(0) < \ldots < \lambda_p(0)$ and satisfying (3.2) with functions $b, g, h : \mathbf{R} \to \mathbf{R}$ such that b, g^2, h^2 are Lipschitz continuous and g^2h^2 is convex or in class $\mathcal{C}^{1,1}$. Then the first collision time τ is infinite a.s.*

Proof: We define $U = -\sum_{i<j}\log(\lambda_j - \lambda_i)$ on $t \in [0, \tau]$. Applying Itô formula, using (3.2) and the fact that $d\lambda_i d\lambda_j = \delta_{ij}4g^2(\lambda_i)h^2(\lambda_i)dt$, we obtain

$$dU = \sum_{i<j}\frac{d\lambda_i - d\lambda_j}{\lambda_j-\lambda_i} + \frac{1}{2}\frac{d\left\langle\lambda_i,\lambda_i\right\rangle + d\left\langle\lambda_j,\lambda_j\right\rangle}{(\lambda_j-\lambda_i)^2} = dM + dA^{(1)} + dA^{(2)} + dA^{(3)},$$

where

$$dM = 2\sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)d\nu_i - g(\lambda_j)h(\lambda_j)d\nu_j}{\lambda_j - \lambda_i},$$

$$dA^{(1)} = \sum_{i<j} \frac{b(\lambda_i) - b(\lambda_j)}{\lambda_j - \lambda_i}\, dt,$$

$$dA^{(2)} = 2\sum_{i<j} \frac{(g^2(\lambda_j) - g^2(\lambda_i))(h^2(\lambda_j) - h^2(\lambda_i))}{(\lambda_j - \lambda_i)^2}\, dt,$$

$$dA^{(3)} = \sum_{i<j} \frac{1}{\lambda_j - \lambda_i} \sum_{k\neq i, k\neq j} \left(\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} - \frac{G(\lambda_j, \lambda_k)}{\lambda_j - \lambda_k} \right) dt$$

$$= \sum_{i<j<k} \frac{G(\lambda_j, \lambda_k)(\lambda_k - \lambda_j) - G(\lambda_i, \lambda_k)(\lambda_k - \lambda_i) + G(\lambda_i, \lambda_j)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} dt.$$

We will show that the finite variation part of U is bounded on any interval $[0, t]$. Lipschitz continuity of b, g^2, and h^2 implies that $|A_t^{(1)}| \le K p(p-1)t/2$ and $|A_t^{(2)}| \le K^2 p(p-1)t$, where K is a constant appearing in the Lipschitz condition. Observe also that if for every x, y, z, we set

$$H(x, y, z) = [(g^2(x) - g^2(z))(h^2(y) - h^2(z)) + (g^2(y) - g^2(z))(h^2(x) - h^2(z))](y - x),$$

then $H(x, y, z) = (G(x, y) - G(x, z) - G(y, z) + G(z, z))(y - x)$ and

$$H(x, y, z) + H(y, z, x) - H(x, z, y) = 2(z - y)G(y, z) - 2(z - x)G(x, z)$$
$$+ 2(y - x)G(x, y) + G(x, x)(z - y) - G(y, y)(z - x) + G(z, z)(y - x).$$

Using the last equality and the fact that $|H(x, y, z)| \le 2K^2|(y - x)(z - y)(z - x)|$, we can write $2dA^{(3)} = dA^{(4)} + dA^{(5)}$, where $0 \le A_t^{(4)} \le K^2 p(p-1)(p-2)t/6$ and

$$dA_t^{(5)} = \sum_{i<j<k} \frac{G(\lambda_j, \lambda_j)(\lambda_k - \lambda_i) - G(\lambda_i, \lambda_i)(\lambda_k - \lambda_j) - G(\lambda_k, \lambda_k)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} dt$$

$$= \sum_{i<j<k} \left(\frac{G(\lambda_j, \lambda_j) - G(\lambda_i, \lambda_i)}{\lambda_j - \lambda_i} - \frac{G(\lambda_k, \lambda_k) - G(\lambda_j, \lambda_j)}{\lambda_k - \lambda_j} \right) \frac{1}{\lambda_k - \lambda_i} dt.$$

If $G(x, x) = 2g^2(x)h^2(x)$ is convex, then obviously the expression under the last sum and $A^{(5)}$ is non-positive. When $G(x, x)$ is $\mathcal{C}^{1,1}$ (i.e., $|G'(x, x) - G'(y, y)| \le C|x - y|$) then it is bounded by C and $|A_t^{(5)}| \le Ct$.

Since finite-variation part of U is finite whenever t is bounded, applying McKean argument (see Refs. 21 and 22) we obtain that U cannot explode in finite time with positive probability and consequently $\tau = \infty$ a.s. □

Remark 2: Note that if $p = 2$, then the assumptions on g^2h^2 can be dropped since in that case $dA^{(3)} \equiv 0$.

In the modern theory of particle systems it is important to consider and to study β-versions of a particle system given by the SDEs system (3.2), i.e., the solutions of the SDEs system

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \beta \left(b(\lambda_i) + \sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) dt, \quad \beta > 0. \tag{3.14}$$

Note that the system (3.14) is for $\beta \neq 1$ no longer of the form (3.2), because $\beta G(x, y) \neq g^2(x)h^2(y) + g^2(y)h^2(x)$. However, we have the following.

Corollary 1: Let $\Lambda = (\lambda_i)_{i=1\ldots p}$ be a process starting from $\lambda_1(0) < \ldots < \lambda_p(0)$ and satisfying (3.14) with functions $b, g, h : \mathbf{R} \to \mathbf{R}$ such that b, g^2, h^2 are Lipschitz continuous and g^2h^2 is convex or in class $\mathcal{C}^{1,1}$. If $\beta \geq 1$, then the first collision time τ is infinite a.s.

Proof: The proof is similar to the proof of Theorem 5, with the decomposition $dU = dM + dA^{(1)} + dA^{(2)} + dA^{(3)}$ given by

$$dM = 2\sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)d\nu_i - g(\lambda_j)h(\lambda_j)d\nu_j}{\lambda_j - \lambda_i},$$

$$dA^{(1)} = \beta \sum_{i<j} \frac{b(\lambda_i) - b(\lambda_j)}{\lambda_j - \lambda_i}\, dt,$$

$$dA^{(2)} = 2\sum_{i<j} \frac{(g^2(\lambda_j) - g^2(\lambda_i))(h^2(\lambda_j) - h^2(\lambda_i))}{(\lambda_j - \lambda_i)^2}\, dt + 2(1-\beta)\sum_{i<j} \frac{G(\lambda_i, \lambda_j)}{(\lambda_j - \lambda_i)^2},$$

$$dA^{(3)} = \beta \sum_{i<j} \frac{1}{\lambda_j - \lambda_i} \sum_{k\neq i, k\neq j} \left(\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} - \frac{G(\lambda_j, \lambda_k)}{\lambda_j - \lambda_k} \right) dt.$$

The estimates of M, $A^{(1)}$, $A^{(3)}$ and of the first term of $A^{(2)}$ are identical as in the proof of Theorem 5. The term $2(1-\beta)\sum_{i<j} \frac{G(\lambda_i,\lambda_j)}{(\lambda_j-\lambda_i)^2}$ is less or equal 0 for $\beta \geq 1$, so $A^{(2)}$ cannot explode to $+\infty$ and neither can U.

Remark 3: The condition $\beta \geq 1$ is optimal in Corollary 1. It is known (Refs. 16 and 26) that the Dyson Brownian motion defined as a solution of the SDEs system

$$dY_i = d\nu_i + \beta \sum_{k\neq i} \frac{1/2}{Y_i - Y_k} dt, \quad i = 1, \ldots, p$$

has collisions for $\beta < 1$. Note that taking $g(x) = 1/2$, $h(x) = 1$, $b(x) = 0$, and $\beta = 1$ the Dyson SDEs system is of the form (3.2) and a general Dyson SDEs system is the β-version of the $\beta = 1$ case.

Observe that Theorem 5 holds also in the complex case. The β-version of Eq. (3.9) is defined by

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \frac{\beta}{2}\left(b(\lambda_i) + 2\sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) dt. \tag{3.15}$$

Corollary 2. Under the hypotheses of Theorem 5, the solutions of the SDEs system (3.9), i.e., the eigenvalues of the process X_t on $\mathcal{H}_p$, verify $\tau = \infty$ a.s. It is also true for their β-versions (3.15) with $\beta \geq 1$.

Proof: Note that the system (3.9) is equal to the system (3.14) with the same g and h, $b/2$ instead of b, and $\beta = 2$. □

D. Spectral matrix Yamada-Watanabe theorem

Theorem 6: *Consider the matrix SDE (3.1) on $\mathcal{S}_p$,*

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt,$$

where $g, h, b : \mathbf{R} \to \mathbf{R}$ and $X_0 \in \tilde{\mathcal{S}}_p$. Suppose that

$$|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x-y|), \quad x, y \in \mathbf{R}, \tag{3.16}$$

where ρ: $(0, \infty) \to (0, \infty)$ *is a Borel function such that* $\int_{0^+} \rho^{-1}(x)dx = \infty$, *that the function* $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$ *is locally Lipschitz and strictly positive on* $\{x \neq y\}$ *and that* b *is locally Lipschitz. Then, for* $t < \tau$, *the pathwise uniqueness holds for the eigenvalue and eigenvector processes of* X_t, *solutions of the SDEs system (3.2) and (3.3).*

Remark 4: The hypothesis in Theorem 6 on the strict positivity of $G(x, y)$ off the diagonal $\{x = y\}$ is equivalent to the condition that g and h have not more than one zero and their zeros are not common.

Remark 5: In the matrix SDE (3.1), the functions g and h appear only in the martingale part, whereas in Eqs. (3.2) and (3.3) they intervene also in the finite variation part. That is why a Lipschitz condition on the symmetrized function $g^2 \oplus h^2$ cannot be avoided in a spectral matrix Yamada-Watanabe theorem on $\mathcal{S}_p$.

Proof: We diagonalize $X_0 = h_0\lambda_0 h_0^T$. We will show that Eqs. (3.2) and (3.3) have unique strong solutions when $\Lambda_0 = \lambda_0$ and $H_0 = h_0$. The functions

$$b_i(\lambda_1, \dots, \lambda_p) = b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k},$$

$$c_{ij}(\lambda_1, \dots, \lambda_p, h_{11}, h_{12}, \dots, h_{pp}) = \delta_{kj} h_{ik} \frac{\sqrt{G(\lambda_j, \lambda_k)}}{\lambda_j - \lambda_k},$$

$$d_{ij}(\lambda_1, \dots, \lambda_p, h_{11}, h_{12}, \dots, h_{pp}) = -\frac{1}{2} h_{ij} \sum_{k \neq j} \frac{G(\lambda_j, \lambda_k)}{(\lambda_k - \lambda_j)^2}$$

are locally Lipschitz continuous on $D = \{0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_p\} \times [-1, 1]^r$, $r = p^2$. Thus, they can be extended from the compact sets

$$D_m = \{0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_p < m, \lambda_{i+1} - \lambda_i \geq 1/m\} \times [-1, 1]^r$$

to bounded Lipschitz continuous functions on $\mathbf{R}^{p+r}$. We will denote by b_i^m, c_{ik}^m, and d_{ij}^m such extensions for $m = 1, 2, \dots$.

We consider the following system of SDE (recall that $\beta_{kj} = \beta_{jk}$):

$$d\lambda_i^m = 2g(\lambda_i^m)h(\lambda_i^m)d\nu_i + b_i^m(\Lambda^m)dt, \quad i = 1, \dots, p,$$

$$dh_{ij} = \sum_{k \neq j} c_{ij}^m(\Lambda^m, H)\, d\beta_{kj}(t) + d_{ij}^m(\Lambda^m, H)\, dt, \quad 1 \leq i, j \leq p.$$

Since $|g(x)h(x) - g(y)h(y))|^2 \leq \rho(|x - y|)$ and $\int_{0^+} \rho(x)^{-1}dx = \infty$, by Theorem 2 with $q = \frac{1}{2}p(p-1)$, we obtain that there exists a unique strong solution of the above-given system of SDEs. Using the fact that $D_m \subset D_{m+1}$, $\lim_{m\to\infty} D_m = D$ and the standard procedure we get that there exists a unique strong solution (Λ_t, H_t) of the systems (3.2) and (3.3) up to the first exit time from the set D. This time is almost surely equal to τ, the first collision time of the eigenvalues. □

Theorems 6 and 5 imply the following global strong existence result for eigenvalues and eigenvectors of a matrix SDE on the space $\mathcal{S}_p$.

Corollary 3: Suppose that b, g^2, h^2 *are Lipschitz continuous,* g^2h^2 *is convex or in class* $\mathcal{C}^{1,1}$ *and that* $G(x, y)$ *is strictly positive on* $\{x \neq y\}$. *Then the system of SDEs (3.2) and (3.3) for eigenvalue and eigenvector processes of the matrix process on* $\mathcal{S}_p$ *given by (3.1) admits a unique strong solution on* $[0, \infty)$.

Proof: Recall that if a non-negative function F is Lipschitz continuous, then $\sqrt{F}$ is 1/2-Hölder continuous. Observe that if g^2 and h^2 are Lipschitz continuous, then g^2h^2 is locally Lipschitz and gh is 1/2-Hölder. Thus (3.16) is verified and Theorem 6 applies on $[0, \tau)$. By Theorem 5, $\tau = \infty$ almost surely. □

Remark 6. Theorem 6 and Corollary 3 establish the pathwise uniqueness and strong existence of eigenvalue and eigenvector processes Λ_t and H_t of the process X_t. It is an open question whether the pathwise uniqueness and strong existence hold for the matrix SDE (3.1) itself. Note that the process X_t takes values in the space $\mathcal{S}_p$ of dimension $p(p+1)/2$, whereas the Brownian matrix in the SDE (3.1) contains p^2 independent Brownian motions. Thus, we have a redundance phenomenon in the matrix SDE (3.1). The SDEs system (3.2) and (3.3) for (Λ_t, H_t) has the advantage to be non-redundant, because it contains exactly $p(p+1)/2$ independent Brownian motions.

In the light of Theorem 6 and Corollary 3, it is natural to conjecture the pathwise uniqueness and strong existence for the matrix SDE (3.1). Otherwise, the most precise description of the matrix process X_t would be given by the SDEs for its eigenvalue and eigenvector processes (3.2) and (3.3) and not by the matrix SDE (3.1), despite its simplicity and because of the redundance described above.

IV. APPLICATIONS

A. Noncolliding particle systems of squared Bessel processes

In a recent paper by Katori and Tanemura,[17] particle systems of squared Bessel processes $\mathrm{BESQ}^{(\nu)}$, $\nu > -1$, interacting with each other by *long ranged repulsive forces* are studied. If there are N particles, their positions $X_i^{(\nu)}$ are given by the following system of SDEs, see Ref. 17, p. 593,

$$\begin{aligned} dX_i^{(\nu)}(t) &= 2\sqrt{X_i^{(\nu)}(t)}dB_i(t) + 2(\nu+1)dt + 4X_i^{(\nu)}(t)\sum_{j\neq i}\frac{dt}{X_i^{(\nu)}(t)-X_j^{(\nu)}(t)} \\ &= 2\sqrt{X_i^{(\nu)}(t)}dB_i(t) + 2(\nu+N)dt + 2\sum_{j\neq i}\frac{X_i^{(\nu)}(t)+X_j^{(\nu)}(t)}{X_i^{(\nu)}(t)-X_j^{(\nu)}(t)}dt,\ i=1,\dots,N \end{aligned}$$

with a collection of independent standard Brownian motions $\{B_i(t), i = 1, \dots, N\}$ and, if $-1 < \nu < 0$, with a reflection wall at the origin. Theorem 4 implies that the processes $X_i^{(\nu)}(t)$ are the eigenvalues of a complex Wishart (or Laguerre) process, with shape parameter $\delta = \nu + N$, see the end of Sec. III B. It may be also seen as a β-version of the real Wishart eigenvalue process, with $\beta = 2$.

Theorem 7: *The system of SDEs for a particle system of N squared Bessel processes* $\mathrm{BESQ}^{(\nu)}$, *with* $0 \le X_i^{(\nu)}(0) < X_2^{(\nu)}(0) < \dots < X_N^{(\nu)}(0)$ *admits a unique strong solution on* $[0, \infty)$ *for* $\nu \ge -1$.

Proof: Like for a Squared Bessel process on $\mathbf{R}^+$, one must start with the following system of SDEs:

$$dY_i^{(\nu)}(t) = 2\sqrt{|Y_i^{(\nu)}(t)|}dB_i(t) + 2(\nu+N)dt + 2\sum_{j\neq i}\frac{|Y_i^{(\nu)}(t)|+|Y_j^{(\nu)}(t)|}{Y_i^{(\nu)}(t)-Y_j^{(\nu)}(t)},\ i=1,\dots,N,$$

which is well defined on $\mathbf{R}^N$ up to the first collision time τ. We suppose that $0 \le Y_1^{(\nu)}(0) < Y_2^{(\nu)}(0) < \dots < Y_N^{(\nu)}(0)$. It follows from Corollary 2 that the collision time for the processes $(Y_i^{(\nu)}(t))$, $i = 1, \dots, N$ is $\tau = \infty$ a.s.

First suppose that $\nu > -1$. Theorem 2 applied to the last system, with a standard use of localization techniques as in the proof of Theorem 6, gives the existence of a pathwise unique strong solution $(Y_i^{(\nu)}(t))$. It remains to show that $Y_1^{(\nu)}(t) \ge 0$ for all $t > 0$.

Denote

$$b_1(t) = \nu + N + \sum_{j\neq 1}\frac{|Y_1^{(\nu)}(t)|+|Y_j^{(\nu)}(t)|}{Y_1^{(\nu)}(t)-Y_j^{(\nu)}(t)}.$$

We define two stopping times

$$\vartheta = \inf\{t > 0 |\, Y_1^{(\nu)}(t) < 0\};$$

$$\kappa = \inf\{t > \vartheta \,|\, b_1(t) = 0\}.$$

Suppose that $\mathbf{P}(\vartheta < \infty) > 0$. Then there exists $T > 0$ such that $\mathbf{P}(\vartheta < T) > 0$. As $Y_1^{(\nu)}(\vartheta) = 0$ and $b_1(\vartheta) = \nu + N - (N - 1) = \nu + 1 > 0$, we see that if $\vartheta < \infty$, then $\kappa > \vartheta$.

It follows from Ref. 25, Lemma 3.3, p. 389 that the local time $L^0(Y_1^{(\nu)}) = 0$. Using Tanaka's formula,[25] VI (1.2) we obtain for $t \geq 0$,

$$\mathbf{E}(Y_1^{(\nu)}((\vartheta + t) \wedge \kappa \wedge T))^- = -\mathbf{E} \int_{\vartheta\wedge T}^{(\vartheta+t)\wedge\kappa\wedge T} \mathbf{1}_{\{Y_1^{(\nu)}(s)\leq 0\}} dY_1^{(\nu)}(s)$$

$$= -2\mathbf{E} \int_{\vartheta\wedge T}^{(\vartheta+t)\wedge\kappa\wedge T} \mathbf{1}_{\{Y_1^{(\nu)}(s)\leq 0\}} b_1(s) ds \leq 0.$$

In the last inequality, we used the fact that $b_1(s) > 0$ when $\vartheta \leq s < \kappa$. Thus,

$$Y_1^{(\nu)}((\vartheta + t) \wedge \kappa \wedge T) \geq 0$$

for $t > 0$ which contradicts the definition of ϑ. We deduce that $\vartheta = \infty$ almost surely.

In the case $\nu = -1$, let $T_0 = \inf\{t > 0 |\, Y_1^{(\nu)}(t) = 0\}$. Observe that if $T_0 < \infty$, then $b_1(T_0) = 0$. Define $\tilde{Y}_1^{(\nu)}(t) = Y_1^{(\nu)}(t)$ when $t < T_0$ and $\tilde{Y}_1^{(\nu)}(t) = 0$ when $t \geq T_0$. Then $(\tilde{Y}_1^{(\nu)}, Y_2^{(\nu)}, \ldots, Y_N^{(\nu)})$ is a solution of the same SDE system as $(Y_1^{(\nu)}, Y_2^{(\nu)}, \ldots, Y_N^{(\nu)})$. Consequently, by Theorem 2, we have $Y_1^{(\nu)} = \tilde{Y}_1^{(\nu)} \geq 0$. □

B. Wishart stochastic differential equations

Wishart processes on $\overline{\mathcal{S}_p^+}$ are matrix analogues of Squared Bessel processes on $\mathbf{R}^+$. Wishart processes with shape parameter n (which corresponds to the dimension of a BESQ on $\mathbf{R}^+$) are simply constructed as $X_t = N_t^T N_t$, where N_t is an $n \times p$ Brownian matrix. Let $\alpha > 0$ and $B = (B_t)_{t\geq 0}$ be a Brownian $p \times p$ matrix. We write $\sqrt{X_t}$ in the spectral action sense of $g(X_t)$ with $g(x) = \sqrt{x}$, explained in (1.1). $\sqrt{X_t}$ is the symmetric matrix such that its square equals X_t. The Wishart stochastic differential equation for a Wishart process with a shape parameter α is

$$\begin{cases} dX_t = \sqrt{X_t} dB_t + dB_t^T \sqrt{X_t} + \alpha I dt \\ X_0 = x_0. \end{cases} \tag{4.1}$$

It was introduced by Bru[3] by first writing the SDE for $X_t = N_t^T N_t$ and next replacing the parameter n by α. It was shown in Ref. 3 that if $x_0 \in \overline{\mathcal{S}_p^+}$ and $\alpha > p - 1$, then there exists a unique weak solution of (4.1). Also according to Ref. 3, the conditions $\alpha \geq p + 1$ and $x_0 \in \mathcal{S}_p^+$ imply that (4.1) has a unique strong solution. We reinforce considerably these results.

Our methods apply to the following matrix stochastic differential equation:

$$dY_t = \sqrt{|Y_t|} dB_t + dB_t^T \sqrt{|Y_t|} + \alpha I dt, \tag{4.2}$$

where $\alpha \in \mathbf{R}$, $Y_0 = y_0 \in \tilde{S}_p$, and $|Y_t|$ is defined in the spectral action sense of $f(Y_t)$ with $f(x) = |x|$, explained in (1.1).

We have $g(x) = \sqrt{|x|}$, $h(x) = 1$, and $G(x, y) = |x| + |y|$ for $x, y \in \mathbf{R}$. These functions satisfy the hypotheses of Theorems 5 and 6.

By Theorem 3, the eigenvalues of the generalized Wishart process Y_t verify the following system of SDEs:

$$d\lambda_i = 2\sqrt{|\lambda_i|} d\nu_i + \left(\alpha + \sum_{k\neq i} \frac{|\lambda_i| + |\lambda_k|}{\lambda_i - \lambda_k} \right) dt.$$

First, using Theorem 5 we obtain the following.

Corollary 4: For $\alpha \in \mathbf{R}$ *and* $\lambda_1(0) < \lambda_2(0) < \ldots < \lambda_p(0)$, *the eigenvalues* $\lambda_i(t)$ *never collide, i.e., the first collision time* $\tau = \infty$ *almost surely.*

Next, Theorem 6 implies the following.

Corollary 5: The SDEs system for the eigenvalues and eigenvectors (Λ_t, H_t) *corresponding to the generalized Wishart SDE (4.2) with* $Y_0 = y_0 \in \tilde{\mathcal{S}}_p$ *has a unique strong solution on* $[0, \infty)$ *for any* $\alpha \in \mathbf{R}$.

In order to consider Eq. (4.1), we must prove the non-negativity of the smallest eigenvalue of the process Y_t, when starting from a non-negative value.

Proposition 1: If $\alpha \geq p - 1$ *and* $\lambda_1(0) \geq 0$, *then the process* $\lambda_1(t)$ *remains non-negative.*

Proof: We argue as in the proof of Theorem 7. □

Consequently, using the unicity of solutions in Theorem 6, we obtain the following.

Corollary 6: Consider the Wishart SDE (4.1) with x_0 *such that* $0 \leq \lambda_1(0) < \lambda_2(0) < \ldots < \lambda_p(0)$. *Then the corresponding system of SDEs for eigenvalue and eigenvector processes* (Λ_t, H_t) *has a unique strong solution on* $[0, \infty)$ *for* $\alpha \geq p - 1$.

Remark 7: Bru[3] showed that for $\alpha > p - 1$ the Wishart processes have the absolutely continuous Wishart laws which are very important in multivariate statistics, see, e.g., the monograph of Muirhead.[23] The singular Wishart processes corresponding to $\alpha = 1, \ldots p - 1$ are obtained as $X_t = N_t^T N_t$, where N_t is an $\alpha \times p$ Brownian matrix. Then $X_0 = N_0^T N_0$ has eigenvalue 0 of multiplicity $p - \alpha$ so $x_0 \notin \tilde{\mathcal{S}}_p$.

Remark 8: An important perturbation of the Wishart SDE (4.1) is the equation for the Wishart process with constant drift $c > 0$, which may be also viewed as a squared matrix Ornstein-Uhlenbeck process

$$dX_t = \sqrt{X_t}dB_t + dB_t^T\sqrt{X_t} + \alpha I dt + cX_t dt, \quad X_0 \in \tilde{\mathcal{S}}_p. \tag{4.3}$$

This equation has the form (3.1) with $g(x) = \sqrt{x}$, $h(x) = 1$ and $b(x) = \alpha + cx$. By Theorems 5 and 6, the SDEs system for its eigenvalue and eigenvector processes has a unique strong solution with $t \in [0, \infty)$ for any $\alpha \geq p - 1$, $c > 0$ and $0 \leq \lambda_1(0) < \lambda_2(0) < \ldots < \lambda_p(0)$. More general squared matrix Ornstein-Uhlenbeck processes were first studied by Bru[3] and recently by Mayerhofer *et al.*[22] Our spectral strong existence and uniqueness result for (4.3) is not covered by these papers.

Remark 9: The existence and pathwise unicity of strong solutions for the Wishart SDE (4.1) for $\alpha \geq p - 1$ remains an open problem. The difficulty of proving it is related to a redundance in the SDE (4.1), cf. Remark 6. On the other hand, our result on the strong existence and pathwise unicity of eigenvalues and eigenvectors of X_t supports the conjecture of the existence and pathwise unicity of strong solutions for the Wishart SDE (4.1) for $\alpha \geq p - 1$.

C. Matrix Jacobi processes

Let 0_p and I_p be zero and identity $p \times p$ matrices. Define $\mathcal{S}_p[0, I] = \{X \in \mathcal{S}_p | \, 0_p \leq X \leq I_p\}$. Denote by $\hat{\mathcal{S}}_p[0, I] = \{X \in \mathcal{S}_p | \, 0_p < X < I_p\}$ and by $\tilde{\mathcal{S}}_p[0, I]$, the set of matrices in $\mathcal{S}_p[0, I]$ with distinct eigenvalues. A matrix Jacobi process of dimensions (q, r), with $q \wedge r > p - 1$, and with values in $\mathcal{S}_p[0, I]$, was defined and studied by Doumerc[10] as a solution of the following matrix SDE, with respect to a $p \times p$ Brownian matrix B_t,

$$\begin{cases} dX_t = \sqrt{X_t}dB_t\sqrt{I_p - X_t} + \sqrt{I_p - X_t}dB_t^T\sqrt{X_t} + (qI_m - (q+r)X_t)dt \\ X_0 = x_0 \in \mathcal{S}_p[0, I]. \end{cases} \tag{4.4}$$

In Ref. 10, Theorem 9.3.1, p. 135, it was shown that if $q \wedge r \geq p + 1$ and $x_0 \in \hat{\mathcal{S}}_p[0, I]$, then (4.4) has a unique strong solution in $\hat{\mathcal{S}}_p[0, I]$. In the case $q \wedge r \in (p - 1, p + 1)$ and $x_0 \in \tilde{\mathcal{S}}_p[0, I]$, the existence of a unique solution in law was proved in Ref. 10. Our methods allow one to strengthen the results of Doumerc.

Corollary 7: Let $q \wedge r \geq p - 1$ and $x_0 \in \tilde{\mathcal{S}}_p[0, I]$. Then the SDEs system for the eigenvalue and eigenvector processes for the matrix SDE (4.4) has a unique strong solution for $t \in [0, \infty)$.

Proof: We apply Theorems 5 and 6 with $g(x) = \sqrt{|x|}$, $h(x) = \sqrt{|1 - x|}$ and $b(x) = q - (q + r)x$. Next we prove similarly as in the proof of Theorem 7 that $0 \leq \lambda_1(t) < \ldots < \lambda_p(t) \leq 1$.

D. β-Wishart and β-Jacobi processes

Let $\beta > 0$. One calls a β-Wishart process a solution of the system of SDEs,

$$d\lambda_i = 2\sqrt{\lambda_i}d\nu_i + \beta\left(\alpha + \sum_{k\neq i}\frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}\right)dt. \tag{4.5}$$

The β-Wishart processes were studied by Demni.[7] In the theory of random matrices and its physical applications, the β-Wishart processes are related to Chiral Gaussian Ensembles, which were introduced as effective (approximation) theoretical models describing energy spectra of quantum particle systems in high energy physics. Usually, a symmetry of Hamiltonian is imposed and it fixes the value of β to be 1, 2, or 4, respectively, in real symmetric, Hermitian and symplectic cases. On the other hand, from the point of view of statistical physics, β is regarded as the inverse temperature, $\beta = 1/(k_B T)$, and should be treated as a continuous positive parameter. In this sense, the β-Wishart systems are statistical mechanics models of "log-gases" (The strength of the force between particles is proportional to the inverse of distances. Then the potential, which is obtained by integrating the force, is logarithmic function of the distance. So the system is called a "log-gas"). For more information on log-gases, see the recent monograph of Forrester.[11]

In Ref. 7, the existence and uniqueness of strong solutions of the SDE system (4.5) was established for $\beta > 0$ and $\alpha > p - 1 + \frac{1}{\beta}$. Lépingle[20] observed the last result in the classical Wishart case $\beta = 1$. Our Theorem 2 and Corollary 1, together with comparison techniques like in the proof of Theorem 7, imply the following result, not covered by results of Demni and Lépingle.

Corollary 8: The SDE system (4.5) with $0 \leq \lambda_1(0) < \lambda_2(0) < \ldots < \lambda_p(0)$ has a unique strong solution for $t \in [0, \infty)$, for any $\alpha \geq p - 1$ and $\beta \geq 1$.

The β-Jacobi processes (λ_i), $i = 1, \ldots, p$ are $[0, 1]^p$-valued processes generalizing processes of eigenvalues of matrix Jacobi processes defined by (4.4),

$$d\lambda_i = 2\sqrt{\lambda_i(1 - \lambda_i)}d\nu_i + \beta\left(q - (q + r)\lambda_i + \sum_{k\neq i}\frac{\lambda_i(1 - \lambda_k) + \lambda_k(1 - \lambda_i)}{\lambda_i - \lambda_k}\right)dt. \tag{4.6}$$

Indeed, for $\beta = 1$ the formula (4.6) was shown in Ref. 10 and it follows directly from Theorem 3. β-Jacobi processes were recently studied by Demni in Ref. 8. He showed that the system (4.6) has a unique strong solution for all time t when $\beta > 0$ and $q \wedge r > p - 1 + 1/\beta$. As an application of Theorem 2, Theorem 5 and the comparison techniques like in the proof of Theorem 7, we improve this result when $\beta \geq 1$.

Corollary 9: The SDE system (4.6) with $0 \leq \lambda_1(0) < \lambda_2(0) < \ldots < \lambda_p(0) \leq 1$ has a unique strong solution for $t \in [0, \infty)$, for any $\beta \geq 1$ and $q \wedge r \geq p - 1$.

Remark 10: It would be interesting to extend our generalization of the Yamada-Watanabe theorem to the SDEs considered by Cépa-Lépingle.[4]

The Wishart eigenvalue processes are radial Dunkl processes. Existence and unicity problems for SDEs for important classes of radial Dunkl processes were studied by Demni,[6] using Ref. 4.

The natural counterpart of the Dunkl theory in the negatively curved setting is the theory of hypergeometric Laplacians of Heckman and Opdam, connected with the Cherednik operators, which are the analogues of the Dunkl operators in the flat case. The stochastic processes generated by Laplacians of Heckman and Opdam were studied in Ref. 27 and are called Heckman–Opdam or Cherednik processes.

The Jacobi eigenvalues processes being an important example of the radial Cherednik processes, we conjecture that the strong existence and unicity would hold for radial Cherednik processes.

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