Multipartite entanglement and high precision metrology

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DPG Meeting, Hannover, 21 March 2013

Outline

1. **Motivation**
   - Why the connection between multipartite entanglement and Fisher information is important?

2. **Metrology and multipartite entanglement**
   - Quantum Fisher information
   - Properties of the Quantum Fisher information
   - Quantum Fisher information and entanglement

Why the connection between multipartite entanglement and Fisher information is important?

- Genuine multipartite entanglement appears often in quantum information.
- While bipartite entanglement is quite well understood, the role of multipartite entanglement is not so clear.
- Thus, it is very interesting if we can show that it has a central role in metrology.

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Metrology and multipartite entanglement in the literature

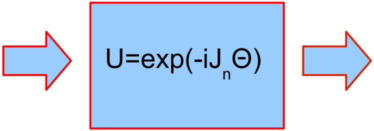
- One of the important applications of entangled multipartite quantum states is sub-shotnoise metrology.
  V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).

- Multipartite entanglement, not simple nonseparability, is needed for extreme spin squeezing, which can be applied in spectroscopy and atomic clocks.
  A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001).

- Not all entangled states are useful for phase estimation, at least in a linear interferometer.
  P. Hyllus, O. Gühne, and A. Smerzi, 82, 012337 (2009).

Quantum Fisher information

- Let us consider the following process:



- The dynamics described above is $\varrho_{\rm out} = e^{-i\theta J_{\vec{n}}} \varrho e^{+i\theta J_{\vec{n}}}$.
- We would like to determine the angle θ by measuring $\varrho_{\rm out}$.

Quantum Fisher information II

Quantum Cramér-Rao bound

The phase estimation sensitivity is limited as

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q[\varrho, J_{\vec{n}}]}},$$

where F_Q is the quantum Fisher information, ϱ is a quantum state and $J_{\vec{n}}$ is a collective angular momentum component.

- The Braunstein-Caves quantum Fisher information is

$$F[\varrho, X] = \sum_{ij} \frac{2(\lambda_i - \lambda)^2}{\lambda_i + \lambda_j} |X_{ij}|^2.$$

C.W. Helstrom, *Quantum Detection and Estimation Theory* (1976),
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (1982).

Outline

Properties of the Quantum Fisher information

Two important properties:

1. For a pure state ϱ, we have $F[\varrho, J_l] = 4(\Delta J_l)^2_\varrho$.
2. $F[\varrho, J_l]$ is convex in the state, that is
   $F[p_1\varrho_1 + p_2\varrho_2, J_l] \leq p_1 F[\varrho_1, J_l] + p_2 F[\varrho_2, J_l]$.

It also follows that $F[\varrho, J_l] \leq 4(\Delta J_l)^2_\varrho$.

C.W. Helstrom, *Quantum Detection and Estimation Theory* (1976).

A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (1982).

S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72, 3439 (1994).
L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).

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Quantum Fisher information and entanglement

Pezzé, Smerzi, PRL 2009

For N-qubit separable states we have

$$F_Q[\varrho, J_l] \leq N.$$

Here, $J_l = \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)}$ where $\sigma_l^{(k)}$ are the Pauli spin matrices. The maximum for the left-hand side is N^2.

Thus, for separable states

$$\Delta\theta \geq \frac{1}{\sqrt{N}},$$

while for entangled states

$$\Delta\theta \geq \frac{1}{N}.$$

Quantum Fisher information and entanglement II

Observation 1

For N-qubit separable states we have

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq 2N. \quad (1)$$

- Eq. (1) is a condition for the average sensitivity of the interferometer. All states violating Eq. (1) are entangled.

GT, PRA 85, 022322 (2012); P. Hyllus et al., PRA 85, 022321 (2012).

Quantum Fisher information and entanglement III

Observation 2

For quantum states we have the bound

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N(N+2). \quad (2)$$

GHZ states and N-qubit symmetric Dicke states with $\frac{N}{2}$ excitations saturate Eq. (2).

- Dicke states have been investigated recently in several experiments.
- In general, pure symmetric states for which $\langle J_l \rangle = 0$ for $l = x, y, z$ saturate Eq. (2).

GT, PRA 85, 022322 (2012); P. Hyllus et al., PRA 85, 022321 (2012).

Quantum Fisher information and multipartite entanglement

Next, we will consider k-producible or k-entangled states:

***Observation* 3**

For N-qubit k-producible states states

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq nk(k+2) + (N-nk)(N-nk+2).$$

where n is the integer part of $\frac{N}{k}$. For the $k = N - 1$ case, this bound can be improved

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N^2 + 1. \tag{3}$$

Eq. (3) is also the inequality for biseparable states. Any state that violates Eq. (3) is genuine multipartite entangled.

Quantum Fisher information and multipartite entanglement II

Fact

Genuine multipartite entanglement, not simple nonseparability is needed to achieve maximum sensitivity in a linear interferometer.

Quantum Fisher information and multipartite entanglement III

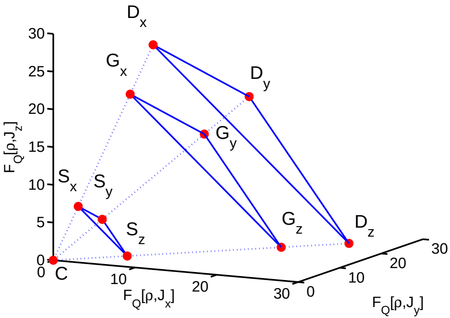


Figure: Points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space for $N = 6$.

- Points corresponding to separable states are not above the $S_x - S_y - S_z$ plane.
- Points corresponding to biseparable states are not above the $G_x - G_y - G_z$ plane.

Which part of the space corresponds to quantum states? - Points

- A completely mixed state

$$\varrho_C = \frac{\mathbb{1}}{2^N}.$$

corresponds to the point $C(0, 0, 0)$.

- States corresponding to the point $S_x(0, N, N)$ is

$$|\Psi\rangle_{S_l} = |+\frac{1}{2}\rangle_x^{\otimes N/2} \otimes |-\frac{1}{2}\rangle_x^{\otimes N/2}.$$

S_y and S_z are similar.

Which part of the space corresponds to quantum states? - Points II

- D_z : N-qubit symmetric Dicke state with $\frac{N}{2}$ excitations.

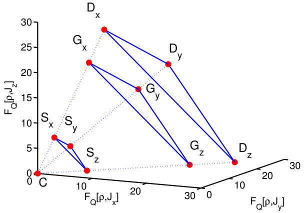
$$|\mathcal{D}_N^{(N/2)}\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \{|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\},$$

where $\sum_k \mathcal{P}_k$ denotes summation over all possible permutations.

- N-qubit GHZ states

$$|\Psi\rangle_{GHZ_z} = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N} \right).$$

Which part of the space corresponds to quantum states? - 2D polytopes



- For all points in the S_x, S_y, S_z polytope, there is a corresponding pure product state for even N.

- For given $F[\varrho, J_l]$ for $l = x, y, z$, such a state is defined as

$$\varrho = \left[\frac{\mathbb{1}}{2} + \frac{1}{2}\sum_{l=x,y,z} c_l\sigma_l\right]^{\otimes N/2} \otimes \left[\frac{\mathbb{1}}{2} - \frac{1}{2}\sum_{l=x,y,z} c_l\sigma_l\right]^{\otimes N/2},$$

where $c_l^2 = 1 - \frac{F_Q[\varrho, J_l]}{N}$, where $\sum_l c_l^2 = 1$.

Which part of the space corresponds to quantum states? - 2D polytopes II

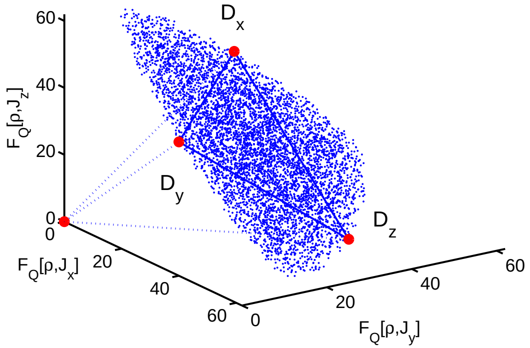


Figure: Randomly chosen points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space corresponding to states $|\Psi(\alpha_x, \alpha_y, \alpha_z)\rangle$ for $N = 8$.

- All the points are in the plane of D_x, D_y and D_z.

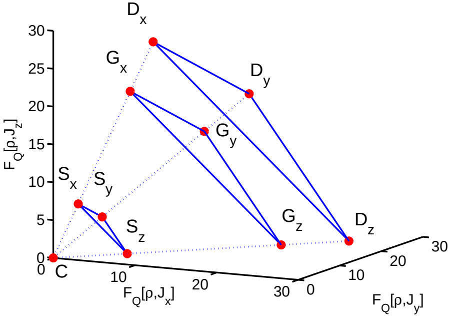
Which part of the space corresponds to quantum states? - 3D polytopes

- A pure state mixed with the completely mixed state

$$\varrho^{(\mathrm{mixed})}(p) = p\varrho + (1-p)\frac{\mathbb{1}}{2^N}$$

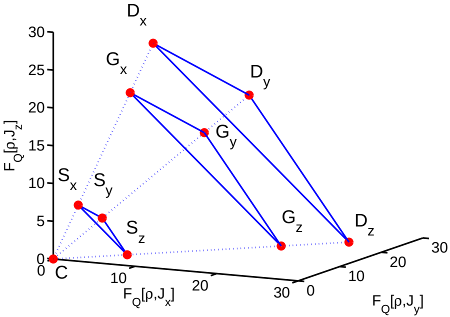
- The states $\varrho^{(\mathrm{mixed})}(p)$ are on a straight line on our figures.

Which part of the space corresponds to quantum states? - 3D polytopes II



Observation 5. If N is even, then there is a separable state for each point in the S_x, S_y, S_z, C polytope.

Which part of the space corresponds to quantum states? - 3D polytopes III



Observation 6. If N is divisible by 4, then for all the points of the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope, there is a quantum state with genuine multipartite entanglement.

Summary

- We defined entanglement conditions in terms of the quantum Fisher information.
- We showed that genuine multipartite entanglement is needed for maximum metrological sensitivity.

See:
G. Tóth, PRA 85, 022322 (2012).

Similar paper: Hyllus et al, PRA 85, 022321(2012); Krischek et al., PRL 107, 080504 (2011).

THANK YOU FOR YOUR ATTENTION!

European Research Council
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