



**Multifractal analysis of minimum bias events in
 $\sqrt{s} = 630 \text{ GeV } \bar{p}p$ collisions**

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(submitted to Zeit. Phys. C.)

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Abstract

A search for multifractal structures, in analogy with multifractal theories, is performed on UA1 minimum bias events. A downward concave multifractal spectral function, $f(\alpha)$ (where α is the Lipschitz-Hölder exponent), indicates that there are self-similar cascading processes, governing the evolution from the quark to the hadron level, in the final states of hadronic interactions. $f(\alpha)$ is an accurate measure of the bin to bin fluctuations of any observable. It is shown that the most sensitive comparison between data and the Monte Carlo models, GENCL and PYTHIA 4.8 can be made using $f(\alpha)$. It is found that these models do not fully reproduce the behaviour of the data.

1 Introduction

In low p_T physics, distributions averaged over a large event sample are usually investigated. Bialas and Peschanski [1], however, prompted by sharp spikes in single cosmic ray events, proposed the study of event to event particle density fluctuations in rapidity windows of decreasing bin width. The conjectured power law scaling behaviour of the factorial moments is called intermittency in analogy with the description of bursts of turbulence in the theory of chaos.

The search for intermittency in UA1 minimum bias data has not only stimulated further experimental analysis, but has also yielded a statistically significant intermittent behaviour at collider energies [2, 3]. Intermittent behaviour has subsequently been reported in e^+e^- annihilation [4], hadron - hadron [5], hadron - nucleus [6] and nucleus - nucleus [6, 7] collisions.

The power law behaviour, $(\frac{1}{\delta\eta})^a$ ($a > 0$), in the factorial moments indicates the occurrence of events with fluctuations in their final state particle distributions, implying that some degree of a fractal pattern has emerged [8, 9, 10].

A systematic formalism to investigate hadronic multiparticle production in the framework of a multifractal concept was proposed recently by R.C.Hwa[10]. Although this approach has been performed successfully in many fields of physics, in particle physics it is a new and yet unexplored application. The aim of Hwa's studies is to verify the validity of basic scaling properties, which occur in multifractal theories, when applied to particle production in order to obtain insight into the dynamical mechanism responsible for the hadronisation process of hadron - hadron interactions. Such scaling laws allow for the interpretation of cascading as self-similar processes [11] in analogy with geometrical objects such as fractals (for instance the Cantor set) [9].

It is interesting to know whether the measured fractal behaviour is of the (mono)fractal or multifractal type, i.e. whether the generalized dimensions coincide or not, respectively, for different powers q . From factorial moments, generalized dimensions of only positive integer order, greater than or equal to two,

can be obtained. Hwa's approach is based on the general theory of multifractals [12] and allows for generalized dimensions of real positive or negative orders that emphasize peaks or dips in the distributions respectively. Multifractal spectral functions, $f(\alpha)$, can be studied by varying q , and serve as a measure of the multifractal structure in the data sample [13, 14, 15].

The purpose of our analysis is to answer the following main questions:

- can one find fractal or multifractal properties in data and models ?
- do the data and models contain self-similar and universal behaviour ?
- to what extent is the data sample reproduced by models ?

A short description of the experimental preliminaries is given in section 2. In section 3 we present briefly the multifractal theory and discuss its applicability to our data. Section 4 describes the multifractal properties found in the data. A comparison to the Monte Carlo models GENCL [16] and PYTHIA 4.8 [17] is given in section 5. The universality and self-similarity properties [11] of both the data and the models are discussed in section 6. A summary of this paper can be found in section 7.

2 Experimental Preliminaries

The data sample used in this analysis is from the 1985 UA1 data taking and consists of 159154 events collected with a minimum bias trigger [18]. It has already been used for investigating intermittency with factorial moments at $\sqrt{s}=630$ GeV [2, 3, 19] .

The pseudorapidity (η) and azimuthal angle (ϕ) of charged particle tracks are measured by the UA1 central detector drift chambers (CD) [20]. The particle trajectories, obtained by the pattern recognition algorithm, were subjected to criteria for good track quality: sufficient projected track length ($\geq 40cm$), a sufficient number of track points, a good χ^2 , and transverse momentum $p_t \geq 0.15$ GeV/c. However, in the algorithm, there was a small probability that the respective track segments were not merged correctly (especially if tracks passed from one CD module into a neighbouring module), in which case these tracks became apparently separate (ghost) tracks. A procedure (called ghost finder) has been developed [2, 3, 19] in order to eliminate such spurious tracks.

Track finding acceptance tables as a function of η , ϕ , and p_T had been determined using earlier data [21], exploiting the isotropy requirement in ϕ for each η bin and each p_T interval. The 1985 minimum bias data of UA1 [19] has been confirmed using this procedure and is therefore used, as it is the best available data, for this analysis. There remains, however, a region around $\eta = 0$ and $\phi = 0$ in which the acceptance is exactly zero for tracks satisfying the good quality requirements because this phase space segment is aligned with the direction of

the CD wires and the magnetic dipole field. Fig. 1 shows how the η distribution is affected by the acceptance characteristics of the CD, and the quality to which it can be corrected.

The intervals $1.5 \leq |\eta| < 3.0$, are adopted here for avoiding the central acceptance zero domain and for the reasonable flatness of the particle density. This region is only slightly influenced by acceptance corrections (most correction factors are close to 1). It has thus been selected in order to keep the systematic uncertainties in the data as small as possible. In [2, 3] a list of possible systematic effects in the data is given. An investigation of them shows that the problem reduces to two main effects: the occurrence of split tracks and the fact that the acceptance tables do not contain corrections for multiplicity distortions in different regions of the azimuth angle ϕ of the CD. These are the largest sources of systematic error, with other contributions having negligible effect by comparison. Therefore we have calculated a multiplicity correction factor

$$R = \frac{\langle n_{ch} \rangle_1}{\langle n_{ch} \rangle_2} \quad (1)$$

in several multiplicity bands, $\langle n_{ch} \rangle_i$ being the average charged multiplicity in two ϕ regions ($i = 1 : 45^\circ \leq |\phi| \leq 135^\circ, i = 2 : -45^\circ \leq \phi \leq 45^\circ, 135^\circ \leq \phi \leq 225^\circ$). Fig. 2 shows R versus five multiplicity ranges for the intervals $-1.5 \leq \eta < 1.5$ and $1.5 \leq |\eta| < 3.0$. We find that for $1.5 \leq |\eta| < 3.0$ R is less than 1.15 for $n_{ch} \geq 70$ and decreasing for lower multiplicities. For $-1.5 \leq \eta < 1.5$ to the contrary, R extends to approximately 2.4 for $n_{ch} \geq 70$ [19]. In the interval $1.5 \leq |\eta| < 3.0$, therefore, the multiplicity corrections are also very small.

The event multiplicities used in this analysis are the result of corrections deduced from the application of both the ghost finder and the acceptance tables.

3 Multifractality Concept

3.1 Multifractality for Infinite Multiplicity

In a distribution $N(\eta)$ given in the form of a histogram, maxima and minima are observed as a function of the bin size $\delta\eta$. According to the theory of multifractals [12] these fluctuations of $N(\eta)$ can be mapped onto a smooth function $f(\alpha)$. α is the Lipschitz -Hölder exponent and determines the scaling properties of the probabilities

$$p_j = k_j/n \quad ; \quad \left(\sum_{j=1}^M k_j = n \quad , \quad \sum_{j=1}^M p_j = 1 \right) \quad (2)$$

to find k_j particles in bin j of $N(\eta)$, n being the total number of entries in $N(\eta)$ and M the number of non-empty bins. The spectral function $f(\alpha)$ is a function of the moments

$$G_q = \sum_{j=1}^M p_j^q \quad (3)$$

The exponents q extend over all real numbers. In the case of self-similar fluctuations, one observes

$$G_q \propto (\delta\eta)^{\tau_q} \quad \text{for } \delta\eta \rightarrow 0. \quad (4)$$

From the derivation of the generalized exponents τ_q , α_q is determined [10, 12]:

$$\alpha_q = \frac{d\tau_q}{dq} \quad (5)$$

and by Legendre transformations one gets the spectral function

$$f(\alpha_q) = q\alpha_q - \tau_q \quad (6)$$

The properties of $f(\alpha_q)$ for multifractal behaviour of $N(\eta)$ are defined by [10, 12]:

$$\frac{df(\alpha_q)}{d\alpha_q} = q \quad , \quad \frac{d^2 f(\alpha_q)}{d\alpha_q^2} < 0; \quad (7)$$

This downward concave form of $f(\alpha_q)$ is characteristic for multifractals. One finds

- $f(\alpha_q)$ is downward concave
- $f(\alpha_q = \alpha_0)$ is maximal
- $f(\alpha_q) < f(\alpha_0)$, for $q \neq 0$

This behaviour of $f(\alpha_q)$ is shown in fig. 3 together with the special case $f(\alpha_q) = \alpha = 1$ for all q if there are absolutely no fluctuations in $N(\eta)$. The width of $f(\alpha_q)$ is a measure of the size of the fluctuations and the value $f(\alpha_0) < 1$ is a measure of the number of empty bins. For $q > 1$ ($q < 1$) $f(\alpha_q)$ is a quantitative measure of the maxima (minima) of $N(\eta)$ in a very accurate way, because it contains an infinite set of G_q - moments.

The generalized dimensions of the multifractals are [10]

$$D_q = \frac{\tau_q}{q-1} = \frac{1}{q-1} \lim_{\delta\eta \rightarrow 0} \frac{\ln G_q}{\ln \delta\eta} \quad (8)$$

- $D_0 = f(\alpha_0)$... fractal dimension
- $D_1 = f(\alpha_1) = \alpha_1$... information dimension
- $D_2 = 2\alpha_2 - f(\alpha_2)$... correlation dimension
- $D_3 = 0.5(3\alpha_3 - f(\alpha_3))$

⋮

3.2 Multifractality for Finite Multiplicity

In multiparticle production, however, $N(\eta)$ can be identified by the pseudorapidity distribution, and the applicability of the multifractality concept is a posteriori justified by the resulting $f(\alpha_q)$ spectrum [10]. It should be noted that the limit $\delta\eta \rightarrow 0$ is not reliable since the lower limit of k_j for nonempty bins is one particle. The decrease of $\delta\eta$ beyond this limit does not yield more information since the summation in the G_q -moments has to be taken only over non-empty bins. The properties of equation (7) have however been deduced for the mathematical limit which would correspond in particle physics to an infinite number of final state particles and scaling behaviour for $\delta\eta \rightarrow 0$. Finite multiplicities thus seem to cast doubt on the strict applicability, in the mathematical sense, of the theory of multifractals to multiparticle production. The scaling behaviour, expected only for $\delta\eta \rightarrow 0$, could however be approached even at finite $\delta\eta$. Multifractality in multiparticle production is therefore asserted if the spectral function $f(\alpha_q)$ has the properties of equation (7) (fig. 3), irrespective of $\delta\eta$ being infinitesimal or not.

Since we consider events with finite multiplicities it is recommended to improve the result by averaging [10] over many events:

$$\langle \ln G_q(\mu, \nu) \rangle = N^{-1} \sum_{i=1}^N \ln \sum_{j=1}^M p_{ij}^q \quad (9)$$

$$\langle \tau_q(\mu, \nu) \rangle = \frac{\langle \ln G_q(\mu, \nu) \rangle|_{\mu_2} - \langle \ln G_q(\mu, \nu) \rangle|_{\mu_1}}{\ln \delta\eta|_{\mu_2} - \ln \delta\eta|_{\mu_1}} = \frac{-\Delta \langle \ln G_q(\mu, \nu) \rangle}{\Delta \mu \ln 2} \quad (10)$$

where $n = 2^\nu$ is the number of particles per η interval, $M = 2^\mu$ is the number of non-empty bins and N is the number of events. Furthermore we have

$$\langle \alpha_q(\mu, \nu) \rangle = \frac{d \langle \tau_q(\mu, \nu) \rangle}{dq} \quad (11)$$

$$\langle f(\alpha_q(\mu, \nu)) \rangle = q \langle \alpha_q(\mu, \nu) \rangle - \langle \tau_q(\mu, \nu) \rangle \quad (12)$$

$$D_q(\mu, \nu) = \frac{\langle \tau_q(\mu, \nu) \rangle}{q - 1} \quad (13)$$

4 Multifractal Structure in the Data

Using the data discussed in section 2 as input for formulae (9), (10), (11) and (12) an investigation of the multifractal structure has been performed. We consider the union of the two pseudorapidity intervals $1.5 \leq |\eta| < 3.0$. In fig. 4 the behaviour of $\langle \ln G_q \rangle$ versus μ as a function of the power q of the moments is displayed ($\mu = 1, 2, 3, 4$ means that the η interval is divided into 2, 4, 8, 16 equal bins). According to the power of the moments, a considerable dispersion of $\langle \ln G_q \rangle$ is observed, which is strongest for $\mu = 1$. In contrast, the scaling behaviour is

expected to have only one slope independently of μ . By decreasing $\delta\eta$ more and more non-empty bins contain only one particle. The scaling properties of the G_q - moments are lost for bins containing only one entry. In this case we have

$$G_q = n^{1-q} = \text{const.} \quad (14)$$

Therefore multifractal behaviour can be observed only as long there are more entries per bin, $\delta\eta$ being rather large (depending on n).

Fig. 5 shows $\langle\tau_q\rangle$ and $\langle\alpha_q\rangle$ for $\mu = 1, 2$ as a function of q , and fig. 6 gives the behaviour of the spectral function $\langle f(\alpha_q) \rangle$ versus $\langle\alpha_q\rangle$ for $\mu = 1$ and 2. For $\mu \geq 3$ the multifractal behaviour (7) breaks down due to the lack of average multiplicity in the subintervals $\delta\eta$. We find for $\mu = 1$ and 2 that the behaviour of the experimental average multifractal spectrum is compatible with the one expected from theory (fig. 3).

According to formula (13) we find the following fractal, information and correlation dimensions:

$\mu = 1 :$ $D_0 = 0.768 \pm 0.001$ $D_1 = 0.678 \pm 0.001$ $D_2 = 0.622 \pm 0.001$	$\mu = 2 :$ $D_0 = 0.678 \pm 0.001$ $D_1 = 0.645 \pm 0.001$ $D_2 = 0.611 \pm 0.001$
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Since the dimensions D_q are all different ($1 > D_0 > D_1 > D_2 > \dots$) and $f(\alpha_q)$ is downward concave we find for $\mu = 1$ and 2 multifractal structure in the data in the sense defined at the end of the previous section.

In order to discuss the multifractality in greater detail we present in fig. 7 $\langle f(\alpha_q) \rangle$ versus $\langle\alpha_q\rangle$ this time for the restricted interval $1.5 \leq \eta < 3.0$. It is found that for $\mu = 1$ and 2 the multifractal behaviour is in qualitative agreement with fig. 6 but there are considerable quantitative differences. The generalized dimensions are now:

$\mu = 1 :$ $D_0 = 0.669 \pm 0.001$ $D_1 = 0.607 \pm 0.001$ $D_2 = 0.565 \pm 0.001$	$\mu = 2 :$ $D_0 = 0.472 \pm 0.001$ $D_1 = 0.459 \pm 0.001$ $D_2 = 0.442 \pm 0.001$
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The multifractal behaviour of the pseudorapidity distribution $N(\eta)$ is thus a function of the size of the η interval and of the number of subdivisions. This behaviour can be understood in terms of the multiplicity, which is also a function of the total η interval; the larger the considered total interval the greater the number of particles observed and the less empty bins will occur, for instance for $\mu = 1$. This is expressed in $f(\alpha_q)$ by a fractal dimension D_0 closer to one ($D_0(\mu = 1) > D_0(\mu = 2)$). For 4 subdivisions we expect more empty bins than for 2, and so D_0 is decreasing with increasing μ . For 4 subdivisions the bin to bin

fluctuation is smaller than for 2, and so the corresponding $f(\alpha_q)$ is narrower for 4 bins than for 2 bins.

All the errors quoted in this paper are of statistical origin. The systematic uncertainties contained in the data, have been discussed in the intermittency paper [2]. It has been shown that the main sources for systematic corrections are multiplicity losses in different azimuthal regions of the CD (fig. 2) which have not been taken into account in the acceptance tables, and the splitting of particle tracks (ghosts). We have repeated our calculations of $\langle f(\alpha_q) \rangle$ for the following two cases:

- acceptance corrections not taken into account but ghost finder used
- acceptance corrections taken into account but ghost finder not used

In fig. 8 we present these $\langle f(\alpha_q) \rangle$ versus $\langle \alpha_q \rangle$ results together with $\langle f(\alpha_q) \rangle$ of fig. 6 (acceptance tables taken into account and ghost finder used). The deviations of both recalculations from the fully corrected case are very small. We therefore conclude that the systematic uncertainties, although numerically unknown, are very small and will not change the conclusions drawn from the comparison with models in the following sections. It is important to note that the choice of the $|\eta|$ - interval (1.5 - 3.0) has been motivated by the necessity to keep systematic corrections small. That we find multifractal structures only for $\mu = 1$ and 2 corresponding to relatively large subdivisions, $\delta\eta = 1.5(0.75)$ and $0.75(0.375)$ respectively, may serve as explanation that split tracks are of negligible importance.

5 Comparison to Models

Having shown the similarity of the experimental multifractal spectrum to the mathematically conjectured one, it is interesting to investigate which models can quantitatively reproduce the experimental results. $f(\alpha_q)$ and τ_q give a precise quantitative description of the fluctuations of $N(\eta)$ and therefore provide an accurate quantitative comparison for models. The UA5 Monte Carlo program GENCL [16], known to be the best model for describing η distributions at collider energies [22], and the LUND program PYTHIA 4.8 [17] have been used as input to equations (9) to (13).

GENCL includes the production and decay of resonances and clusters. Particle multiplicity and transverse momentum distributions, short range rapidity correlations, and the dependence of the average transverse momentum on the event multiplicity are very accurately reproduced by it [22].

PYTHIA is based on the LUND - string fragmentation mechanism, including the production and decay of K_s^0 , Λ , vector and pseudoscalar resonances (e.g. K_{890}^* , ω , η , η' , ϕ), spin 1/2 baryons ("Σ - like", "Λ - like") and spin 3/2 baryons. It also includes hard parton - parton scattering according to perturbative QCD

with jet production and multiple hard interactions in one event. This model reproduces the observed particle multiplicity distribution, as well as low- p_T jet and multijet production [23].

The results obtained from these models are included in figs. 4 - 7. It can be seen in figs. 6 and 7 that GENCL and PYTHIA also show multifractal structure for 2 and 4 subdivisions in qualitative agreement with the data. However, there are quantitative differences in the shape of $f(\alpha_q)$. For $1.5 \leq |\eta| < 3.0$ we find for instance the following generalized dimensions:

GENCL:

$\mu = 1 :$	$\mu = 2 :$
$D_0 = 0.818 \pm 0.001$	$D_0 = 0.706 \pm 0.001$
$D_1 = 0.728 \pm 0.001$	$D_1 = 0.672 \pm 0.001$
$D_2 = 0.671 \pm 0.001$	$D_2 = 0.638 \pm 0.001$

PYTHIA:

$\mu = 1 :$	$\mu = 2 :$
$D_0 = 0.853 \pm 0.001$	$D_0 = 0.728 \pm 0.001$
$D_1 = 0.77 \pm 0.001$	$D_1 = 0.689 \pm 0.001$
$D_2 = 0.718 \pm 0.001$	$D_2 = 0.652 \pm 0.001$

Therefore we conclude that GENCL and PYTHIA do not reproduce the data completely [14]. From fig. 8 we deduce that the systematic uncertainties do not account for the differences between the data and the models.

In pseudorapidity space the spectral function $f(\alpha_q)$ is a much more sensitive measure of the discrepancies between the data and the models than the G_q -moments, τ_q or α_q , as can be seen in figs. 4 to 7.

6 Universality and Self-Similarity

In section 4 we found that the multifractal structure depends on the size of the pseudorapidity interval (or the multiplicity expressed by ν) and on the number of subdivisions expressed by μ . In order to study this dependence in more detail and to establish a comparison between different multiplicities and subdivisions, a universality relation depending only on $\mu - \nu$ and involving the quantities

$$\Gamma_q(\mu - \nu) = \ln\langle G_q(\mu, \nu) \rangle - \ln\langle G_q(\nu, \nu) \rangle \quad (15)$$

has been derived [11]. This universality relation originates from contrasting the analysis of one event with extremely high multiplicity to that of many events with finite multiplicities. The essential assumption for obtaining equation (15) is self-similarity. We have shown in figs. 6 and 7 that for 2 subdivisions ($\mu = 1$) there is multifractal behaviour. If one divides each of the 2 bins again into two bins and continues this process until many bins are obtained, then the breaking

down of the content of each successive bin into two new bins can be expressed by the same multiplicity splitting function $\Psi(x,n)$ as that which describes the first subdivision (x being the fraction of particles in one of any two subdivisions and n being the number of particles before dividing the phase space cell). This amounts to having multifractality at any higher splitting level. Self-similarity is then given by the equality of the splitting process at any splitting level of an interval into two subintervals. The test of the self-similarity property would consist in showing that, for all possible values of μ and ν , the results depend only on $\mu - \nu$ and q [11]. This behaviour is examined conveniently in a diagram $\Gamma_q(\mu - \nu)$ versus $\mu - \nu$.

Fig. 9 presents $\Gamma_q(\mu - \nu)$ versus $\mu - \nu$ for $\nu = 2, 3, 4$, $\mu = 1, 2, 3, 4$ and $q = \pm 5$ for UA1 data in the interval $1.5 \leq |\eta| < 3.0$. One observes small deviations from universal lines. For $q < 5$, the deviations from universality are less pronounced because a higher power q enhances deviations more than a lower q . Selfsimilarity seems to be present to a great extent in the data [14].

A comparison to GENCL and PYTHIA is also performed in fig. 9. In this figure, data and model agree. Note that the main difference between this and the previous analysis is that the whole multiplicity distribution was folded for fig. 4 to 7, whereas in this section only events with constant multiplicities are considered ($\nu = 2, 3, 4$ corresponding to $n_{ch} = 4, 8, 16$). We observe that for $\nu = 2$ and $\mu = 1$ in fig. 10a the disagreement between data and models is strongest. For $\nu = 3$ and $\mu = 2$ in fig. 10a and also for $\nu = 3$ and $\mu = 1$ in fig. 10b there is still some, though less, disagreement, and for $\nu = 4$ there is even less disagreement between data and models (agreement with GENCL is better than that with PYTHIA). For low multiplicity events there seems to be stronger disagreement between data and models. The disagreement in fig. 6 is therefore mainly due to low multiplicities which represent the bulk of the events. The comparison with GENCL and PYTHIA in fig. 10 gives much less agreement than that found in fig. 9. This proves again that $f(\alpha_q)$ is the more sensitive quantity for comparison between data and model.

Multifractal structure is found for $n = 16$ ($\nu = 4$) and $\mu = 1, 2, 3$ ($\nu = 4, \mu = 2, 3$ are shown in fig. 10); for $n = 8$ ($\nu = 3$); and $\mu = 1, 2$ (fig. 10) and for $n = 4$ ($\nu = 1$) and $\mu = 1, 2$ (not shown). All other cases do not show a downward concave $\langle f(\alpha_q) \rangle$. For $\mu - \nu = -3$ we find for $\nu = 4, \mu = 1$ a downward concave $\langle f(\alpha_q) \rangle$ (not shown). The fractal dimension in this case is $D_0 = 0.9979 \pm 0.0011$. This means that there are no empty bins ($D_0 = 1$ is the dimension of a line). For $\mu - \nu = -2$ we find for $\nu = 3, \mu = 1$ and $\nu = 4, \mu = 2$ a downward concave $\langle f(\alpha_q) \rangle$ (fig. 10b). For $\mu - \nu = -1$ we find for $\nu = 2, \mu = 1$; $\nu = 3, \mu = 2$; and $\nu = 4, \mu = 3$ a downward concave $\langle f(\alpha_q) \rangle$ (fig. 10a). For $\mu - \nu = 0$ we find for $\nu = 2, \mu = 2$ a downward concave $\langle f(\alpha_q) \rangle$ (not shown). Here the fractal dimension is $D_0 = 0.5719 \pm 0.0011$, which means that there is fractal structure in the η -distribution for $n_{ch} = 4$. For $\mu - \nu = 1$ and 2 , no multifractal structure is observed.

Universality should exhibit itself in both the shape of the spectral functions,

$f(\alpha_q)$, and in their coincidence for constant $\mu - \nu$ [11]. Figs. 10a and b contain all these cases and show that the second expectation is not well realized, neither in the data [15] nor in the models.

Universal behaviour is found in terms of equation (15) but it is not clear from these results that, for $\mu - \nu = \text{constant}$, the same multifractality is observed for all possible μ and ν .

7 Summary

We have performed an analysis of 159154 minimum bias events in order to find multifractal structures. The answers to the questions formulated in the introduction can be given in the following way:

- according to the results presented in figs. 6, 7 and 10 we find multifractal structure in the data and in the Monte Carlo models GENCL and PYTHIA 4.8 for two and four subdivisions of the considered η interval. The corresponding spectral functions show a behaviour compatible with the one predicted by theory ((7), fig. 3). The breakdown of the multifractal behaviour for $\mu > 2$ is due to low average multiplicity in the considered η - interval. This is suggested by fig. 10a, where for $\mu = 3$ we find multifractal behaviour only for $\nu = 4$ ($n_{ch} = 16$).
- For large bin sizes $\mu = 1$ and 2 scaling behaviour is found, suggesting a self-similar cascading. It depends on the size of the η - interval and on μ (dependence of the values of the generalized dimensions D_q on μ and η is observed (section 4)). Non-scaling contaminations from bins containing only one entry is becoming more and more dominant with μ increasing (section 4). Fig. 9 shows that $\Gamma_q(\mu - \nu)$ versus $\mu - \nu$ populates universal lines almost perfectly. According to the universality relation (15) self-similarity should also be found for much finer binning. The multifractal spectral functions $f(\alpha_q)$, for $\mu - \nu = \text{const.}$ do not coincide in fig. 10 [11]. This shows that $f(\alpha_q)$ versus α_q for $\mu - \nu = \text{const.}$ is a more sensitive measure of the universality. This property of the data and the models therefore remains inconclusive.
- The comparison of the multifractal spectra of data, GENCL and PYTHIA 4.8, shows that (fig. 6,7,10):
 - GENCL agrees better with the data than PYTHIA; this reflects the fact that GENCL reproduces average distributions like multiplicity, pseudorapidity, transverse momentum, two-particle correlations, etc. much better than PYTHIA.
 - for low multiplicities $\nu=2$ and 3, the disagreement between data and models is more visible than for higher multiplicities $\nu = 4$.

Therefore no quantitative agreement between data and these models is found.

Finally it is emphasized that both the data and the models contain statistical and possibly dynamical fluctuations, with the interesting physics contained in the latter. The procedure necessary to filter out these fluctuations from the data is a task which still remains; some ideas have recently been suggested [24]. A real test of the models can only be performed if the multifractal spectral functions can be obtained for the dynamical fluctuations alone.

Acknowledgements

We are grateful to R.C.Hwa, C.B.Chiu and W.Florkowski for many suggestions and exchange of information, which was essential to obtain the presented results. We are also grateful to R.Frühwirth for checking the error calculation.

We are thankful to the management and staff of CERN and of all participating institutes for their vigorous support of the experiment. The following funding agencies have contributed to this program:

Fonds zur Förderung der Wissenschaftlichen Forschung, Austria.
Valtion luonnontieteellinen toimikunta, Suomen Akatemia, Finland.
Institut National de Physique Nucléaire et de Physique des
Particules and Institut de Recherche Fondamentale (CEA), France.
Bundesministerium für Forschung und Technologie, Fed. Rep.
Germany.
Istituto Nazionale di Fisica Nucleare, Italy.
Science and Engineering Research Council, United Kingdom.
Stichting Voor Fundamenteel Onderzoek der Materie, The
Netherlands.
Centro de Investigaciones Energetica, Medioambientales y
Tecnologicas (CIEMAT), Spain.
Department of Energy, USA.

Thanks are also due to the following people who have worked with the collaboration in the preparations for and data collection on the runs described here: L. Baumard, F. Bernasconi, D.Brozzi, V. Cecconi, L. Dumps, G. Fetchenhauer, G. Gallay, S. Lazic, J.C.Michelon, S.Pavlon and L. Pollet.

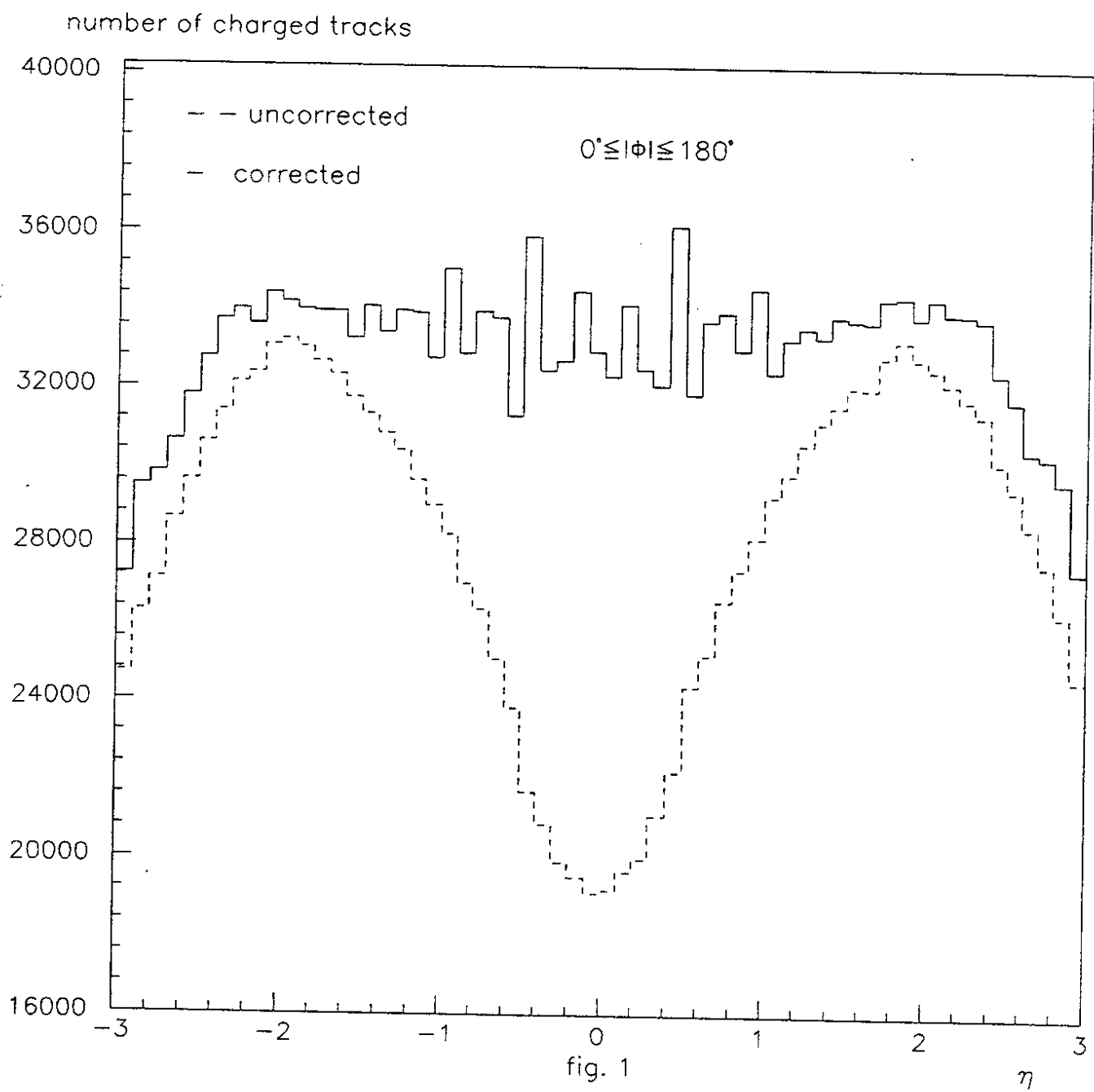
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Figure Captions

- fig. 1: The corrected distribution of pseudorapidity η compared to the uncorrected one, for all azimuthal angles ϕ .
- fig. 2: The ratio $R = \frac{\langle n_{ch} \rangle_1}{\langle n_{ch} \rangle_2}$ in 5 multiplicity bins ($n_{ch} \leq 15, 16 \leq n_{ch} \leq 30, 31 \leq n_{ch} \leq 50, 51 \leq n_{ch} \leq 70, n_{ch} \geq 70$) for 3 different η intervals.
- fig. 3: The multifractal spectral function $f(\alpha)$ versus α for flat $N(\eta)$ and for fluctuating $N(\eta)$.
- fig. 4: The quantity $\langle \ln G_q \rangle$ versus μ as a function of q for data and the Monte Carlo programs GENCL and PYTHIA 4.8.
- fig. 5: The Lipschitz - Hölder exponents $\langle \alpha_q \rangle$ and the generalized exponents $\langle \tau_q \rangle$ versus q for data , GENCL and PYTHIA 4.8.
- fig. 6: For both intervals $1.5 \leq |\eta| < 3.0$ the multifractal spectral function $\langle f(\alpha_q) \rangle$ versus $\langle \alpha_q \rangle$ for $\mu = 1$ and 2 for the data, GENCL and PYTHIA 4.8.
- fig. 7: For the interval $1.5 \leq \eta < 3.0$, the multifractal spectral function $\langle f(\alpha_q) \rangle$ versus $\langle \alpha_q \rangle$ for $\mu = 1$ and 2 for the data, GENCL and PYTHIA 4.8.
- fig. 8: The multifractal spectral function $\langle f(\alpha_q) \rangle$ versus $\langle \alpha_q \rangle$ for 3 cases:
- (a) data corrected with acceptance tables and ghost finder
 - (b) data corrected with acceptance tables only
 - (c) data corrected with ghost finder only
- fig. 9: The universality function $\Gamma_q(\mu - \nu)$ versus $\mu - \nu$ for data, GENCL and PYTHIA 4.8. (since the models are very close together it has not been attempted to label them, but GENCL is again reproducing the data better)
- fig. 10a: For $\mu - \nu = -1$ a comparison of $\langle f(\alpha_q) \rangle$ for data, GENCL and PYTHIA 4.8 versus $\langle \alpha_q \rangle$.
- fig. 10b: For $\mu - \nu = -2$ a comparison of $\langle f(\alpha_q) \rangle$ for data, GENCL and PYTHIA 4.8 versus $\langle \alpha_q \rangle$.



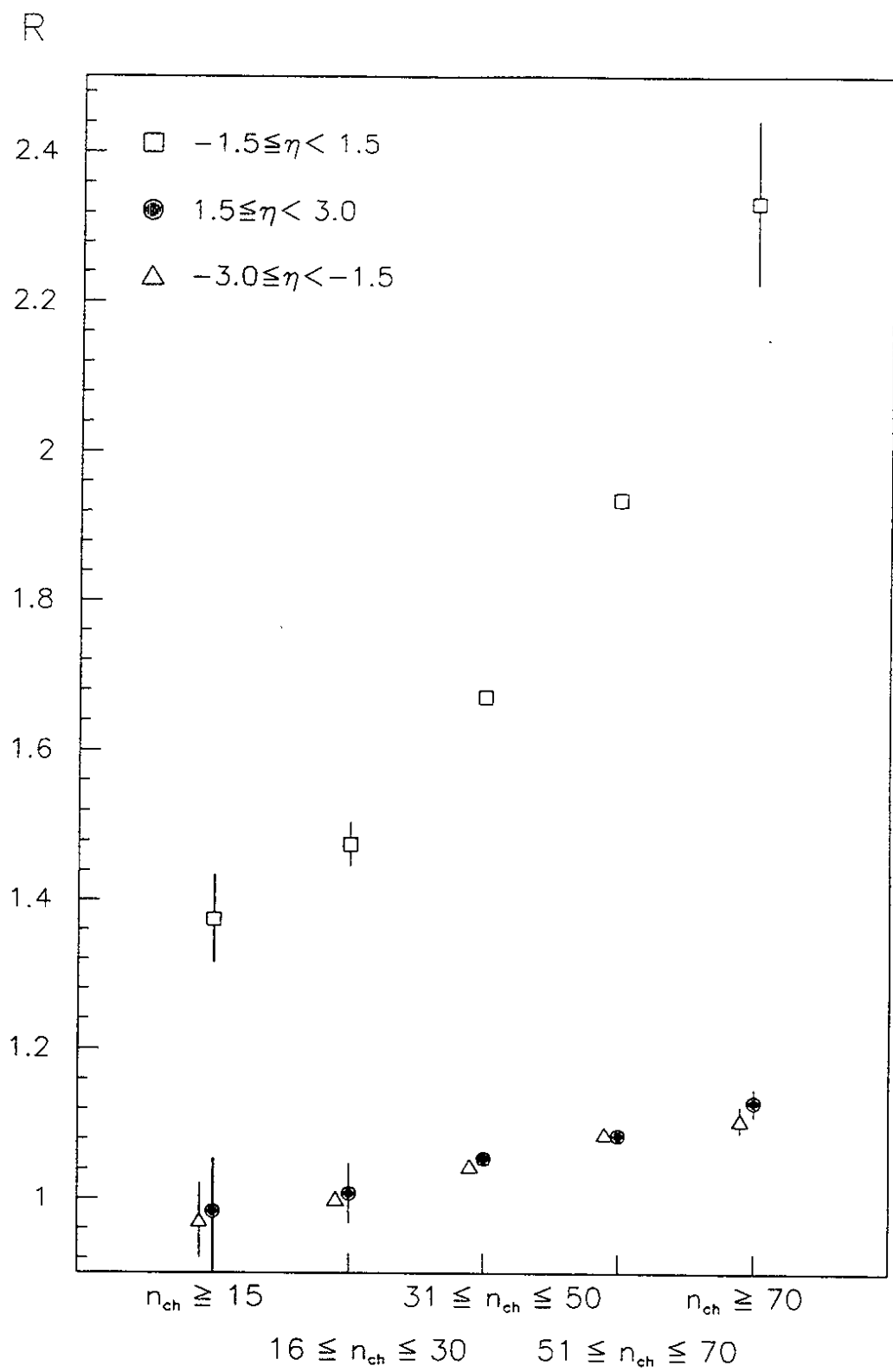


fig. 2

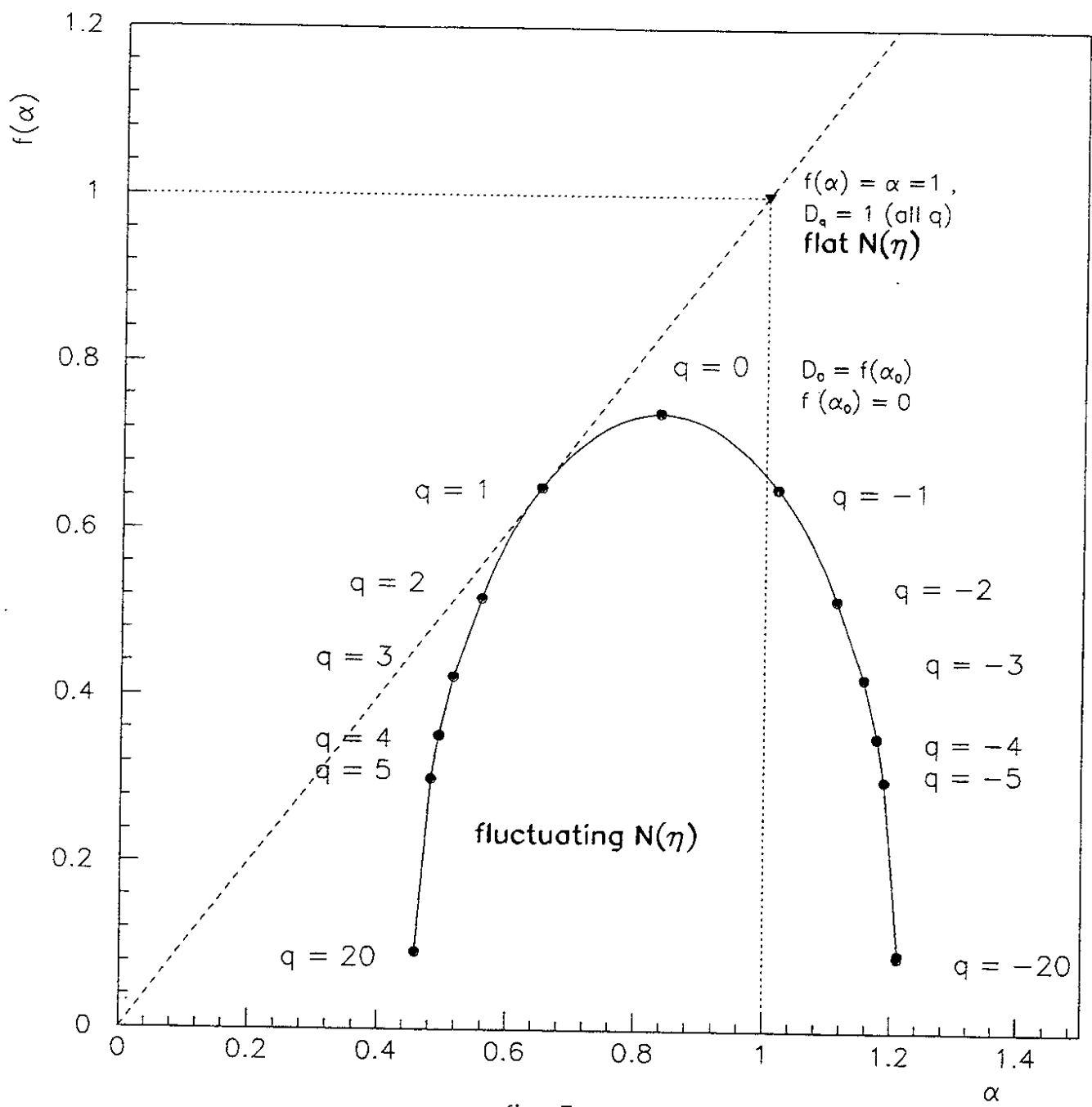
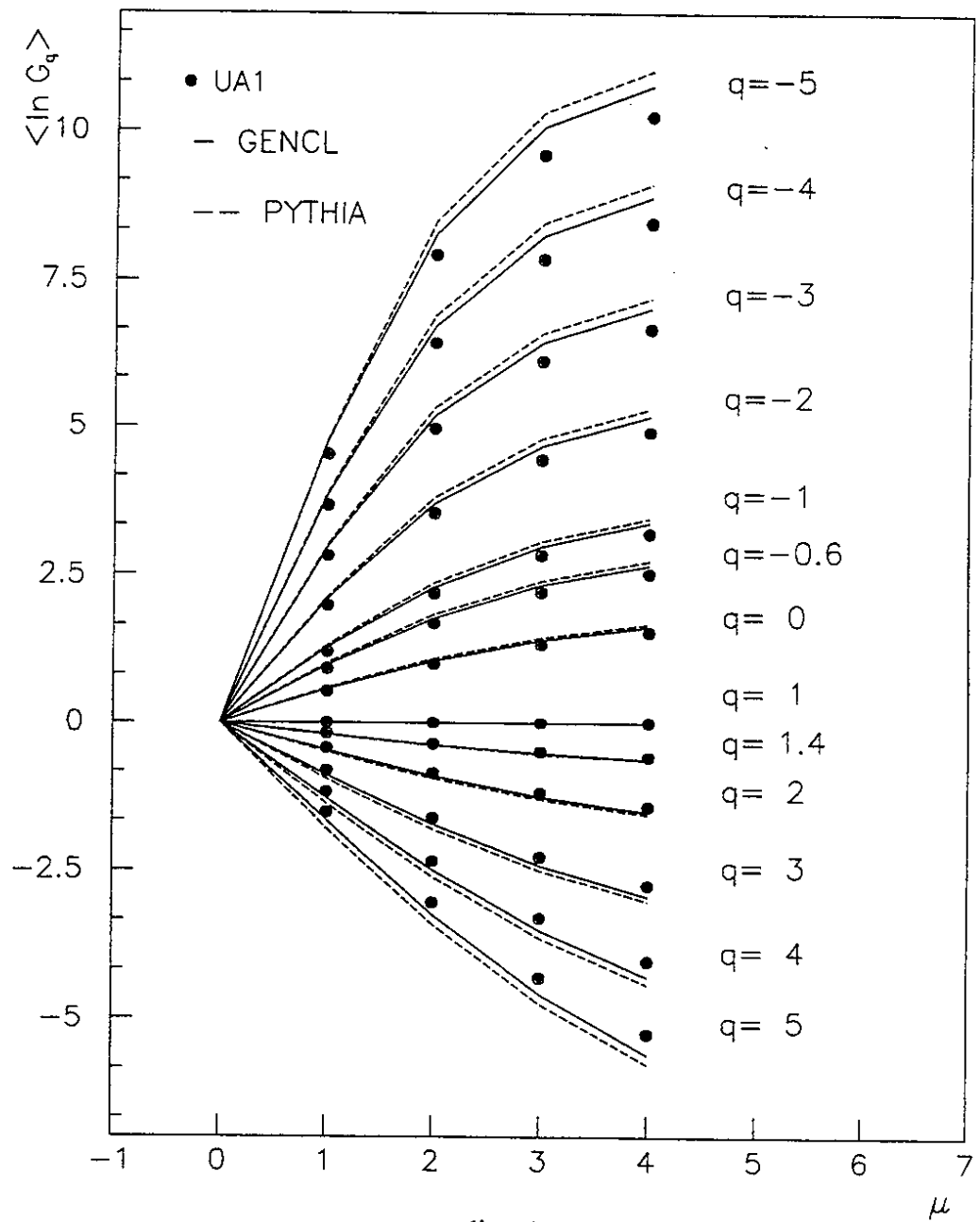


fig. 3



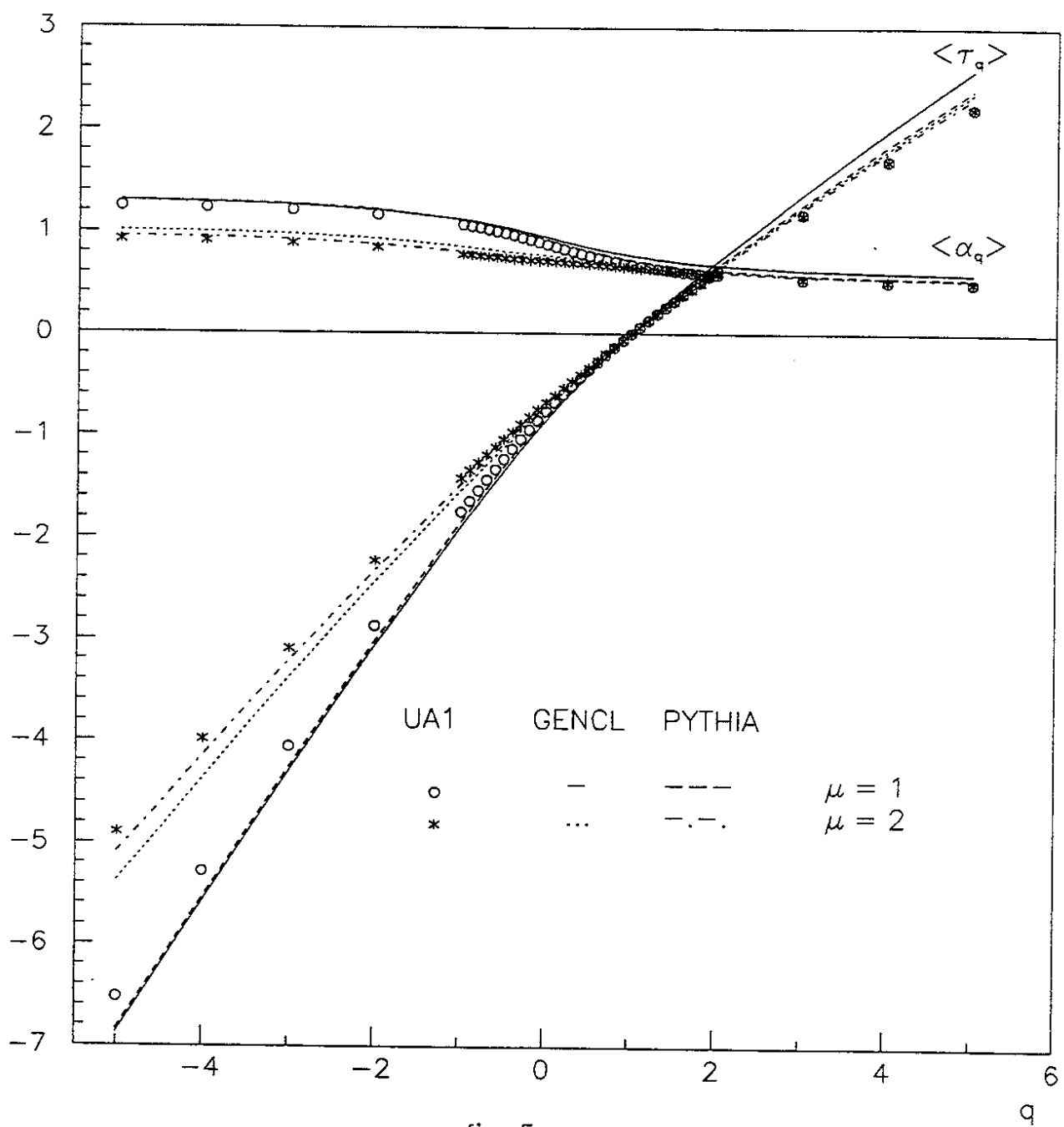


fig. 5

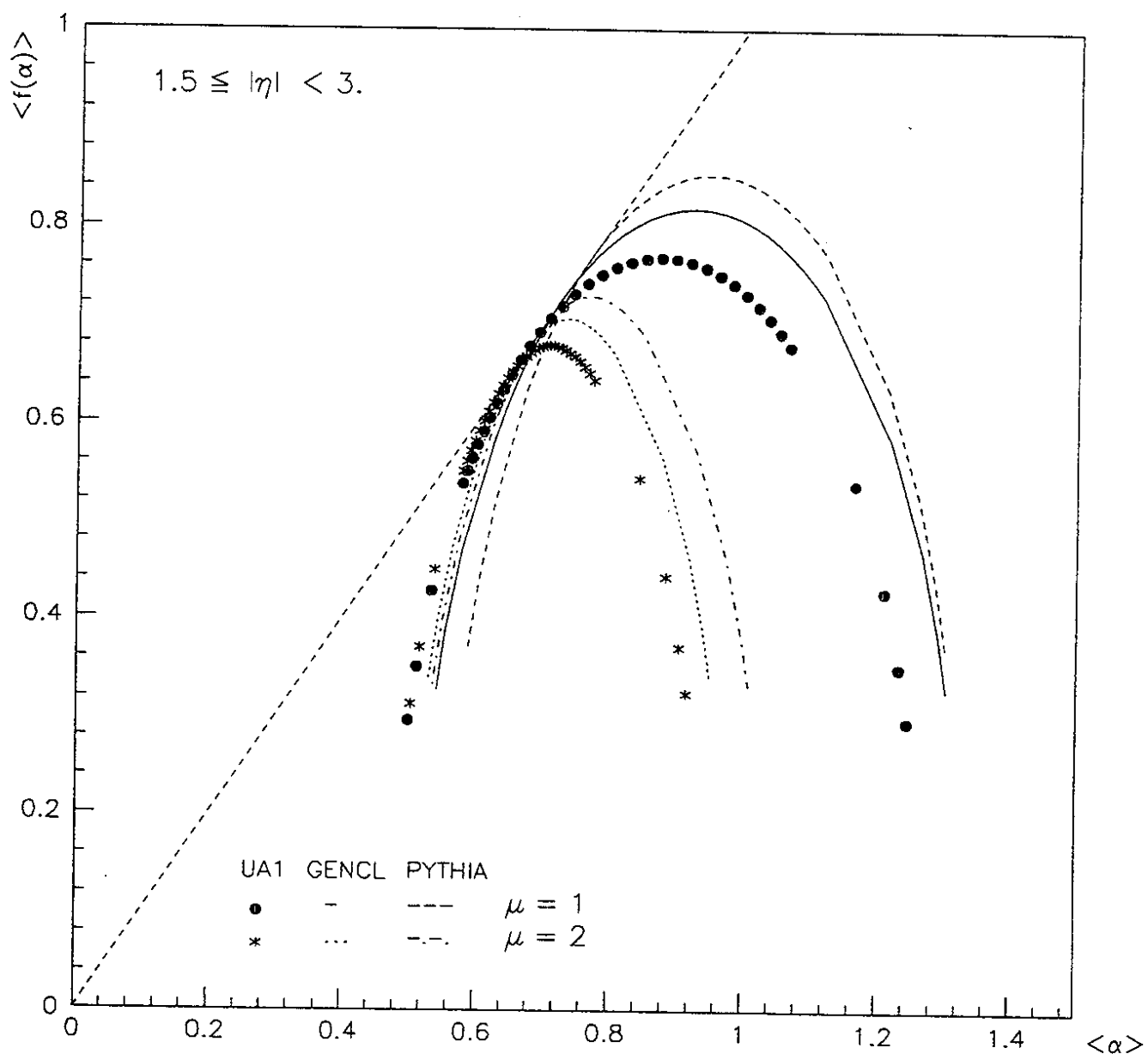


fig. 6

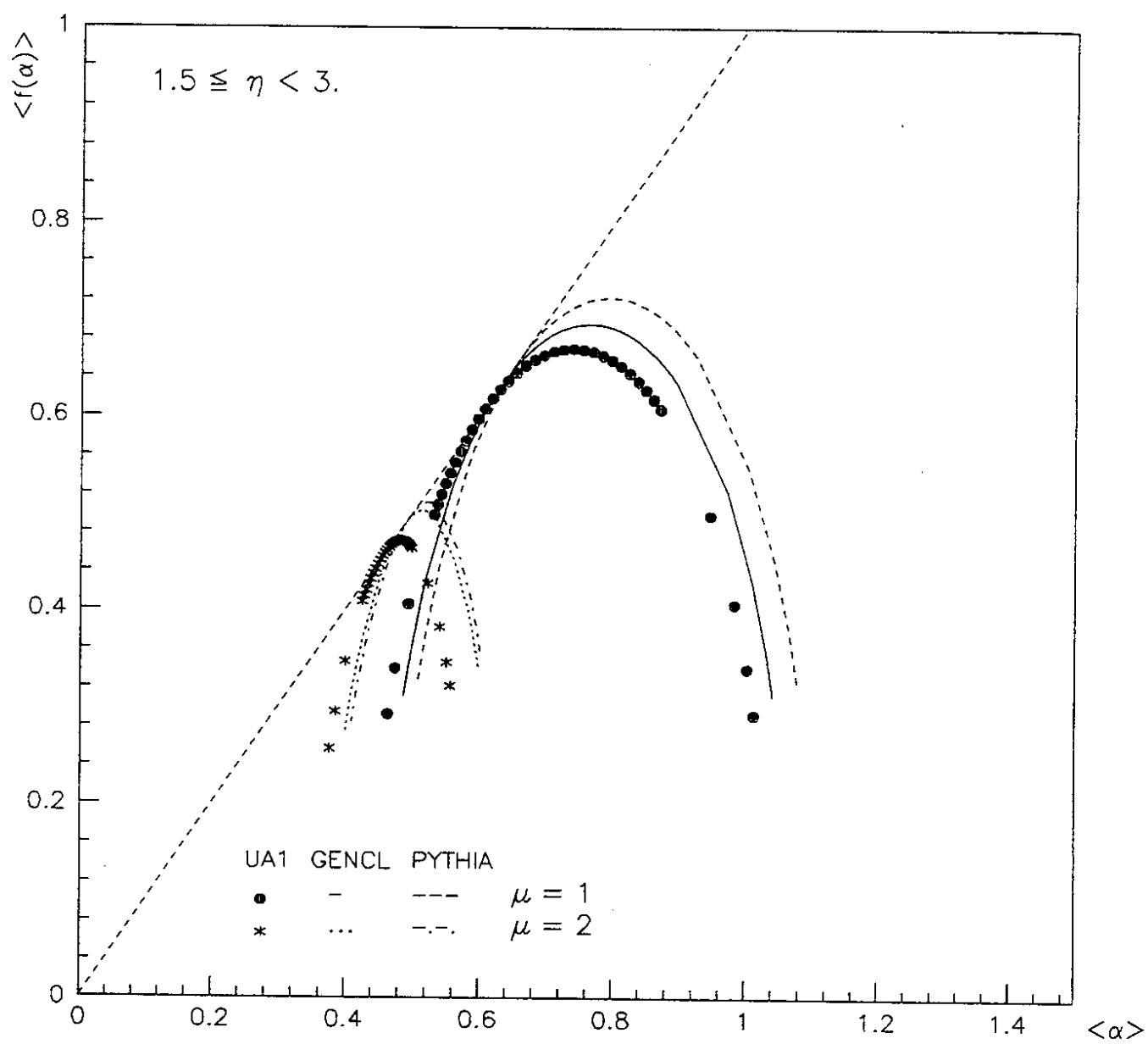


fig. 7

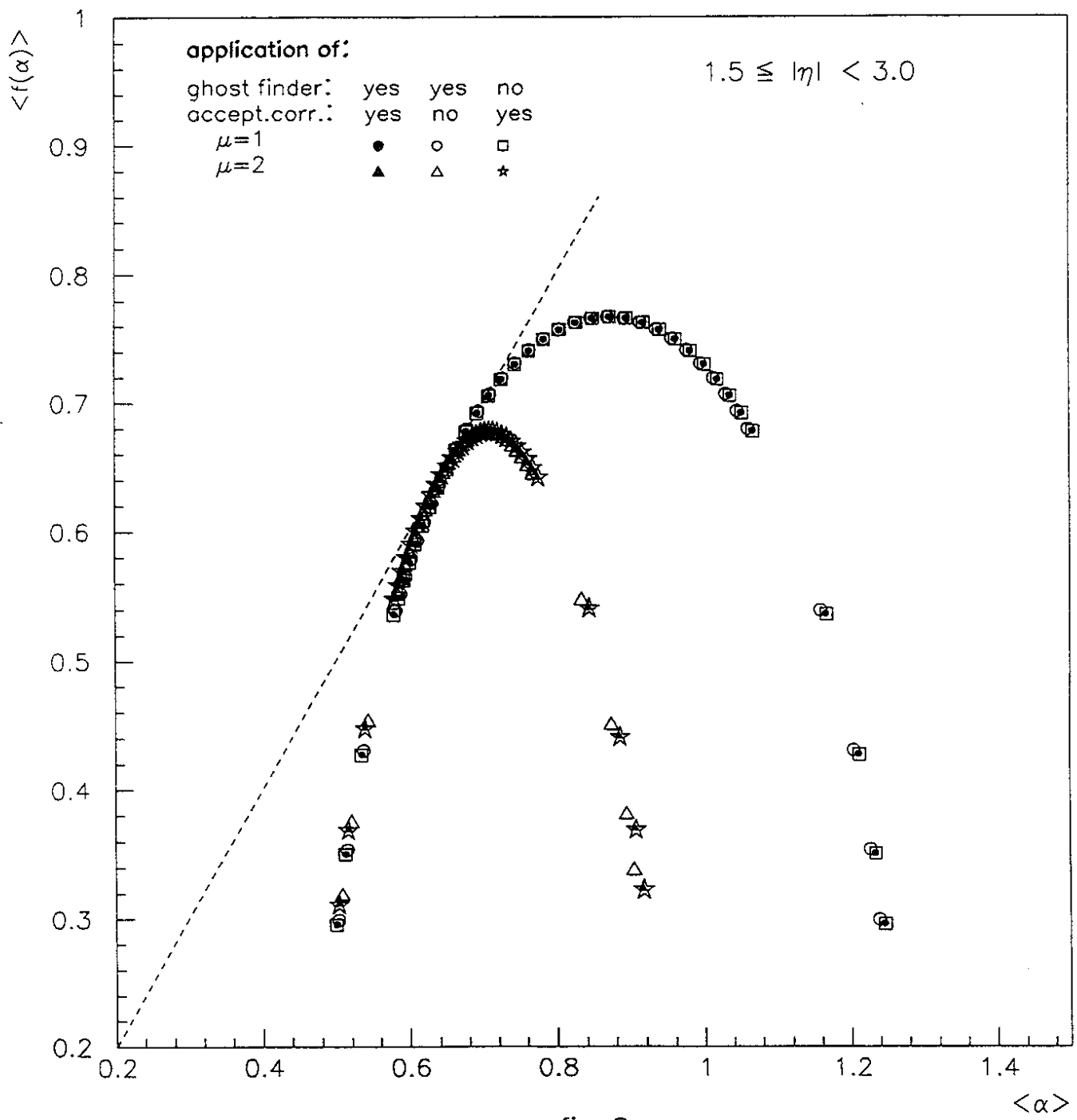


fig. 8

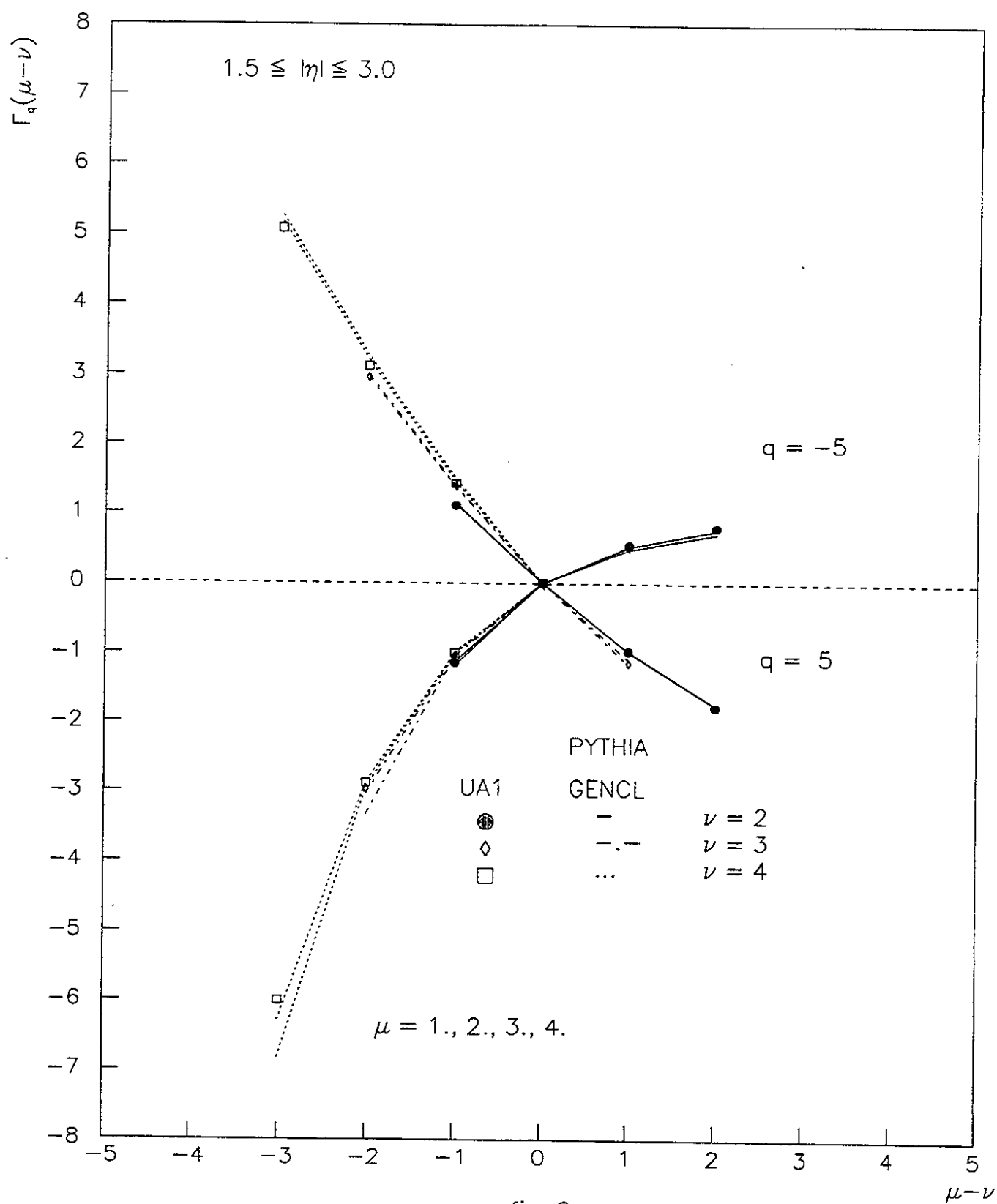


fig. 9

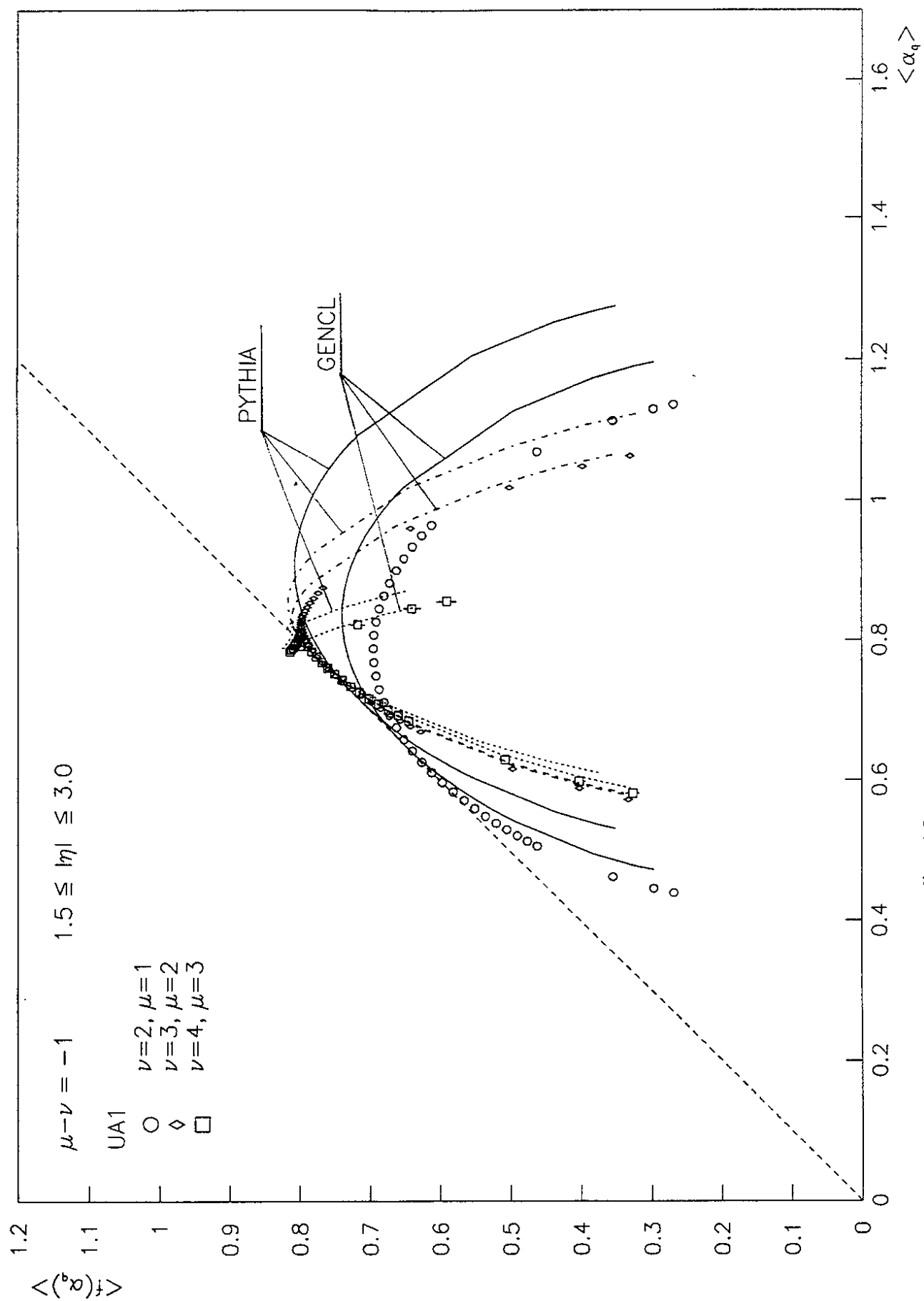


fig. 10a

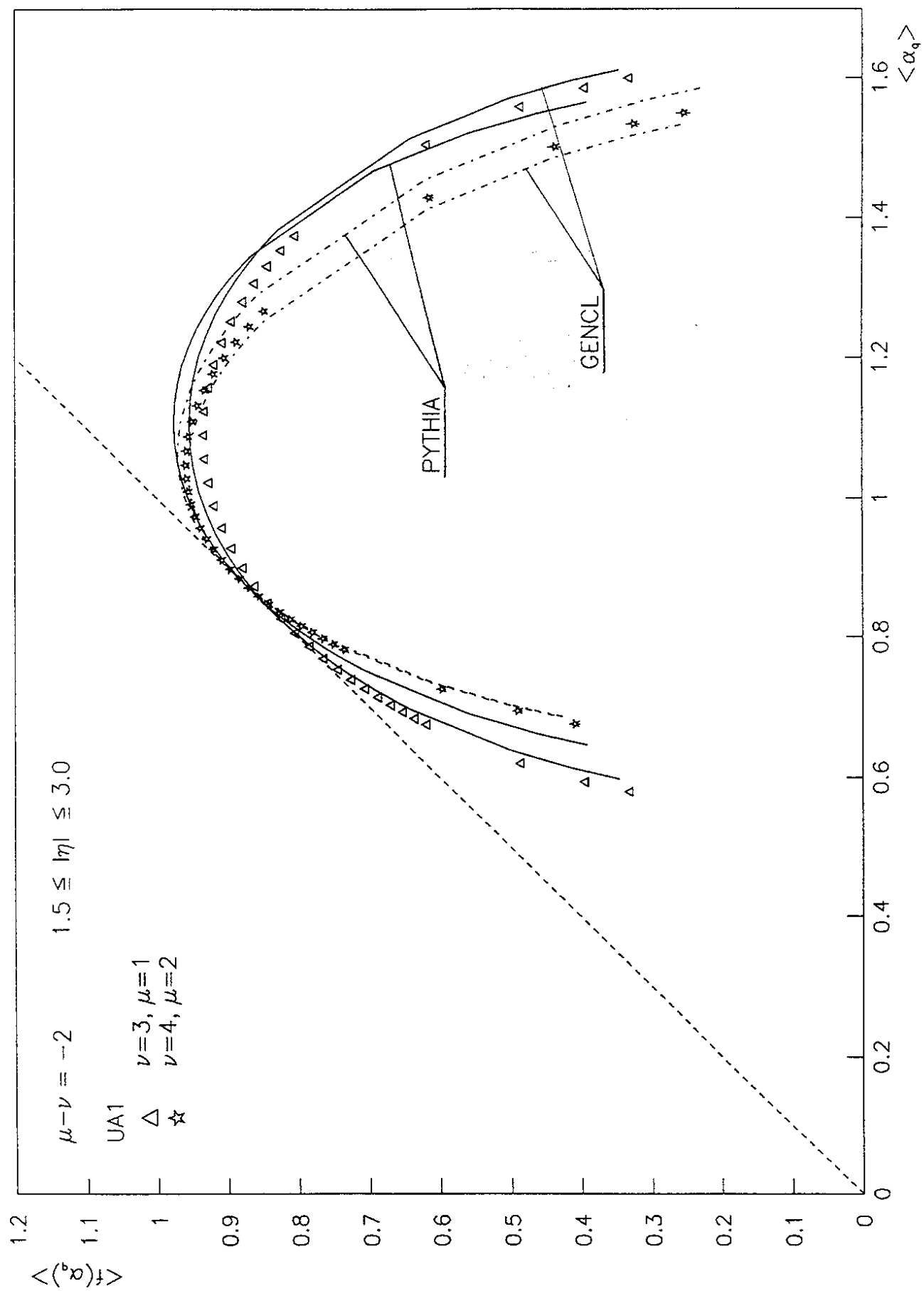


fig. 10b