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MULTILINEAR PROOFS FOR TWO THEOREMS ON CIRCULAR AVERAGES

BY

DANIEL M. OBERLIN (TALLAHASSEE, FLORIDA)

Let λ be Lebesgue measure on the unit circle in $\mathbb{R}^2$ and, for small $\delta > 0$, let $A(\delta)$ be the annulus $\{1-\delta \le |x| \le 1+\delta\}$ in $\mathbb{R}^2$. Denote by $\|f\|_p$ the norm of a function f in $L^p(\mathbb{R}^2)$ and by $\widehat{f}$ the Fourier transform of f. The purpose of this note is to present new proofs for two known results:

THEOREM 1. *There is a constant C such that*

$$\|\lambda * f\|_3 \le C\|f\|_{3/2}$$

for $f \in L^{3/2}(\mathbb{R}^2)$.

THEOREM 2. *There is a constant C such that*

$$\|\widehat{f}\|_{L^{4/3}(A(\delta))} \le C\delta^{3/4}|\log\delta|^{1/4}\|f\|_{4/3}$$

for $f \in L^{4/3}(\mathbb{R}^2)$.

Theorem 1 is a special case of a result of Strichartz [S] while Theorem 2 is due to Tomas [T]. The ground common to the statements of Theorems 1 and 2 is that they both deal with circular (or annular) averages. The similarity between the proofs we give is that both are effected with multilinear interpolation. The proof presented here for Theorem 1 utilizes a device of Christ [C], while the original proof is based on interpolation with an analytic family of operators. Our proof of Theorem 2 rests on the multilinear Riesz–Thorin theorem and seems a little simpler than the original argument.

In what follows, C denotes a positive constant which may vary from line to line.

Proof of Theorem 1. An argument analogous to that on pp. 227–228 of [C] shows that it is enough to establish the estimate

$$\Big|\int_{\mathbb{R}^2} \lambda * f_1(x)\lambda * f_2(x)\lambda * f_3(x)\,dx\Big| \le C\|f_1\|_1\|f_2\|_{2,1}\|f_3\|_{2,1} \tag{1}$$

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for functions f_i on $\mathbb{R}^2$. Here $\|\cdot\|_{2,1}$ denotes a Lorentz norm. It is really enough to establish (1) when f_1 is replaced by a point mass at an arbitrary point in $\mathbb{R}^2$ and f_2, f_3 are characteristic functions of measurable subsets of $\mathbb{R}^2$. Using the notations $e^{i\theta}$ for $(\cos\theta,\ \sin\theta)$ and $|E|$ for the Lebesgue measure of a measurable $E\subseteq\mathbb{R}^2$, we see then that it suffices to show that

$$\int_0^{2\pi}\lambda*\mathbf{1}_{E_1}(e^{i\theta})\lambda*\mathbf{1}_{E_2}(e^{i\theta})\,d\theta\le C|E_1|^{1/2}|E_2|^{1/2}\qquad\text{if } E_1\,,E_2\subseteq\mathbb{R}^2\,,$$

or that

$$\Big(\int_0^{2\pi}[\lambda*\mathbf{1}_E(e^{i\theta})]^2d\theta\Big)^{1/2}\le C|E|^{1/2}\qquad\text{if } E\subseteq\mathbb{R}^2\,.$$

This, in turn, is equivalent to establishing the estimate

$$\Big|\int_0^{2\pi}\lambda*\mathbf{1}_E(e^{i\theta})g(\theta)\,d\theta\Big|\le C|E|^{1/2}\|g\|_{L^2(d\theta)}\tag{2}$$

for $E\subset\mathbb{R}^2$ and functions g on $[0,2\pi)$.

The transformation $T:(\theta,\phi)\mapsto e^{i\theta}+e^{i\phi}$ is essentially a two-to-one mapping of $[0,2\pi)\times[0,2\pi)$ onto $\{|x|\le 2\}$. Thus the change of variables formula gives

$$\int_0^{2\pi}\lambda*\mathbf{1}_E(e^{i\theta})g(\theta)\,d\theta=\int_0^{2\pi}\int_0^{2\pi}\mathbf{1}_E(e^{i\theta}+e^{i\phi})g(\theta)\,d\phi\,d\theta$$
$$=\int_{|x|\le 2}\mathbf{1}_E(x)[\,\widetilde{g}_1(x)\omega_1(x)+\widetilde{g}_2(x)\omega_2(x)]\,dx$$

where if $(\theta_1\,,\phi_1)$ and $(\theta_2\,,\phi_2)$ are the inverse images of x under T, chosen so that $0\le\theta_1<\pi$, say, then

$$\widetilde{g}_i(x)=g(\theta_i),\qquad \omega_i(x)=|\sin(\theta_i-\phi_i)|^{-1}\qquad\text{for } i=1,2\,.$$

Thus (2) will follow from

$$\|\widetilde{g}_i\omega_i\|_{L^{2,\infty}(\mathbb{R}^2)}\le C\|g\|_{L^2(d\theta)}\,.\tag{3}$$

But, for $s>0$,

$$|\{x:\widetilde{g}_i(x)\omega_i(x)>s|\le\iint_{\{|g(\theta)|>s|\sin(\theta-\phi)|\}}|\sin(\theta-\phi)|\,d\phi\,d\theta\le Cs^{-2}\|g\|^2_{L^2(d\theta)}\,.$$

This establishes (3) and completes the proof of Theorem 1.

Proof of Theorem 2. By duality it is enough to show that if f is supported on $A(\delta)$, then

$$\|\widehat{f}\|_4\le C\delta^{3/4}|\log\delta|^{1/4}\|f\|_4\,.\tag{4}$$

And it will actually suffice to establish (4) under the assumption that f is supported in

$$\widetilde{A}(\delta) \doteq \{re^{i\theta} : 1-\delta \le r \le 1+\delta,\ 0 \le \theta \le 1/8\}$$

and that $0 < \delta < 1/8$. Using the Plancherel theorem in the usual way to express $\|\widehat{f}\|_4$ in terms of f, we see that (4) is a consequence of

$$(5)\qquad \left|\int\int\int f_1(x-y)f_2(y)f_3(x-z)f_4(z)\,dx\,dy\,dz\right| \le C\delta^3|\log\delta|\prod_{i=1}^{4}\|f_i\|_4$$

for functions f_i supported on $\widetilde{A}(\delta)$. But (5) will follow from the multilinear Riesz–Thorin theorem and the four estimates obtained by replacing $\prod_{i=1}^4 \|f_i\|_4$ in (5) with $\|f_j\|_1 \prod_{i\ne j}\|f_i\|_\infty$. The case $j=1$ is typical, so we will show that

$$(6)\qquad \left|\int\int\int f_1(x-y)f_2(y)f_3(x-z)f_4(z)\,dx\,dy\,dz\right| \le C\delta^3|\log\delta|\,\|f_1\|_1\prod_{i=2}^{4}\|f_i\|_\infty .$$

It is enough to establish (6) when f_1 is replaced by a point mass at some $x_0 \in \widetilde{A}(\delta)$ and when each $\|f_i\|_\infty = 1$. Then the LHS of (6) will be largest when each f_i is the characteristic function of $\widetilde{A}(\delta)$. Writing A for $\widetilde{A}(\delta)$, we see that (6) reduces to

$$(7)\qquad \int\int \mathbf{1}_A(x-x_0)\,\mathbf{1}_A(x-z)\,\mathbf{1}_A(z)\,dx\,dz \le C\delta^3|\log\delta| \quad \text{if } x_0\in A .$$

Assume for a moment that

$$(8)\qquad \int \mathbf{1}_A(x-x_0)\,\mathbf{1}_A(x-z)\,dx \le C\min\{\delta, \delta^2/|x_0-z|\} \quad \text{if } x_0, z\in A .$$

Then the LHS of (7) is bounded by a multiple of

$$\delta \int_{|x_0-z|<10\delta} dz + \delta^2 \int_{|x_0-z|\ge 10\delta} \mathbf{1}_A(z)\frac{dz}{|x_0-z|} \le C\delta^3|\log\delta|.$$

Thus Theorem 2 will be proved as soon as (8) is established. But (8) follows from

$$(9)\qquad |x_1 + A(\delta)\cap x_2 + A(\delta)| \le C\frac{\delta^2}{|x_1-x_2|} \quad \text{if, say, } 10\delta \le |x_1-x_2| \le \frac{1}{2} .$$

Here $x_1 + A(\delta)$ is the translate of $A(\delta)$ by $x_1\in\mathbb{R}^2$ and $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^2$. Under the assumptions on x_1 and x_2,

$$x_1 + A(\delta)\cap x_2 + A(\delta)$$

is a union of two sets, each of which is a rigid motion of the set in Figure 1.

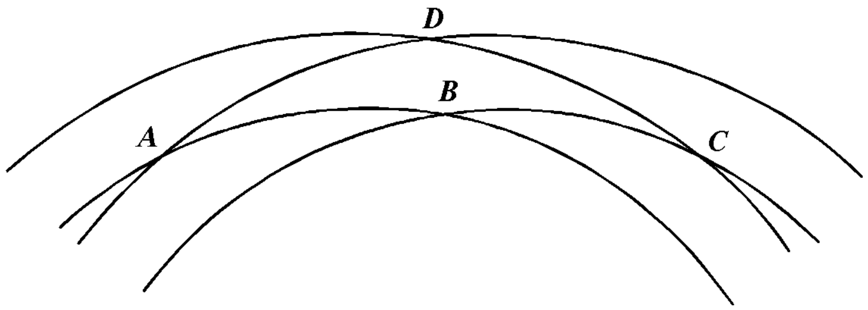

Fig. 1

Trigonometry shows that the segment AC has length $4\delta/|x_1 - x_2|$ and BD has length

$$[(1+\delta)^2 - |x_1 - x_2|^2/4]^{1/2} - [(1-\delta)^2 - |x_1 - x_2|^2/4]^{1/2}.$$

This last expression is bounded by $C\delta$ since $0 < \delta < 1/8$ and since $|x_1 - x_2| \le 1/2$. Thus (9) follows and the proof of Theorem 2 is complete.

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DEPARTMENT OF MATHEMATICS
THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306
U.S.A.

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