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Abstract

Generalizing the approach of a previous work [15] the authors present multilevel preconditioners for three-dimensional (3D) elliptic problems discretized by a family of Rannacher Turek non-conforming finite elements.

Preconditioners based on various multilevel extensions of two-level finite element methods (FEM) lead to iterative methods which often have an optimal order computational complexity with respect to the number of degrees of freedom of the system. Such methods were first presented in [6, 7], and are based on (recursive) two-level splittings of the finite element space.

An important point to make is that in the case of non-conforming elements the finite element spaces corresponding to two successive levels of mesh refinement are not nested in general. To handle this, a proper two-level basis is required to enable us to fit the general framework for the construction of two-level preconditioners for conforming finite elements and to generalize the method to the multilevel case.

The first major contribution of the paper is the derived estimates of the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality which is shown to allow the efficient multilevel extension of the related two-level preconditioners. Representative numerical tests well illustrate the optimal complexity of the resulting iterative solver, also for the case of non-smooth coefficients. The second important achievement concerns the experimental study of AMLI solvers applied to the case of voxel FEM simulation. Here the coefficient jumps are resolved on the finest mesh only and therefore the classical CBS inequality based convergence theory is not directly applicable. The obtained results, however, demonstrate the efficiency of the proposed algorithms in this case also, as illustrated by an example of microstructure analysis of bones.

KEY WORDS: non-conforming FEM, multilevel preconditioners, hierarchical basis, CBS constant.

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1 Introduction

In this paper we consider the elliptic boundary value problem

$$\begin{aligned} Lu \equiv -\nabla \cdot (\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})) &= f(\mathbf{x}) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma_D, \\ (\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})) \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{1}$$

where Ω is a polyhedral domain in $\mathbb{R}^3$, $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, the coefficient matrix $\mathbf{a}(\mathbf{x})$ is symmetric positive definite and uniformly bounded in Ω, $\mathbf{n}$ is the outward unit vector normal to the boundary $\Gamma = \partial\Omega$, and $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. We assume that the elements of the diffusion coefficient matrix $\mathbf{a}(\mathbf{x})$ are piecewise smooth functions on $\bar{\Omega}$.

The weak formulation of the above problem reads as follows:
given $f \in L^2(\Omega)$ find $u \in \mathcal{V} \equiv H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$\mathcal{A}(u,v) = (f,v) \quad \forall v \in H^1_D(\Omega), \quad \text{where} \quad \mathcal{A}(u,v) = \int_\Omega \mathbf{a}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}. \tag{2}$$

We assume that the domain Ω is discretized by the partition $\mathcal{T}_h$ which is obtained by a proper refinement of a given coarser partition $\mathcal{T}_H$. We assume also that $\mathcal{T}_H$ is aligned with the discontinuities of the coefficient $\mathbf{a}(\mathbf{x})$ so that over each element $E \in \mathcal{T}_H$ the coefficients of $\mathbf{a}(\mathbf{x})$ are smooth functions.

The variational problem (2) is then discretized using the finite element method, i.e., the continuous space $\mathcal{V}$ is replaced by a finite dimensional subspace $\mathcal{V}_h$. Then the finite element formulation is: find $u_h \in \mathcal{V}_h$, satisfying

$$\mathcal{A}_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathcal{V}_h, \quad \text{where} \quad \mathcal{A}_h(u_h, v_h) = \sum_{e \in \mathcal{T}_h} \int_e \mathbf{a}(e)\nabla u_h \cdot \nabla v_h d\mathbf{x}. \tag{3}$$

Here $\mathbf{a}(e)$ is a piecewise constant symmetric positive definite matrix, defined by the integral averaged values of $\mathbf{a}(\mathbf{x})$ over each element from the coarser triangulation $\mathcal{T}_H$. We note that in this way strong coefficient jumps across the boundaries between adjacent finite elements from $\mathcal{T}_H$ are allowed.

The resulting discrete problem to be solved is then a linear system of equations

$$A_h \mathbf{u}_h = \mathbf{f}_h, \tag{4}$$

with A_h and $\mathbf{f}_h$ being the corresponding global stiffness matrix and global right-hand side, and h being the discretization (meshsize) parameter of the underlying partition $\mathcal{T}_h$ of Ω.

The aim of this paper is to investigate multilevel preconditioners of optimal complexity for solving the system (4). The general setting and some well-known results for the case of conforming finite elements are summarized in the rest of this section. The next sections are devoted to the study of two-level and multilevel preconditioners for the case of non-conforming Rannacher-Turek finite elements. A unified hierarchical splitting of the FEM spaces is developed, followed by a local analysis that results in uniform estimates of the angle between the "coarse" space and its complementary space. The numerical results that are presented towards the end of the paper are completed by some concluding remarks.

1.1 Two-level setting

We are concerned with the construction of a two-level preconditioner M for A_h, such that the spectral condition number $\varkappa(M^{-1}A_h)$ of the preconditioned matrix $M^{-1}A_h$ is uniformly bounded w.r.t. mesh size h and possible coefficient jumps (if the averaging of coefficients on the macro elements is used).

The classical theory for constructing optimal order two-level preconditioners was first developed in [4, 9], see also [3]. The general framework requires to define two nested finite element spaces $\mathcal{V}_H \subset \mathcal{V}_h$, that correspond to two consecutive (regular) mesh refinements. The well studied case of conforming linear finite elements is the starting point in the theory of two- and multilevel methods.

Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be two successive mesh refinements of the domain Ω, which correspond to $\mathcal{V}_H$ and $\mathcal{V}_h$. Let $\{\phi_H^{(k)}, k = 1, 2, \cdots, N_H\}$ and $\{\phi_h^{(k)}, k = 1, 2, \cdots, N_h\}$ be the standard finite element nodal basis functions. We split the meshpoints $\mathbf{N}_h$ from $\mathcal{T}_h$ into two groups: the first group contains the nodes $\mathbf{N}_H$ from $\mathcal{T}_H$ and the second one consists of the rest, where the latter are the newly added node-points $\mathbf{N}_{h\backslash H}$ from $\mathcal{T}_h \backslash \mathcal{T}_H$. Next we define the so-called hierarchical basis functions

$$\{\widetilde{\phi}_h^{(k)}, k = 1, 2, \cdots, N_h\} = \{\phi_H^{(l)} \text{ on } \mathcal{T}_H\} \cup \{\phi_h^{(m)} \text{ on } \mathcal{T}_h \backslash \mathcal{T}_H\}. \tag{5}$$

Let then $\widetilde{A}_h$ be the corresponding hierarchical stiffness matrix. Under the splitting (5) both matrices A_h and $\widetilde{A}_h$ admit in a natural way a two-by-two block structure

$$A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \} \mathbf{N}_{h\backslash H} \\ \} \mathbf{N}_H \end{matrix}, \qquad \widetilde{A}_h = \begin{bmatrix} A_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & A_H \end{bmatrix} \begin{matrix} \} \mathbf{N}_{h\backslash H} \\ \} \mathbf{N}_H \end{matrix}. \tag{6}$$

Remark 1.1 Clearly, the hierarchical stiffness matrix $\widetilde{A}_h$ is more dense than A_h and therefore its action on a vector is computationally more expensive. However, there exists a sparse transformation matrix J, which enables us in practical implementations to work with A_h, since $\widetilde{A}_h = J A_h J^T$.

1.2 Two-level preconditioners and the strengthened Cauchy-Bunyakowski-Schwarz inequality

The key role in the derivation of optimal convergence rate estimates of two- and multilevel methods plays the constant γ in the so-called strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality, associated with the angle between the two subspaces of the splitting. More precisely, the value of the upper bound for $\gamma \in (0, 1)$ is a part of the construction of various multilevel extensions of the related two-level methods.

Consider a general matrix A, which is assumed to be symmetric positive definite and partitioned as in (6). The quality of this partitioning is characterized by the corresponding strengthened CBS inequality constant:

$$\gamma = \sup_{\mathbf{v}_1 \in \mathbb{R}^{n_1 - n_2},\, \mathbf{v}_2 \in \mathbb{R}^{n_2}} \frac{\mathbf{v}_1^T A_{12} \mathbf{v}_2}{\left(\mathbf{v}_1^T A_{11} \mathbf{v}_1\right)^{1/2} \left(\mathbf{v}_2^T A_{22} \mathbf{v}_2\right)^{1/2}}, \tag{7}$$

where $n_1 = |\mathbf{N}_h|$ and $n_2 = |\mathbf{N}_H|$ denote the cardinality of the sets $\mathbf{N}_h$ and $\mathbf{N}_H$, respectively.

Consider now the two-level preconditioners (a detailed description can be found, e.g., in [3]) to A under the assumptions

$$A_{11} \le C_{11} \le (1 + \delta_1) A_{11} \text{ and } A_{22} \le C_{22} \le (1 + \delta_2) A_{22}. \tag{8}$$

The inequalities (8) are in a positive semidefinite sense where C_{11} and C_{22} are symmetric and positive definite matrices for some positive constants δ_i, $i = 1, 2$.

The additive preconditioner M_A and the multiplicative preconditioner M_F are then introduced as

$$M_A = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}, \text{ and } M_F = \begin{bmatrix} C_{11} & 0 \\ A_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I_1 & C_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{9}$$

respectively. When $C_{11} = A_{11}$ and $C_{22} = A_{22}$, then the following estimates hold (for some details on various two-level estimates, see, e.g., [3]):

$$\varkappa(M_A^{-1}A) \leq \frac{1+\gamma}{1-\gamma}, \text{ and } \varkappa(M_F^{-1}A) \leq \frac{1}{1-\gamma^2}.$$

In the hierarchical bases context $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of the finite element space $\mathcal{V}_h$ spanned, respectively, by the basis functions at the new nodes $\mathbf{N}_{h\setminus H}$ and by the basis functions at the old nodes $\mathbf{N}_H$. For the strengthened CBS inequality constant, there holds that

$$\gamma = \cos(\mathcal{V}_1,\, \mathcal{V}_2) = \sup_{u\,\in\,\mathcal{V}_1,\; v\,\in\,\mathcal{V}_2} \frac{\mathcal{A}(u,v)}{\sqrt{\mathcal{A}(u,u)\mathcal{A}(v,v)}} \tag{10}$$

where $\mathcal{A}(\cdot,\cdot)$ is the bilinear form which appears in the variational formulation of the original problem. As shown in [4], the constant γ can be estimated locally over each (macro) finite element $E \in \mathcal{T}_H$, which means that $\gamma = \max\limits_E \gamma_E$, where

$$\gamma_E = \sup_{u\,\in\,\mathcal{V}_1(E),\; v\,\in\,\mathcal{V}_2(E)} \frac{\mathcal{A}_E(u,v)}{\sqrt{\mathcal{A}_E(u,u)\mathcal{A}_E(v,v)}}, \quad v \neq \text{const}.$$

The spaces $\mathcal{V}_k(E)$ above contain the functions from $\mathcal{V}_k$ restricted to E and $\mathcal{A}_E(u,v)$ corresponds to $\mathcal{A}(u,v)$ restricted over the element E of $\mathcal{T}_H$ (see also [16]).

We stress here, that the above technique is originally developed and straightforwardly applicable for conforming finite elements and nested finite element spaces only, i.e., when $\mathcal{V}_H \subset \mathcal{V}_h$.

2 Rannacher-Turek finite elements

Non-conforming finite elements based on *rotated* multilinear shape functions were introduced by Rannacher and Turek [22] as a class of simple elements for the Stokes problem. More generally, recent activities in the development of efficient solution methods for non-conforming finite element systems are inspired by their attractive properties as a stable discretization tool for ill-conditioned problems.

The cube $[-1,1]^3$ is used as a reference element $\hat{e}$ to define the isoparametric rotated trilinear element $e \in \mathcal{T}_h$. Let $\psi_e : \hat{e} \to e$ be the trilinear bijective mapping between the reference element $\hat{e}$ and e. The polynomial space of shape functions $\hat{\phi}_i$ on the reference element $\hat{e}$ is defined by

$$\hat{\mathcal{P}} := \{\hat{\phi}_i : 1 \leq i \leq 6\} = \text{span}\{1, x, y, z, x^2 - y^2, y^2 - z^2\},$$

and the shape functions ϕ_i on e are computed from $\hat{\phi}_i$ via the relations

$$\{\phi_i\}_{i=1}^6 = \{\hat{\phi}_i \circ \psi_e^{-1}\}_{i=1}^6,$$

where $\circ$ means the superposition of functions $\hat{\phi}_i$ and ψ_e^{-1}.

In the following, two different discretization variants, i.e., two different sets of shape functions $\hat{\phi}_i$ are considered. For the variant MP (mid point), $\{\hat{\phi}_i\}_{i=1}^6$ are found by the point-wise interpolation condition

$$\hat{\phi}_i(b_\Gamma^j) = \delta_{ij},$$

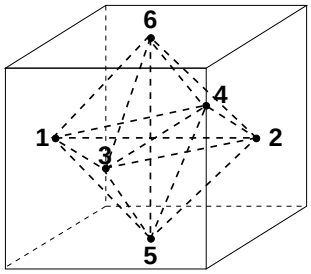


Figure 1: Node numbering and connectivity pattern of the reference element $\hat{e}$.

where $b_\Gamma^j, j = 1, 6$ are the centers of the faces of the cube $\hat{e}$. Then,

$$\hat{\phi}_1(x,y,z) = \frac{1-3x+2x^2-y^2-z^2}{6}, \quad \hat{\phi}_2(x,y,z) = \frac{1+3x+2x^2-y^2-z^2}{6},$$
$$\hat{\phi}_3(x,y,z) = \frac{1-x^2-3y+2y^2-z^2}{6}, \quad \hat{\phi}_4(x,y,z) = \frac{1-x^2+3y+2y^2-z^2}{6},$$
$$\hat{\phi}_5(x,y,z) = \frac{1-x^2-y^2-3z+2z^2}{6}, \quad \hat{\phi}_6(x,y,z) = \frac{1-x^2-y^2+3z+2z^2}{6}.$$

Alternatively, the variant MV (mean value) corresponds to integral mean-value interpolation condition

$$|\Gamma_{\hat{e}}^j|^{-1} \int_{\Gamma_{\hat{e}}^j} \hat{\phi}_i d\Gamma_{\hat{e}}^j = \delta_{ij},$$

where $\Gamma_{\hat{e}}^j$ are the faces of the reference element $\hat{e}$. This leads to

$$\hat{\phi}_1(x,y,z) = \frac{2-6x+6x^2-3y^2-3z^2}{12}, \quad \hat{\phi}_2(x,y,z) = \frac{2+6x+6x^2-3y^2-3z^2}{12},$$
$$\hat{\phi}_3(x,y,z) = \frac{2-3x^2-6y+6y^2-3z^2}{12}, \quad \hat{\phi}_4(x,y,z) = \frac{2-3x^2+6y+6y^2-3z^2}{12},$$
$$\hat{\phi}_5(x,y,z) = \frac{2-3x^2-3y^2-6z+6z^2}{12}, \quad \hat{\phi}_6(x,y,z) = \frac{2-3x^2-3y^2+6z+6z^2}{12}.$$

Let us consider the model isotropic problem with diagonal coefficient matrix

$$\mathbf{a}(\mathbf{x}) = a(e) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

In what follows we will assume that all elements in the triangulation are cubes with mesh size h. Then the element stiffness matrices, corresponding to the variants MP and MV are given by

$$A_e^{MP} = a(e)\frac{2h}{9} \begin{bmatrix} 17 & -1 & -4 & -4 & -4 & -4 \\ -1 & 17 & -4 & -4 & -4 & -4 \\ -4 & -4 & 17 & -1 & -4 & -4 \\ -4 & -4 & -1 & 17 & -4 & -4 \\ -4 & -4 & -4 & -4 & 17 & -1 \\ -4 & -4 & -4 & -4 & -1 & 17 \end{bmatrix}, \quad A_e^{MV} = a(e)2h \begin{bmatrix} 3 & 1 & -1 & -1 & -1 & -1 \\ 1 & 3 & -1 & -1 & -1 & -1 \\ -1 & -1 & 3 & 1 & -1 & -1 \\ -1 & -1 & 1 & 3 & -1 & -1 \\ -1 & -1 & -1 & -1 & 3 & 1 \\ -1 & -1 & -1 & -1 & 1 & 3 \end{bmatrix}.$$

3 Hierarchical two-level splittings

Let us consider two consecutive discretizations $\mathcal{T}_H$ and $\mathcal{T}_h$. It is obvious that $\mathcal{V}_H$ and $\mathcal{V}_h$ are not nested in this case, which is well illustrated by Figure 2.

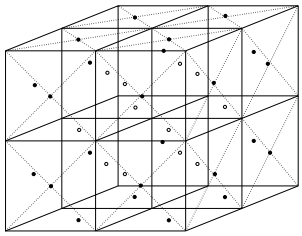


Figure 2: One macro element.

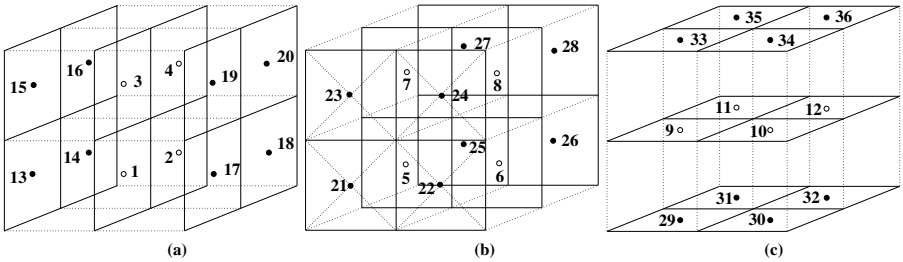


Figure 3: Node numbering in macro element

3.1 "First reduce" (FR) two-level splitting

We follow the idea of [12, 15, 20] to define an algebraic two-level preconditioner. For that reason, let $\varphi_E = \{\phi_i(x,y)\}_{i=1}^{36}$ be the macro-element vector of the nodal basis functions and A_E be the macro-element stiffness matrix corresponding to $E \in \mathcal{T}_h$. The global stiffness matrix A_h is given by

$$A_h = \sum_{E \in \mathcal{T}_h} A_E$$

where the summation is understood as the FEM assembly procedure. Next, we introduce the following macro-element level transformation matrix J_E in the 2×2 block diagonal form

$$J_E = \frac{1}{4} \begin{bmatrix} 4I & \\ & J_{E,22} \end{bmatrix}, \tag{12}$$

where I is the 12×12 identity matrix and

$$J_{E,22} = \begin{bmatrix} P & & & & & \\ & P & & & & \\ & & P & & & \\ & & & P & & \\ & & & & P & \\ & & & & & P \\ E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{bmatrix}. \tag{13}$$

Each block E_i is a 6×4 zero matrix except for its i-th row which is composed of all ones, and

$$P = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The matrix J_E defines locally a two-level hierarchical basis $\widetilde{\varphi}_E$, namely, $\widetilde{\varphi}_E = J_E \varphi_E$. The hierarchical two-level macro-element stiffness matrix is then obtained as

$$\widetilde{A}_E = J_E A_E J_E^T,$$

and the related global stiffness matrix reads as

$$\widetilde{A}_h = \sum_{E \in \mathcal{T}_h} \widetilde{A}_E.$$

We split now the two-level stiffness matrix $\widetilde{A}_h$ into 2×2 block form

$$\widetilde{A}_h = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{14}$$

where $\widetilde{A}_{11}$ corresponds to interior nodal unknowns with respect to the macro elements $E \in \mathcal{T}_h$. The first step of the "First Reduce" (FR) algorithm is to eliminate these unknowns. For this purpose we factor $\widetilde{A}_h$, i.e.,

$$\widetilde{A}_h = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{21} & B \end{bmatrix} \begin{bmatrix} I_1 & \widetilde{A}_{11}^{-1} \widetilde{A}_{12} \\ 0 & I_2 \end{bmatrix}, \tag{15}$$

where $B = \widetilde{A}_{22} - \widetilde{A}_{21} \widetilde{A}_{11}^{-1} \widetilde{A}_{12}$ stands for the Schur complement of this elimination step.
Next we consider a two-level splitting of the matrix B in the block form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{16}$$

where the first block corresponds to the differences of three different couples of basis functions from each macro-element face. The matrix B_{22} corresponds to the sum of basis functions from each macro-element face and can be associated with the coarse grid. It is important to note that

$$\ker(B_{E;22}) = \ker(A_e) = \text{span}\{(1,1,1,1,1,1)^T\}$$

which allows us to apply a local analysis to estimate the constant γ corresponding to the splitting defined by the block partition (16).

For our analysis we proceed as follows:

Step 1: We observe that the upper-left block of $\widetilde{A}_h$ is a block-diagonal matrix. The diagonal entries of $\widetilde{A}_{11}$ are 12×12 blocks, related to the interior points $\{1, 2, \ldots, 12\}$, cf. Figure 2, which are not connected to nodes in other macro elements. Thus, the corresponding unknowns can be eliminated exactly, i.e., to be done locally. Therefore, we first compute the local Schur

complements arising from static condensation of the "interior degrees of freedom" and obtain the (24×24) matrix B_E. Next we split B_E as

$$B_E = \begin{bmatrix} B_{E,11} & B_{E,12} \\ B_{E,21} & B_{E,22} \end{bmatrix} \begin{matrix} \}\text{two-level "difference" basis functions} \\ \}\text{two-level "sum" basis functions} \end{matrix}$$

written again in two-by-two block form.

Step 2: We are now in a position to estimate the CBS constant corresponding to the 2×2 splitting of B. Following the general theory, it suffices to compute the minimal eigenvalue of the generalized eigenproblem

$$S_E \mathbf{v}_E = \lambda_E^{(1)} B_{E,22} \mathbf{v}_E, \quad \mathbf{v}_E \neq \mathbf{c} := (c, c, \ldots, c)^T,$$

where $S_E = B_{E,22} - B_{E,21} B_{E,11}^{-1} B_{E,12}$, and then

$$\gamma^2 \leq \max_{E \in \mathcal{T}_h} \gamma_E^2 = \max_{E \in \mathcal{T}_h} (1 - \lambda_E^{(1)}). \tag{17}$$

3.2 Two-level splitting by differences and aggregates (DA)

Similarly to the FR case, the DA splitting is easily described for one macro element. If $\phi_1, \ldots, \phi_{36}$ are the standard nodal basis functions for the macro element, then we define

$$\begin{aligned}
\mathcal{V}(E) &= \text{span}\,\{\phi_1, \ldots, \phi_{36}\} = \mathcal{V}_1(E) \oplus \mathcal{V}_2(E), \\
\mathcal{V}_1(E) &= \text{span}\,\{\phi_1, \ldots, \phi_{12},\ \phi_{14} + \phi_{16} - (\phi_{13} + \phi_{15}),\ \phi_{15} + \phi_{16} - (\phi_{13} + \phi_{14}), \\
&\qquad \phi_{13} + \phi_{16} - (\phi_{14} + \phi_{15}),\ \ldots, \phi_{34} + \phi_{36} - (\phi_{33} + \phi_{35}), \\
&\qquad \phi_{35} + \phi_{36} - (\phi_{33} + \phi_{34}),\ \phi_{33} + \phi_{36} - (\phi_{34} + \phi_{35})\} \\
\mathcal{V}_2(E) &= \text{span}\,\{\phi_{13} + \phi_{14} + \phi_{15} + \phi_{16} + \sum_{j=1}^{12} \beta_{1j} \phi_j,\ \ldots, \\
&\qquad \phi_{33} + \phi_{34} + \phi_{35} + \phi_{36} + \sum_{j=1}^{12} \beta_{6j} \phi_j\}.
\end{aligned}$$

Using the related transformation matrix

$$J_E = \frac{1}{4} \begin{bmatrix} 4I & \\ J_{E,21} & J_{E,22} \end{bmatrix}, \tag{18}$$

where I is 12×12 identity matrix, $J_{E,22}$ is the same as (13),

$$J_{E,21} = \begin{bmatrix} \mathbf{0} \\ \mathcal{B} \end{bmatrix}$$

and $\mathcal{B} = (\beta_{ij})_{6 \times 12}$. The vector of the macro-element basis functions $\varphi_E = \{\phi_i\}_{i=1}^{36}$ is transformed to a new hierarchical basis $\tilde{\varphi}_E = \{\tilde{\phi}_i\}_{i=1}^{36} = J_E \varphi_E$. Accordingly, J_E transforms the macro-element stiffness matrix into a hierarchical form

$$\tilde{A}_E = J_E A_E J_E^T = \begin{bmatrix} \tilde{A}_{E,11} & \tilde{A}_{E,12} \\ \tilde{A}_{E,21} & \tilde{A}_{E,22} \end{bmatrix} \begin{matrix} \tilde{\phi}_i \in \mathcal{V}_1(E) \\ \tilde{\phi}_i \in \mathcal{V}_2(E) \end{matrix}. \tag{19}$$

Following the local definitions, for the whole finite element space $\mathcal{V}_h$ with the standard nodal finite element basis $\varphi = \{\phi_h^{(i)}\}_{i=1}^{N_h}$ we can similarly construct the new hierarchical basis $\tilde{\varphi} = \{\tilde{\varphi}_h^{(i)}\}_{i=1}^{N_h}$ and the corresponding splitting

$$\mathcal{V}_h = \mathcal{V}_1 \oplus \mathcal{V}_2 \,. \tag{20}$$

The transformation J such that $\tilde{\varphi} = J\varphi$, can be used for transformation of the stiffness matrix A_h to hierarchical form $\tilde{A}_h = JA_hJ^T$, which allows preconditioning by the two-level preconditioners based on the splitting (20).

Now, we are in a position to analyze the constant $\gamma = \cos(\mathcal{V}_1, \mathcal{V}_2)$ for the splitting (20). Again, as in the previous section, we would like to perform this analysis locally, by considering the corresponding problems on macro elements. For this purpose we need to have satisfied the condition

(i) $\ker(\tilde{A}_{E,22}) = \ker(A_e)$,

which is equivalent to

$$\sum_{i=1}^{6} \beta_{ij} = 1 \,, \qquad \forall j \in \{1, 2, \ldots, 12\}. \tag{21}$$

There are obviously various DA splittings satisfying the condition (i). See for some more details about aggregation based preconditioners in the review paper [11].

When the two-level algorithm is recursively generalized to the multilevel case, it could be desirable if

(ii) $\tilde{A}_{E,22}$ is proportional to A_e.

It seems to be rather complicated to find a parameter matrix $\mathcal{B}$, which satisfies the condition (ii) in the general case of Rannacher-Turek trilinear finite elements.

4 Uniform estimates of the CBS constant

We study in this section both splitting algorithms, FR and DA, for both variants MP and MV of rotated trilinear finite elements.

4.1 FR algorithm

Following (17) we compute the local CBS constant and derive the following global estimates for the isotropic model problem on a mesh composed of cubic elements. The bounds are uniform with respect to the size of the discrete problem and any possible jumps of the coefficients.

Variant MP: For the FR splitting we have

$$\lambda_E^{(1)} = \frac{13}{21} \,, \qquad \gamma_E^2 = 1 - \lambda_E^{(1)} = \frac{8}{21},$$

and therefore

$$\gamma_{MP}^2 \leq \frac{8}{21}. \tag{22}$$

Variant MV: For the FR splitting we further have

$$\lambda_E^{(1)} = \frac{1}{2} \,, \qquad \gamma_E^2 = 1 - \lambda_E^{(1)} = \frac{1}{2},$$

and therefore

$$\gamma_{MV}^2 \leq \frac{1}{2}. \tag{23}$$

Let us remind once again, that the obtained estimates hold theoretically for the two-level algorithm only. This is because the matrix B_{22} is only associated with the coarse discretization $e \in \mathcal{T}_H$ and is not proportional to the related element stiffness matrix A_e. As we will see later, the CBS constants have a very stable behavior in the FR multilevel setting, which has been verified numerically, cf. Table 1 and Figure 4.

4.2 DA algorithm

Due to the isotropy of the model problem, the non-zero part $\mathcal{B}$ of the down-left block $J_{E,21}$ of the transformation matrix J_E can be simplified to the form

$$\mathcal{B} = \begin{bmatrix} a & a & a & a & b & c & b & c & b & c & b & c \\ a & a & a & a & c & b & c & b & c & b & c & b \\ b & c & b & c & a & a & a & a & b & b & c & c \\ c & b & c & b & a & a & a & a & c & c & b & b \\ b & b & c & c & b & b & c & c & a & a & a & a \\ c & c & b & b & c & c & b & b & a & a & a & a \end{bmatrix}. \tag{24}$$

The condition (i) is equivalent to

$$a + b + c = 1.$$

Let us write the condition (ii) in the form

$$\tilde{A}_{E,22} = pA_e. \tag{25}$$

Then, (ii) is reduced to a system of three nonlinear equations for (a, b, c), with a parameter p. For the relatively less complicated 2D case, a similar approach was recently proposed in [15]. Now it appears that in the 3D case the system for (a, b, c) has again a solution if $p \in [p_0, \infty)$ for some $p_0 > 0$. In such a case, we can optimize the parameter p, so that the CBS constant is minimal. The obtained results are summarized below. For the related analysis we have used symbolic computations with the computer algebra program MATHEMATICA.

Variant MP:

Lemma 4.1 *There exists a DA two-level splitting satisfying the condition (ii), if and only if,*

$$p \geq \frac{3}{14}.$$

Then, the obtained solutions for (a, b, c) are invariant with respect to the local CBS constant

$$\gamma_E^2 = 1 - \frac{1}{8p},$$

and for the related optimal splitting

$$\gamma_{MP}^2 \leq \frac{5}{12}. \tag{26}$$

Although the statements of Lemma 4.1. look very simply, the midterm derivations are rather technical, which is just illustrated by the following expressions of four different solutions for (a, b, c):

$$\left(\frac{94}{273} - \frac{2}{21}\,\xi(p) - \frac{3\,\eta(p)}{26\sqrt{2}}\,,\ \frac{10 - 26\,\xi(p)}{273} + \frac{3\sqrt{2}\,\eta(p)}{52}\,,\ \frac{5 + 8\,\xi(p)}{42}\right),$$

$$\left(\frac{94}{273} + \frac{2}{21}\,\xi(p) + \frac{3\,\eta(p)}{26\sqrt{2}}\,,\ \frac{10 + 26\,\xi(p)}{273} - \frac{3\sqrt{2}\,\eta(p)}{52}\,,\ \frac{5 - 8\,\xi(p)}{42}\right),$$

$$\left(\frac{94}{273} - \frac{2}{21}\,\xi(p) + \frac{3\,\eta(p)}{26\sqrt{2}}\,,\ \frac{10 - 26\,\xi(p)}{273} - \frac{3\sqrt{2}\,\eta(p)}{52}\,,\ \frac{5 + 8\,\xi(p)}{42}\right),$$

$$\left(\frac{94}{273} + \frac{2}{21}\,\xi(p) - \frac{3\,\eta(p)}{26\sqrt{2}}\,,\ \frac{10 + 26\,\xi(p)}{273} + \frac{3\sqrt{2}\,\eta(p)}{52}\,,\ \frac{5 - 8\,\xi(p)}{42}\right),$$

where $\xi(p) = \sqrt{-3 + 14\,p}$ and $\eta(p) = \sqrt{-21 + 104\,p}$.

Variant MV: The same approach is applied to get the estimates below.

Lemma 4.2 *There exists a DA two-level splitting satisfying the condition (ii), if and only if,*

$$p \geq \frac{1}{4}.$$

Then, the obtained solutions for (a, b, c) are invariant with respect to the local CBS constant

$$\gamma_E^2 = 1 - \frac{1}{8p},$$

and for the related optimal splitting

$$\gamma_{MV}^2 \leq \frac{1}{2}. \tag{27}$$

5 Multilevel preconditioning

The multilevel methods have evolved from two-level methods. The straightforward recursive extension leads to the class of hierarchical basis (HB) methods for which the condition number grows in general exponentially with the number of levels ℓ. Therefore, in order to obtain multilevel preconditioners of both additive or multiplicative type, which have optimal convergence rate, i.e.,

$$\varkappa(M^{(\ell)^{-1}} A) = O(1),$$

and optimal computational complexity (linearly proportional to the number of degrees of freedom n_ℓ at the finest discretization level), HB preconditioners are combined with various types of stabilization techniques.

One particular purely algebraic stabilization technique is the so-called Algebraic Multilevel Iteration (AMLI) method, where a specially constructed matrix polynomial P_{β_k} of degree β_k is used on some (all) levels $k = 1, \cdots, \ell$. The AMLI methods are originally introduced and studied in a multiplicative form, see [6, 7].

Starting from the coarsest mesh (level 0) with $M_F^{(0)} = A^{(0)}$, the basic idea is to apply the two-level preconditioner (9) recursively at all levels $k = 1, 2, \ldots, \ell$ of mesh refinement, i.e.,

$$M_F^{(k)} = \begin{bmatrix} C_{11}^{(k)} & 0 \\ \widetilde{A}_{21}^{(k)} & C_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & C_{11}^{(k)^{-1}} \widetilde{A}_{12}^{(k)} \\ 0 & I \end{bmatrix}. \tag{28}$$

Here $C_{11}^{(k)}$ is some preconditioner for the upper left block of the (hierarchical) stiffness matrix

$$\widetilde{A}^{(k)} = \begin{bmatrix} \widetilde{A}_{11}^{(k)} & \widetilde{A}_{12}^{(k)} \\ \widetilde{A}_{21}^{(k)} & \widetilde{A}_{22}^{(k)} \end{bmatrix}$$

at level k and the matrix $C_{22}^{(k)}$ is implicitly defined by the equation

$$C_{22}^{(k)^{-1}} = \left[I - P_\beta \left(M_F^{(k-1)^{-1}} \widetilde{A}^{(k-1)} \right) \right] \widetilde{A}^{(k-1)^{-1}} \tag{29}$$

where $M_F^{(k-1)}$ and $\widetilde{A}^{(k-1)}$ denote the multiplicative preconditioner and the (aggregated in the case of DA) stiffness matrix corresponding to level $(k-1)$, respectively, and $\widetilde{A}^{(0)} = A^{(0)}$ by definition. Then, as well known from theory [6, 7], a properly shifted Chebyshev polynomial P_{β_k} of degree β_k, satisfying the conditions

$$0 \leq P_{\beta_k}(t) < 1, \quad 0 < t \leq 1, \quad P_{\beta_k}(0) = 1,$$

can be used in order to stabilize the condition number of the linear AMLI preconditioner.

The main result from this analysis is that the AMLI preconditioner has optimal computational complexity, if $\beta_k = \beta$ and

$$\frac{1}{\sqrt{1-\gamma^2}} < \beta < \tau, \tag{30}$$

where $\tau \approx \dfrac{n_{k+1}}{n_k}$ is the reduction factor of the number of degrees of freedom.

In the case DA, $\gamma_{MP}^2 < 5/12$, $\gamma_{MV}^2 < 1/2$, and the optimality condition is reached for polynomial degrees $\beta \in \{2, 3, \ldots, 7\}$.

Now, let us turn back to the case FR. The multilevel behavior of the CBS constant is studied numerically. This means, that at the current coarsening step, the role of the element stiffness matrix is played by the related last computed block $B_{E,22}$. The obtained results are shown below in both, table and graphic form.

The computed (local) estimates for γ^2 for the FR algorithm are always smaller than the related ones for the DA algorithm. One can also observe a nice one-side convergence to the value of $\theta \approx 0.39238$ for both, MP and MV, cases, see Figure 4; Similar results were reported in [15] for 2D problems.

The conclusion of the considerations in this section is that the DA splitting provides better opportunities for a systematic theoretical analysis. However, the practical value of the counterpart approach FR seems to compensate for this advantage.

Remark 5.1 It is important to note, that the CBS constant is not only used to analyze the related two-level and multilevel preconditioners. It is also involved in the construction of the acceleration matrix polynomial P_β. In other words, the smaller γ means the faster the PCG method convergence in a very general setting.

Table 1: Multilevel behavior of γ^2 for "First reduce" algorithm

variant	ℓ	$\ell-1$	$\ell-2$	$\ell-3$	$\ell-4$	$\ell-5$
MP	0.38095	0.39061	0.39211	0.39234	0.39237	0.39238
MV	0.5	0.4	0.39344	0.39253	0.39240	0.39238

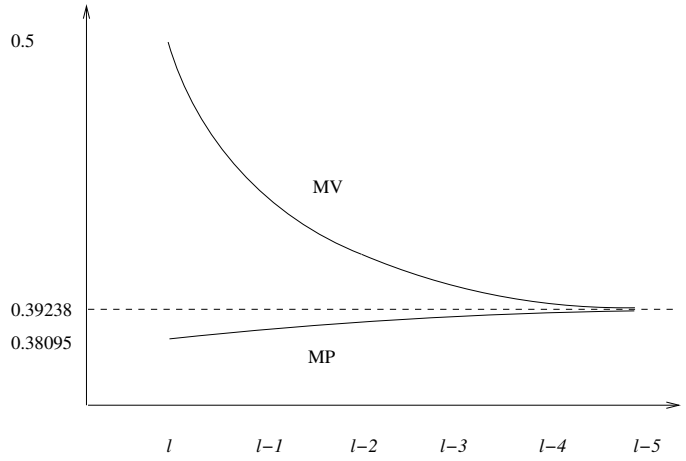


Figure 4: Multilevel behavior of γ^2 for "First reduce" algorithm

Remark 5.2 According to our local estimates for γ there is evidence to suggest that the symmetric preconditioner of block-diagonal (additive) form yields an optimal order AMLI method for the DA approach provided third- to seventh-order stabilization polynomials are employed, i.e., the optimality condition

$$\sqrt{\frac{1+\gamma}{1-\gamma}} < \beta < \tau \tag{31}$$

is met for both – MP and MV – discretization variants if $\beta \in \{3, 4, \ldots, 7\}$.

Stabilization techniques for additive multilevel iteration methods and nearly optimal order parameter-free block-diagonal preconditioners of AMLI-type are discussed in References [2, 5, 21].

Regarding the computational complexity of the multilevel algorithms we remark that the left-hand side inequalities in (30) and (31) assure that the condition number will be bounded uniformly in the number of levels whereas the right-hand side inequalities allow to estimate the computational work $w^{(\ell)}$ that is required for one application of the preconditioner at level ℓ of the finest discretization, i.e., $w^{(\ell)} \leq c\,(N^{(\ell)} + \beta\, N^{(\ell-1)} + \ldots + \beta^{\ell}\, N^{(0)}) < c\, N/(1 - \frac{\beta}{\tau})$, where $N = N^{(\ell)}$. The work for the construction of the proposed AMLI preconditioners is also proportional to N. This can easily be seen by observing that

- the matrices $A^{(k)}$, $0 \leq k \leq \ell$, have at most 11 nonzero entries per row,
- every two-level transformation $J^{(k)}$ is the identity for interior unknowns,
- in case of the FR splitting the remaining rows of $J^{(k)}$, which are given according to $J_{E,22}$, see (13), have 4 nonzero entries per row,

- in case of the DA splitting the remaining rows of $J^{(k)}$ are given according to $[J_{E,21}, J_{E,22}]$, which results in 4 or at most 28 nonzeros per row,
- the costs for the elimination of the interior nodal unknowns is $\mathcal{O}(N^{(k)})$,
- the (global) product $\widetilde{A}^{(k)} = J^{(k)} A^{(k)} J^{(k)T}$ requires $\mathcal{O}(N^{(k)})$ operations,
- alternatively, the hierarchical basis matrix $\widetilde{A}^{(k)}$ can be assembled from the local contributions $\widetilde{A}_E^{(k)} = J_E A_E^{(k)} J_E^T$, at total costs of $\mathcal{O}(N^{(k)})$ operations, which include the computation of all local matrix products.

Clearly, the storage requirement for the preconditioner is $\mathcal{O}(N)$ as well.

6 Numerical results

We solve the model problem (1) using the preconditioned conjugate gradient (PCG) method combined with the multiplicative variant of the multilevel preconditioner based on either DA or FR splitting.

6.1 Jump in coefficients aligned with coarse mesh

The computational domain is $\Omega = (0,1)^3$ in the first example and both discretization variants, MP and MV, are considered. The mesh size is varied in the range $h = 1/8$ to $h = 1/128$ resulting in 512 to $2\,097\,157$ finite elements with $1\,728$ to $6\,340\,608$ nodes, respectively. For any element e in $\mathcal{T}_h$ the matrix $\mathbf{a}(e)$ in (3) is defined by $\mathbf{a}(e) := \alpha(e) \cdot I$, where the following situation of a jump in the coefficient $\alpha = \alpha(e)$ is considered:

$$\alpha(e) = \left\{ \begin{array}{ll} 1 & \text{in } (I_1 \times I_1 \times I_1) \bigcup (I_2 \times I_2 \times I_1) \bigcup (I_2 \times I_1 \times I_2) \bigcup (I_1 \times I_2 \times I_2) \\ \varepsilon & \text{elsewhere} \end{array} \right\},$$

where $I_1 = (0, 0.5]$ and $I_2 = (0.5, 1)$, and $\varepsilon = 10^{-3}$. All computations are performed on a Fujitsu Siemens Primergy RX600 S3 workstation with four dual core Intel Xeon MP processors (3.4 GHz) with 64 GB RAM. Table 2 summarizes the number of PCG iterations that reduce the residual norm by a factor 10^8 when performing the V-cycle AMLI. In Table 3 we list the corresponding results for the linear AMLI W-cycle employing the matrix polynomial $Q_1(t) = (1 - P_2(t))/t = q_0 + q_1 t$ for stabilizing the condition number. In accordance with the analysis in [6, 7] we use the coefficients

$$q_0 = \frac{2}{\sqrt{1-\gamma^2}}, \quad q_1 = -\frac{1}{1-\gamma^2}. \tag{32}$$

It is notable that this choice, although theoretically founded for the situation of exact inversion of the pivot block A_{11} only, still yields satisfying results in our case where we use an approximate inversion of A_{11}, i.e., an incomplete factorization based on a drop tolerance (ILU(tol)); We set the drop tolerance tol to 10^{-3} in all the experiments presented in this paper.[1] Finally, Table 4 summarizes the results for the (variable-step) non-linear AMLI method stabilized by two inner generalized conjugate gradient iterations at every intermediate level, cf., [8, 17, 21] (and using a direct solve on the coarsest mesh with mesh size $h^{-1} = 4$, as in the other tests).

[1] During the computation of the triangular incomplete factors of a matrix M the entries smaller in magnitude than the local drop tolerance (given by the product of the drop tolerance tol and the diagonal entry of the corresponding row i of M) are dropped from the appropriate factor [23].

Table 2: Linear AMLI V-cycle: number of PCG iterations

MP:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	9	12	16	20	24
	$\varepsilon = 10^{-3}$	9	12	16	20	25
FR:	$\varepsilon = 1$	8	11	14	18	22
	$\varepsilon = 10^{-3}$	8	11	14	18	22

MV:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	12	17	22	29	38
	$\varepsilon = 10^{-3}$	12	17	22	30	39
FR:	$\varepsilon = 1$	10	14	17	21	26
	$\varepsilon = 10^{-3}$	10	14	17	21	26

Table 3: Linear AMLI W-cycle: number of PCG iterations

MP:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	9	10	10	10	10
	$\varepsilon = 10^{-3}$	9	10	10	10	10
FR:	$\varepsilon = 1$	8	9	9	9	9
	$\varepsilon = 10^{-3}$	8	9	9	9	9

MV:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	12	15	15	16	16
	$\varepsilon = 10^{-3}$	12	15	16	16	16
FR:	$\varepsilon = 1$	10	12	12	12	12
	$\varepsilon = 10^{-3}$	10	12	12	12	12

Table 4: Non-linear AMLI W-cycle: number of (outer) GCG iterations

MP:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	9	9	9	9	9
	$\varepsilon = 10^{-3}$	9	10	10	10	10
FR:	$\varepsilon = 1$	8	9	9	9	9
	$\varepsilon = 10^{-3}$	8	9	9	9	9

MV:	h^{-1}	8	16	32	64	128
DA:	$\varepsilon = 1$	12	12	12	12	12
	$\varepsilon = 10^{-3}$	12	12	12	12	12
FR:	$\varepsilon = 1$	10	11	11	11	11
	$\varepsilon = 10^{-3}$	10	11	11	11	11

As the theory predicts the preconditioners are perfectly robust with respect to jump discontinuities of the coefficients $\mathbf{a}(e)$ if they do not occur inside any element of the coarsest mesh partition. The results slightly favor the FR approach, and, they illustrate well the optimal complexity of the iterative solvers, when using a W-cycle (linear or non-linear), for both of the splittings and for both discretization variants.

6.2 Random distribution of jump in coefficients

The remaining experiments deal with examples where the coefficient functions are rough in the sense that their variations (jumps) need to be resolved on the finest mesh. In these tests we use the FR basis transformation in combination with the non-linear AMLI W-cycle method, i.e., two inner GCG iterations at all coarse levels (except the coarsest one). The number of outer iterations that we report in Table 5 reduce the residual by a factor 10^6. Here, the coefficient $\alpha(e)$ is constant elementwise only; It is initialized randomly, taking either of the values 1 or ε, where 1 occurs with some fixed probability p. By comparing the results shown in Tables 4 and 5, we observe that in general, the solver is not robust with respect to jump discontinuities (aligned with the finest mesh partition). However, for a fixed value of ε the solution process still remains of optimal order of computational complexity, no matter how large the jumps are. We want to emphasize this advantage of the non-linear AMLI over the linear algorithm, which does not achieve a stabilization of the condition number (and thus no longer results

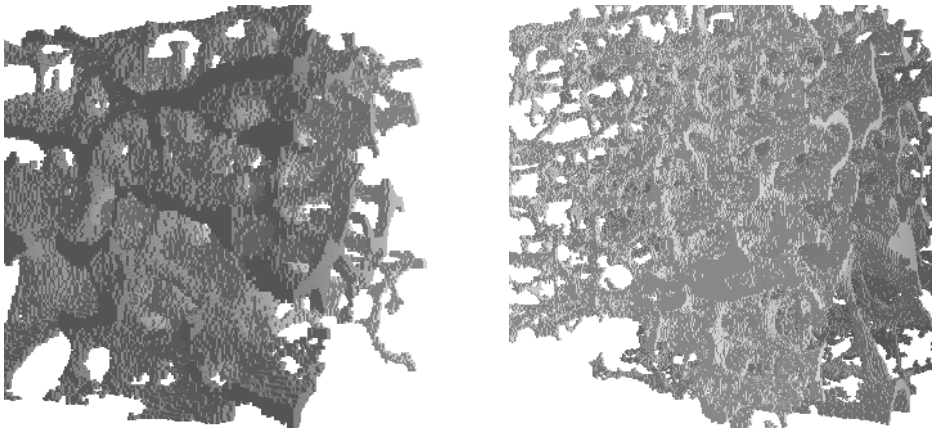
Table 5: Non-linear AMLI W-cycle: GCG iterations for problem with random coefficients

$p = 1/2$					
FR-MV: h^{-1}	8	16	32	64	128
$\varepsilon = 10^{-1}$	9	9	9	9	9
$\varepsilon = 10^{-2}$	21	22	22	21	21
$\varepsilon = 10^{-3}$	42	59	58	56	54
$p = 1/10$					
FR-MV: h^{-1}	8	16	32	64	128
$\varepsilon = 10^{-1}$	9	9	9	9	9
$\varepsilon = 10^{-2}$	17	22	22	22	22
$\varepsilon = 10^{-3}$	29	60	55	50	50

in an optimal-order method) for problems with large coefficient jumps that can only be resolved on the finest mesh.

6.3 An example related to the microstructure of human bones

Finally we consider another similar problem but with a real-life background. Here the distribution of large and small coefficients corresponds to the distribution of the solid and the liquid phase of a human lumbar vertebral body L3. The voxel size in each direction (in-slice pixel size and distance from slice to slice) is 37 μm. The data is extracted from a micro-CT scan.[2] In order to save memory (and CPU



(a) 128 slices (128^3 voxels) (b) 256 slices (256^3 voxels)

Figure 5: Cube extracted from lumbar vertebral body.

time) the number of orthogonal search directions in the GCG algorithm at the fine-grid level is reduced to 10 (instead of 20 which we used in the previous examples). Again, the relative residual is 10^{-6}. In Table 6 we list the CPU time for the setup and for the solution of the linear system with the non-linear

[2]Bone 3D Project Team (ESA MAP Project AO-99-030, ESTEC Contract #14592/00/NL/SH): Vertebral Body Data Set ESA29-99-L3, http://bone3d.zib.de/data/2005/ESA29-99-L3/

AMLI method based on the FR approach. Table 7 summarizes the corresponding number of W-cycle iterations. Both discretization variants, MP and MV, are considered. The number of outer iterations to achieve the same relative residual only slightly increases when comparing the random coefficient distribution to that of the bone structure. In fact, this increase is due to the reduction of orthogonal search directions in the GCG algorithm (at the fine-grid level) from 20 to 10. The total solution time reported in Table 6 indicates the optimal complexity of the method. Though the CPU time for solving the largest problem ($h^{-1} = 256$) is in the range of two ($\varepsilon = 10^{-1}$) to six ($\varepsilon = 10^{-3}$) thousand seconds this seems to be acceptable in view of the huge number of degrees of freedom ($50\,528\,256$ DOF) and in view of the fact that we used a serial code.

Table 6: CPU time (variant FR)

MP					
setup time					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	0.21	2.00	17.4	146.7	1259
$\varepsilon = 10^{-2}$	0.22	1.97	17.1	144.5	1237
$\varepsilon = 10^{-3}$	0.22	1.95	17.0	144.0	1230
solution time					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	0.04	0.63	6.8	68.4	700
$\varepsilon = 10^{-2}$	0.12	1.42	16.1	162.1	1662
$\varepsilon = 10^{-3}$	0.36	4.44	44.5	449.8	4487
MV					
setup time					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	0.23	2.13	18.6	157.2	1386
$\varepsilon = 10^{-2}$	0.22	2.08	18.3	154.4	1361
$\varepsilon = 10^{-3}$	0.22	2.06	18.1	153.4	1337
solution time					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	0.05	0.66	7.0	70.8	757
$\varepsilon = 10^{-2}$	0.12	1.49	16.7	167.9	1779
$\varepsilon = 10^{-3}$	0.34	4.30	45.8	465.8	4818

Table 7: Non-linear AMLI W-cycle: GCG iterations (variant FR) for bone structure

MP					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	8	9	9	9	9
$\varepsilon = 10^{-2}$	22	21	22	23	22
$\varepsilon = 10^{-3}$	73	66	61	61	61

MV					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	9	9	9	9	9
$\varepsilon = 10^{-2}$	21	21	22	22	22
$\varepsilon = 10^{-3}$	67	61	61	61	61

The amount of fill-in occurring in the ILU(tol) factorization that is used for the inexact solves with the pivot blocks A_{11} is shown in Table 8; the *fill-in quotient* appearing in this table is obtained from dividing the number of nonzero entries in the incomplete factor L (or U) by the number of nonzero entries in the lower (or upper) triangular part of A_{11}, i.e., this quotient equals one for an ILU(0) factorization.

Table 8: ILU(tol) max. fill-in quotient (rel. to ILU(0)) for $\mathrm{tol} = 10^{-3}$

MP					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	2.01	2.10	2.15	2.17	2.18
$\varepsilon = 10^{-2}$	1.96	2.05	2.10	2.12	2.15
$\varepsilon = 10^{-3}$	1.94	2.04	2.10	2.12	2.14
MV					
h^{-1}	16	32	64	128	256
$\varepsilon = 10^{-1}$	2.36	2.56	2.70	2.76	2.81
$\varepsilon = 10^{-2}$	2.26	2.45	2.60	2.65	2.70
$\varepsilon = 10^{-3}$	2.23	2.42	2.55	2.60	2.65

A speedup potential is the use of a *cheaper* approximate inverse of the pivot blocks. As we see from Table 8 the computation and each application of the ILU(tol) preconditioner (with $\mathrm{tol} = 10^{-3}$) has about two to three times the operation count as compared to the ILU(0) preconditioner (which can be used in case of smooth coefficients only).

Finally we want to remark that the FR splitting has the favorable property to improve the angle between the induced FE subspaces when applied recursively. The maximum of the local $\gamma^2_{E,\max}$ over all macro elements is shown in Table 9 on the first five subsequent levels. At least to some extent,

Table 9: Multilevel behavior of $\gamma^2_{E,\max}$ for FR approach

MP					
$h^{-1} = 128$	ℓ	$\ell-1$	$\ell-2$	$\ell-3$	$\ell-4$
$\varepsilon = 10^{-1}$	0.84775	0.82695	0.80832	0.77615	0.65715
$\varepsilon = 10^{-2}$	0.98275	0.97812	0.97557	0.96826	0.91927
$\varepsilon = 10^{-3}$	0.99825	0.99775	0.99748	0.99665	0.99038
MV					
$h^{-1} = 128$	ℓ	$\ell-1$	$\ell-2$	$\ell-3$	$\ell-4$
$\varepsilon = 10^{-1}$	0.88270	0.83248	0.80981	0.77636	0.65708
$\varepsilon = 10^{-2}$	0.98684	0.97910	0.97565	0.96838	0.92088
$\varepsilon = 10^{-3}$	0.99867	0.99785	0.99749	0.99667	0.99079

this makes the optimality of the method plausible also in case of non-smooth coefficients. However, a deeper understanding will require further theoretical investigations.

7 Concluding remarks

In this paper, we presented a general setting of hierarchical splittings of the Rannacher-Turek non-conforming FEM spaces. New estimates for the constant in the strengthened CBS inequality are derived for the considered model problem.

Let us summarize the following important issues. The DA algorithm allows for a direct extension of the γ estimate to the multi-level case if the condition (ii) holds. The derived local bounds are close to their counterparts in the 2D case, cf. [15]. In particular, they secure the possibility of stabilizing the condition number by second degree polynomials (2-fold W-cycles) when using the multiplicative and by third degree polynomials (3-fold W-cycles) when using the additive variant of the multilevel method. However, in general, it is hard to say anything about the existence of a DA splitting satisfying condition (ii). For the FR algorithm, the theoretically derived estimates of the CBS constant γ are not directly applicable to the multilevel case but the recursively computed γ-s show a very promising behavior which could be advantageous from a practical point of view. Moreover, as it has recently been shown by two of the authors, the FR splitting always yields a larger angle between the subspaces (a smaller γ) than the DA splitting, when applied to one and the same stiffness matrix; It has even been shown that the FR splitting is the best splitting based on differences and aggregates in the sense of minimizing the related CBS constant, see [18]. The numerical tests fully confirm the theoretical estimates. Moreover, they indicate some self-stabilization of the FR algorithm and the non-linear AMLI which is especially important for the case of strong coefficient jumps that are not aligned with the coarse(st) mesh partition.

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