SIAM J. MATH. ANAL.
Vol. 28, No. 4, pp. 944–970, July 1997

010

MULTIFRACTAL FORMALISM FOR FUNCTIONS PART I: RESULTS VALID FOR ALL FUNCTIONS*

S. JAFFARD†

Abstract. The multifractal formalism for functions relates some functional norms of a signal to its "Hölder spectrum" (which is the dimension of the set of points where the signal has a given Hölder regularity). This formalism was initially introduced by Frisch and Parisi in order to numerically determine the spectrum of fully turbulent fluids; it was later extended by Arneodo, Bacry, and Muzy using wavelet techniques and has since been used by many physicists. Until now, it has only been supported by heuristic arguments and verified for a few specific examples. Our purpose is to investigate the mathematical validity of these formulas; in particular, we obtain the following results:

• The multifractal formalism yields for any function an upper bound of its spectrum.

• We introduce a "case study," the *self-similar functions*; we prove that these functions have a concave spectrum (increasing and then decreasing) and that the different formulas allow us to determine either the whole increasing part of their spectrum or a part of it.

• One of these methods (the wavelet-maxima method) yields the complete spectrum of the self-similar functions.

We also discuss the implications of these results for fully developed turbulence.

Key words. multifractal formalism, self-similarity, wavelet transform

AMS subject classifications. Primary, 26A15; Secondary, 76F99

PII. S0036141095282991

1. Introduction and statement of results.

One-dimensional multifractal measures have been the object of many investigations by mathematicians and theoretical physicists (see, for instance, [5], [7], [12], [23], and the references therein). Basically, such measures have very different "scalings" from point to point, i.e., for such a measure μ, if I is an interval, the quantity $\mu(I)$ scales like $|I|^\alpha$, where the exponent α differs very much following the position of the center of the interval I. Such measures are important because they are natural measures carried by some strange attractors and thus appear in the modeling of many natural phenomena (diffusion-limited aggregates, invariant measures of dynamical systems, voltage drop across a random transistor network, etc.; see [2] and the references therein).

It may happen that the natural, fractal-like object that one wants to understand is not a set or a measure but a function. The study of multifractal functions has proved important in several domains of physics. Examples include plots of random walks, interfaces developing in reaction-limited growth processes, and turbulent velocity signals at inertial range (see [3]). The relevant mathematical tool studied in this context is the *Hölder spectrum*, also frequently called *spectrum of singularities*; this function associates with each positive α the Hausdorff dimension of the set where F is approximately Hölder of order α (in a sense to be made precise). The most important example where one would like to determine the spectrum of singularities of a function is the velocity of fully developed turbulence. The reason is that turbulent flows are not spatially homogeneous: the irregularity of the velocity seems to differ widely from point to point. This phenomenon, called "intermittency," suggests that the determination of the Hölder spectrum of the velocity of the fluid might be

* Received by the editors March 13, 1995; accepted for publication (in revised form) April 5, 1996.
http://www.siam.org/journals/sima/28-4/28299.html

† CMLA, École Normale Supérieure Cachan, 61 Avenue du Président Wilson, 94235 Cachan cedex, France and Département de Mathématiques, Faculté des Sciences et Technologie, Université Paris XII, 61 Avenue du Gal. de Gaulle, 94010 Créteil cedex, France (jaffard@cmla.ens-cachan.fr).

a nontrivial function and thus would yield important information on the nature of turbulence.

The first problem in this ambitious program is the numerical determination of the spectrum. Obviously, it is almost impossible to deduce it from the mathematical definition since it involves the successive determination of several intricate limits. The only method is to find some "reasonable" assumptions under which the spectrum could be derived using only "averaged quantities" (which should be numerically stable) extracted from the signal. Such formulas for the spectrum can be guessed heuristically using similarities with statistical physics. Frisch and Parisi [14] proposed, in one dimension, a formula using the L^p modulus of continuity of the velocity along one axis. Arneodo, Bacry, and Muzy (in [2], [3], and [26]) proposed, also in one dimension, other formulas based on the wavelet transform of the signal, and they proved their formulas' validity when the function considered is the indefinite integral of a multinomial measure or a C^∞ perturbation of such a measure. The origin of this method may be traced to the seminal work of Mandelbrot [23], and it has been used a great deal by physicists (see for instance [4], [12], [24], and the references therein), so the scope of its mathematical validity has become an important issue.

Our purpose in this paper is twofold:

• In Part I, we give some general results concerning the multifractal formalism. We show that for any function, it yields an upper bound of its Hölder spectrum, but we also show via some explicit counterexamples that, in general, it does not yield the exact spectrum.

• In Part II, we introduce and study a model case, "self-similar functions," and prove that the multifractal formalism holds for these functions. Examples of such functions include the indefinite integrals of self-similar measures, but they also include widely oscillating, several-dimensional functions—two requirements which are obviously needed, for instance, in any realistic model of turbulence.

Before describing the multifractal formalism, we need to recall some definitions and notation concerning the Hölder regularity of functions.

Suppose that α is a positive real number; a function $F : \mathbb{R}^m \to \mathbb{R}$ is $C^\alpha(x_0)$ if there exists a polynomial P of degree less than α such that

$$|F(x) - P(x - x_0)| \le C|x - x_0|^\alpha \tag{1.1}$$

and F belongs to $\Gamma^\alpha(x_0)$ if (see [18])

$$\begin{cases} \forall \beta > \alpha, & F \notin C^\beta(x_0), \\ \forall \beta < \alpha, & F \in C^\beta(x_0). \end{cases}$$

A function F is C^α (or $C^\alpha(\mathbb{R}^m)$) if (1.1) holds for any x in $\mathbb{R}^m$, the constant C being uniform. (Using *this* definition C^1 means Lipschitz.) We also need the two following definitions which assert (in two slightly different ways) that the singularity of F at x_0 can be measured on a "large" set near x_0. We denote by $\text{mes}A$ the Lebesgue measure of a set A.

Definition 1.1. *Let $\alpha > -m$; a point x_0 is a strong α-singularity of F if there* *exist* $C, C' > 0$ *such that* $\forall P$ *polynomial of degree at most* α, $\forall j, \exists A_j, B_j$,

$$\text{(1.2)}\quad \begin{cases} \text{mes}A_j \geq C2^{-mj}, \qquad \text{mes}B_j \geq C2^{-mj}, \\ \forall x \in A_j \cup B_j, \quad |x - x_0| \leq 2^{-j}, \\ \forall x \in A_j,\, \forall y \in B_j, \quad (F(x) - P(x - x_0)) - (F(y) - P(y - x_0)) \geq C'2^{-\alpha j}. \end{cases}$$

Note that if $\alpha < 1$, the last condition reduces to $F(x) - F(y) \geq C'2^{-\alpha j}$. The *wavelet transform* of a function F is defined as follows:

$$C(a, b) = \frac{1}{a^m} \int F(t)\psi\left(\frac{t-b}{a}\right)dt,$$

where ψ is a radial function with moments of order less than K vanishing and with derivatives of order less than K having fast decay (with a K "large enough" depending on the properties of F that we want to analyze).

DEFINITION 1.2. *A point* x_0 *is a wavelet* α*-singularity of* F *if there exist wavelet coefficients* $C(a_n, b_n)$ *in a cone pointing towards* x_0 *(i.e.,* $\mid b_n - x_0 \mid \leq Ca_n$*) such that* $a_n \to 0$, $a_n/a_{n+1} \leq C$, *and*

$$|C(a_n, b_n)| \geq Ca_n^\alpha. \tag{1.3}$$

We will prove in section 2 that the two previous definitions are related and that if F is $C^\alpha(x_0)$ and x_0 is a wavelet α-singularity of F, then x_0 is a strong α-singularity of F.

We can now define the object of our study.

DEFINITION 1.3. *The Hölder spectrum of a function* F *is the function* $d(\alpha)$ *defined for each* $\alpha \geq 0$ *as follows:*

$d(\alpha)$ *is the Hausdorff dimension of the set of points* x_0 *where* F *belongs to* $\Gamma^\alpha(x_0)$.

Remark. We will sometimes also call the function $D(\alpha)$, which is the packing dimension of the strong α-singularities, the *packing dimension spectrum*.

The two definitions of dimension that we use will be recalled when needed. Note that $d(\alpha)$ and $D(\alpha)$ are defined point by point. We will consider mainly $d(\alpha)$ except in section 4 of Part I and section 6 of Part II.

We are now in a position to describe the methods used by Frisch and Parisi on one side and Arneodo, Bacry, and Muzy on the other in order to determine the spectrum of singularities of functions.

- The *structure function* method first requires the computation of

$$S_q(l) = \int_{\mathbb{R}^m} |F(x + l) - F(x)|^q dx.$$

Assuming that the order of magnitude of $S_q(l)$ is $|l|^{\zeta(q)}$ when $l \to 0$, the Hölder spectrum is computed using the formula

$$d(\alpha) = \inf_q (q\alpha - \zeta(q) + m). \tag{1.4}$$

(We will define $\zeta(q)$ precisely below.)

- In the *wavelet-transform integral* method, one computes

$$\tilde{Z}(a, q) = \int_{\mathbb{R}^m} |C(a, b)|^q db,$$

and then if the order of magnitude of $\tilde{Z}(a,q)$ is $a^{\eta(q)}$,

$$d(\alpha) = \inf_q (q\alpha - \eta(q) + m). \tag{1.5}$$

• In order to describe the *wavelet-transform maxima* method, we first have to introduce the notion of a *line of maxima*; consider for a given $a' > 0$ the local maxima of the function $b \to C(a', b)$; generically, they belong to a line of maxima $b = l(a)$ defined in a small left-neighborhood $[a'', a']$ of a' by the condition that $b \to C(a, b)$ has a local maximum for $b = l(a)$. Usually, one cannot choose $a'' = 0$ because the lines of maxima have ramifications called "fingerprints." The wavelet-transform maxima method first requires the computation of

$$Z(a,q) = \sum_l \sup_{(b=l(a))} |C(a,b)|^q, \tag{1.6}$$

where l is a line of maxima of the wavelet transform defined on $[a'', a']$ and where the sum is taken on all lines of local maxima defined in left-neighborhoods $[a'', a']$ of a'. If the order of magnitude of $Z(a,q)$ is $a^{\theta(q)}$, then

$$d(\alpha) = \inf_q (qh - \theta(q)). \tag{1.7}$$

Numerically, according to [3], the most reliable method seems to be the last one, probably because the restriction of the computation to the maxima insures that small errors are less likely to be taken into account since at the maxima, they are relatively less important. More generally, methods that involve the wavelet transform are numerically more stable, probably because they involve only averaged quantities and not the direct values of the function. The use of such quantities has been advocated in [15]. The structure function method involves only order-one differences so that it is clearly unfit for computing the spectrum $d(\alpha)$ when α is larger than 1.

Since the scalings assumed above do not necessarily hold, we use the following definitions. Let

$$\zeta(q) = \liminf_{l \to 0} \frac{\log S_q(l)}{\log |l|}; \tag{1.8}$$

$$\eta(q) = \liminf_{a \to 0} \frac{\log \tilde{Z}(a,q)}{\log a}; \tag{1.9}$$

$$\theta(q) = \liminf_{a \to 0} \frac{\log Z(a,q)}{\log a}, \tag{1.10}$$

The multifractal formalism may seem surprising at first glance because it relates pointwise behaviors to global estimates. Before giving some mathematical explanations for it, it may be enlightening to give the heuristic classical argument from which it is derived. Although this argument cannot be transformed into a correct mathematical proof, it at least shows why these formulas can be expected to hold, and a careful study of its defects shows under which type of additional conditions it should be mathematically correct.

We calculate the contribution of singularities of order α to the integral

$$\int_{\mathbb{R}^m} |F(x+l) - F(x)|^q dx.$$

Near a singularity of order α, we have, in a small box of size $|l|$,

$$|F(x+l) - F(x)|^q \sim |l|^{\alpha q}.$$

If the dimension of these singularities is $d(\alpha)$, it means that there are about $|l|^{-d(\alpha)}$ such boxes, each of volume $|l|^m$, so that the total contribution to the integral is $|l|^{\alpha q + m - d(\alpha)}$. The real order of magnitude of the integral is given by the largest contribution, which, since $l \to 0$ is given by the smallest exponent, is such that

$$\zeta(q) = \inf_\alpha (\alpha q + m - d(\alpha)). \tag{1.11}$$

This formula is not the one that we are looking for since we know $\zeta(q)$ and are looking for $d(\alpha)$, but if it holds and if d is concave (we will see that in general this assumption need not be verified; however in many cases it is), $d(\alpha)$ is recovered by an inverse Legendre transform formula which yields (1.4). Of course, if $d(\alpha)$ is not concave, one expects the right-hand side of (1.4) to yield only the convex hull of the spectrum.

In all cases, (1.11) is more likely to hold because the concavity problem does not appear there. (A straightforward application of Young's formula shows that $\zeta(q)$ is always concave.)

In the first part of this paper, the following results will be proved.

THEOREM 1.4. *If $q > 1$ and $\zeta(q) < q$, then $\zeta(q) = \eta(q)$ for any function F. In general, these functions need not be related to $\theta(q)$.*

If F is a function of one real variable, and $0 < \eta(1) < 1$, the box dimension of the graph of F is $2 - \eta(1)$.

The following upper bound holds for any function F such that $\eta(p) > m$ $\forall p$:

$$d(\alpha) \leq \inf_p (m - \eta(p) + \alpha p). \tag{1.12}$$

Also, without any assumption on η,

$$D(\alpha) \leq \inf_p (m - \eta(p) + \alpha p).$$

In general, (1.12) *cannot be an equality*; *more precisely, let $d(\alpha)$ be a Riemann-integrable positive function on $\mathbb{R}^+$. There exists F_1 and F_2 which share the same function η, but the spectrum of F_1 is $d(\alpha)$ and F_2 is C^∞ except at the origin* (*so that its spectrum is equal to $-\infty$ everywhere except at one point*).

Some counterexamples will show that a smooth function (with a large $\eta(p)$) may nonetheless be such that $\theta(p)$ can be arbitrarily small. (The case $\theta(p) = -\infty$ $\forall p > 0$ can even happen.) The wavelet-transform maxima method need not be correct, even in the more precise framework of self-similar functions, where the other methods will work. However, after a slight modification, it yields the correct spectrum for self-similar functions. The mathematical problem with using (1.6) is that the lines of maxima can be too close to each other. In that case, we should instead keep for each interval of width a only one line passing through this interval that yields the largest contribution. However, the reader will see that the mathematical counterexamples where $\eta(q) \neq \theta(q)$ are very contrived, and the author's belief is that for practical applications, (1.5) and (1.7) have the same range of validity.

The last assertion in the theorem is stronger than the mere failure of the Legendre transform formulas. It asserts that there is not enough information in the function η to determine the spectrum. In particular, contrary to a common belief, the fact that

η is not linear does not imply that the signal has a multifractal structure. It also shows that, mathematically, "any" function $d(\alpha)$ can be a spectrum. It is surprising to notice that in several different fields of application, this does not seem to be the case. The spectra computed numerically have always the same shape—roughly speaking, the upper part of an ellipse. This is actually the shape we will find for self-similar functions. There can be several explanations to this analogy. Either (a) these physical signals satisfy some "scaling-invariance" properties which makes them fit in the framework (perhaps generalized in some ways) of self-similar functions or (b) a pessimistic explanation could be that, these spectra being (perhaps wrongly) calculated using a Legendre transform, the convex hull of the true spectrum is actually calculated and not the spectrum itself—hence this "generic" concave shape. We will also see that these examples answer the following problem raised by Daubechies and Lagarias in [9], which is somehow converse to the multifractal formalism: Is η the Legendre transform of $m-d(\alpha)$? Positive answers to this problem find fewer applications than the multifractal formalism since in practice one wants to obtain $d(\alpha)$ knowing $\eta(p)$ or $\zeta(p)$ and not the converse; nonetheless, it might hold more generally (see [9]). The problem raised by Daubechies and Lagarias is to find explicit counterexamples. We will see that in most cases, F_1 and F_2 are such counterexamples.

One of the referees of this paper raised the problem of a relationship between $\theta(q)$ and $\eta(q)$ such as

$$\theta(q) \leq \eta(q) - m.$$

This is true for self-similar functions satisfying the closed-set condition because then the regions where the wavelet transform is large (and these are the regions taken into account to estimate $\eta(q)$) are isolated so that there must exist a local maximum of the wavelet transform in the neighborhood. In general, however, we have no answer to this problem.

We now define self-similar functions by analogy with self-similar sets.

Recall that a set K is *strictly self-similar* if it is a finite union of disjoint subsets $K_1, \dots, K_d$ which can be deduced from K by similitudes. For instance, the triadic Cantor set and the Van Koch curve are self-similar. These sets have been widely studied, as have the measures supported on them. They play an important role in the modeling of several physical phenomena (see, for instance, [5], [16], [12], and [3]).

Suppose that F is continuous and compactly supported and let Ω be the bounded open subset of $\mathbb{R}^n$ such that $\bar{\Omega} = \text{supp}(F)$. The intuitive idea of a self-similar function is that there should exist disjoint subsets $\Omega_1, \dots, \Omega_d$ of Ω such that the graph of F restricted to each Ω_i is a "contraction" of the graph of F, modulo a certain error, which is supposed to be smooth. First, suppose that "smooth" means Lipschitz, and let us formalize this definition.

There should exist similitudes $(S_i)_{i=1,\dots,d}$ such that if $S_i(\Omega) = \Omega_i$,

$$\begin{aligned} &\forall i, \quad \Omega_i \subset \Omega, \\ &\Omega_i \cap \Omega_j = \phi \text{ if } i \neq j, \\ \forall x \in \Omega_j, \quad &F(x) = \lambda_j F(S_j^{-1}(x)) + g_j(x) \quad \text{with } g_j \text{ Lipschitz on } \bar{\Omega}_j. \end{aligned} \tag{1.13}$$

We suppose that S_i are *contractions*, i.e., the product of an isometry by the mapping $x \to \mu_i x$, where $\mu_i < 1$.

Equation (1.13) does not tell how F behaves outside Ω_i. We make the assumption that it is smooth, i.e., Lipschitz, outside $\bigcup \Omega_i$.

Since $F(S_j^{-1}(x)) = 0$ if $x \notin \Omega_j$,

$$F(x) = \sum_{i=1}^{d} \lambda_j F(S_j^{-1}(x)) + g(x),$$

where $g = g_j$ on Ω_j, $g = F$ outside $\bigcup \Omega_j$, and g is obviously continuous since F is continuous; since it is Lipschitz on $\bigcup \bar{\Omega}_j$ and outside $\bigcup \Omega_j$, g is uniformly Lipschitz.

This equation holds for any Lipschitz function F (use it as a definition for g when all $\lambda_j = 0$) so that it is interesting only if F is not uniformly Lipschitz, and in that case, we will be interested in determining the points where F is C^α for $\alpha < 1$.

We will generalize this model slightly by assuming that g is $C^k(\mathbb{R}^m)$ and not necessarily compactly supported but also that the derivatives of g of order less than k have fast decay. The same remark shows that in this case, we should suppose that F is not $C^k(\mathbb{R}^m)$, and we will be interested in determining where F is C^α for $\alpha < k$. We will thus use the following definition.

Definition 1.5. *A function $F : \mathbb{R}^m \to \mathbb{R}$ is self-similar (of order $k \in \mathbb{R}^+$) if the three following conditions hold:*

• *There exists a bounded open set Ω and $S_1, \ldots, S_d$ contractive similitudes such that*

$$S_i(\Omega) \subset \Omega, \tag{1.14}$$

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad \text{if } i \neq j. \tag{1.15}$$

(The S_i's are the product of an isometry with the mapping $x \to \mu_i x$, where $\mu_i < 1$.)

• *There exists a C^k function g such that g and its derivatives of order less than k have fast decay and F satisfies*

$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x). \tag{1.16}$$

• *The function F is not uniformly C^k in a certain closed subset of Ω.*

Recall that g has fast decay if

$$\forall n \in \mathbb{N}, \quad |g(x)| \leq \frac{C_n}{(1+|x|)^n}.$$

Let

$$\alpha_{\min} = \inf_{i=1,\ldots,d} \left(\frac{\log \lambda_i}{\log \mu_i} \right), \qquad \alpha_{\max} = \sup_{i=1,\ldots,d} \left(\frac{\log \lambda_i}{\log \mu_i} \right).$$

We use this notation because $\alpha_{\min}$ will turn out to be the smallest pointwise Hölder regularity exponent of F and $\alpha_{\max}$ the largest (lower than k). Let τ be the function defined by

$$\sum_{i=1}^{d} \lambda_i^a \mu_i^{-\tau(a)} = 1.$$

Some results concerning the multifractal formalism for self-similar functions are summed up in the following theorem and will be proved in the second part of the paper.

THEOREM 1.6. *Suppose that F is self-similar. If $\alpha_{\min} > 0$, the function $d(\alpha)$ vanishes outside $[\alpha_{\min}, \alpha_{\max}] \cup [k, +\infty)$ and is analytic and concave on $[\alpha_{\min}, \alpha_{\max}]$. Its maximal value $d_{\max}$ on this interval satisfies*

$$\sum \mu_i^{d_{\max}} = 1.$$

Let α_0 be the value for which this maximum is attained. First, suppose that g is C^∞. If $\alpha \leq \alpha_0$, $d(\alpha)$ can be obtained by computing the Legendre transform of either $\eta(q) - m$ or $\zeta(q) - m$.

If g is only C^k, let p_0 be defined by $\eta(p_0) = kp_0$ and let $\alpha_1 < \alpha_0$ be the value of the inverse Legendre transform of $\eta(q) - m$ at p_0; if $\alpha \leq \alpha_1$, $d(\alpha)$ can be obtained by computing the Legendre transform of either $\eta(q) - m$ or $\zeta(q) - m$.

Without any assumption on $\alpha_{\min}$, if $\sum \mid \lambda_j \mid \mu_j^m < 1$, the same results hold if we replace $d(\alpha)$ by $D'(\alpha)$, the packing dimension of the wavelet α-singularities (or by $D(\alpha)$ if g and λ_i are positive and if furthermore the separated open-set condition holds).

We will also prove that in some cases, the wavelet-maxima method can be modified so that it yields the whole spectrum of self-similar functions (see Theorem 2.2 in Part II).

Corollary 8.5 in Part II of this paper will extend this result to a larger class of functions than self-similar functions.

Before we begin to study the multifractal formalism for functions, we show its relationship to the multifractal formalism for measures. We recall that if μ is a probability measure on $[0,1]$, one defines

$$\tau(q) = \lim_{j \to +\infty} \frac{1}{j \log 2} \log \sum \left(\mu\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j} \right] \right) \right)^q$$

and

$$E_\alpha = \left\{ t : \ \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \to \alpha \right\},$$

where $I_n(t)$ is the interval $[k/2^j, (k+1)/2^j]$ which contains t. The multifractal formalism for measures asserts that the dimension of E_α is the Legendre transform of τ (see, for instance, [5] and [12] for mathematical results concerning this assertion). Let F be an indefinite integral of μ ($F(x) = \mu([0,x])$). Clearly,

$$t \in E_\alpha \Leftrightarrow |F(x+h) - F(x)| \sim h^\alpha$$

and

$$\sum \left(\mu\left(\left[\frac{k}{2^j}, \frac{k+1}{2^j} \right] \right) \right)^q \sum \left| F\left(\frac{k}{2^j}\right) - F\left(\frac{k+1}{2^j}\right) \right|^q \sim 2^j \int |F(x+2^{-j}) - F(x)|^q dx.$$

Thus if F is the indefinite integral of a probability measure supported on $[0,1]$, the two multifractal formalisms are identical. However, in dimensions larger than one or for functions that are not of bounded variation, the multifractal formalism for functions cannot be obtained as a consequence of the multifractal formalism for measures.

Our purpose in Part I of this paper is to prove Theorem 1.4. In section 2, we make explicit the relation between the size of the wavelet transform and the local regularity of the function. In section 3, we identify the quantities $S_q(l)$ or $Z(a,q)$ with some functional norms, thus proving the first point of Theorem 1.4. In section 4, we prove the upper estimate for the dimensions of singularities and the formula for the box dimension of the graph of F. In section 5, we study the wavelet-maxima method. In section 6, we construct counterexamples to the validity of the multifractal formalism in all generality.

The two parts of this paper can be read independently. Some results of this paper have been announced in [18], [19], and [20].

2. Regularity, singularities, and two-microlocalization. The results of Theorems 1.4 and 1.6 relate the pointwise behavior of a function to estimates on its wavelet transform. Our purpose in this section is to recall existing results on this topic and prove new ones concerning either negative exponents α or strong α-singularities. We first recall the basic properties of the wavelet transform.

Let ψ be in $C^{k+1}(\mathbb{R}^m)$, radial, with moments of order less than $k+1$ vanishing, and such that the derivatives of ψ of order less than $k+1$ have fast decay. The wavelet transform of F is defined by

$$C(a,b)(F) = \frac{1}{a^m}\int_{\mathbb{R}^m} F(t)\psi\left(\frac{t-b}{a}\right)dt, \tag{2.1}$$

and if $C(\psi) = \int |\hat{\psi}(\xi)|^2 d\xi/|\xi|$, F is recovered from its wavelet transform by

$$F(t) = C(\psi)\int_{a>0}\int C(a,b)(F)\psi\left(\frac{t-b}{a}\right)db\frac{da}{a^{m+1}}.$$

An intuitive idea is that a large wavelet coefficient means that the corresponding function locally has an oscillation at the corresponding scale of a corresponding amplitude. Although there does not seem to be a straightforward relationship between the two notions, Propositions 2.2 and 2.5 can be seen as a mathematical formulation of this idea. The following results can be found in [25] and [17]. Suppose that $s > 0$.

• $F \in C^s(\mathbb{R}^m)$ if and only if

$$|C(a,b)(F)| \le Ca^s. \tag{2.2}$$

(Recall that if $s = 1$, the space $C^s(\mathbb{R}^m)$ must be replaced by the Zygmund class, which is composed of the continuous functions F such that $|F(x+h)+F(x-h)-2F(x)| \le Ch$, or, more generally, if s is a positive integer, then it must be replaced by the corresponding indefinite integrals of the Zygmund class.)

• If $F \in C^s(x_0)$, then

$$|C(a,b)(F)| \le Ca^s\left(1+\frac{|b-x_0|}{a}\right)^s. \tag{2.3}$$

• If (2.3) holds and if $F \in C^\varepsilon(\mathbb{R}^m)$ for an $\varepsilon > 0$, there exists a polynomial P such that if $|x-x_0| \le 1/2$,

$$|F(x) - P(x-x_0)| \le C|x-x_0|^s \log\left(\frac{1}{|x-x_0|}\right). \tag{2.4}$$

Due partly to physical motivations (the study of the velocity of turbulent fluids, for instance), we do not want to consider only bounded functions, and thus we want to be able to consider points where F has a singularity (i.e., in a neighborhood in which it is unbounded). We first want to obtain results similar to (2.3) or (2.4) for singularities. A first problem is the definition of singularities that we should adopt.

The following definition is a straightforward generalization of (1.1) to negative exponents.

DEFINITION 2.1. *Suppose that* $-m < s \le 0$. *F is $C^s(x_0)$ if*

$$|F(x)| \le C|x - x_0|^s. \tag{2.5}$$

We have to make the assumption $-m < s \le 0$ because if $s \le m$, F might not be locally integrable and thus might not be a distribution. In that case, no computation on F (such as defining wavelet coefficients) would make sense. We will nonetheless see later how to define singularities of order less than $-m$.

We now relate (2.5) to conditions on the wavelet transform of F. We first check that if (2.5) holds, then

$$|C(a,b)(F)| \le Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^s; \tag{2.6}$$

First, suppose that ψ is supported in $B(0,1)$. Then

$$|C(a,b)| \le \frac{C}{a^m} \int_{B(b,a)} |x - x_0|^s dx,$$

where $B(x,r)$ is the ball centered at x of radius r. If $|b - x_0| \ge 2a$ and $x \in B(b,a)$, then $|x - x_0| \sim |b - x_0|$ and $|C(a,b)|$ is bounded by $(C/a^m)4^m a^m |b - x_0|^s$. Otherwise, $|x - x_0| \sim a$ and the integral is bounded by $(C/a^m)4^m a^m a^s$, and hence we have (2.6). The general case holds because condition (2.6) does not depend on the particular wavelet chosen (see [21]).

Note that we will often use the notation $a \sim b$ for positive quantities, which will always mean that the quotient a/b is bounded from below and above by positive constants.

If (2.6) holds, one can easily check that it implies no regularity for F. In that case, of course, we refuse to make a minimal smoothness assumption like $F \in C^\varepsilon(\mathbb{R}^m)$, which was needed in a similar situation in order to get (2.4). Let us show intuitively how to obtain a converse estimate. Suppose that $\text{supp}\psi \subset B(0,1)$, (2.5) holds, and $|\nabla F(x)| \le C|x - x_0|^{s-1}$; we further have $|C(a,b)| \le Ca|b - x_0|^{s-1}$ for $|b - x_0| > a$.

Conversely, one can easily check that this last estimate together with (2.6) implies that $|F(x)| \le C|x - x_0|^s$. We actually prove a slightly more general result.

PROPOSITION 2.2. *Let* $-m < s \le 0$. *If* $|F(x)| \le C|x - x_0|^s$, *then*

$$|C(a,b)(F)| \le Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^s.$$

Conversely, suppose that $\exists s' < s$ *such that*

$$|C(a,b)(F)| \le Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^{s'}. \tag{2.7}$$

Then $|F(x)| \le C|x - x_0|^s$.

Proof. We already proved the first part. Suppose that (2.7) holds. Using the reconstruction formula for F,

$$|F(t)| \le C \int \left[\int_{B(t,a)} |C(a,b)| db\right] \frac{da}{a^{m+1}}.$$

If $|t-x_0| \geq 2a$, $|b-x_0| \geq a$ and the right-hand side is bounded by

$$C \int_{a \leq \frac{|t-x_0|}{2}} a^{s-s'} |t-x_0|^{s'} \frac{a^m}{a^{m+1}} da \leq C|t-x_0|^s.$$

If $|t-x_0| \leq 2a$, $|b-x_0| \leq 4a$ and we get the bound

$$C \int_{a \geq \frac{|t-x_0|}{2}} a a^s a^m \frac{da}{a^{m+1}} \leq C|t-x_0|^s.$$

Hence Proposition 2.2 follows. ∎

Let us now recall the following definition of the two-microlocal spaces $C^{s,s'}(x_0)$ (see [17]):

$$F \in C^{s,s'}(x_0) \Longleftrightarrow |C(a,b)| \leq C a^s \left(1 + \frac{|b-x_0|}{a}\right)^{-s'}. \tag{2.8}$$

Proposition 2.2 generalizes to negative exponents the continuous embeddings

$$C^s(x_0) \hookrightarrow C^{s,-s'}(x_0) \quad \text{if } s' < s, \tag{2.9}$$

proved in [21], so that it also yields a justification of Definition 2.1 (and thus to the definition of strong α-singularities when $\alpha \leq 0$).

The problem of defining Hölder exponents for $s \leq -m$ is not straightforward. As mentioned before, we cannot consider only conditions such as $|F(x)| \leq C|x-x_0|^s$ since this does not imply that F is a distribution. The following definition has sometimes been proposed:

$$F \in C^s(x_0) \Longleftrightarrow (-\Delta)^{-\frac{[s]}{2}} F \in C^{s-[s]}(x_0). \tag{2.10}$$

There are two problems with this definition. The first is that it is not consistent with the definition for $s > 0$. Let us present an example. Consider $F(x) = x^{1/2}\cos(1/x)$; the integral of F is $O(x^{5/2})$ at the origin. Nonetheless, we would not consider F to be a $C^{3/2}$ function at the origin. Furthermore, this definition is also not consistent with the "natural" definition (2.5) when $-n < \alpha \leq 0$ for essentially the same reasons (we leave this verification to the reader). In order to go further, we interpret (2.10) as a two-microlocal condition. It implies $(-\Delta)^{-\frac{[s]}{2}} F \in C^{s-[s],-s+[s]}(x_0)$ so that $F \in C^{s,-s+[s]}(x_0)$. This condition is very far from $f \in C^{s,-s}$ which because of Proposition 2.2 should be "close" to the condition $F \in C^s(x_0)$. We show how to obtain a definition which is consistent with the definition for $s > -m$ and with the imbeddings in (2.9).

First, note that if s' is positive, $C^{s,s'}(x_0) \hookrightarrow C^s(\mathbb{R}^m)$, where by extension we define for a negative s

$$C^s(\mathbb{R}^m) = \dot{B}^{s,\infty}_{\infty} = \{F : \ |C(a,b)| \leq C a^s\}.$$

Thus the condition $F \in C^s(x_0)$, where s is negative, implies a global (negative) regularity for F. For $s \leq -m$, we will suppose that this regularity holds, which will guarantee that F is a distribution. In [11], Eyink proposed the definition $f \in C^{s,-s}(x_0)$. The advantage is that Proposition 4.1 can immediately be extended, which one uses with this definition of a pointwise Hölder exponent. The drawback is that this condition implies no pointwise regularity, even for positive s. Thus we adopt the following definition.

DEFINITION 2.3. *Suppose that $s \leq -m$. F belongs to $C^s(x_0)$ if $F \in \dot{B}_{\infty}^{s,\infty}$ and if F restricted to $\mathbb{R}^m - \{x_0\}$ is a function that satisfies*

$$|F(x)| \leq C|x - x_0|^s.$$

Note that this definition is slightly redundant since any function defined on $\mathbb{R}^m - \{x_0\}$ is the restriction of a distribution (defined on $\mathbb{R}^m$) which belongs to $\dot{B}_{\infty}^{s,\infty}$.

If we define $\dot{C}^s(\mathbb{R}^m) = \dot{B}_{\infty}^{s,\infty}(\mathbb{R}^m)$, we have the surprising continuous embedding

$$\dot{C}^s(x_0) \hookrightarrow \dot{C}^s(\mathbb{R}^m),$$

which goes in the opposite direction than it would for positive s.

This definition coincides with Definition 2.1 when $-m < s \leq 0$ since in that case the function F itself is the corresponding distribution, so

$$|F(x)| \leq C|x - x_0|^s \implies F \in \dot{B}_{\infty}^{s,\infty}.$$

Suppose that $F \in C^s(x_0)$. If $|b - x_0| \geq 2a$, as in the case where $s > -m$, we get $|C(a, b)| \leq C|b - x_0|^s$. Since $|C(a, b)| \leq a^s$ by hypothesis, we see that $C^s(x_0) \hookrightarrow C^{s,-s}(x_0)$.

PROPOSITION 2.4. *Using the previous definition of negative Hölder regularity, if $s \leq -m$, the following embeddings hold:*

$$\begin{cases} F \in C^s(x_0) \Rightarrow F \in C^{s,-s}(x_0), \\ F \in C^{s,-s'}(x_0) \quad \textit{for an } s' < s \Rightarrow F \in C^s(x_0). \end{cases} \tag{2.11}$$

The proof of the second implication is similar to the case where $s > -m$. It is interesting to check that some distributions which "should" belong to these generalized Hölder spaces satisfy these conditions. For instance, the distribution $p.p.(1/x)$ defined by

$$\left\langle p.p. \left(\frac{1}{x}\right) \mid \phi \right\rangle = \lim_{\epsilon \to 0} \int_{\mathbb{R} - [-\epsilon, \epsilon]} \frac{\phi(x)}{x} dx$$

is C^{-1} at 0 and $f.p.(1/x^2)$ defined by

$$\left\langle f.p. \left(\frac{1}{x^2}\right) \mid \phi \right\rangle = \lim_{\epsilon \to 0} \left(\int_{\mathbb{R} - [-\epsilon, \epsilon]} \frac{\phi(x)}{x^2} dx - \frac{2\phi(0)}{\epsilon} \right)$$

is C^{-2} at the origin. We leave these verifications as an exercise.

We now prove the following proposition, which relates the size of the wavelet transform to the existence of strong α-singularities when the wavelet used is compactly supported.

PROPOSITION 2.5. *Suppose that F is $C^\alpha(x_0)$ and that x_0 is a wavelet α-singularity of F. Then x_0 is a strong α-singularity of F.*

For the sake of simplicity, we restrict our focus to the case where $0 < \alpha < 1$. Suppose that F is $C^\alpha(x_0)$ and that x_0 is not a strong α-singularity of F. Let $\epsilon > 0$

be fixed and $a > 0$ be such that (1.3) does not hold for any (a, b) in the cone over x_0. For any x in the ball $B(x_0, Ca)$ (except on an exceptional set E_a of measure at most ϵa^m), we have

$$|F(x) - \bar{F}| \le \epsilon a^\alpha,$$

where $\bar{F}$ is the mean value of F in the ball $B(x_0, Ca)$. Also, if $x \in E_a$,

$$|F(x) - \bar{F}| \le |x - x_0|^\alpha.$$

If the support of the wavelet $\psi((x-b)/a)$ is included in $B(x_0, Ca)$,

$$|C(a,b)| = \frac{1}{a^m} \left| \int F(x)\psi\left(\frac{x-b}{a}\right) dx \right| \le \frac{1}{a^m} \int_{B(x_0,Ca)} |F - \bar{F}|,$$

the integral on E_a is bounded by $a^\alpha \epsilon a^m$ and outside E_a by $\epsilon a^\alpha a^m$, so $|C(a,b)| \le 2C\epsilon a^\alpha$ and (1.3) does not hold. Hence we have a contradiction, and thus Proposition 2.5 holds.

The condition that F is $C^\alpha(x_0)$ is necessary in Proposition 2.5, as shown by the following counterexample. Suppose that ψ (perhaps after a translation) is compactly supported in an interval of the form $[2^l, 2^{l+1}]$, and suppose that the $2^{j/2}\psi(2^j x - k)$'s form an orthonormal wavelet basis of $L^2(\mathbb{R})$ (see [8] for such functions). Let I be an interval such that $\psi(x) \ge C > 0$ on I. Define $F(x) = \sum_j 2^{-(\alpha-1)j} 1_{A_j}(x)$, where $A_j = 2^{-j} I_j$ and I_j is a subinterval of I of length 2^{-j}. Then clearly $2^{-j} \int F(x)\psi(2^j x)dx \ge C2^{-\alpha l j}$ but F has no strong singularity at 0 (but is only $C^{\alpha-1}(0)$).

3. Some functional norm estimates. We first show the link between quantities such as $S_p(l)$ or $\tilde{Z}(a,q)$ and Sobolev or Besov-type norms. We recall a few definitions and characterizations.

Suppose that $s \in \mathbb{R}$ and $p, q > 0$. A function F belongs to the homogeneous Besov space $B_p^{s,q}$ if

$$\int_{a>0} \left[\int |C(a,b)|^p db \right]^{q/p} \frac{da}{a^{sq+1}} < +\infty \tag{3.1}$$

(which follows directly from [25]).

Since $\eta(p)$ is the infimum of all numbers τ verifying, for a small enough,

$$\tilde{Z}(a,p) \left(= \int |C(a,b)|^p db \right) \le Ca^\tau,$$

we see that if $p > 0$,

$$\eta(p) = \sup\{\tau : F \in B_p^{\tau/p,\infty}\}. \tag{3.2}$$

A similar characterization exists for the function $\zeta(p)$. The spaces $H^{s,p}$ introduced by Nikol'skii (see [1] or [27]) are defined as follows.

Let $s \ge 0$. If s is not an integer, $s = m + \sigma$ with m integer and $0 < \sigma < 1$. Let $p \ge 1$, $F \in H^{s,p}$ if $F \in L^p$ and for any multiindex α such that $|\alpha| = m$,

$$\int \frac{|\partial^\alpha F(x+h) - \partial^\alpha F(x)|^p}{|h|^{\sigma p}} dx \le C. \tag{3.3}$$

Recall that $\zeta(p)$ is the lim sup of the numbers ξ such that

$$S_p(h) \left(= \int |F(x+h) - F(x)|^p dx \right) \leq C h^{\xi(p)}$$

for h small enough. Thus if $p \geq 1$, $\zeta(p) = \sup\{s : F \in H^{s/p,p}\}$.

Of course, we see here that the formula in the structure function method must be modified as follows in order to be consistent with (3.3): If $\zeta(p)$ is less than 1, the formula is all right; if it is equal to 1, one should use the same formula but with the gradient of F; and so on until $\zeta(p)$ falls between two integers. (Note that this procedure is obviously difficult to handle numerically if $\zeta(p)$ is large.)

The following embeddings hold if $p \geq 1$:

$$\forall \epsilon > 0, \quad H^{s+\epsilon,p} \hookrightarrow B_p^{s,\infty} \hookrightarrow H^{s-\epsilon,p} \tag{3.4}$$

(because (3.4) holds between $H^{s,p}$ and $W^{s,p}$ spaces (see [1]), between $W^{s,p}$ and $L^{p,s}$ spaces (see [1]), and between $L^{p,s}$ and $B^{s,\infty}$ spaces (see [21] or [25])). Thus, if $p > 1$, $\zeta(p) = \eta(p)$ and the function η can be defined by

$$\eta(p) = \sup\{s : F \in B_p^{s/p,p}\} = \sup\{s : F \in L^{p,s/p}\} \tag{3.5}$$

(where $L^{p,s}$ is defined for $s > 0$ by $f \in L^{p,s} \iff f \in L^p$ and $(-\Delta)^{s/2} f \in L^p$), and if $0 < p \leq 1$, it can be defined by the first equality only, so the last characterization of $\eta(p)$ in (3.5) is again a straightforward consequence of Sobolev-type embeddings.

Proposition 3.1. *The following characterizations hold:*

$$\forall p > 0, \quad \eta(p) = \sup\{s : F \in B_p^{s/p,\infty}\},$$
$$\forall p > 1, \quad \eta(p) = \zeta(p) = \sup\{s : F \in H^{s/p,p}\} = \sup\{s : \ F \in L^{s/p,p}\}.$$

Remark. The number $\eta(2)$ can be interpreted as follows:

$$\eta(2) = \sup \left\{ s : \int |\hat{F}(\xi)|^2 (1 + |\xi|^2)^{s/2} d\xi \leq C \right\}.$$

This holds because $B_2^{s,2} = L^{2,s}$, and $\forall q, q', q''$, $B_p^{s+\epsilon,q} \subset B_p^{s,q'} \subset B_p^{s-\epsilon,q''}$, so

$$\begin{aligned} \eta(2) &= \sup\{s : F \in B_2^{s,\infty}\} = \sup\{s : F \in B_2^{s,2}\} \\ &= \sup\{s : F \in L^{2,s/2}\} = \sup \left\{ s : \int |\hat{F}(\xi)|^2 (1 + |\xi|^2)^{s/2} d\xi \leq C \right\}. \end{aligned}$$

Note that this result differs from [3], where the interpretation given for $\eta(2)$ is $|\hat{F}(\xi)|^2 \sim |\xi|^{-\eta(2)-2}$. Nonetheless, the interpretation given in [3] is correct provided that such a scaling holds. An interpretation of $\eta(1)$ of very different nature will be given in section 4.

We will show in section 6 that "any" function $d(\alpha)$ can be a Hölder spectrum. It is interesting to notice that this is not the case with the function $\eta(p)$, which because of the Sobolev imbeddings between $L^{p,s}$ spaces cannot be arbitrary. Since $L^{p,s} \subset L^{t,q}$ if $t \leq s$ and $q = mp/(m - (s-t)p)$ (see [1]), if $q \geq p$,

$$\frac{\eta(q) - \eta(p)}{q - p} \geq \frac{\eta(p) - m}{p}. \tag{3.6}$$

In particular, we see that $\eta'(p) \geq \eta(p) - m/p$. Conversely, it is easy to check that any function $\eta(p)$ that satisfies (3.6) can be associated with a function F so that (3.6) characterizes all possible functions $\eta(p)$.

4. Upper bounds for dimensions of spectrums. A first problem that we meet is that of which mathematical definition of "dimension" we should use. The physical literature is often unclear about this point, sometimes using the term *Hausdorff dimension* but computing it using coverings by boxes of the same size. Of course, a given set of points (a potential "set of Hölder singularities" of our function) can have very different dimensions depending on the definition considered. We will see that the "good definition" depends on the kind of singularities that we look for. For Hölder singularities, we will get bounds on Hausdorff dimensions, and for strong α-singularities, we will get bounds on packing dimensions. An important difference between the two settings is that in the first we necessarily have to suppose some minimal uniform regularity for F, which is not required in the second. We first recall the definition of the Hausdorff dimension and Hausdorff measure.

Let $A \subset \mathbb{R}^n$ and R_ε be the set of all coverings of A by sets of diameter at most ε. Let

$$M(\varepsilon, d) = \inf_{r \in R_\varepsilon} \sum_{A_i \in r} (\operatorname{diam} A_i)^d.$$

Then by definition,

$$d - \operatorname{Mes}(A) = \limsup_{\varepsilon \to 0} \; M(\varepsilon, d)$$

is the d-dimensional Hausdorff measure. The *Hausdorff dimension* of A is

$$D = \inf\{d : d - \operatorname{Mes}(A) = 0\} = \sup\,\{d : d - \operatorname{Mes}(A) = +\infty\}.$$

If the coverings are done using only balls or only dyadic cubes, we obtain an equivalent quantity for the d-measure, and thus D is not changed.

PROPOSITION 4.1. *Let $s - m/p > 0$ and $p > 0$. If $F \in B_p^{s,\infty}$, $d(\alpha) \le m - (s-\alpha)p$. Thus if $\eta(p)$ satisfies $\eta(p) > m \; \forall p$, $d(\alpha) \le \inf_p (m - \eta(p) + \alpha p)$.*

This proposition is reminiscent of [5], where Brown, Michon, and Peyrière proved similar results for measures (in dimension 1). If $s \le m/p$, a function in $L^{p,s}$ or $B_p^{s,\infty}$ can be infinite on a dense set and thus smooth at no point (see [21]), so that no such result can hold if we do not make the assumption $s - m/p > 0$.

Proof of Proposition 4.1. we use a slight modification of the two-microlocal space, for convenience. We thus define

$$F \in C_p^{s,s'}(x_0) \quad \text{if and only if } |C_{j,k}| \le C 2^{-(\frac{m}{2}+s)j} j^{2/p} (1 + |2^j x - k|)^{-s'}. \tag{4.1}$$

We will prove that if $F \in B_p^{s,\infty}$, then $d > 0$. Outside a set of d-measure 0, $F \in C_p^{s-m-d/p,-d/p}(x)$. Thus if $0 < s - m/p < \alpha < s$, the set $\{x : \; F \notin C^\alpha(x)\}$ has Hausdorff dimension at most $m - (s-\alpha)p$, and Proposition 4.1 follows.

Let $F \in B_p^{s,\infty}$. Then

$$\forall j, \quad \sum_k |C_{j,k}|^p \, 2^{(ps + \frac{mp}{2} - m)j} \le C. \tag{4.2}$$

Let d be such that $0 < d \le n$ and $B_{j,k}$ be the ball centered on $k2^{-j}$ and of size

$$\operatorname{diam}(B_{j,k}) \; = \; |C_{j,k}|^{p/d} 2^{\alpha j} j^{-2/d},$$

where α is such that

$$-d\alpha + ps + \frac{mp}{2} - m \;=\; 0.$$

Then (4.2) can be rewritten as

$$\forall j, \quad \sum_k (\operatorname{diam} B_{j,k})^d \leq \frac{c}{j^2}. \tag{4.3}$$

Let $A_j = \bigcup_k B_{j,k}$. Equation (4.3) implies that the d-measure of $A = \limsup A_j$ is 0. If $x \notin \limsup A_j$, $\exists\, j_0,\ \forall j \geq 0,\ \forall k,\ x \notin B_{j,k}$ so that

$$|x - k2^{-j}| \geq C|C_{j,k}|^{p/d}\, 2^{\alpha j} j^{-2/d}.$$

Hence

$$\forall j \geq j_0, \quad |C_{j,k}| \leq C2^{-(m/2+s-m/p)j}|x - k2^{-j}|^{d/p} j^{2/p}$$

and thus $F \in C_p^{s-m-d/p\,,-d/p}(x)$ (because (4.1) automatically holds for $j \leq j_0$). Hence Proposition 4.1 follows.

One can wonder if similar bounds (or equalities) hold for dimensions of strong α-singularities. This problem is important for the following reasons. Recall that the multifractal formalism was introduced for the study of turbulence. In [6], Caffarelli, Kohn, and Nirenberg obtained a bound on the dimension of (possible) singularities in Navier–Stokes equations that is actually a bound on the packing dimension of "strong α-singularities" following the definition that we gave (with $\alpha = 0$).

Another reason to obtain bounds for dimensions of strong singularities is that when global regularity conditions (which imply that F is continuous) no longer hold, no result such as Proposition 4.1 can be proved. Even in the strict framework of self-similar functions, we will see in Part II that no such bounds exist. Since for applications we clearly want to be able to consider unbounded functions (for instance, the velocity of a turbulent fluid may be unbounded), it is important to obtain some positive results in that case.

Our purpose is to prove that if F belongs to $W^{s,p}$, given $\alpha < s$, the set of points x where F has a strong α-singularity has a small packing dimension. We first recall the definition of the packing dimension of a subset of $\mathbb{R}^m$ (see [12]).

Let $J > 0$ and Λ_J be the set of dyadic cubes of size 2^{-J} which contain a point of E. Define

$$m_d(E) \;=\; \lim_{J\to+\infty} \sum_{\lambda\in\Lambda_J} 2^{-dJ} \;=\; \Lambda_j^\sharp\, 2^{-dJ}$$

(where $\Lambda_j^\sharp$ denotes the cardinality of Λ_j) and

$$\operatorname{mes}_d(E) \;=\; \inf_{E\subset\cup E_n} \sum_n m_d\,(E_n).$$

The *box dimension* of E is the value of d for which $m_d\,(E)$ falls from $+\infty$ to 0. This dimension is also called the potential dimension by some physicists. It is the only one that is numerically easy to compute because it does not involve optimal coverings.

The *packing dimension* of E is the value of d for which $\mathrm{mes}_d(E)$ falls from $+\infty$ to 0. It is clearly no larger than the box dimension.

PROPOSITION 4.2. *Let $F \in W^{s,p}(\mathbb{R}^m)$ and α be such that $-m < \alpha \leq 1$. The packing dimension of the strong α-singularities of F is bounded by $m - (s-\alpha)p$ so that if $D(\alpha)$ is the packing dimension of strong α-singularities and $-m < \alpha \leq 1$, then*

$$D(\alpha) \leq \inf_p (m - \eta(p) + \alpha p). \tag{4.4}$$

Such a result is in many cases more satisfactory than Proposition 4.1 since we do not have to make the assumption of a minimal Hölder regularity of F. Actually, the numerical estimation of the upper bound for $D(\alpha)$ when $\alpha = 0$ using (4.4) could be a way to check whether a stronger result than the one obtained by Caffarelli, Kohn, and Nirenberg in [6] holds.

Of course, a way to avoid the problem of unbounded functions could be to consider indefinite integrals or perhaps iterated indefinite integrals of the velocity, but such quantities would have no direct physical interpretation.

We first describe the functional setting that we use. We will give bounds on the packing dimension of strong α-singularities in the Sobolev spaces $W^{s,p}$. Recall that (see [1])

$$\text{if } 0 < s < 1, \quad f \in W^{s,p} \Leftrightarrow f \in L^p \quad \text{and} \quad \int\int \frac{|f(x+t) - f(x)|^p}{|t|^{m+sp}}\, dx\, dt \leq +\infty. \tag{4.5}$$

For $s \geq 1$, these spaces can be defined as follows. First, if $0 < s < 2$, they can be defined by replacing $|f(x+t) - f(x)|$ by $|f(x+t) + f(x-t) - 2f(x)|$ in (4.5), and if $\alpha \geq 2$, $f \in W^{s,p} \Leftrightarrow f \in L^p$ and $\forall i = 1, \ldots, n$, $\partial f / \partial x_i \in W^{s-1,p}$ (see [1]).

The fact that these spaces are defined by a condition on the L^p-modulus of continuity $\omega_p(t) = \|f(\cdot + t) - f(\cdot)\|_p$ will yield an easy direct estimate on the packing dimension of the strong α-singularities. (The intuitive idea is that if x_0 is such a singularity, the contribution for x close to x_0 to the integral $\int_{x\in\mathbb{R}^m} |f(x+t) - f(x)|^p\, dx$ is large.) The spaces $L^{p,s}$ and $W^{s,p}$ are closely related since (see [1]) $W^{s,2} = L^{2,s}$ and $L^{p,s} \subset W^{s,p'}$ if $p > p'$ and $W^{s,p} \subset L^{p',s}$ if $p > p'$. Thus the bound of $D(\alpha)$ given by Theorem 1.4 is a consequence of Proposition 4.2 which we now prove. Define $E_{l,m}$ as the set of points x_0 such that (1.2) holds with

$$2^{-l} \leq C < 2 \cdot 2^{-l} \quad \text{and} \quad 2^{-n} \leq C' < 2 \cdot 2^{-n}.$$

The set of strong α-singularities of F is $\bigcup_{l,n} E_{l,n}$. Let l and n be fixed. Let Λ_j be the set of dyadic cubes of size 2^{-j} such that $\Lambda_j \cap E_{l,n} \neq \emptyset$. If $\lambda \in \Lambda_j$, there exist $A_j, B_j \subset 3\lambda$ such that (1.2) holds (where 3λ is the cube that has the same center as λ and is three times larger). We restrict the integral (4.5) to $x \in 3\lambda$, $x + t \in 3\lambda$. The integral on this set is thus bounded from below by

$$2^{-2l} 2^{-2mj} \frac{(2^{-n} 2^{-\alpha j})^p}{(2^{-j})^{m+sp}}.$$

If we sum up for all $\lambda \in \Lambda_j$, each integral is taken at most 4^m times. Thus

$$\Lambda_j^\sharp\, 2^{-2l}\, 2^{-np}\, 2^{-j[\alpha p + m - sp]} \leq 4^m \|f\|^p_{W^{s,p}}$$

and thus if $d > \alpha p + m - sp$, $d - \text{mes}(E_{l,m}) = 0$ so that $d - \text{mes}(\bigcup_{l,n} E_{l,n}) = 0$. Hence Proposition 4.2 follows.

We now check that if F is a function of one real variable, the box dimension of the graph of F is exactly $2 - \eta(1)$ if $\eta(1)$ is between 0 and 1. This is a straightforward consequence of the following result (see [10] or [13]).

PROPOSITION 4.3. *Suppose* $0 < \gamma < 1$ *and* $F : [0,1] \to \mathbb{R}$ *is continuous. Then the box dimension of the graph of* F *is exactly* $2 - \gamma$ *if and only if*

$$F \in \bigcap_{\alpha < \gamma} B_1^{\alpha,\infty} \Big\backslash \bigcup_{\beta > \gamma} B_1^{\beta,\infty}.$$

Thus the result holds because $\eta(p) = \sup\{s : F \in B_p^{s/p,\infty}\}$.

5. The wavelet-maxima method. Our purpose in this section is to show that the wavelet-maxima method can yield a function $\theta(q)$ which is much smaller than $\eta(q) - m$ so that in general the multifractal formalism cannot hold using this method. Via our counterexamples, we will show how to slightly modify its definition so that $\theta(q) = \eta(q) - m$. Our specific study of the wavelet-maxima method is justified by its numerical importance. Arneodo, Bacry, and Muzy compared the three numerical methods in cases where the Hölder spectrum is known analytically (self-similar functions, Riemann's function), and they clearly showed (in a personal communication) that the wavelet-maxima method is the most accurate.

The reason why the wavelet-maxima method may fail is easy to understand intuitively if we relate it to the wavelet-transform integral method. The two quantities $\int_{\mathbb{R}^m} |C(a,b)|^q db$ and $a \sum_{\ell \in \mathcal{L}(a)} \sup_{(b,a') \in \ell} |C(a',b)|^q$ have the same order of magnitude if the spacing between the maxima is approximately a since then the second term is a Riemann sum of the first term. Thus the counterexamples that we will construct will have maxima with spacing much smaller than a, and if we slightly modify the wavelet-maxima method by imposing the restriction that we select only one maximum (or, say, C maxima) in an interval of length a, then the multifractal formalism will hold.

In order to give some insight into the pitfalls of the wavelet-maxima method, we begin by describing an example where the maxima accumulate in certain regions. Not surprisingly, this example involves chirps.

LEMMA 5.1. *Suppose that* ψ *is compactly supported on* $[0,l]$*, has a vanishing integral and* m *first vanishing moments, and satisfies*

$$\exists \epsilon > 0 \quad \psi(x) = x^m \quad \forall x \in [0,\epsilon].$$

(This is the case, for instance, if ψ *is a spline.) There exists a function* F *that is compactly supported and arbitrarily smooth and a sequence* $a_n \to 0$ *such that for all values of* n*, the wavelet transform* $C(a_n,b)$ *has infinite maxima.*

We first construct F such that this property holds for a small interval of values of the dilation parameter a. The general case will be obtained by a superposition argument. Let

$$F(x) = x^k \sin\left(\frac{1}{x^l}\right) \phi(x),$$

where ϕ is C^∞ except at the origin and supported on $[0,1]$, $\phi(x) = 1$ $\forall x \in [0,1/2]$, and ϕ is such that the integral and the first m moments of F vanish. After dilating

ψ, we can suppose that it is equal to x^m on the interval $[0,1]$. Then if $x \leq 1/4$ and $a \in [1/2,1]$,

$$\begin{aligned}\frac{1}{a}\int F(t)\psi\left(\frac{x-t}{a}\right)dt &= \frac{1}{a}\int_x^1 F(t)\left(\frac{x-t}{a}\right)^m dt \\ &= -\frac{1}{a}\int_0^x F(t)\left(\frac{x-t}{a}\right)^m dt = -\frac{1}{a}\int_0^x t^k\left(\frac{x-t}{a}\right)^m \sin(t^{-l})dt\end{aligned}$$

Integrating m times by parts, we obtain either

$$\int F(t)\psi(x-t)dt = a^{-m-1}x^{k+m(l+1)}\sin\frac{1}{x^l} + o(x^{k+m(l+1)})$$

or

$$\int F(t)\psi(x-t)dt = a^{-m-1}x^{k+m(l+1)}\cos\frac{1}{x^l} + o(x^{k+m(l+1)})$$

depending on the parity of m. In all cases, the wavelet transform of F has for $a \in [1/2,1]$ an infinity of lines of maxima. The general case is obtained by considering the function

$$G(x) = \sum_{j=0}^{\infty} 2^{-mj}F(2^j(x-l)),$$

where l is larger than the size of the support of ψ.

This example also shows that one should be careful when using the wavelet-maxima method since the superposition of a small smooth function can completely perturbate the lines of maxima.

We now show that the two functions $\theta(q)$ and $\eta(q)-1$ can differ dramatically so that even in cases where the multifractal formalism holds when using the wavelet integral method, it may prove wrong when using the wavelet-maxima method. To this end, we will construct a smooth function F (so that $\eta(p)$ will take the maximal value that is compatible with the smoothness of the wavelet) such that $\theta(q) = -\infty\ \forall q$. This example will use a wavelet with one vanishing moment. However, we will show how to modify it in order to deal with wavelets with a given number of vanishing moments. We will also show in Part II that F can be a self-similar function (which will provide a case where the multifractal formalism holds using the wavelet integral method and does not hold using the wavelet-maxima method).

PROPOSITION 5.2. *Let ψ be even and compactly supported* (*say on* $[-1,1]$) *and satisfy*

$$\int \psi(x)dx = 0 \quad \textit{and} \quad \int x\psi(x)dx = 1.$$

There exists a C^∞ compactly supported function F such that $\theta(q) = -\infty\ \ \forall q > 0$.

Proof. The idea of the proof is to construct a function g whose wavelet transform is equal to, say, 1 on an interval and to perturbate it by adding another function whose wavelet transform is extremely small but oscillates extremely fast, thus creating a huge number of new maxima which take values close to 1.

Let g be a C^∞ odd function supported by $[-3,3]$ such that $g(x)=1$ on $[1,2]$. Let

$$h_j(x) = 2^{-j^2}g(2^{j+4}(x-8)) + 2^{-j^4}\sin(2^{j^3}\pi x)\phi(2^j x),$$

where ϕ is a C^∞ function supported on $[1/2,1]$ that verifies $\phi(x+3/4)=\phi(3/4-x)$ and $\forall x \in [9/16,15/16]$, $\phi(x)=1$. Let F be the indefinite integral of $\sum_{j\geq 0} h_j(x)$. Since h_j has a vanishing integral, F is C^∞ and compactly supported. Let G be the indefinite integral of g. F and the series $\sum_{j\geq 0} 2^{-j^2}2^{-j-4}G(2^{j+4}(x-8))$ will have the same function η. (Here the calculation will yield $\eta(p)=p$ because this series is a C^∞ function and the wavelet used will have only one vanishing moment.)

Note that

$$\frac{1}{a}\int \sin(\omega x)\psi\left(\frac{x-b}{a}\right)dx\, \sin(\omega b)\hat{\psi}(\omega a).$$

(Here $\omega = 2^{j^3}$.) For a given value of j, we choose a in the interval $[1/100.2^{-j}, 1/10.2^{-j}]$ such that $\hat{\psi}(\omega a)$ does not vanish (which is possible since $\hat{\psi}(\omega a)$ is an analytic function of a).

Integrating by parts, one checks that on an interval of length at least $2^{-j-4}5/16$, the wavelet transform of $2^{-j^2}2^{-j-4}G(2^{j+4}(x-8))$ takes a constant value equal to 2^{-j^2}. Thus on the same interval, the wavelet transform of F is $2^{-j^2}+2^{-j^4}\sin(2^{j^3}b)\hat{\psi}(2^{j^3}a)$. Thus it has about $2^{-j-4}2^{j^3}$ maxima, and

$$\sum_{\max}|C(a,b)|^q \sim 2^{-j-4}2^{j^3}2^{-j^2 q}.$$

Since j can be chosen arbitrarily large, the result is proved. $\square$

Note that we could have chosen a wavelet with a given number of vanishing moments. In that case, we would have integrated g not once but the corresponding number of times. The important fact is that the wavelet transform of G should locally be constant. The reader will also easily check that we could have imposed a given function η for F.

6. Counterexamples to the multifractal formalism. We define $\mathcal{C}$ as the class of functions that can be written as the supremum of a countable set of functions of the form $c1_{[a,b]}(x)$ (where we can have $a=b$). Thus Riemann-integrable functions belong to $\mathcal{C}$, but so do, for instance, the indicatrix function of the rationals (but not the indicatrix function of the irrationals).

PROPOSITION 6.1. *Let $d(s):]0,+\infty[\to [0,m]$ be a function in $\mathcal{C}$. There exist two continuous functions G_1 and $G_2 : \mathbb{R}^m \to \mathbb{R}$ that share the same function $\eta(q)$ such that $d(s)$ is the Hölder spectrum of G_1 while G_2 is C^∞ except at the origin, so its spectrum vanishes everywhere.*

We construct these functions when the space dimension is $m=1$. The generalization to the multidimensional case is straightforward.

We first construct G_1 when $d(s)=cs1_{a,b}(s)$, where $0<a\leq b<\infty$ and $cb\leq 1$. We will actually use three other parameters α, β, and γ, where $a=\gamma$, $b=\beta\gamma$, and $c=1/(\alpha\beta\gamma)$ so that $\gamma>0$, $\beta\geq 1$, and $\alpha\geq 1$. We thus define $G_1 = F^{(\alpha,\beta,\gamma)}$. The general case will be obtained using a simple "superposition" procedure of the $F^{(\alpha,\beta,\gamma)}$.

We will explicitly construct G_1 by defining its coefficients on an orthonormal wavelet basis. The function G_2 will then be obtained by just moving at each scale the location of the nonvanishing wavelet coefficients of G_1. We use an orthonormal wavelet basis in the Schwartz class (see [25]), and the functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x-k), \quad j,k\in \mathbf{Z},$$

are an orthonormal basis of $L^2(\mathbb{R})$. Sometimes we will index the wavelets $\psi_{j,k}$ or the wavelet coefficients $C_{j,k}$ $(= \int F\psi_{j,k})$ by the dyadic intervals $\lambda = [k2^{-j}, (k+1)k2^{-j}]$.

Let Λ be the collection of all dyadic intervals of length at most 1. We will construct a subcollection $\Lambda(\alpha,\beta) \subset \Lambda$ and consider the following "lacunary" wavelet series:

$$F(x) = \sum_{\lambda \in \Lambda(\alpha,\beta)} 2^{-(\gamma+1/2)j}\psi_\lambda(x). \tag{6.1}$$

The construction of $\Lambda(\alpha,\beta)$ is performed as follows. Define $\Lambda(\alpha,\beta) = \bigcup_{m\geq 1}\Lambda_m^{(\alpha,\beta)}$, where $\Lambda_m^{(\alpha,\beta)}$ is the set of intervals λ or couples (j,k) such that $j = [\alpha\beta m]$ and

$$2^{-j}k = \epsilon_1 l_1 + \cdots + \epsilon_m l_m \in F_m, \quad \epsilon_1, \ldots, \epsilon_m \in \{0,1\}, \quad l_n = 2^{-[\alpha n]},$$

$[x]$ is the entire part of x and thus $k = 2^{[\alpha\beta m]}(\pm l_1 \pm \cdots \pm l_m)$ is an integer since $[\alpha\beta m] \geq [\alpha n]$.

PROPOSITION 6.2. *The function F defined by* (6.1) *belongs to the global Hölder space $C^\gamma(\mathbb{R})$ so that if $s < \gamma$, the set $E^{(s)}$ of points x_0 where $f \in \Gamma^s(x_0)$ is empty. If $\gamma \leq s \leq \beta\gamma$, the Hausdorff dimension of $E^{(s)}$ is $s/\alpha\beta\gamma$. If $s > \beta\gamma$, $E^{(s)}$ is empty.*

The characterization of the space C^γ on the wavelet coefficients is

$$|C_{j,k}| \leq C2^{-(\gamma+1/2)j}$$

(a simple rewriting of (2.2) in the orthonormal basis setting). Thus F belongs to $C^\gamma(\mathbb{R})$ and the spectrum of F vanishes for $s < \gamma$.

LEMMA 6.3. *A point x_0 belongs to $E^{(s)}$ if and only if*

$$\mathrm{dist}(x_0, F_m) = \eta_m 2^{-\frac{\alpha\beta\gamma}{s}m}, \quad (F_m = \{\pm l_1 \pm l_2 \pm \cdots \pm l_m\}) \tag{6.2}$$

with

$$\liminf_{m\to\infty} \eta_m 2^{-m\epsilon} = 0 \quad \textit{for any } \epsilon > 0 \tag{6.3}$$

and

$$\liminf_{m\to\infty} \eta_m 2^{m\epsilon} = +\infty \quad \textit{for any } \epsilon > 0. \tag{6.4}$$

Proof. If F is $C^s(x_0)$, the rewriting of (2.3) yields

$$|C_{j,k}| \leq C2^{-(s+1/2)j}(1 + |2^j x_0 - k|)^s. \tag{6.5}$$

Conversely, from (2.3), we deduce that if (6.5) holds and if F is $C^\epsilon(\mathbb{R})$ for an $\epsilon > 0$, then there exists a polynomial P such that

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^s \log\left(\frac{1}{|x - x_0|}\right).$$

Thus we see that F is $C^{s-\epsilon}(x_0)$ $\forall\epsilon > 0$ if $\forall\epsilon > 0$, $\forall\lambda \in \Lambda(\alpha,\beta)$,

$$2^{-(\gamma+1/2)j} \leq C2^{-(s-\epsilon+1/2)j}(1 + |2^j x_0 - k|)^{s-\epsilon} = C2^{-j/2}(2^{-j} + \mathrm{dist}(x_0,\lambda))^{s-\epsilon}.$$

Conversely, if $\exists\lambda \in \Lambda(\alpha,\beta)$ corresponding to arbitrary large values of j such that

$$2^{-(\gamma+1/2)j} \geq C2^{-j/2}(2^{-j} + \mathrm{dist}(x_0,\lambda))^{s+\epsilon},$$

F does not belong to $C^{s+\epsilon}(x_0)$ $\forall\epsilon > 0$. These two conditions can be written as

$$\limsup_{\lambda\in\Lambda(\alpha,\beta)} 2^{-\gamma j}(2^{-j} + \mathrm{dist}(x_0,\lambda))^{-s-\epsilon} = +\infty \quad \text{for any } \epsilon > 0 \tag{6.6}$$

and

$$\limsup_{\lambda\in\Lambda(\alpha,\beta)} 2^{-\gamma j}(2^{-j} + \mathrm{dist}(x_0,\lambda))^{-s+\epsilon} < \infty \quad \text{for any } \epsilon > 0. \tag{6.7}$$

Condition (6.6) can also be written as

$$2^{-j} + \mathrm{dist}(x_0,\lambda) = \eta(\lambda)2^{-\frac{\gamma}{s+\epsilon}j},$$

where $\liminf \eta(\lambda) = 0$. Since $s \geq \gamma$, $2^{-j} = o(2^{-\frac{\gamma}{s+\epsilon}j})$ and the only condition to be checked is

$$\mathrm{dist}(x_0,\lambda) = \eta(\lambda)2^{-\frac{\gamma}{s+\epsilon}j}.$$

Since $\lambda \in \Lambda_m^{(\lambda,\beta)}$, this condition is equivalent to (6.3).

The same proof shows that (6.7) becomes (6.4). Hence we have Lemma 6.3.

We now define a compact K_α and sets $E_{\alpha,\delta}$. K_α will be composed of the limit points of the F_m, and the $E_{\alpha,\delta}$'s will be subsets of K_α.

Let K_α be the compact set of the sums $\sum_1^\infty \epsilon_j l_j$, where $\epsilon_j = \pm 1$. Another equivalent definition is

$$K_\alpha = \bigcap_1^\infty (F_m + [-\lambda_m, \lambda_m]),$$

where

$$\lambda_m = l_{m+1} + l_{m+2} + \cdots.$$

Note that the sets $G_m = F_m + [-\lambda_m, \lambda_m]$ form a decreasing sequence of compact sets.

Let $G_m^{(\beta)}$ be defined by $G_m^{(\beta)} = F_m + [-\lambda_m^\beta, \lambda_m^\beta]$ and let $E_{\alpha,\beta}$ be the set of points that belong to infinite $G_m^{(\beta)}$'s. Since $\beta \geq 1$, $G_m^{(\beta)} \subset G_m$ so that $E_{\alpha,\beta} \subset K_\alpha$ and, of course, $E_{\alpha,\beta} = K_\alpha$ if $\beta = 1$.

The idea of the construction that we made is as follows. We have placed "large" wavelet coefficients on F_m so that on these sets the function F is exactly Γ^γ, but at points which are at a certain distance on F_m (measured by their belonging to certain $G_m^{(\beta)}$'s), these "large" wavelet coefficients create "weaker" singularities (corresponding to an exponent larger than γ).

LEMMA 6.4. *If $\gamma \leq s < \beta\gamma$, then* (6.3) *is equivalent to $x \in \bigcap_{\delta<\frac{\beta\gamma}{s}} E_{\alpha,\delta}$, while if $s \geq \beta\gamma$, it is equivalent to $x \in K_\alpha$. Condition* (6.4) *is equivalent to $x \notin \bigcup_{\delta>\frac{\beta\gamma}{s}} E_{\alpha,\delta}$, while if $s > \beta\gamma$, it is equivalent to $x \notin K_\alpha$.*

Proof. If (6.3) holds and if $\delta < \beta\gamma/s$, let us check that $x \in E_{\alpha,\delta}$. To this end, we choose $\epsilon > 0$ such that $\delta < \beta\gamma/s - \epsilon$. Then

$$\mathrm{dist}(x_0, F_m) = \eta_m 2^{-\frac{\alpha\beta\gamma}{s}m} = o(2^{-(\frac{\alpha\beta\gamma}{s}-\epsilon)m})$$

so that $\mathrm{dist}(x_0, F_m) = o(l_m^\delta) \leq \lambda_m^\delta$ (because $\lambda_m \sim l_m$) for infinite values of m. Thus $x \in E_{\alpha,\delta}$.

Conversely, if $x \in E_{\alpha,\delta}$, $\operatorname{dist}(x, F_m) \leq \lambda_m^\delta$ so that $\operatorname{dist}(x, F_m) \leq C2^{-\alpha\delta m}$ for infinite values of m. If $\delta > \beta\gamma/s - \epsilon$, we get (6.3). When $s \geq \beta\gamma$, we observe that if $\eta_m > 0$ is an arbitrary sequence such that $\liminf \eta_m = 0$ and if

$$x \in \bigcap_{m \geq 1} F_m + [-\eta_m, \eta_m],$$

then $x \in K_\alpha$. This is because K_α is a compact set, and if $x \notin K_\alpha$, then $\operatorname{dist}(x, K_\alpha) = \eta > 0$ so that $\operatorname{dist}(x, F_m) \geq \eta$; hence we have a contradiction. Condition (6.3) is thus equivalent to $x \in K_\alpha$ as soon as $s \geq \beta\gamma$. The proof of the second part of the lemma is similar.

LEMMA 6.5. *The Hausdorff dimension of $E_{\alpha,\beta}$ is $1/\alpha\beta$. If $\gamma \leq s \leq \beta\gamma$, the Hausdorff dimension of $E^{(s)}$ is $s/\alpha\beta\gamma$. If $s > \beta\gamma$, the set $E^{(s)}$ is empty.*

The set $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta} = \bigcap_{m \geq 1} E_{\alpha,\beta}^{(m)} \quad \text{where} \quad E_{\alpha,\beta}^{(m)} = G_m^{(\beta)} \cup G_{m+1}^{(\beta)} \cup \cdots \quad \text{and} \quad G_m^{(\beta)} = F_m + [-\lambda_m^\beta, \lambda_m^\beta].$$

For any $\epsilon > 0$, we can cover $E_{\alpha,\beta}$ by the intervals I_q that appear in $G_n^{(\beta)}$, $n \geq m$. For a fixed n, there are 2^n such intervals of length $\sim 2^{-\alpha n\beta}$ so that if $d > 1/\alpha\beta$, $\sum |I_q|^d \leq C$, where C does not depend on ϵ. Thus the Hausdorff dimension of $E_{\alpha,\beta}$ is bounded by $1/\alpha\beta$.

Now suppose that $\gamma \leq s \leq \beta\gamma$. Then

$$E^{(s)} = \left(\bigcap_{\delta < \frac{\beta\gamma}{s}} E_{\alpha,\delta}\right) \Big\backslash \left(\bigcup_{\delta > \frac{\beta\gamma}{s}} E_{\alpha,\delta}\right) \quad \text{if} \ \gamma \leq s < \beta\gamma,$$

while if $s = \beta\gamma$,

$$E^{(s)} = K_\alpha \Big\backslash \left(\bigcup_{\delta > 1} E_{\alpha,\delta}\right).$$

Checking is done the same way in both cases, so we suppose that $\gamma \leq s < \beta\gamma$. Thus $E^{(s)} \subset E_{\alpha,\delta}$ for all $\delta < \beta\gamma/s$ so that $\dim(E^{(s)}) \leq s/\alpha\beta\gamma$. Hence we have the two upper bounds for the Hausdorff dimensions in Lemma 6.5.

In order to obtain the lower bounds, we use a standard procedure. We construct a probability measure μ that is supported on $E_{\alpha,\beta}$ and has certain "scalings."

We now construct this measure.

Let $m_1 < m_2 < \cdots$ be an increasing sequence of integers that tends to ∞ quickly enough that for any $n \geq 1$, $m_{n+1} \geq \exp(m_n)$, and now let

$$K_{(\alpha,\beta)} = \bigcap_{n \geq 1} \tilde{G}_{m_n}^{(\beta)} \tag{6.8}$$

with

$$\tilde{G}_m^{(\beta)} = F_m + [-2^{-[\alpha\beta m]}, 2^{-[\alpha\beta m]}]. \tag{6.9}$$

This means that $\tilde{G}_m^{(\beta)}$ is a finite union of dyadic intervals, and the dyadic intervals that form $\tilde{G}_{m+1}^{(\beta)}$ will either be disjoint of those composing $\tilde{G}_m^{(\beta)}$ or included in them (just because they are dyadic intervals of smaller length). We have $[\alpha\beta m] \geq \beta[\alpha m]$, and the set $\tilde{G}_m^{(\beta)}$ is included in $G_m^{(\beta)}$ so that $K_{(\alpha,\beta)} \in E_{\alpha,\beta}$.

Let N_n be the number of intervals of length $2.2^{-[\alpha\beta m_n]}$ that can be found in $H_n = \tilde{G}_{m_1}^{(\beta)} \cap \cdots \cap \tilde{G}_{m_n}^{(\beta)}$, and let μ_n be the probability measure which on each of these N_n intervals takes the value $2^{[\alpha\beta m_n]}(2N_n)^{-1}dx$. We can easily check that $\mu_n \rightharpoonup \mu$ when $n \to \infty$, where $\mu = \mu_{(\alpha,\beta)}$ is supported by $K_{(\alpha,\beta)}$.

LEMMA 6.6. *There exists C such that $\forall I$ of length $|I| \leq 1/2$,*

$$\mu(I) \leq C|I|^{1/\alpha\beta} \log\frac{1}{|I|}. \tag{6.10}$$

Proof. We first estimate N_n. H_n is composed of N_n intervals of length $2^{-[\alpha\beta m_n]}$. When constructing H_{n+1}, we split each of these intervals into $2^{m_{n+1}-\beta m_n+\epsilon_n}$ intervals, where $|\epsilon_n| \leq 2$. Thus $N_{n+1} = N_n 2^{m_{n+1}-\beta m_n+\epsilon_n}$.

Now let I be an interval and define n by $2^{-\alpha\beta m_n} \leq |I| < 2^{-\alpha\beta m_{n-1}}$.

Consider the two cases $2^{-\alpha\beta m_n} \leq |I| < 2^{-\alpha m_n}$ and $2^{-\alpha m_n} \leq |I| < 2^{-\alpha\beta m_{n-1}}$. In the first one, I intersects at most two of the intervals that compose H_n so that

$$\mu(I) \leq CN_n^{-1} \leq C' 2^{-m_n+O(m_{n-1})} \leq C|I|^{1/\alpha\beta}\log(|I|)$$

since $m_n \geq \exp(m_{n-1})$. In the second case, suppose that $|I| \sim 2^{-\alpha j}$. Thus $\beta m_{n-1} \leq j \leq m_n$. I meets at most 2^{m_n-j} intervals so that $\mu(I) \leq 2^{m_n-j}/N_n$, but $N_n = N_{n-1}2^{m_n-\beta m_{n-1}+\epsilon_n}$. Thus

$$\begin{aligned}\mu(I) &\leq \frac{2^{-j}2^{\beta m_{n-1}-\epsilon_n}}{N_{n-1}} \leq C|I|^{1/\alpha\beta}2^{j/\beta}2^{-j}2^{\beta m_{n-1}}2^{-m_{n-1}+O(m_{n-2})} \\ &\leq C|I|^{1/\alpha\beta}2^{\frac{(\beta-1)}{\beta}(\beta m_{n-1}-j)+O(m_{n-2})}\end{aligned}$$

so that $\mu(I) \leq C|I|^{1/\alpha\beta}\log(|I|)$. Hence we have Lemma 6.6. □

We now prove the lower bounds in Lemma 6.5. We use the following slight modification of Hausdorff measure. Let $A \subset \mathbb{R}^m$ and R_ε be the set of all coverings of A by sets of diameter at most ε. Let

$$M(\varepsilon, d) \;=\; \inf_{r\in R_\varepsilon} \sum_{A_i\in r} (\operatorname{diam} A_i)^d \log\left(\frac{1}{(\operatorname{diam} A_i)}\right)$$

and let

$$d - \operatorname{mes}(A) \;=\; \limsup_{\varepsilon\to 0}\, M(\varepsilon, d)$$

be this "modified" d-dimensional Hausdorff measure. Of course, this modification does not change the Hausdorff dimension of A, which is

$$D \;=\; \inf\{d : d - \operatorname{mes}(A) = 0\} = \;\sup\,\{d : d - \operatorname{mes}(A) = +\infty\}.$$

We conclude with the following classical proposition (cf. [12]).

PROPOSITION 6.7. *Let $\mathcal{H}^s$ be the modified Hausdorff measure of dimension s. Let μ be a probability measure on $\mathbb{R}^m$, $F \in \mathbb{R}^m$. If* $\limsup_{r\to 0}\mu(B(x,r))/r^s\log(1/r) < C\ \forall x \in F$,

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{C}.$$

The first lower bound in Lemma 6.5 is thus a consequence of Lemma 6.6 and Proposition 6.7. The $\mathcal{H}^{1/\alpha\beta}$ measure of $E_{\alpha,\beta}$ is strictly positive, and thus the Hausdorff dimension of $E_{\alpha,\beta}$ is at least $1/\alpha\beta$.

We show that $\dim(E^{(s)}) \geq s/\alpha\beta\gamma$. Let μ be the probability measure $\mu_{\alpha,\beta\gamma/s}$. We check that

$$E^{(s)} \supset E_{\alpha,\beta\gamma/s} \Big\backslash \left(\bigcup_{\delta > \beta\gamma/s} E_{\alpha,\delta} \right) \tag{6.11}$$

and

$$\mu(E_{\alpha,\delta}) = 0 \quad \text{ for any } \delta > \frac{\beta\gamma}{s};$$

since the union of these sets can be written as a countable union, the measure of their union vanishes so that the measure of $E^{(s)}$ is the same as the measure of $E_{\alpha,\beta\gamma/s}$, which is strictly positive. Hence we have the last point of Lemma 6.5.

We now prove the general case in Proposition 6.1.

Let $E_1, E_2, \dots$ be disjoint subsets of $\mathbb{R}$ and suppose that $E_k \subset [a_k, b_k]$, where the $[a_k, b_k]$'s are disjoint. Let d_k be the Hausdorff dimension of E_k. Then the Hausdorff dimension of $\bigcup_{k\geq 1} E_k$ is $\sup(d_k)$.

We return to the function $F_{(\alpha,\beta,\gamma)}$. Clearly, $F_{(\alpha,\beta,\gamma)}$ has fast decay and is C^∞ outside of a compact set. After replacing $F_{(\alpha,\beta,\gamma)}(x)$ by $F_{(\alpha,\beta,\gamma)}(px+q)$, we can, without changing the spectrum of $F_{(\alpha,\beta,\gamma)}$, suppose that it is C^∞ outside any given interval $[a,b]$. Let $F_k(x) = f_{(\alpha_k,\beta_k,\gamma_k)}(x)$ be a sequence of functions as in Proposition 6.2 and consider the corresponding spectra $d_k(s)$. We can suppose that the singular supports of the $F_k(x)$'s are included in $[2^{-k-1}, 2^{-k}]$. (The singular support of a function is the closure of the set where this function is not C^∞.) We can also replace F_k by $\epsilon_k F_k$, where $\epsilon_k > 0$ tends to 0. Then let $G_1 = \sum_0^\infty \epsilon_k F_k$, and $d(s)$ is the supremum of the $d_k(s)$'s. The function G_1 thus constructed satisfies the requirements of Proposition 6.1, since we can easily check that a supremum of a countable set of functions of the form $ax1_{[b,c]}(x)$ is also a supremum of functions of the form $a1_{[b,c]}(x)$.

The construction of G_2 is now very easy. We remark that at each level j, the number of nonvanishing wavelet coefficients of $F_{(\alpha,\beta,\gamma)}$ is $o(2^j)$. Thus the same property holds for G_1 itself if we have chosen the contraction factors p (defined above) to be large enough. We now consider a function G_2 that has at each level j the same nonvanishing wavelet coefficients as G_1 but situated at different dyadic intervals. We group them in the smallest possible interval I_j centered at the origin. Thus the quantity (4.2) is the same for G_1 and G_2 so that these two functions share the same $B_p^{s,\infty}$ norm and hence the same function η. Nonetheless, if $x \neq 0$, there are a finite number of nonvanishing wavelet coefficients in a certain interval centered at x because the length of I_j tends to 0. Thus F_2 is C^∞ at x.

We now check that G_1 and G_2 are counterexamples to the following problem raised in [9]: Is η or ζ the Legendre transform of $m - d(\alpha)$?

Consider the function F defined by (6.1). At each level $j = [\alpha\beta m]$, it has 2^m wavelet coefficients equal to $2^{-(\gamma+1/2)j}$ so that for this j,

$$\left(\sum_k |C_{j,k}|^p \right)^{1/p} \sim 2^{(-(\gamma+1/2)+1/p\alpha\beta)j}$$

so that

$$\|F\|_{B_p^{s,\infty}} \sim 2^{js} 2^{j(\frac{1}{2}-\frac{1}{p})} \left(\sum_k |C_{j,k}|^p \right)^{1/p} \sim 2^{j(s-\gamma+\frac{c\gamma}{p}-\frac{1}{p})}$$

and $\eta(p) = ap + 1 - ca$.

Thus $\eta(p)$ is linear and does not depend on b so that it clearly can be the Legendre transform of neither $cs1_{[a,b]}(s)$ (when $a \neq b$) nor the function 0. Thus in general, neither F_1 nor F_2 satisfies that η or ζ is the Legendre transform of its spectrum.

7. Acknowledgments. The author is thankful to Alain Arneodo, Emmanuel Bacry, Benoit Mandelbrot, and Yves Meyer for many enlightening discussions.

REFERENCES

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1978.
[2] A. Arneodo, E. Bacry, J. F. Muzy, *The thermodynamics of fractals revisited with wavelets*, Phys. A, 213 (1995), pp. 232–275.
[3] E. Bacry, A. Arneodo, J. F. Muzy, *Singularity spectrum of fractal signals from wavelet analysis: Exact results*, J. Statist. Phys., 70 (1993), p. 314.
[4] R. Benzi, L. Biferale, and G. Parisi, *On intermittency in a cascade model for turbulence*, Phys. D, 65 (1993), pp. 163–171.
[5] G. Brown, G. Michon, and J.Peyrière, *On the multifractal analysis of measures*, J. Statist. Phys., 66 (1992), pp. 775–790.
[6] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math., 35 (1982), p. 771.
[7] P. Collet, J. Lebowitz, and A. Porzio, *The dimension spectrum of some dynamical systems* J. Statist. Phys., 47 (1987), pp. 609–644.
[8] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math., 41 (1988), pp. 909–996.
[9] I. Daubechies and J. Lagarias, *On the thermodynamic formalism for multifractal functions*, in The State of Matter, M. Aizenman and H. Araki, eds., World Scientific, Singapore, 1994, pp. 213–265.
[10] A. Deliu and B. Jawerth, *Geometrical dimension versus smoothness*, Constr. Approx., 2 (1992), pp. 211–222.
[11] G. Eyink, *Besov spaces and the multifractal hypothesis*, J. Stat. Phys., 78 (1995), pp. 353–375.
[12] I. Falconer, *Fractal Geometry,* John Wiley, New York, 1990.
[13] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood–Paley theory and the study of function spaces*, CBMS Ser. 79, AMS, Providence, RI, 1991.
[14] U. Frisch and G. Parisi, *Fully developed turbulence and intermittency*, in Proc. Enrico Fermi International Summer School in Physics, North–Holland, Amsterdam, 1985, pp. 84–88.
[15] M. Holschneider, *Fractal wavelet dimensions and localization*, Comm. Math. Phys., 160 (1994), pp. 457–473.
[16] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., 30 (1981), pp. 713–747.
[17] S. Jaffard, *Pointwise smoothness, two-microlocalization and wavelet coefficients*, Pub. Mat., 35 (1991), pp. 155–168.
[18] S. Jaffard, *Sur la dimension de Hausdorff des points singuliers d'une fonction*, C. R. Acad. Sci. Paris Sér. I Math., 314 (1991), pp. 31–36 (in French).
[19] S. Jaffard, *Construction de fonctions multifractales ayant un spectre de singularités prescrit*, C. R. Acad. Sci. Paris Sér. I Math., 315 (1992), pp. 19–24 (in French).
[20] S. Jaffard, *Formalisme multifractal pour les fonctions* C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), pp. 745–750 (in French).
[21] S. Jaffard and Y. Meyer, *Wavelet methods for Pointwise regularity and local oscillations of functions*, Mem. Amer. Math. Soc., 123 (1996).
[22] L. Olsen, *A multifractal formalism*, Adv. Math., to appear.
[23] B. Mandelbrot, *Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier*, J. Fluid Mech., 62 (1974), p. 331.
[24] C. Meneveau and K. R. Sreenivasan, *Measurement of $f(\alpha)$ from scaling of histograms and applications to dynamical systems and fully developed turbulence*, Phys. Lett. A, 137 (1989), pp. 103–112.

[25] Y. Meyer, *Ondelettes et opérateurs*, Hermann, Paris, 1990.
[26] J. F. Muzy, A. Arneodo, and E. Bacry, *A multifractal formalism revisited with wavelets*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 4 (1994), p. 245.
[27] S. M. Nicolskii, *Extension of functions of several variables preserving differential properties*, Amer. Math. Soc. Trans. (2), 83 (1969), pp. 159–188.