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MULTIFRACTAL DIMENSIONS AND SCALING EXPONENTS FOR STRONGLY BOUNDED RANDOM CASCADES

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The multifractal structure of a measure refers to some notion of dimension of the set which supports singularities of a given order α as a function of the parameter α. Measures with a nontrivial multifractal structure are commonly referred to as *multifractals*. Multifractal measures are being studied both empirically and theoretically within the statistical theory of turbulence and in the study of strange attractors of certain dynamical systems. Conventional wisdom suggests that various definitions of the multifractal structure of random cascades exist and coincide. While this is rigorously known to be the case for certain deterministic cascade measures, the same is not true for random cascades. The purpose of this paper is to pursue this theory for a class of random cascades. In addition to providing a new role for the modified cumulant generating function (structure function) studied by Mandelbrot, Kahane and Peyrière, the results have implications for the theoretical interpretation of empirical data on turbulence and rainfall distributions.

1. Introduction. For a suitable notion of dimension, the *multifractal* structure of a (possibly random) measure μ on $\mathbb{R}^d$, or more generally a metric space (T, ρ), refers to the dimensions of the sets

$$F(\alpha) := \{\mathbf{x} \in T: \mu B_\delta(\mathbf{x}) \sim \delta^\alpha \text{ as } \delta \to 0\}, \tag{1.1}$$

as a function of the parameter α. Here $B_\delta(\mathbf{x})$ denotes a (closed) ball of radius $\delta > 0$ located at $\mathbf{x}$ and $\mu B_\delta(\mathbf{x}) \sim \delta^\alpha$ means

$$\lim_{\delta \to 0} \frac{\log \mu B_\delta(\mathbf{x})}{\log \delta} = \alpha.$$

In the event that the measure μ has a continuous positive density on $T = \mathbb{R}^d$, say, then any reasonable definition of dimension will provide a (spiked) function of α which takes the value d at $\alpha = d$, and is otherwise the dimension of the empty set. In the case of certain singular measures one may obtain a less trivial dimension curve. For a simple example, let $p + q = 1$,

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$0 < p \neq q < 1$, and take μ to be the measure on $[0,1]$ defined by the prescription

$$\mu\left(\left[\sum_{j=1}^{n}\sigma_j 2^{-j}, \sum_{j=1}^{n}\sigma_j 2^{-j} + 2^{-n}\right)\right) = p^{\sum_{j=1}^{n}\sigma_j} q^{n-\sum_{j=1}^{n}\sigma_j}, \tag{1.2}$$

where $\sigma_j \in \{0,1\}$, $j = 1,\dots,n$, $n \geq 1$. Let $\delta = 2^{-n}$. Write $x = (\sigma_1, \sigma_2, \dots)$ in binary with the usual convention for uniqueness. The mass of $B_\delta(x) = [\sum_{j=1}^{n}\sigma_j 2^{-j}, \sum_{j=1}^{n}\sigma_j 2^{-j} + 2^{-n})$ is $p^{\sum_{j=1}^{n}\sigma_j} q^{n-\sum_{j=1}^{n}\sigma_j} = (2^{-n})^{\alpha}$ for

$$\alpha = \alpha_n(x) := -\frac{1}{n}\sum_{j=1}^{n}\sigma_j \log_2 p - \left(1 - \frac{1}{n}\sum_{j=1}^{n}\sigma_j\right)\log_2 q.$$

According to a classic result of Eggleston (1949), for a fixed $0 \leq \gamma \leq 1$, the subset of numbers x in $[0,1]$ for which $(1/n)\sum_{j=1}^{n}\sigma_j \to \gamma$ as $n \to \infty$ has Hausdorff dimension $-\gamma\log_2\gamma - (1-\gamma)\log_2(1-\gamma)$. Therefore, the Hausdorff dimension of $F(\alpha)$ will be given by

$$-\gamma(\alpha)\log_2\gamma(\alpha) - (1-\gamma(\alpha))\log_2(1-\gamma(\alpha)),$$

where $\gamma \equiv \gamma(\alpha)$ is the solution to the equation

$$-\gamma\log_2 p - (1-\gamma)\log_2 q = \alpha. \tag{1.3}$$

We shall refer to this example as the *deterministic binomial cascade*.

While this conveys the spirit, the theory depends on the dimension function which one has in mind. In practice, a value of "dimension" is theoretically computed and/or empirically estimated from a variety of ostensibly different vantage points. Often it is in the form of a scaling exponent which is then interpreted as a dimension. In this paper we shall consider various versions which are defined as follows.

Hausdorff dimension. Let B_r^i denote an ith ball of radius r in an arbitrary cover of $F(\alpha)$ by countably many balls as indicated below. Then

$$h(\alpha) := \inf\left\{\theta \geq 0 : \lim_{\delta\to 0} \inf_{\bigcup_{i=1}^{\infty} B_r^i \supseteq F(\alpha),\, r\leq\delta} \sum_{i=1}^{\infty} r^{\theta} = 0\right\}. \tag{1.4}$$

Box dimension. Assuming that the indicated limit exists, let

$$b(\alpha) := \lim_{\delta\to 0}\frac{\log m(\delta,\alpha)}{-\log\delta}, \tag{1.5}$$

where $m(\delta,\alpha)$ is the smallest number of balls of radius at most δ required to cover $F(\alpha)$.

Singularity spectrum. This is a scaling exponent for the "size" of the set of singularities of order α defined as follows. Suppose that μ is compactly supported and let $\Delta_{\mathbf{k}}$ denote the δ-mesh cube of side length δ located at the integer lattice site $\mathbf{k}$; say by the lower left corner. Let

$$N_\delta(\alpha) := \#\{\mathbf{k}: \mu(\Delta_{\mathbf{k}}) > \delta^\alpha\}. \tag{1.6}$$

Then, assuming that the indicated limits exist, define

$$f(\alpha) := \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{\log[N_\delta(\alpha + \varepsilon) - N_\delta(\alpha - \varepsilon)]}{-\log \delta}. \tag{1.7}$$

Rényi dimension. Assuming that the indicated limit exists, let

$$(h - 1)D(h) \equiv \tau(h) := \lim_{\delta \to 0} \frac{\log M_\delta(h)}{-\log \delta}, \tag{1.8}$$

where

$$M_\delta(h) := \sum_{\mathbf{k}}{}' \{\mu(\Delta_{\mathbf{k}})\}^h, \tag{1.9}$$

the prime indicating a sum over those cubes $\Delta_{\mathbf{k}}$ which meet the support of μ and are significant only in the case $h \le 0$. The quantity $D(h)$ is related to *Rényi information*; cf. Rényi (1970). It occurs in the physics literature in Grassberger (1983), Hentschel and Procaccia (1983) and Paladin and Vulpiani (1984) under the name of *generalized dimension*. The exponent function $\tau(h)$ will be referred to as a *Rényi exponent* here.

There is a somewhat simple formalism based on Legendre transform duality and large deviation rates which makes various connections between the exponents and dimensions indicated above appear to be quite plausible; for example, see Jensen, Kadanoff, Libchaber, Procaccia and Stavans (1985), Halsey, Jensen, Kadanoff, Procaccia and Shraiman (1986), Mandelbrot (1988), Tel (1988), Grassberger (1983), Hentschel and Procaccia (1983) and Paladin and Vulpiani (1984). However, it appears that essentially the only case in which the computations and connections have been worked out rigorously are for certain deterministic measures, most notably the *deterministic multinomial cascades* defined by a measure μ, depending on parameters $b \in \mathbb{N}$, $\mathbf{p} = (p_0, \dots, p_{b-1})$, $p_i \ge 0$, $\sum_{i=0}^{b-1} p_i = 1$, as follows. Let $J = [0,1]$ denote the unit interval and let $J(\sigma)$, $\sigma = 0, 1, 2, \dots, b-1$, denote a partition of J into b subintervals of lengths b^{-1}. Inductively, given $J(\sigma_1, \dots, \sigma_n)$, $\sigma_i \in \{0, 1, 2, \dots, b-1\}$, let $J(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$, $\sigma_{n+1} \in \{0, 1, \dots, b-1\}$ denote the partition of $J(\sigma_1, \dots, \sigma_n)$ into b subintervals of sidelength $b^{-(n+1)}$. Define a set function μ by $\mu(J) := 1$ and for subsets $J(\sigma_1, \dots, \sigma_n)$ of J by

$$\mu(J(\sigma_1, \dots, \sigma_n)) := p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n}, \qquad \sigma_i \in \{0, 1, \dots, b-1\}. \tag{1.10}$$

Then μ has a unique extension to a probability measure on the Borel subsets of J which is referred to as the *multinomial cascade*; see Figure 1. The effective partition number $\bar{b} := \#\{i: p_i > 0\}$ may be viewed as a large-scale *intermittancy parameter* for the multinomial cascade. For this class of examples the rigorous theory connecting the various exponents and dimensions

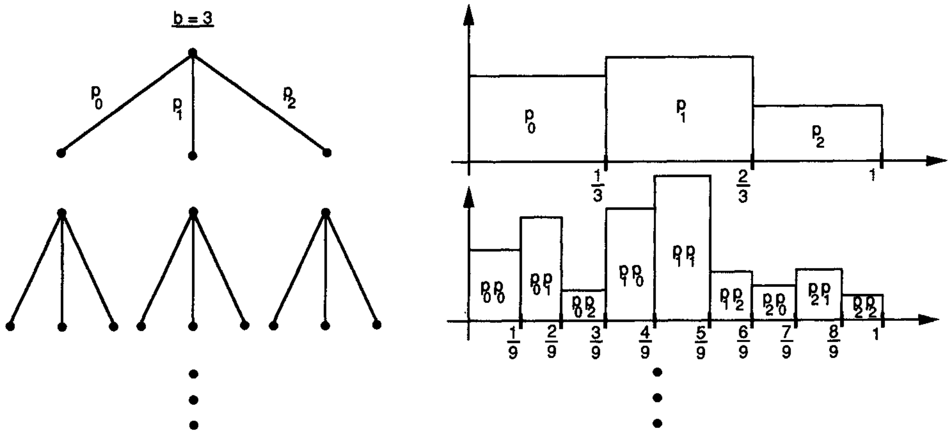

FIG. 1. *Trinomial cascade.*

defined above is quite complete; for example, see Brown, Michon and Peyrière (1990), Tel (1988), Falconer (1990) and Cawley and Mauldin (1991).

The random cascades are essentially obtained by replacing the factor p_{σ_i} in (1.10) by $W(\sigma_1, \dots, \sigma_i)b^{-1}$, where the $W(\sigma_1, \dots, \sigma_i)$ are i.i.d. mean one nonnegative random variables, say distributed as W. The random variable W is referred to as the *cascaded variable* and its distribution is referred to as the *cascaded distribution*.

Frisch and Parisi (1985), who invented the term "multifractal," first suggested that certain cascade models found in the statistical theory of turbulence and extensively studied by Kolmogorov (1941, 1962), Novikov and Stewart (1964), Yaglom (1966), Mandelbrot (1974), Frisch, Sulem and Nelkin (1978) and more recently Benzi, Paladin, Parisi and Vulpiani (1984) and Meneveau and Sreenivasan (1987) should provide important examples of measures having a nontrivial multifractal structure in the sense being described here. However, the authors know of no examples within the theory of random cascades where the formalism connecting the exponents and dimensions is worked out rigorously.

Recent data analysis and computer simulation have also provided evidence of similar structure in the spatial distribution of rainfall; see Schertzer and Lovejoy (1987), Lovejoy and Schertzer (1990) and Gupta and Waymire (1990, 1992). Interestingly, hierarchical random measure representations of spatial rainfall distributions have been prominent in the hydrologic and atmospheric sciences since their introduction by Le Cam (1961).

Our main objective is to extend the results illustrated by the deterministic multinomial cascade to the random cascades. For the problems to be addressed here, we take $d = 1$ without loss of generality. In the next section the random cascade is defined, some preliminary results are noted and the main results of this paper are stated. In Section 3 we compute $h(\alpha)$ and $b(\alpha)$. In Section 4 the singularity set is studied from the point of view of the Rényi exponents $\tau(h)$. In particular, we establish the existence of $\tau(h)$ and further show it to coincide with a basic structure function under certain conditions on the distribution of the cascaded variables. The connections between the results of these computations and the spectrum of singularities $f(\alpha)$ are largely understood in terms of certain soft analysis results. These are discussed in some concluding remarks in Section 5. However, some simple counterexamples to the existence and Legendre transform duality between exponents given in Section 2 should be noted. In any case, all of the dimensions and exponents can be related to the spectrum of singularities under certain conditions which, in the generality of this paper, are in some sense sharp. Most significant to practical applications of the results given here is the insight gained into the way in which singularity spectra may determine the random cascade model via a transform of the underlying cascaded distribution. The implications of our results for the uniqueness problem are also noted in the final remarks.

2. Random cascades: Preliminaries and statements of results. Let $\overline{T} := \{0, 1, \dots, b-1\}^N$ and regard each $\sigma := (\sigma_1, \sigma_2, \dots) \in \overline{T}$ as providing the successive vertices $(\sigma_1), (\sigma_1, \sigma_2), (\sigma_1, \sigma_2, \sigma_3), \dots, (\sigma_1, \sigma_2, \dots, \sigma_n), \dots$ of a unique path through the b-ary tree T rooted at the vertex $\emptyset$. The parameter b is referred to as the *cascade branching number*. It is convenient to give $\overline{T}$ the product topology metrized by $\rho(\sigma, \eta) = b^{-\mathbf{a}(\sigma, \eta)}$, where $\mathbf{a}(\sigma, \eta) :=$ #vertices common to σ, η and represents the common ancestry of σ, η. Then $\overline{T}$ may be viewed as the completion (or boundary) of the countable graph T. Also, $B_{b^{-n}}(\sigma) = J(\sigma_1, \sigma_2, \dots, \sigma_n) = [\sum_{j=1}^{n} \sigma_j b^{-j}, \sum_{j=1}^{n} \sigma_j b^{-j} + b^{-n})$. In particular, this makes the computation of the Hausdorff dimension with respect to the tree metric the same as using the Euclidean metric on the unit interval; see Kahane (1985), pages 128–131, and Furstenberg (1970).

Let $W(\sigma_1, \dots, \sigma_n)$, for

$$n = 1, 2, \dots, (\sigma_1, \dots, \sigma_n) \in T := \bigcup_{n=1}^{\infty} \{0, 1, \dots, b-1\}^n,$$

be i.i.d. nonnegative mean one random variables and weight the vertices by $W_n(\sigma) \equiv W(\sigma_1, \dots, \sigma_n)$. Then the mass per unit volume (density) $\mu(B_{b^{-n}}(\sigma))/b^{-n}$ is obtained as the product $W_1(\sigma)W_2(\sigma) \cdots W_n(\sigma)$ along the path determined by the first n generations of σ. Define

$$\psi_n(\mathbf{x}) = W_1(\sigma)W_2(\sigma) \cdots W_n(\sigma), \qquad \mathbf{x} \in J(\sigma_1, \dots, \sigma_n), \quad \sigma = (\sigma_1, \dots, \sigma_n, \dots). \tag{2.1}$$

Then the sequence of *random measures* defined by the density (Radon-Nikodym derivative) $\mu_n(dx) = \psi_n(x)\,dx$, $n \geq 1$, is easily checked to a.s. have a weak* limit μ_∞, since for each bounded continuous function f on J, the sequence $\{\int_J f\,d\mu_n\}$ is an L_1-bounded martingale with respect to the sequence $\mathscr{F}_n := \sigma\{W(\sigma_1, \sigma_2, \dots, \sigma_n)\colon \sigma_i \in \{0, 1, 2, \dots, b-1\}\}$.

The problems of nondegeneracy of the limit measure, divergence of moments and the a.s. calculation of the dimension of the support of μ_∞ are

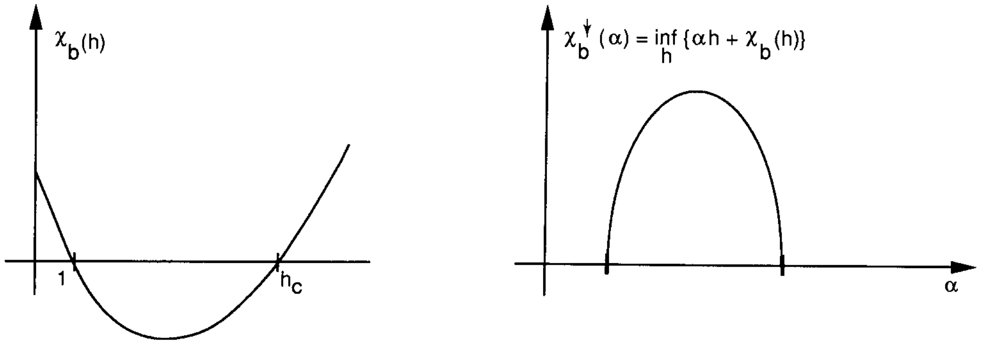

FIG. 2. *The MKP function and related Legendre transform.*

treated in Mandelbrot (1974) and Kahane and Peyrière (1976). In particular, the basic structure function, to be referred to as the *MKP-function*, analyzed to obtain solutions to these problems is a "modified cumulant generating function" defined by

$$\chi_b(h) = \log_b EW^h - (h - 1). \tag{2.2}$$

In particular, one has the following theorem; see Figure 2.

THEOREM 2.1 [Kahane and Peyrière (1976)]. *Let W denote a random variable which has the common distribution of the $W(\sigma)$'s and MKP-function $\chi_b(h)$.*

(i) (Nondegeneracy) *If* $-D = \chi_b'(1-) \equiv EW \log_b W - 1 < 0$, *then* $E_{\mu_\infty}([0,1]) > 0$, *and conversely.*

(ii) (Divergence of moments) *Let $h > 1$. Then $Z_\infty := \mu_\infty([0,1])$ has a finite moment of order h if and only if $h < h_c = \sup\{h \ge 1: \chi_b(h) < 0\}$. Moreover, $EZ_\infty^h < \infty$ for all $h > 0$, if and only if W is essentially bounded by b and $P(W = b) < 1/b$.*

(iii) (Size of support) *Assume that $E(Z_\infty \log Z_\infty) < \infty$. Then μ_∞ is a.s. supported by the random set*

$$\operatorname{supp}(\mu_\infty) = \left\{\sigma \in \overline{T}: \lim_{n\to\infty} \frac{\log_b \mu_\infty B_{b^{-n}}(\sigma)}{n} = EW \log_b W - 1\right\}$$

of Hausdorff dimension $D = -\chi_b'(1) = 1 - EW \log_b W$.

REMARK 2.1. (i) The nondegeneracy asserted in Theorem 2.1(i) is also equivalent to $EZ_\infty = 1$.

(ii) Related results are also given by Ben (1987) and Williams (1990) under a more general condition on the elements of the cascade of $J = [0,1]$. In particular, the i.i.d. mean one assumption is replaced to cover the case when

$$E\frac{1}{b}\sum_{\sigma=0}^{b-1} W(\sigma) = 1, \tag{2.3}$$

where the nonnegative random vectors

$$(W(0), W(1), \dots, W(b-1)), (W(0,0), W(0,1), \dots, W(0,b-1)),$$
$$(W(1,0), W(1,1), \dots, W(1,b-1)), (W(2,0), W(2,1), \dots, W(2,b-1)), \dots,$$
$$(W(\sigma_1, \dots, \sigma_n, 0), W(\sigma_1, \dots, \sigma_n, 1), W(\sigma_1, \dots, \sigma_n, 2), \dots,$$
$$W(\sigma_1, \dots, \sigma_n, b-1)), \dots$$

are i.i.d. and, while integrals of continuous functions need not form a martingale, the integrals of indicators of subgrid cells do form a martingale and this is enough to prove existence of a limit cascade measure. However, for the results here we assume that the cascaded variables are i.i.d. throughout.

In view of Theorem 2.1 the tails of the total mass distribution depend critically on the essential supremum of the cascaded variables. Similarly, probabilities of small values of the total mass depend on small values of the cascaded variables in a way which will also be made more precise below and in later sections. To avoid certain technical problems in analyzing the singularities along the lines indicated in the previous section, some of which appear to be serious, we shall primarily focus on cascaded variables of the type described in the following definition.

DEFINITION 2.1. A nonnegative cascaded variable W will be said to be *strongly bounded below* if $P(W \geq a) = 1$ for some positive number a. W will be said to be *strongly bounded above* if $P(W < b) = 1$, where b is the cascade branching number.

As do Kahane and Peyrière (1976), we rely on the following theorem of Billingsley (1965) for the computation of Hausdorff dimension. In preparation first note Billingsley's generalization of Hausdorff dimension given as follows. Let μ be a probability measure on $[0,1]$ and define the μ-dimension of a Borel set F by

$$(2.4)\quad \dim_\mu(F) = \inf\left\{\theta \geq 0: \lim_{\delta \to 0} \inf_{\bigcup_{i=1}^{\infty} B_i \supseteq F,\ \mu(B_i) \leq \delta} \sum_{i=1}^{\infty} \mu^\theta(B_i) = 0\right\}.$$

Observe that if $0 < \mu(F) \leq 1$, then $\dim_\mu(F) = 1$.

THEOREM 2.2 [Billingsley (1965)]. *Let μ and ν be probability measures on $[0,1]$. If*

$$F \subseteq \left\{\sigma \in \overline{T}: \lim_{n \to \infty} \frac{\log \mu B_{b^{-n}}(\sigma)}{\log \nu B_{b^{-n}}(\sigma)} = \gamma\right\},$$

then

$$\dim_\nu(F) = \gamma \dim_\mu(F).$$

The following formulas are basic to the study of random cascades.

PROPOSITION 2.3. *Let Δ_k^i, $i = 1, 2, \dots, b^k$, be an arbitrary enumeration of the* kth *generation b-adic subintervals $J(\sigma_1, \dots, \sigma_k)$ of $[0,1]$. Then*:

$$\text{(i)} \qquad \mu_n(\Delta_k^i) \overset{\text{dist}}{=} Z_{n-k}^{(k)}(i)\mu_k(\Delta_k^i), \qquad n \geq k,\ i = 1, \dots, b^k,$$

where $Z_{n-k}^{(k)}(i)$ is distributed as the total mass, $\mu_{n-k}([0,1])$, and is independent of $\mu_k(\Delta_k^i)$.

$$\text{(ii)} \qquad \mu_\infty(\Delta_k^i) \overset{\text{dist}}{=} Z_\infty^{(k)}(i)\mu_k(\Delta_k^i), \qquad n \geq k,\ i = 1, \dots, b^k,$$

where $Z_\infty^{(k)}(i)$ is distributed as the total mass, $\mu_\infty([0,1])$, and is independent of $\mu_k(\Delta_k^i)$.

PROOF. The proof of (ii) follows from (i) by taking a limit. To prove part (i), proceed as follows. First write

$$\Delta_k^i = J(\bar{\sigma}_1, \dots, \bar{\sigma}_k).$$

Then

$$\begin{aligned} \mu_n(\Delta_k^i) &= \sum_{\sigma_{k+1}, \dots, \sigma_n} \mu_n(J(\bar{\sigma}_1, \dots, \bar{\sigma}_k, \sigma_{k+1}, \dots, \sigma_n)) \\ &= W(\bar{\sigma}_1)W(\bar{\sigma}_1, \bar{\sigma}_2) \cdots W(\bar{\sigma}_1, \dots, \bar{\sigma}_k)b^{-k} \\ &\quad \times \sum_{\sigma_{k+1}, \dots, \sigma_n} W(\bar{\sigma}_1, \dots, \bar{\sigma}_k, \sigma_{k+1}) \\ &\quad \cdots W(\bar{\sigma}_1, \dots, \bar{\sigma}_k, \sigma_{k+1}, \dots, \sigma_n)b^{n-k} \\ &\overset{\text{dist}}{=} \mu_k(\Delta_k^i)Z_{n-k}(\bar{\sigma}_1, \dots, \bar{\sigma}_k). \end{aligned}$$

This defines $Z_{(n-k)}^{(k)}(i) := Z_{n-k}(\bar{\sigma}_1, \dots, \bar{\sigma}_k)$. □

An important special case of Proposition 2.3(ii) which will be used to study the distribution of total mass is the identity

$$Z_\infty \overset{\text{dist}}{=} b^{-1} \sum_{i=0}^{b-1} W_i Z_\infty^{(1)}(i), \tag{2.5}$$

where the $Z_\infty^{(1)}(i)$, $i = 0, 1, \dots, b-1$, are i.i.d. distributed as Z_∞.

The representation furnished by Proposition 2.3 may be viewed jointly for $i = 1, 2, \dots, b^k$. While marginally the quantities $\mu_k(\Delta_k^i)$, $i = 1, 2, \dots, b^k$, are identically distributed they are, of course, not independent. The correlations among the masses at various scales may be computed from the tree structure of the cascade. However, the $Z_{n-k}^{(k)}(i)$, $i = 1, 2, \dots, b^k$, and the $Z_\infty^{(k)}(i)$, $i = 1, 2, \dots, b^k$, respectively, are i.i.d. In fact, for each k one may define i.i.d. random measures $\zeta_\infty^{(k)}$ on $[0,1]$, distributed as $b^{-k}\mu_\infty$ such that

$$\mu_\infty(\Delta_k^{(i)}) \overset{\text{dist}}{=} \zeta_\infty^{(k)}(\Delta_k^{(i)})\mu_k(\Delta_k^{(i)}). \tag{2.6}$$

For the methods of this paper an important use of the above representation is in the analysis of the behavior of the tails of the total mass distribution as given by the following results.

PROPOSITION 2.4. *Assume that* $\chi_b'(1) < 0$. *If* $P(W = 0) = 0$, *then* $P(Z = 0) = P(\mu_\infty(\Delta_n^i) = 0) = 0$. *On the other hand, suppose that the distribution of the cascaded random variable* W *has an atom at* 0, *say* $P(W = 0) = r > 0$. *Let* $\gamma = P(Z_\infty = 0)$. *Then*

$$P\big(\mu_\infty(\Delta_n^i) = 0\big) = 1 - (1-\gamma)(1-r)^n,$$

where γ *is the smallest positive solution to*

$$\gamma = \big(r + (1-r)\gamma\big)^b.$$

PROOF. The condition on the MKP-function guarantees nondegeneracy of μ_∞ and $\gamma < 1$. Let us first calculate γ. In view of the basic recursion, one has

$$Z_\infty \overset{\text{dist}}{=} b^{-1} \sum_{i=0}^{b-1} W_i Z_\infty(i),$$

where $Z_\infty(i)$'s are i.i.d., independent of the W's, and distributed as Z_∞. Thus,

$$\gamma = \{P(WZ_\infty = 0)\}^b = \{r + \gamma - r\gamma\}^b.$$

In particular, notice that this makes γ a fixed point of the probability generating function of the binomial distribution with parameters r and b. It is the smallest positive solution since, by convexity of the function $\gamma \to \{r + \gamma - r\gamma\}^b$, the only solution $\gamma < 1$ is the smallest positive solution. Now, let $W_1, \dots, W_n$ denote the cascade variables along the path leading to Δ_n^i. Then by Proposition 2.3 one has

$$\begin{aligned}
&P\big(\mu_\infty(\Delta_n^i) = 0\big)\\
&\quad = P\left(\prod_{j=1}^{n} W_j Z_\infty(i) = 0\right)\\
&\quad = P\left(\bigcup_{j=1}^{n} \{W_j = 0\} \cup \{Z_\infty(i) = 0\}\right)\\
&\quad = P\left(\bigcup_{j=1}^{n} \{W_j = 0\}\right) + \gamma - \gamma P\left(\bigcup_{j=1}^{n} \{W_j = 0\}\right)\\
&\quad = (1-\gamma) P\left(\bigcup_{j=1}^{n} \{W_j = 0\}\right) + \gamma\\
&\quad = (1-\gamma)\sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le j_1 < \cdots < j_k \le n} P\big(W_{j_1} = 0, \dots, W_{j_k} = 0\big) + \gamma\\
&\quad = (1-\gamma)\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} r^k + \gamma\\
&\quad = (1-\gamma)\{1 - (1-r)^n\} + \gamma.
\end{aligned}$$

□

COROLLARY 2.5. (i) *If the cascaded variable W is strongly bounded above, then Z_∞ has moments of all positive orders.* (ii) *If the cascaded variable W is strongly bounded below, then there are positive constants c_0, c_1 such that*

$$\phi(r) = Ee^{-rZ_\infty} \le c_0 e^{-c_1 r^\gamma},$$

with

$$0 < \gamma = \frac{\log b}{\log b - \log a} < 1.$$

In particular, Z_∞ has negative moments of all orders.

PROOF. The assertion (i) is a special case of Theorem 2.1(ii). To prove (ii), one proceeds as follows. From (2.4), we have

$$\phi(br) = (Ee^{-rWZ})^b \le (Ee^{-arZ})^b = \phi^b(ar).$$

Let $\psi(r) = \phi(r^{1/\gamma})$, $(b/a)^\gamma = b$. Then

$$\psi(br) = \psi\left(\left(\frac{b}{a}\right)^\gamma r\right) = \phi\left(\frac{b}{a} r^{1/\gamma}\right) \le \phi^b(r^{1/\gamma}) = \psi^b(r).$$

Let $\chi(r) = \log \psi(r)$, $\chi_1 = \chi(1) = \log \phi(1) < 0$. Then $\chi(b) \le b\chi_1$, $\chi(b^{n+1}) \le b\chi(b^n) \le b^{n+1}\chi_1$. Thus,

$$\limsup_{n\to\infty} \frac{\chi(b^n)}{b^n} \le \chi_1.$$

Given $r > 0$, let $n(r)$ be a nonnegative integer for which $b^{n(r)} \le r < b^{n(r)+1}$. Then, since $\chi(r)$ is monotonically decreasing,

$$\frac{\chi(r)}{r} \le \frac{\chi(b^{n(r)})}{b^{n(r)+1}}$$

and therefore

$$\limsup_{r\to\infty} \frac{\chi(r)}{r} \le \frac{\chi_1}{b}.$$

Thus, there exist $0 < c_1$, $|c| < \infty$ such that for all $r \ge 0$,

$$\chi(r) \le c - c_1 r.$$

Take $c_0 = e^c$, then for all $r \ge 0$, $\psi(r) \le c_0 e^{-c_1 r}$ and therefore $\phi(r) \le c_0 e^{-c_1 r^\gamma}$. □

We will close this section with a few simple examples and counterexamples to the Legendre transform formalism and then with the statements of our main results.

EXAMPLE 2.1. The heuristic idea behind the Legendre transform formalism is that leading term asymptotics suggest

$$M_\delta(h) \sim \sum_\alpha \delta^{\alpha h}\delta^{-f(\alpha)} \sim \delta^{\min\{h\alpha - f(\alpha)\}}, \tag{2.7}$$

and therefore

$$\tau(h) = -\min\{h\alpha - f(\alpha)\} = \max\{-h\alpha - (-f(\alpha))\} = (-f)^*(-h), \tag{2.8}$$

$$f(\alpha) = \min\{\alpha h + \tau(h)\} = -\tau^*(-\alpha), \tag{2.9}$$

where the asterisk denotes Legendre transform. The following simple examples show that the duality is more delicate than the formalism shows. Let $\mu = \frac{1}{2}\lambda_1 \times \delta_0 + \frac{1}{2}\lambda_2$, where λ_1 is one-dimensional Lebesgue measure on $[0, 1]$, λ_2 is two-dimensional Lebesgue measure on $[0, 1] \times [0, 1]$ and δ_0 is the Dirac unit mass measure at 0. Then

$$h(\alpha) = \begin{cases} 1, & \text{if } \alpha = 1, \\ 2, & \text{if } \alpha = 2, \\ 0, & \text{otherwise}; \end{cases}$$

$$f(\alpha) = \begin{cases} 1, & \text{if } \alpha = 1, \\ 2, & \text{if } \alpha = 2, \\ -\infty, & \text{otherwise}. \end{cases}$$

Note that these functions are not convex. On the other hand,

$$\tau(h) = \begin{cases} -2(h-1), & \text{if } h \le 1, \\ -(h-1), & \text{if } h \ge 1, \end{cases}$$

$$\min_h\{\alpha h + \tau(h)\} = \begin{cases} \alpha, & \text{if } 1 \le \alpha \le 2, \\ -\infty, & \text{otherwise}. \end{cases}$$

Since Lebesgue measure may be viewed as a cascade measure, this simple example can be reformulated as a multinomial cascade. In particular, one can also easily construct continuous singular measures similarly to this by a base 5 deterministic multinomial cascade; see Brown, Michon and Peyrière (1990).

EXAMPLE 2.2. While the Hausdorff dimension of a Borel set will always exist, the same is not the case for the exponent functions $\tau(h)$, $f(\alpha)$. To obtain an example, simply let K_n be the union of 2^n subintervals of $[0, 1]$ of equal lengths l_n, where $l_{n+1} < \frac{1}{2}l_n$, $l_0 = 1$, but such that l_n has two subsequences converging to 0 at distinct rates. Assign measure 2^{-n} to each of the intervals of K_n and note that this extends to a measure concentrated on $K = \bigcap K_n$.

The main results of this paper can be stated as follows.

THEOREM 2.6 (Hausdorff dimension). *Assume that the cascaded random variable W is strongly bounded above and below and has mean* 1. *Let α be* *such that* $\chi_b^\dagger(\alpha) := \inf_h\{\alpha h + \chi_b(h)\} > 0$. *Let*

$$\rho(h) = \frac{EW^h \log_b W}{EW^h}.$$

Then

$$\rho(h) = 1 - \alpha$$

has a unique solution $h = \beta$. *If*

$$E\left(\frac{W}{\|W\|_\infty}\right)^\beta > \frac{1}{b},$$

then

$$h(\alpha) = \dim F(\alpha) = \chi_b^\dagger(\alpha).$$

THEOREM 2.7 (Rényi exponent). *Assume that W is strongly bounded above and below and that* $EW^{2h}/(EW^h)^2 < b$. *Then, with probability* 1,

$$\tau(h) := \lim_{n\to\infty} \frac{\log M_n(h)}{n \log b} = \chi_b(h).$$

THEOREM 2.8 (Spectrum of singularites). *Assume that W is strongly bounded above and below and that* $EW^{2h}/(EW^h)^2 < b$ *for all h. If the spectrum of singularities* $f(\alpha)$ *exists, then*

$$\bar{f}(\alpha) = \chi_b^\dagger(\alpha)$$

and

$$\chi_b(h) = (-f)^*(-h),$$

where $\bar{f}(\alpha)$ *denotes the closed convex hull of f.*

3. Computation of Hausdorff and box dimensions $h(\alpha), b(\alpha)$.

We assume throughout that μ_∞ is nontrivial, that is, $EW \log_b W < 1$ (cf. Theorem 2.1).

LEMMA 3.1. *If Z is a positive random variable having finite moments of all orders* $h > 1$, *then for any* $\varepsilon > 0$,

$$\sum_n b^n P(\log Z > n\varepsilon) < \infty.$$

PROOF. Choose $h > 1$ such that

$$e^{h\varepsilon} > b.$$

Then by Markov's inequality,

$$P(\log Z > n\varepsilon) = P(Z > e^{n\varepsilon}) \le \frac{EZ^h}{(e^{\varepsilon h})^n}. \qquad \square$$

LEMMA 3.2. *If Z is a positive random variable such that $\phi(r) := Ee^{-rZ} \le c_0 e^{-c_1 r^\gamma}$, $r > 0$, for some $0 < \gamma < 1$, $c_0, c_1 > 0$, then*

$$\sum_n b^n P(\log Z < -n\varepsilon) < \infty.$$

PROOF. Let $\eta = e^{-n\varepsilon}$. Then, for $r > 0$,

$$\phi(r) = Ee^{-rZ} \ge e^{-r\eta} P(Z < \eta),$$

so that

$$P(Z < \eta) \le e^{r\eta}\phi(r) \le c_0 e^{(r\eta - c_1 r^\gamma)}.$$

Taking the infimum over r, we obtain

$$P(Z < \eta) \le c_0 e^{\inf(r\eta - c_1 r^\gamma)}.$$

In particular, therefore, one gets by minimizing the indicated exponent

$$P(Z < \eta) \le c_0 \exp\{-c'\eta^{-\gamma/(1-\gamma)}\},$$

where

$$c' = c_1^{1/(1-\gamma)}\gamma^{\gamma/(1-\gamma)}(1 - \gamma^{1/(1-\gamma)}) > 0. \qquad \square$$

REMARK 3.1. The condition of Lemma 3.2 may be relaxed to $\phi(r) := Ee^{-rZ} \le c_0 e^{-c_1(\log r)^{1+\delta}}$, $r > 0$, for some $\delta > 0$, $c_0, c_1 > 0$, without changing the conclusion. However, the stronger assumption is already satisfied under the conditions on the cascaded variables which will be assumed here; cf. Proposition 2.5. In considering extensions of these results, one should note that if, for example, W is uniform on $[0, 2]$, then Z_∞ can easily be checked using (2.5) to have a gamma distribution with

$$\phi(r) = \left(\frac{2}{2 + r}\right)^2 \le c_0 e^{-c_1 \log r},$$

that is, $\delta = 0$.

For a positive mean one random variable W, define

$$\rho(\beta) = E\frac{W^\beta \log_b W}{EW^\beta} = \frac{d}{d\beta}\log_b EW^\beta \tag{3.1}$$

when indicated moments exist (also see Figure 3).

LEMMA 3.3. *If W is a positive mean one random variable which is strongly bounded above and below and such that $P(W = 1) < 1$, then $\rho(\beta)$ is increasing and continuous, $\rho(0) = E\log_b W < 0$, $\rho(1) = EW\log_b W > 0$, and*

(i) $$\lim_{\beta \to \infty} \rho(\beta) = \log_b \|W\|_\infty,$$

(ii) $$\lim_{\beta \to -\infty} \rho(\beta) = \log_b \|W\|_{-\infty},$$

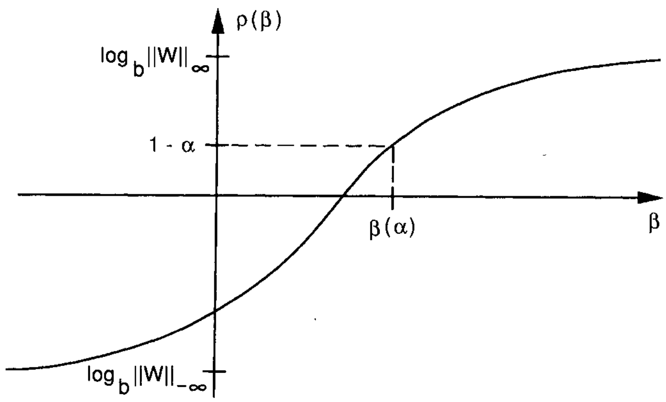

FIG. 3. *Graph of* $\rho(\beta)$.

where $\|W\|_\infty$ *and* $\|W\|_{-\infty}$ *denote the essential supremum and essential infimum of W.*

PROOF. The signs of $\rho(0)$, $\rho(1)$ follow from Jensen's inequality. The numerator and denominator of $\rho(\beta)$ are continuous and the denominator is positive. To see that $\rho(\beta)$ is increasing, simply observe that

$$\frac{d\rho}{d\beta} = \frac{EW^\beta EW^\beta (\log_b W)^2 - \left(EW^\beta \log_b W\right)^2}{\left(EW^\beta\right)^2}$$

$$= \frac{E\left[W^\beta\left(\log_b W - \dfrac{EW^\beta \log_b W}{EW^\beta}\right)^2\right]}{EW^\beta} > 0.$$

The indicated limits as $\beta \to \pm\infty$ follow by dividing the numerator and denominator by $(\|W\|_{\pm\infty} \mp \varepsilon)^\beta$, respectively. □

As before, the MKP-function is given by

$$\chi_b(h) = \log_b EW^h - (h - 1). \tag{3.2}$$

One may easily check that $\chi_b(h)$ is convex with $\chi_b(1) = 0$. Define

$$\chi_b^\dagger(\alpha) := \inf\{\alpha h + \chi_b(h)\} = -\chi_b^*(-\alpha), \tag{3.3}$$

where * denotes the Legendre transform.

LEMMA 3.4. *Let W be a strongly bounded mean one cascaded variable. Then for any* α *satisfying* $\chi_b^\dagger(\alpha) > 0$, *the equation*

$$\rho(\beta) = 1 - \alpha$$

has a unique solution $\beta \equiv \beta(\alpha)$. *Moreover, for* $\beta = \beta(\alpha)$,

$$\chi_b^\dagger(\alpha) = \alpha\beta + \chi_b(\beta)$$

and

$$E\left(\frac{W^\beta}{EW^\beta}\log_b\frac{W^\beta}{EW^\beta}\right) < 1.$$

Also, there is a $\delta > 0$ *such that for* $h = 1 + \delta$,

$$\frac{EW^{\beta h}}{(EW^\beta)^h} < b^{h-1}.$$

PROOF. First observe that

$$\chi_b^\dagger(\alpha) > 0$$

implies

$$\log_b\frac{b}{\|W\|_\infty} = 1 - \log_b\|W\|_\infty \le \alpha \le 1 - \log_b\|W\|_{-\infty} = \log_b\frac{b}{\|W\|_{-\infty}},$$

since, for all $h > 0$,

$$0 < \alpha h + \chi_b(h) = \alpha h + \log_b EW^h - h + 1 = \{\alpha - 1 + \log_b\|W\|_h\}h + 1,$$

and therefore, for all $h > 0$,

$$\alpha - 1 + \log_b\|W\|_h > -\frac{1}{h}.$$

Let $h \to \infty$ to get the indicated lower bound on α, and then consider the similar inequalities for $h < 0$ to get the upper bound.

So unique solvability follows from considerations in Lemma 3.3. Also,

$$\frac{d}{dh}(\alpha h + \chi_b(h)) = \alpha - 1 + \frac{EW^h\log_b W}{EW^h} = 0$$

at

$$\rho(\beta) = 1 - \alpha.$$

Therefore,

$$\chi_b^\dagger(\alpha) = \alpha\beta + \chi_b(\beta).$$

So

$$\begin{aligned}
E\left(\frac{W^\beta}{EW^\beta}\log_b\frac{W^\beta}{EW^\beta}\right) &= \beta\rho(\beta) - \log_b EW^\beta \\
&= \beta(1-\alpha) - \log_b EW^\beta \\
&= 1 - \left[\beta\alpha + \log_b EW^\beta - (\beta - 1)\right] \\
&= 1 - \inf(\alpha h + \chi_b(h)) \\
&= 1 - \chi_b^\dagger(\alpha) < 1.
\end{aligned}$$

To complete the proof of the lemma, observe that

$$\frac{EW^{\beta h}}{(EW^{\beta})^h} < b^{h-1}$$

if and only if

$$\log EW^{\beta h} < h \log(bEW^{\beta}) - \log b.$$

The values of the left and right sides of the equality agree at $h = 1$. The right-hand side is a line of slope $\log(bEW^{\beta})$ and the left-hand side has the smaller slope $EW^{\beta} \log W^{\beta}/EW^{\beta}$ since

$$E\left(\frac{W^{\beta}}{EW^{\beta}} \log_b \frac{W^{\beta}}{EW^{\beta}}\right) < 1. \qquad \square$$

Observe from Lemma 3.4 that for such cascaded variables W,

$$\chi_b^{\dagger}(\alpha) > 0 \tag{3.4}$$

if and only if, for $\rho(\beta) = 1 - \alpha$,

$$E\left(\frac{W}{b^{1-\alpha}}\right)^{\beta} > \frac{1}{b}. \tag{3.5}$$

Let W be a strongly bounded cascaded variable with mean 1. For each α such that $\chi_b^{\dagger}(\alpha) > 0$ construct a *dual cascade* with cascaded variables distributed as

$$W_{\beta} := \frac{W^{\beta}}{EW^{\beta}},$$

where

$$\rho(\beta) = 1 - \alpha,$$

by replacing the values of $W(\sigma_1, \dots, \sigma_k)$ with $W_{\beta}(\sigma_1, \dots, \sigma_k)$, sample point by sample point, for $(\sigma_1, \dots, \sigma_k) \in T$. Let $\mu_{\infty, \beta}$ denote the resulting cascade measure and let $Z_{\infty, \beta}$ denote the total mass. In view of Lemma 3.4 and Theorem 2.1(i), the measure is nontrivial if $\chi_b^{\dagger}(\alpha) > 0$. Also by Lemma 3.4 and Theorem 2.1(ii) one has

$$EZ_{\infty, \beta}^{1+\delta} < \infty \tag{3.6}$$

and, in particular,

$$EZ_{\infty, \beta} \log Z_{\infty, \beta} < \infty. \tag{3.7}$$

LEMMA 3.5. *Let W be a strongly bounded mean one cascaded variable. Let $\chi_b^{\dagger}(\alpha) > 0$ and $\rho(\beta) = 1 - \alpha$. If*

$$E\left(\frac{W}{\|W\|_{\infty}}\right)^{\beta} > \frac{1}{b},$$

then for all $h \geq 1$,

$$EZ^h_{\infty,\beta} < \infty.$$

PROOF. In view of Theorem 2.1(ii) it is enough to show that for all $h > 1$,

$$\frac{EW^{\beta h}}{(EW^{\beta})^h} < b^{h-1}.$$

As in the proof of Lemma 3.4, for this it is enough to show that for all $h > 1$,

$$\log_b EW^{\beta h} < h \log_b(bEW^{\beta}) - \log_b b.$$

Again the left and right sides agree at $h = 1$, and the right side is a line of slope $\log_b(bEW^{\beta})$. Therefore, it suffices to show that for all $h > 1$ the slope of the left side is less than $\log_b(bEW^{\beta})$. That is, it suffices to show for all $h > 1$,

$$\frac{EW^{\beta h} \log_b W^{\beta}}{EW^{\beta h}} < \log_b(bEW^{\beta}).$$

Now, the left side is the derivative of a convex function and therefore increasing in h. Therefore,

$$\begin{aligned}\sup_{h>1} \frac{EW^{\beta h} \log_b W^{\beta}}{EW^{\beta h}} &= \lim_{h\to\infty} \frac{EW^{\beta h} \log_b W^{\beta}}{EW^{\beta h}}\\ &= \log_b \|W\|_{\infty}^{\beta}\\ &< \log_b(bEW^{\beta}).\end{aligned}$$

□

REMARK 3.2. To see that the condition (3.4), or equivalently (3.5), is in this much generality a sharp condition for Lemma 3.5 to hold, consider the following example. Take, for $0 < q < p < 1$, $p + q = 1$, $b = 2$,

$$W = \begin{cases} 2p, & \text{with probability } \frac{1}{2}, \\ 2q, & \text{with probability } \frac{1}{2}. \end{cases} \tag{3.8}$$

Then

$$E\left(\frac{W}{\|W\|_{\infty}}\right)^{\beta} = E\left(\frac{W}{b^{1-\alpha}}\right)^{\beta}.$$

We are now ready to compute the $h(\alpha)$.

PROOF OF THEOREM 2.6. Let $\chi_b^{\dagger}(\alpha) > 0$ and $\rho(\beta) = 1 - \alpha$. The dual cascade $\mu_{\infty,\beta}$ is nontrivial by Lemma 3.4. In fact, by Proposition 2.4, $\mu_{\infty,\beta}([0,1])$ is positive with probability 1. Also, W_{β} is strongly bounded below since W is strongly bounded above and below, and $Z_{\infty,\beta}$ has finite moments of all orders $h > 1$ by Lemma 3.5. By Proposition 2.5(ii), Lemma 3.1, Lemma 3.2 and the

Borel–Cantelli lemma, it follows that with probability 1,

$$\sup_{1\le i\le b^n} \frac{\log Z_\infty^{(n)}(i)}{n} \to 0 \quad \text{as } n\to\infty$$

and

$$\sup_{1\le i\le b^n} \frac{\log Z_{\infty,\beta}^{(n)}(i)}{n} \to 0 \quad \text{as } n\to\infty,$$

where $Z_\infty(i)$, $1\le i\le b^n$, are i.i.d. distributed as Z_∞ and $Z_{\infty,\beta}(i)$, $1\le i\le b^n$, are i.i.d. distributed as $Z_{\infty,\beta}$. Remove the indicated set $D = D_1 \cup D_\beta$ of probability 0 from Ω, where D_1, D_β denote events where the above limits fail, respectively. Let

$$F_\beta(\alpha) := \left\{\sigma\colon \lim_{n\to\infty} \frac{\log_b \mu_{\infty,\beta} B_{b^{-n}}(\sigma)}{-n} = \chi_b^\dagger(\alpha)\right\}.$$

Then on $\Omega - D$ we have, by the definition of $F(\alpha)$ in Section 2 and Proposition 2.3, $\sigma\in F(\alpha)$ if and only if

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \frac{\log_b \mu_\infty B_{b^{-n}}(\sigma)}{-n}\\
&= \lim_{n\to\infty} \frac{\sum_{j=1}^n \log_b W(\sigma_1,\dots,\sigma_j) - n + \log_b Z_\infty(\sigma_1,\dots,\sigma_n)}{-n}\\
&= 1 - \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log_b W(\sigma_1,\dots,\sigma_j).
\end{aligned}$$

Similarly, on $\Omega - D$, $\sigma\in F_\beta(\alpha)$ if and only if

$$\begin{aligned}
\chi_b^\dagger(\alpha) &= \lim_{n\to\infty} \frac{\log_b \mu_{\infty,\beta} B_{b^{-n}}(\sigma)}{-n}\\
&= \lim_{n\to\infty} \frac{\sum_{j=1}^n \log_b W_\beta(\sigma_1,\dots,\sigma_j) - n + \log_b Z_{\infty,\beta}(\sigma_1,\dots,\sigma_n)}{-n}\\
&= 1 - \lim_{n\to\infty}\frac{\beta}{n}\sum_{j=1}^n \log_b W(\sigma_1,\dots,\sigma_j) + \log_b EW^\beta\\
&= \beta\left(1 - \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log_b W(\sigma_1,\dots,\sigma_j)\right) + \log_b EW^\beta - (\beta-1)\\
&= \beta\left(1 - \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log_b W(\sigma_1,\dots,\sigma_j)\right) + \chi_b(\beta).
\end{aligned}$$

Using Lemma 3.4 and Theorem 2.1, one has $\mu_{\infty,\beta}F_\beta(\alpha) = \mu_{\infty,\beta}([0,1]) > 0$ with probability 1. It therefore follows from Billingsley's Theorem 2.2 that with probability 1,

$$h(\alpha) = \dim F(\alpha) = \dim F_\beta(\alpha) = \chi_b^\dagger(\alpha) \dim_{\mu_\infty, \beta} F_\beta(\alpha) = \chi_b^\dagger(\alpha). \qquad \square$$

REMARK 3.3. It is possible to relax the technical condition required by Lemma 3.5 to purely a condition of strong boundedness if one applies our estimates on the left and right tails of the total mass of Z_∞ to recent Hausdorff dimension results of Lyons and Pemantle (1992).

The computation of the box dimension $b(\alpha)$ is trivial and somewhat uninteresting since it is saturated by the Hausdorff dimension of the tree (for the tree metric). However, we include it for completeness below.

PROPOSITION 3.1. *If $F(\alpha) \neq \emptyset$, then $b(\alpha) = 1$.*

PROOF. Observe that if $\sigma = (\sigma_1, \sigma_2, \dots) \in F(\alpha)$, then for any n and arbitrary $\bar{\sigma}_1, \dots, \bar{\sigma}_n \in \{0, 1, \dots, b-1\}$, the point $(\bar{\sigma}_1, \dots, \bar{\sigma}_n, \sigma_{n+1}, \sigma_{n+2}, \dots)$ must also belong to $F(\alpha)$. Therefore, it takes at least b^n balls of radius at most b^{-n} (in the metric of the tree defined in Section 2) to cover $F(\alpha)$. For arbitrary radius $\delta > 0$ choose n such that $b^{-n-1} < \delta \le b^{-n}$ and apply the same reasoning. $\square$

4. Computation of Rényi exponents $\tau(h)$. We assume throughout that μ_∞ is nondegenerate, that is, for the cascaded random variable $EW \log_b W < 1$ (cf. Theorem 2.1). In this section we shall focus on the Rényi exponents $\tau(h)$ defined in (1.8). Throughout this section Δ_k^i, $i = 1, 2, \dots, b^k$, will continue to denote the b-adic intervals at the kth generation of the cascade. Also, whenever convenient W and Z will be used to denote generic random variables with the distribution of the cascaded variables and the total mass, respectively. Define, taking $\delta = b^{-n}$,

$$N_n(\alpha) := \#\{i : \mu_\infty(\Delta_n^i) > b^{-n\alpha}\} \tag{4.1}$$

and

$$M_n(h) := \sum_{i=1}^{b^n} \mu_\infty^h(\Delta_n^i). \tag{4.2}$$

As a warm-up, to get some insight into the relationship between exponents, assume that the cascaded variables are strongly bounded above and below and note that for $\delta = b^{-n}$, and using Proposition 2.3,

$$\begin{aligned} EM_n(h) &= E \sum_{i=1}^{b^n} \mu_\infty^h(\Delta_n^i) \\ &= b^n b^{-nh} (EW^h)^n EZ_\infty^h \\ &= (b^{1-h} EW^h)^n EZ_\infty^h. \end{aligned} \tag{4.3}$$

In particular,

$$E\frac{\sum_{i=1}^{b^n}\mu_\infty^h(\Delta_i^n)}{(b^{1-h}EW^h)^n} = EZ_\infty^h, \qquad n = 1,2,\dots, \tag{4.4}$$

$$\log EM_n(h) = n\log b\chi_b(h) + \log EZ_\infty^h. \tag{4.5}$$

Now, consider that by the Cramér–Chernoff large-deviation theory [cf. Deuschel and Stroock (1989)] one also has, writing $\Delta_k^i = J(\bar{\sigma}_1,\dots,\bar{\sigma}_k)$ and noting the independence with the prefactors $Z_\infty(i)$,

$$\begin{aligned}
&\frac{1}{n}\log P\big(b^{-n(\alpha+\varepsilon)} < \mu_\infty(\Delta_n^i) \le b^{-n(\alpha-\varepsilon)}\big)\\
&\quad = \frac{1}{n}\log P\Bigg((1-\alpha-\varepsilon)\log b < \frac{1}{n}\sum_{j=1}^{n}\log W(\bar{\sigma}_1,\dots,\bar{\sigma}_j)\\
&\qquad\qquad + \frac{1}{n}\log Z_\infty(i) \le (1-\alpha+\varepsilon)\log b\Bigg)\\
&\quad \sim -\inf_{(1-\alpha-\varepsilon<h/\log b<1-\alpha+\varepsilon)}\chi^*(h),
\end{aligned} \tag{4.6}$$

if $E\log W \notin (\log b - \alpha\log b - \varepsilon\log b, \log b - \alpha\log b + \varepsilon\log b)$. Therefore,

$$\begin{aligned}
&E\{N_n(\alpha+\varepsilon) - N_n(\alpha-\varepsilon)\}\\
&\quad = b^nP\big(b^{-n(\alpha+\varepsilon)} < \mu_\infty(\Delta_n^i) \le b^{-n(\alpha-\varepsilon)}\big)\\
&\quad = \exp\big\{n\log b + \log P\big(b^{-n(\alpha+\varepsilon)} < \mu_\infty(\Delta_n^i) \le b^{-n(\alpha-\varepsilon)}\big)\big\}\\
&\quad \sim \exp\Big\{n\log b - n\inf_{(1-\alpha-\varepsilon<h/\log b<1-\alpha+\varepsilon)}\chi^*(h)\Big\}\\
&\quad = \exp\Bigg\{n\log b\Bigg(1 - \inf_{(1-\alpha-\varepsilon<h/\log b<1-\alpha+\varepsilon)}\frac{\chi^*(h)}{\log b}\Bigg)\Bigg\}.
\end{aligned} \tag{4.7}$$

By convexity, for suitable α values one has for large n and small ε,

$$\begin{aligned}
\frac{\log E\{N_n(\alpha+\varepsilon) - N_n(\alpha-\varepsilon)\}}{n\log b} &\sim 1 - \frac{\chi^*(\log b - \alpha\log b)}{\log b}\\
&= -\chi_b^*(-\alpha).
\end{aligned} \tag{4.8}$$

Thus, the expected value of the Rényi exponents and spectrum of singularities occur in Legendre transform pairs and take values which, under certain additional conditions, we will see in the next section to be almost sure as well!

Proof of Theorem 2.7. First observe that from the proof of Proposition 1.2 one sees that

$$\left\{\frac{\sum_{i=1}^{b^n}\mu_n^q(\Delta_i^n)}{(b^{1-q}EX^q)^n} : n = 1,2,\dots\right\}$$

is an L^1-bounded martingale with respect to $\{\mathscr{F}_n\}$. Therefore, by the submartingale convergence theorem one has that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{b^n} \mu_n^h(\Delta_n^i)}{(EW^h b^{1-h})^n}$$

exists a.s. and defines a random variable Y, say.

Write

$$\frac{\sum_{i=1}^{b^n} \mu_\infty^h(\Delta_n^i)}{(EW^h b^{1-h})^n} = \frac{\sum_{i=1}^{b^n} \mu_n^h(\Delta_n^i) Z_\infty^h(i) - \sum_{i=1}^{b^n} \mu_n^h(\Delta_n^i) EZ_\infty^h}{(EW^h b^{1-h})^n} + \frac{\sum_{i=1}^{b^n} \mu_n^h(\Delta_n^i)}{(EW^h b^{1-h})^n} EZ_\infty^h$$
$$:= A_n + B_n.$$

We now see that $A_n \to 0$ and $B_n \to Y' := YEZ_\infty^h$ with probability 1 as $n \to \infty$. While the latter assertion is obvious by the first observation above, the former follows by observing

$$\begin{aligned}\operatorname{Var}(A_n) &= \frac{E\{\sum_{i=1}^{b^n} \mu_n^h(\Delta_n^i)(Z_\infty^h(i) - EZ_\infty^h)\}^2}{(EW^h b^{1-h})^{2n}} \\ &= \frac{\sum_{i=1}^{b^n} E\mu_n^{2h}(\Delta_n^i) \operatorname{Var}(Z_\infty^h)}{(EW^h b^{1-h})^{2n}} \\ &= \left(\frac{1}{b} \frac{EW^{2h}}{(EW^h)^2}\right)^n \operatorname{Var} Z_\infty^h.\end{aligned}$$

In particular, $EA_n = 0$ and $\sum_n \operatorname{Var}(A_n) < \infty$ under the conditions stated at the outset. Thus, $A_n \to 0$ a.s. by Chebyshev's inequality and the Borel–Cantelli lemma. Now

$$\begin{aligned}\lim_{n\to\infty} \frac{\log \sum_{i=1}^{b^n} \mu_\infty^h(\Delta_n^i)}{n \log b} &= \lim_{n\to\infty} \frac{\log \dfrac{\sum_{i=1}^{b^n} \mu_\infty^h(\Delta_n^i)}{(EW^h b^{1-h})^n} + \log(EW^h b^{1-h})^n}{n \log b} \\ &= \frac{\log(EW^h b^{1-h})}{\log b} = \chi_b(h). \qquad \square\end{aligned}$$

REMARK 4.1. Observe that for the case of the random binomial example the required moment ratio bound is satisfied for all h. That is, for $p + q = 1$, $p, q > 0$,

$$W = \begin{cases} 2p, & \text{w.p. } \frac{1}{2}, \\ 2q, & \text{w.p. } \frac{1}{2}, \end{cases} \tag{4.9}$$

one has

$$\frac{EW^{2h}}{(EW^h)^2} = 2\frac{p^{2h} + q^{2h}}{p^{2h} + q^{2h} + 2p^h q^h}. \tag{4.10}$$

In particular, using Theorems 2.6 and 2.7 and the results of Eggleston's theorem given in Section 1 for the deterministic cascade, one obtains a proof that the dimensions and exponents for the random binomial cascade respectively coincide with those of the deterministic binomial cascade. The same is also true for the multinomial cascades. This has important practical implications for the analysis of turbulence data in Meneveau and Sreenivasan (1987).

REMARK 4.2. Note that if $W \le t$ a.s. and if $P(W = t) = p > 0$, then for $h > 0$,

$$\frac{EW^{2h}}{(EW^h)^2} \le \frac{t^{2h}}{p^2 t^{2h}} = \frac{1}{p^2} < b$$

for

$$p > \frac{1}{\sqrt{b}}. \tag{4.11}$$

REMARK 4.3. For an example of a strongly bounded cascaded distribution where the moment ratio bound is not satisfied take W to be uniform on $[\frac{1}{2}, \frac{3}{2}]$ and h sufficiently large.

REMARK 4.4. The uniqueness problem for random cascades refers to the problem of when the cascaded distribution is uniquely determined by the exponents for the singularity sets. Observe that by Theorem 2.7 it follows that within the class of strongly bounded random cascades satisfying the moment ratio bound for all h, one does get that the cascaded distribution is uniquely determined by the Rényi exponents.

5. Computation of singularity spectrum $f(\alpha)$. This section relates the computations of the previous sections to the spectrum of singularities $f(\alpha)$ of a random cascade μ_∞ whenever $f(\alpha)$ exists (as an extended real-valued function). Notice that since the limit (1.7) defining the spectrum of singularities $f(\alpha)$ for a random cascade $\mu = \mu_\infty$ is measurable with respect to the tail sigma-field of countably many i.i.d. random variables $W(\sigma)$, $\sigma \in T$, the function $f(\alpha)$ is deterministic.

We shall provide two general theorems from which the main theorem will follow. As noted in Example 2.1, the spectrum of singularities need not be concave. However, using the Cauchy–Schwarz inequality in (1.9), one can easily see that it is always the case that the Rényi exponent $\tau(h)$ is a convex function.

THEOREM 5.1. *Let $f\colon \mathscr{R} \to \overline{\mathscr{R}}$ be an arbitrary extended real-valued function and let*

$$\mathscr{E}_0(f) = \{(x,y)\colon y \geq f(x)\}.$$

Now let $\bar{f}$ denote the convex hull of f defined by

$$\bar{f}(x) = \inf\{y\colon (x,y) \in \mathscr{E}(f)\},$$

where $\mathscr{E}(f)$ is the smallest closed convex set containing $\mathscr{E}_0(f)$. Let f^*, $\bar{f}^*$ denote the Legendre transforms of f, $\bar{f}$, respectively, defined by

$$f^*(\lambda) = \sup_x\{\lambda x - f(x)\}, \qquad \bar{f}^*(\lambda) = \sup_x\{\lambda x - \bar{f}(x)\}.$$

Then

$$f^*(\lambda) \equiv \bar{f}^*(\lambda).$$

PROOF. Clearly, $\bar{f}(x) \leq f(x)$. Thus, $f^*(\lambda) \leq \bar{f}^*(\lambda)$ for all λ. Therefore, if $f^*(\lambda) = +\infty$, then $\bar{f}^*(\lambda) = f^*(\lambda)$. Thus, it suffices to show that if $f^*(\lambda) < \infty$, then $\bar{f}^*(\lambda) \leq f^*(\lambda)$. Let λ be such that $f^*(\lambda) < \infty$. Then for all x,

$$f^*(\lambda) \geq \lambda x - f(x),$$

that is, $f(x) \geq \lambda x - f^*(\lambda)$ for all x.

Therefore, $\mathscr{E}_0(f)$ lies above the line $y = \lambda x - f^*(\lambda)$, so that $\mathscr{E}(f)$ also lies above the line $y = \lambda x - f^*(\lambda)$. Thus, $\bar{f}(x) \geq \lambda x - f^*(\lambda)$ for all x. It follows that

$$\bar{f}^*(\lambda) = \sup_x\{\lambda x - \bar{f}(x)\} \leq \sup_x\{\lambda x - (\lambda x - f^*(\lambda))\} = f^*(\lambda). \qquad \square$$

COROLLARY 5.1. *Let $f\colon \mathscr{R} \to \overline{\mathscr{R}}$ be an arbitrary extended real-valued function and let $\bar{f}$ be the convex hull of f. Then $\bar{f} = f^{**}$.*

PROOF. Since $\bar{f}$ is convex and lower semicontinuous, one has $\bar{f}^{**} = \bar{f}$. $\square$

THEOREM 5.2. *Let μ be a nonnegative a.s. finite random measure on $[0,1]$ such that*

$$\inf_\delta \frac{\mu B_\sigma(x)}{-\log\delta} = -\nu < 0 \quad a.s.$$

and for which $f(\alpha)$, a.s. exists as a deterministic extended real-valued function which is not identically $-\infty$. Then $\tau(h)$ exists and is given by

$$\tau(h) = (-f)^*(-h), \qquad \bar{f}(\alpha) = -\tau^*(-\alpha) = \tau^\dagger(\alpha).$$

PROOF. The proof is sample point by sample point after the indicated sets of probability 0 are removed; however, we suppress the dependence on sample points. Recall the definition of $M_\delta(h)$ given by (1.9) for $0 < \delta < 1$. Since the largest number of disjoint subintervals of $[0,1]$ of length δ is $O(\delta^{-1})$, one has $f(\alpha) \leq 1$. If $f(\alpha)$ is finite, then given an arbitrary number $\gamma > 0$ there are positive numbers ε_0, δ_0 such that for $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$,

$$\delta^{\gamma - f(\alpha)} < dN < \delta^{-\gamma - f(\alpha)},$$

where by an obvious abuse of notation we write

$$dN := N_\delta(\alpha + \varepsilon) - N_\delta(\alpha - \varepsilon).$$

So, for such an α, $0 < \varepsilon < \varepsilon_0$, $0 < \lambda\delta_0$, one has for $h \geq 0$,

$$\begin{aligned}\frac{\log M_\delta(h)}{-\log \delta} &\geq \frac{\log \Sigma'_{\{k:\, \delta^{\alpha+\varepsilon} < \mu(\Delta_k) \leq \delta^{\alpha-\varepsilon}\}} \mu^h(\Delta_k)}{-\log \delta} \\ &\geq \frac{\log \delta^{\alpha h + \varepsilon\gamma + \gamma - f(\alpha)}}{-\log \delta} = f(\alpha) - \alpha h - \varepsilon\gamma - \gamma,\end{aligned}$$

and similarly for $h < 0$,

$$\begin{aligned}\frac{\log M_\delta(h)}{-\log \delta} &\geq \frac{\log \Sigma'_{\{k:\, \delta^{\alpha+\varepsilon} < \mu(\Delta_k) \leq \delta^{\alpha-\varepsilon}\}} \mu^h(\Delta_k)}{-\log \delta} \\ &\geq \frac{\log \delta^{\alpha h - \varepsilon\gamma + \gamma - f(\alpha)}}{-\log \delta} = f(\alpha) - \alpha h + \varepsilon\gamma - \gamma.\end{aligned}$$

If $f(\alpha) = -\infty$, then $\log M_\delta(h)/-\log \delta \geq f(\alpha) - \alpha h$. In any case it follows for all α that

$$\liminf_{\delta \to 0} \frac{\log M_\delta(h)}{-\log \delta} \geq f(\alpha) - \alpha h,$$

and therefore

$$\liminf_{\delta \to 0} \frac{\log M_\delta(h)}{-\log \delta} \geq \sup_\alpha \{ f(\alpha) - \alpha h \}.$$

For the reverse inequality again let $\gamma > 0$. Let $\mu[0,1] = Z < \infty$. Notice that since $f(\alpha)$ must have at least one finite value (by hypothesis of the theorem), $\sup_\alpha\{f(\alpha) - \alpha h\} > -\infty$. First consider $h > 0$. Choose A_+ so large that $1 - hA_+ \leq \sup_\alpha\{f(\alpha) - \alpha h\}$, and choose A_- such that $\delta^{A_-} \geq Z$. Then

$$\begin{aligned}\frac{\log M_\delta(h)}{-\log \delta} &\leq \frac{\log\left\{\Sigma'_{\mu^h\{k:\, \mu(\Delta_k) < \delta^{A_+}\}} \mu^h(\Delta_k) + \Sigma'_{\{k:\, \mu(\Delta_k) \geq \delta^{A_+}\}} \mu^h(\Delta_k)\right\}}{-\log \delta} \\ &\leq \frac{\log\left\{O(\delta^{-1})\delta^{hA_+}\right\} + \Sigma'_{\{k:\, \mu(\Delta_k) \geq \delta^{A_+}\}} \mu^h(\Delta_k)\Big\}}{-\log \delta} \\ &\leq \frac{\log\left\{O(1)\delta^{-\sup_\alpha\{f(\alpha) - \alpha h\}} + \Sigma'_{\{k:\, \mu(\Delta_k) \geq \delta^{A_+}\}} \mu^h(\Delta_k)\right\}}{-\log \delta}.\end{aligned}$$

To bound the second sum, partition the interval $[A_-, A_+]$ into r subintervals $A_- = \alpha_0 < \alpha_1 < \cdots < \alpha_r = A_+$ of lengths ε not exceeding ε_0. This induces a partition of $[\delta^{A_+}, \delta^{A_-}]$ which will cover $\{k: \delta^{A_+} \le \mu(\Delta_k) \le k \le \delta^{A_-}\}$. Thus,

$$\frac{\log M_\delta(h)}{-\log\delta} \le \frac{\log\left\{O(1)\delta^{-\sup_\alpha\{f(\alpha)-\alpha h\}} + \sum_{j=1}^{r}\delta^{-\gamma-f(\alpha_j)}\delta^{h\alpha_j - h(\varepsilon/2)}\right\}}{-\log\delta}$$

$$\le \frac{\log\left\{\delta^{-\sup_\alpha\{f(\alpha)-\alpha h\}}\left(O(1) + r\delta^{-\gamma-h(\varepsilon/2)}\right)\right\}}{-\log\delta}.$$

It now follows for $h > 0$ that

$$\limsup_{\delta\to 0}\frac{\log M_\delta(h)}{-\log\delta} \le \sup_\alpha\{f(\alpha) - \alpha h\}.$$

For the cases $h \le 0$ one proceeds similarly by choosing $B > \nu$, A_+ sufficiently small that $1 - hA_+ < \sup_\alpha\{f(\alpha) - \alpha h\}$, and partitioning the interval $[A_+, B]$ into s subintervals of lengths ε not exceeding ε_0 so that with $h < 0$,

$$\frac{\log M_\delta(h)}{-\log\delta} \le \frac{\log\left\{\sum'_{\{k:\,\mu(\Delta_k)<\delta^{A_+}\}}\mu^h(\Delta_k) + \sum'_{\{k:\,\mu(\Delta_k)\ge\delta^{A_+}\}}\mu^h(\Delta_k)\right\}}{-\log\delta}$$

$$\le \frac{\log\left\{\sum'_{\{k:\,\mu(\Delta_k)<\delta^{A_+}\}}\mu^h(\Delta_k) + O(\delta^{-1})\delta^{hA_+}\right\}}{-\log\delta}$$

$$\le \frac{\log\left\{\sum'_{\{k:\,\mu(\Delta_k)<\delta^{A_+}\}}\mu^h(\Delta_k) + O(1)\delta^{-\sup_\alpha\{f(\alpha)-\alpha h\}}\right\}}{-\log\delta}$$

$$\le \frac{\log\left\{\sum_{j=1}^{s}\delta^{-\gamma-f(\alpha_j)}\delta^{h\alpha_j-h(\varepsilon/2)} + O(1)\delta^{-\sup_\alpha\{f(\alpha)-\alpha h\}}\right\}}{-\log\delta}$$

$$\le \frac{\log\left\{\delta^{-\sup_\alpha\{f(\alpha)-\alpha h\}}\left(O(1) + s\delta^{-\gamma-h(\varepsilon/2)}\right)\right\}}{-\log\delta}. \qquad \square$$

To see precisely how the Legendre transform formalism works between the MKP-function χ_b and the spectrum of singularities $f(\alpha)$ under our specialized conditions, let μ_∞ be the random cascade measure for strongly bounded cascaded variables such that $EW^{2h}/(EW^2)^h < b$ for all h. Then, in view of Theorem 2.7, Corollary 5.1 and Theorem 5.2, if the spectrum of singularities $f(\alpha)$ exists for the random cascade, then

$$\bar{f}(\alpha) = \chi_b^\dagger(\alpha) \tag{5.1}$$

and

$$\chi_b(h) = (-f)^*(-h). \tag{5.2}$$

In particular, under at least some conditions, by taking the convex hull of the spectrum of singularites, one obtains a basic structure function of the cascaded variables.

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