

Multiperiod Portfolio Optimization with Many Risky Assets and General Transaction Costs*

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Abstract

We analyze the optimal portfolio policy for a multiperiod mean-variance investor facing a large number of risky assets in the presence of general transaction cost. For proportional transaction costs, we give a closed-form expression for a no-trade region, shaped as a multi-dimensional parallelogram, and show how the optimal portfolio policy can be efficiently computed by solving a single quadratic program. For market impact costs, we show that at each period it is optimal to trade to the boundary of a state-dependent rebalancing region. Finally, we show empirically that the losses associated with ignoring transaction costs may be large.

Keywords: Portfolio optimization, multiperiod utility, no-trade region, market impact.

JEL Classification: G11.

1 Introduction

Merton (1971) showed that an investor who wishes to maximize her utility of consumption should hold a fixed proportion of her wealth on each of the risky assets, and consume at a rate proportional to her wealth.¹ Merton's seminal work relies on the assumptions that the investor has constant relative risk aversion (CRRA) utility, faces an infinite horizon, can trade continuously and (crucially) costlessly. Implementing Merton's policy, however, requires one to rebalance the portfolio weights continuously, and in practice this may result in high or even infinite transaction costs. Ever since Merton's breakthrough, researchers have tried to address this issue by characterizing the optimal portfolio policy in the presence of transaction costs.

Researchers focused first on the case with a single-risky asset. Magill and Constantinides (1976) consider a finite-horizon continuous-time investor subject to proportional transaction costs and for the first time *conjecture* that the optimal policy is characterized by a *no-trade interval*: if the portfolio weight on the risky-asset is inside this interval, then it is optimal not to trade, and if it is outside, then it is optimal to trade to the boundary of this interval. Constantinides (1979) demonstrates the optimality of the no-trade interval policy in a finite-horizon discrete-time setting. Constantinides (1986) considers the Merton framework with a single risky asset and proportional transaction costs, and computes *approximately-optimal* no-trade interval policies by requiring the investor's consumption rate to be a fixed proportion of her wealth, a condition that is not satisfied in general. Davis and Norman (1990) consider the same framework, show that the *optimal* no-trade interval policy exists, and propose a numerical method to compute it. Dumas and Luciano (1991) consider a continuous-time investor who maximizes utility of terminal wealth, and show how to calculate the boundaries of the no-trade interval for the limiting case when the terminal period goes to infinity.

The case with multiple risky assets is less tractable, and the bulk of the existing literature relies on numerical results for the case with only two risky assets. Akian, Menaldi, and Sulem (1996) consider a multiple risky-asset version of the framework in Davis and Norman (1990),

¹Merton's result holds for either an investor facing a constant investment opportunity set, or an investor with logarithmic utility; see also Mossin (1968), Samuelson (1969), and Merton (1969, 1973).

and for the restrictive case where the investor has power utility with relative risk aversion between zero and one² and risky-asset returns are uncorrelated, they show that there exists a unique optimal portfolio policy. They also compute numerically the no-trade region for the case with *two* uncorrelated stocks. Leland (2000) considers the tracking portfolio problem subject to proportional transaction costs and capital gains tax, and proposes a numerical approach to approximate the no-trade region. Muthuraman and Kumar (2006) consider an infinite-horizon continuous-time investor and propose an efficient numerical approach to compute the no-trade region. Their numerical results show that the no-trade region for the case with two risky assets is characterized by four corner points, but these four corner points are not joined by straight lines, although their numerical experiments show that a quadrilateral no-trade region does provide a very close approximation. Lynch and Tan (2010) consider a finite-horizon discrete-time investor facing proportional and fixed transaction costs, and two risky assets with predictable returns. Using numerical dynamic programming, they show that for the case *without* predictability the no-trade region is closely approximated by a parallelogram, whereas for the case with predictability the no-trade region is closely approximated by a convex quadrilateral.³

Most of the aforementioned papers assume an investor with CRRA utility of consumption who faces borrowing constraints. These assumptions render the problem untractable analytically, and hence they generally rely on numerical analysis for the case with two risky assets. A notable exception is the work by Liu (2004), who obtains an analytically tractable framework by making several restrictive assumptions.⁴ Specifically, he considers an investor with constant *absolute* risk aversion (CARA) and access to unconstrained borrowing⁵, who can invest in multiple *uncorrelated* risky assets. For this framework, Liu shows *analytically* that there exists a box-shaped no-trade region.

²Janeček and Shreve (2004) show that relative risk aversion parameters between one and zero lead to intolerably risky behavior.

³Brown and Smith (2011) also consider the case with proportional transaction costs and return predictability. Specifically, they propose several heuristic trading strategies for a finite-horizon discrete-time investor facing proportional transaction costs and multiple assets with predictable returns, and use upper bounds based on duality theory to evaluate the optimality of the proposed heuristics.

⁴Another important exception is Muthuraman and Zha (2008) who use a simulation-based numerical optimization to approximate the optimal portfolio policy of a continuous-time investor who maximizes her long-term expected growth rate for cases with up to seven risky assets. Also, in their early paper Magill and Constantinides (1976) conjecture the existence of a box-shaped no-trade region for the case where the portfolio weights are small.

⁵He does impose constraints to preclude arbitrage portfolio policies.

Recently, Gârleanu and Pedersen (2013), herein G&P, consider a more tractable framework that allows them to provide closed-form expressions for the optimal portfolio policy in the presence of *quadratic* transaction costs. Their investor maximizes the present value of the mean-variance utility of her *wealth changes* at multiple time periods, she has access to unconstrained borrowing, and she faces multiple risky assets with predictable *price changes*. Several features of this framework make it tractable. First, the focus on utility of wealth changes (rather than consumption) plus the access to unconstrained borrowing imply that there is no need to track the investor's total wealth evolution, and instead it is sufficient to track wealth change at each period. Second, the focus on price changes (rather than returns) implies that there is no need to track the risky-asset price evolution, and instead it is sufficient to account for price changes. Finally, the aforementioned features, combined with the use of mean-variance utility and quadratic transaction costs places the problem in the category of linear quadratic control problems, which are tractable.

In this paper, we use the tractable formulation of G&P to study analytically the optimal portfolio policies for general transaction costs. Our portfolio selection framework is both more general and more specific than that considered by G&P. It is more general because we consider a broader class of transaction costs that includes not only quadratic transaction costs, but also the less tractable proportional and market impact costs. It is more specific because, consistent with most of the literature on proportional transaction costs, we consider the case with constant investment opportunity set, whereas G&P focus on the case with predictability. Specifically, we consider independent and identically distributed (iid) price changes, whereas G&P consider predictable price changes.

We make four contributions. Our first contribution is to characterize analytically the optimal portfolio policy for the case with many risky assets and *proportional* transaction costs. Specifically, we provide a closed-form expression for the no-trade region, which is shaped as a multi-dimensional parallelogram, and use the closed-form expressions to show how the size of the no-trade region shrinks with the investment horizon and the risk-aversion parameter, and grows with the level of proportional transaction costs and the discount factor. Moreover, we show how the optimal portfolio policy can be computed by solving a

quadratic program—a class of optimization problems that can be efficiently solved for cases with up to thousands of risky assets.

Our second contribution is to study analytically the optimal portfolio policy in the presence of market impact costs, which arise when the investor makes large trades that distort market prices.⁶ Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded, see Kyle (1985), and thus that market impact costs are quadratic. Under this assumption, Gârleanu and Pedersen (2013) derive closed-form expressions for the optimal portfolio policy within their multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), and Almgren, Thum, Hauptmann, and Li (2005) show that the square root function is more appropriate for modeling market price impact, thus suggesting market impact costs grow at a rate slower than quadratic. Our contribution is to extend the analysis by G&P to a *general case* where we are able to capture the distortions on market price through a power function with an exponent between one and two. For this general formulation, we show *analytically* that there exists a state-dependent *rebalancing region* for every time period, such that the optimal policy at each period is to trade to the boundary of the corresponding rebalancing region.

Our third contribution is to use an empirical dataset with the prices of 15 commodity futures to evaluate the losses associated with ignoring transaction costs and investing myopically, as well as identifying how these losses depend on relevant parameters. We find that the losses associated with either ignoring transaction costs or behaving myopically can be large. Moreover, the losses from ignoring transaction costs increase in the level of transaction costs, and decrease with the investment horizon, whereas the losses from behaving myopically increase with the investment horizon and are unimodal on the level of transaction costs.

Finally, our analysis relies on certain assumptions underlying the G&P framework such as focus on price changes and multiperiod mean-variance utility. Our fourth contribution is to gauge the robustness of our results to the use of a framework with iid returns and CRRA utility. To do this, we consider an investor who maximizes her CRRA utility of terminal

⁶This is particularly relevant for optimal execution, where institutional investors have to execute an investment decision within a fixed time interval; see Bertsimas and Lo (1998) and Engle, Ferstenberg, and Russell (2012)

wealth by investing in a risk-free and a risky asset with iid returns, in the presence of proportional transaction costs. We compute the investor's optimal portfolio policy numerically, and show that the certainty equivalent loss from following the G&P-type portfolio policy is typically smaller than 0.5%.

Our work is related to Dybvig (2005), who considers a single-period investor with mean-variance utility and proportional transaction costs. For the case with multiple risky assets, he shows that the optimal portfolio policy is characterized by a no-trade region shaped as a parallelogram, but the manuscript does not provide a detailed analytical proof. Like Dybvig (2005), we consider proportional transaction costs and mean-variance utility, but we extend the results to a multi-period setting, and show how the results can be rigorously proven analytically. In addition, we consider the case with market impact costs.

This manuscript is organized as follows. Section 2 describes the multiperiod framework under general transaction costs. Section 3 studies the case with proportional transaction costs, Section 4 the case with market impact costs, and Section 5 the case with quadratic transaction costs. Section 6 evaluates the utility loss associated with ignoring transaction costs and with behaving myopically for an empirical dataset on 15 commodity futures. Section 7 studies numerically the robustness of our modelling framework, and Section 8 concludes. Appendix A contains the figures, Appendix B contains the tables, and Appendix C contains the proofs for all results in the paper.

2 General Framework

Our framework is closely related to the one proposed by G&P. Like G&P, we consider an investor who maximizes the present value of the mean-variance utility of excess wealth changes (net of transaction costs), by investing in multiple risky assets and for multiple periods, and who has access to unconstrained borrowing. Moreover, like G&P, we make assumptions on the distribution of risky-asset price changes rather than returns. One difference with the framework proposed by G&P is that, consistent with most of the existing transaction cost literature, we focus on the case with constant investment opportunity set; that is, we consider the case with iid price changes.

As discussed in the introduction, these assumptions render the model tractable. The assumptions seem reasonable in the context of institutional investors who typically operate many different and relatively unrelated investment strategies. Each of these investment strategies represents only a fraction of the institutional investor’s portfolio, and thus focusing on excess wealth changes and assuming unconstrained borrowing is a good approximation. The stationarity of price changes is also a reasonable assumption for institutional investors who often have shorter investment horizons—because they operate each investment strategy only for a few months or at most a small number of years, and discontinue the investment strategy once its performance deteriorates.

The assumptions underlying our framework, however, do not seem suitable (a priori) to model individual investors who finance their lifetime consumption from the proceeds of their investments. These investors face constraints on borrowing, and have long investment horizons (their lifetime), during which one would expect returns, rather than price changes, to be stationary. To gauge the severity of our assumptions for the case of an individual investor, in Section 7 we compute numerically the optimal portfolio policy of an investor who faces a risk-free and a risky asset with iid returns, and who maximizes her CRRA utility of terminal wealth with a ten year horizon. We find that the certainty equivalent loss from using the G&P-type portfolio policy derived from our framework is less than 0.5% for a broad range of problem parameters.

Finally, there are three main differences between our model and the model by G&P. First, we consider a more general class of transaction costs that includes not only quadratic transaction costs, but also proportional and market impact costs. Second, we consider both finite and infinite investment horizons, whereas G&P focus on the infinite horizon case. Finally, we assume iid price changes, whereas G&P focus on the case with predictable price changes. We now rigorously state this assumption.

Assumption 1. *Price changes in excess of the risk-free asset price are independently and identically distributed (iid) with mean vector μ and covariance matrix Σ .*

The investor’s decision in our framework can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1 - \rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p \right], \quad (1)$$

where $x_t \in \mathbb{R}^N$ contains the number of shares of each of the N risky assets held in period t , T is the investment horizon, ρ is the discount factor, and γ is the *absolute* risk-aversion parameter.⁷

The term $\kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p$ is the transaction cost for the t th period, where $\kappa \in \mathbb{R}$ is the transaction cost parameter, $\Lambda \in \mathbb{R}^{N \times N}$ is the symmetric positive semidefinite transaction cost matrix, and $\|s\|_p$ is the p -norm of vector s ; that is, $\|s\|_p^p = \sum_{i=1}^N |s_i|^p$. This term allows us to capture the transaction costs associated with both small and large trades. Small trades typically do not impact market prices, and thus their transaction costs come from the bid-ask spread and other brokerage fees, which are modeled as proportional to the amount traded. Our transaction cost term captures proportional transaction costs for the case with $p = 1$ and $\Lambda = I$, where I is the identity matrix.

Large trades can have both temporary as well as permanent impact on market prices. Market price impact is temporary when it affects a single transaction, and permanent when it affects every future transaction. For simplicity of exposition, we focus on the case with temporary market impact costs, but our analysis can be extended to the case with permanent impact costs following an approach similar to that in Section 4 of G&P. For market impact costs, Almgren, Thum, Hauptmann, and Li (2005) suggest that transaction costs grow as a power function with an exponent between one and two, and hence we consider in our analysis values of $p \in (1, 2]$. The transaction cost matrix Λ captures the distortions to market prices generated by the interaction between the multiple assets. G&P argue that it can be viewed as a multi-dimensional version of Kyle's lambda, see Kyle (1985), and they argue that a sensible choice for the transaction cost matrix is $\Lambda = \Sigma$. We consider this case as well as the case with $\Lambda = I$ to facilitate the comparison with the case with proportional transaction costs.

Finally, the multiperiod mean-variance framework proposed by G&P and the closely related framework described in Equation (1) differ from the traditional dynamic mean-variance approach, which attempts to maximize the mean-variance utility of *terminal wealth*. Part 1 of Proposition 1 below, however, shows that the utility given in Equation (1) is equal

⁷ Because the investment problem is formulated in terms of wealth changes, the mean-variance utility is defined in terms of the *absolute* risk aversion parameter, rather than the *relative* risk aversion parameter. Note that the relative risk-aversion parameter equals the absolute risk-aversion parameter times the wealth.

to the mean-variance utility of the change in excess of *terminal* wealth for the case where the discount factor $\rho = 0$. This shows that the framework we consider is not too different from the traditional dynamic mean-variance approach. Also, a worrying feature of multiperiod mean-variance frameworks is that as demonstrated by Basak and Chabakauri (2010) they are often time-inconsistent: the investor may find it optimal to deviate from the ex-ante optimal policy as time goes by. Part 2 of Proposition 1 below, however, shows that the framework we consider is time consistent.

Proposition 1. *Let Assumption 1 hold, then the multiperiod mean-variance framework described in Equation (1) satisfies the following properties:*

1. *The utility given in Equation (1) is equivalent to the mean-variance utility of the change in excess terminal wealth for the case where the discount factor $\rho = 0$.*
2. *The optimal portfolio policy for the multiperiod mean-variance framework described in Equation (1) is time consistent.*

3 Proportional Transaction Costs

We now study the case where transaction costs are proportional to the amount traded. This type of transaction cost is appropriate to model small trades, where the transaction cost originates from the bid-ask spread and other brokerage commissions. Section 3.1 characterizes analytically the no-trade region and the optimal portfolio policy, and Section 3.2 shows how the no-trade region depends on the level of proportional transaction costs, the risk-aversion parameter, the discount factor, the investment horizon, and the correlation and variance of asset price changes.

3.1 The no-trade region

The investor's decision for this case can be written as:

$$\max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[(1 - \rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\}. \quad (2)$$

The following theorem characterizes the optimal portfolio policy.

Theorem 1. *Let Assumption 1 hold, then:*

1. *It is optimal not to trade at any period other than the first period; that is,*

$$x_1 = x_2 = \dots = x_T. \quad (3)$$

2. *The investor's optimal portfolio for the first period x_1 (and thus for all subsequent periods) is the solution to the following quadratic programming problem:*

$$\min_{x_1} (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (4)$$

$$s.t. \quad \|\Sigma(x_1 - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma} \frac{\rho}{1 - (1 - \rho)^T}. \quad (5)$$

where x_0 is the starting portfolio, and $x^* = \Sigma^{-1}\mu/\gamma$ is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio).

3. *Constraint (5) defines a no-trade region shaped as a parallelogram centered at the target portfolio x^* , such that if the starting portfolio x_0 is inside this region, then it is optimal not to trade at any period, and if the starting portfolio is outside this no-trade region, then it is optimal to trade at the first period to the point in the boundary of the no-trade region that minimizes the objective function in (4), and not to trade thereafter.*

A few comments are in order. First, a counterintuitive feature of our optimal portfolio policy is that it only involves trading in the first period. A related property, however, holds for most of the policies in the literature. Liu (2004), for instance, explains that: “the optimal trading policy involves possibly an initial discrete change (jump) in the dollar amount invested in the asset, followed by trades in the minimal amount necessary to maintain the dollar amount within a constant interval.” The “jump” in Liu’s policy, is equivalent to the first-period investment in our policy. The reason why our policy does not require any rebalancing after the first period is that it relies on the assumption that *prices changes* are iid. As a result, the portfolio and no-trade region in our framework are defined in terms of *number of shares*, and thus no rebalancing is required after the first period because realized price changes do not alter the number of shares held by the investor. Finally, our numerical results in Section 7 show that the certainty equivalent loss from using the G&P-type policy given in Theorem 1 for the case where *returns* are iid is small.

Second, inequality (5) provides a closed-form expression for the no-trade region. This expression shows that it is optimal to trade only if the marginal increment in utility from trading in one of the assets is larger than the transaction cost parameter κ . To see this, note that inequality (5) can be rewritten as

$$\kappa e \leq (\gamma(1-\rho)(1-(1-\rho)^T)/\rho)\Sigma(x_1 - x^*) \leq \kappa e, \quad (6)$$

where e is the N -dimensional vector of ones. Moreover, because Part 1 of Theorem 1 shows that it is optimal to trade only at the first period, it is easy to show that the term in the middle of (6) is the gradient (first derivative) of the discounted multiperiod mean variance utility $\sum_{t=1}^T (1-\rho)^t (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t)$ with respect to x_1 . Consequently, it is optimal to trade at the first period only if the *marginal* increase in the present value of the multiperiod mean-variance utility is larger than the transaction cost parameter κ .

Finally, inequality (5) shows that the no-trade region is a multi-dimensional parallelogram centered around the target portfolio. Mathematically, this property follows from the *linearity* of the first derivative of the multiperiod mean-variance utility with respect to the portfolio x_1 ; that is, the linearity of the middle term in (6). Our result contrasts with the findings of Muthuraman and Kumar (2006), who show (numerically for the case with two risky assets) that the no-trade region is a convex quadrilateral (rather than a parallelogram), and it is *not* centered around the target portfolio. Mathematically, the reason for this is that the CRRA utility they consider results in a value function whose first derivative is *not* linear. In addition, they impose constraints on borrowing, which adds to the nonlinearity of the boundary conditions defining the no-trade region.⁸ Economically, Muthuraman and Kumar (2006) consider an investor who maximizes CRRA utility of intermediate consumption and with constraints on borrowing. Consequently, the investor in Kumar and Muthuraman's framework is more willing to trade (and thus incur higher transaction costs) when she holds *large* positions on the risky assets, in order to guarantee a more stable level

⁸For the case where the investor maximizes her long-term expected growth rate, Muthuraman and Zha (2008) also find that the no-trade region is a quadrilateral that is not centered around the Merton portfolio. Essentially, maximizing the long-term growth rate is similar to maximizing a logarithmic utility function, which again results in a value function that has a nonlinear derivative. For the case with a single risky asset, Constantinides (1986) considers CRRA utility and constrained borrowing and also finds that the no-trade interval is not centered around the Merton portfolio. Dumas and Luciano (1991), on the other hand, consider the case with a single risky asset and an investor with constrained borrowing and CRRA utility of terminal wealth. For the case where the investor's horizon goes to infinity, they find that the no-trade interval *is* centered. Although Dumas and Luciano (1991) consider a nonlinear utility function and constrained borrowing, the focus on terminal wealth when the investment horizon goes to infinity results in a centered no-trade interval.

of wealth to finance her ongoing consumption. As a result, the no-trade region in Kumar and Muthuraman's framework is not centered around the target portfolio, and instead it is biased towards the risk-free asset.

Third, the optimal portfolio policy can be conveniently computed by solving the quadratic program (4)–(5). This class of optimization problems can be efficiently solved for cases with up to thousands of risky assets using widely available optimization software. As mentioned in the introduction, most of the existing results for the case with transaction costs rely on numerical analysis for the case with two risky assets. Our framework can be used to deal with cases with proportional transaction costs and hundreds or even thousands of risky assets. To gain understanding about the quadratic program (4)–(5), Figure 1 depicts the no-trade region defined by inequality (5) and the level sets for the objective function given by (4) for a case with two assets with mean and covariance matrix equal to the sample estimators for two commodity futures on gasoil and sugar, which are part of the full dataset of 15 commodities described in Section 6. The figure shows that the optimal portfolio policy is to trade to the intersection between the no-trade region and the tangent level set, at which the marginal utility from trading equals the transaction cost parameter κ .

3.2 Comparative statics

The following corollary establishes how the no-trade region depends on the level of proportional transaction costs, the risk-aversion parameter, the discount factor, and the investment horizon.

Corollary 1. *The no-trade region for the multiperiod investor satisfies the following properties:*

1. *The no-trade region expands as the proportional transaction parameter κ increases.*
2. *The no-trade region shrinks as the risk-aversion parameter γ increases.*
3. *The no-trade region expands as the discount factor parameter ρ increases.*
4. *The no-trade region shrinks as the investment horizon T increases.*

Part 1 of Corollary 1 shows that, not surprisingly, the size of the no-trade region grows with the transaction cost parameter κ . The reason for this is that the larger the transaction costs, the less willing the investor is to trade in order to diversify. This is illustrated in Panel (a) of Figure 2, which depicts the no-trade regions for different values of the transaction cost parameter κ for the two commodity futures on gasoil and sugar.⁹ Note also that (as discussed in Section 3.1) the no-trade regions for different values of the transaction cost parameter are all centered around the target portfolio.

Part 2 of Corollary 1 shows that the size of no-trade region decreases with the risk aversion parameter γ . Intuitively, as the investor becomes more risk averse, the optimal policy is to move closer to the diversified (safe) position x^* , despite the transaction costs associated with this. This is illustrated in Figure 2, Panel (b), which also shows that, not surprisingly, the target portfolio shifts towards the risk-free asset as the risk-aversion parameter increases.

Part 3 of Corollary 1 shows that the size of the no-trade region increases with the discount factor ρ . This makes sense intuitively because the larger the discount factor, the less important the utility for future periods and thus the smaller the incentive to trade today. This is illustrated in Figure 2, Panel (c).

Finally, Part 4 of Corollary 1 shows that the size of the no-trade region decreases with the investment horizon T . To see this intuitively, note that we have shown that the optimal policy is to trade at the first period and hold this position thereafter. Then, a multiperiod investor with shorter investment horizon will be more concerned about the transaction costs incurred at the first stage, compared with the investor who has a longer investment horizon. Finally, when $T \rightarrow \infty$, the no-trade region shrinks to the parallelogram bounded by $\kappa\rho/((1 - \rho)\gamma)$, which is much closer to the center x^* . When $T = 1$, the multiperiod problem reduces to the single-period problem studied by Dybvig (2005). This is illustrated in Figure 3, Panel (a).

⁹Although we illustrate Corollary 1 using two commodity futures, the results apply to the general case with N risky assets.

The no-trade region also depends on the correlation between assets. Figure 3, Panel (b) shows the no-trade regions for different correlations¹⁰. When the two assets are positively correlated, the parallelogram leans to the left, reflecting the substitutability of the two risky assets, whereas with negative correlation it leans to the right. In the absence of correlations the no-trade region becomes a rectangle.

Finally, the impact of variance on the no-trade region is shown in Panel (c) of Figure 3, where for expositional clarity we have considered the case with two uncorrelated symmetric risky assets. Like Muthuraman and Kumar (2006), we find that as variance increases, the no-trade region moves towards the risk-free asset because the investor is less willing to hold the risky assets. Also the size of no-trade region shrinks as the variance increases because the investor is more willing to incur transaction costs in order to diversify her portfolio.

4 Market Impact Costs

We now consider the case of large trades that may impact market prices. As discussed in Section 2, to simplify the exposition we focus on the case with temporary market impact costs, but the analysis can be extended to the case with permanent impact costs following an approach similar to that in Section 4 of G&P. Almgren, Thum, Hauptmann, and Li (2005) suggest that market impact costs grow as a power function with an exponent between one and two, and hence we consider a general case, where the transaction costs are given by the p-norm with $p \in (1, 2)$, and where we capture the distortions on market price through the transaction cost matrix Λ . For exposition purposes, we first study the single-period case.

4.1 The Single-Period Case

For the single-period case, the investor's decision is:

$$\max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - \kappa \|\Lambda^{1/p}(x - x_0)\|_p^p, \quad (7)$$

where $1 < p < 2$. Problem (7) can be solved numerically, but unfortunately it is not possible to obtain closed-form expressions for the optimal portfolio policy. The following

¹⁰Because change in correlation also makes the target shift, in order to emphasize how correlation affects the shape of the region, we change the covariance matrix Σ in a manner so as to keep the Markowitz portfolio (the target) fixed, similar to the analysis in Muthuraman and Kumar (2006).

proposition, however, shows that the optimal portfolio policy is to trade to the boundary of a *rebalancing region* that depends on the starting portfolio and contains the target or Markowitz portfolio.

Proposition 2. *Let Assumption 1 hold, then if the starting portfolio x_0 is equal to the target or Markowitz portfolio x^* , the optimal policy is not to trade. Otherwise, it is optimal to trade to the boundary of the following rebalancing region:*

$$\frac{\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q}{p\|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (8)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Comparing Proposition 2 with Theorem 1 we observe that there are three main differences between the cases with proportional and market impact costs. First, for the case with market impact costs it is always optimal to trade (except in the trivial case where the starting portfolio coincides with the target or Markowitz portfolio), whereas for the case with proportional transaction costs it may be optimal not to trade if the starting portfolio is inside the no-trade region. Second, the rebalancing region depends on the starting portfolio x_0 , whereas the no-trade region is independent of it. Third, the rebalancing region contains the target or Markowitz portfolio, but it is not centered around it, whereas the no-trade region is centered around the Markowitz portfolio.

Note that, as in the case with proportional transaction costs, the size of the rebalancing region increases with the transaction cost parameter κ , and decreases with the risk-aversion parameter. Intuitively, the more risk averse the investor, the larger her incentives to trade and diversify her portfolio. Also, the rebalancing region grows with κ because the larger the transaction cost parameter, the less attractive to the investor is to trade to move closer to the target portfolio.

The following corollary gives the rebalancing region for two important particular cases. First, the case where the transaction cost matrix $\Lambda = I$, which is a realistic assumption when the amount traded is small, and thus the interaction between different assets, in terms of market impact, is small. This case also facilitates the comparison with the optimal portfolio policy for the case with proportional transaction costs. The second case corresponds to the

transaction cost matrix $\Lambda = \Sigma$, which G&P argue is realistic in the context of quadratic transaction costs.

Corollary 2. *For the single-period investor defined in (7):*

1. *When the transaction cost matrix is $\Lambda = I$, then the rebalancing region is*

$$\frac{\|\Sigma(x - x^*)\|_q}{p\|x - x_0\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (9)$$

2. *When the transaction cost matrix is $\Lambda = \Sigma$, then the rebalancing region is*

$$\frac{\|\Sigma^{1/q}(x - x^*)\|_q}{p\|\Sigma^{1/p}(x - x_0)\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (10)$$

Note that, in both particular cases, the Markowitz strategy x^* is contained in the rebalancing region.

To gain intuition about the form of the rebalancing regions characterized in (9) and (10), Panel (a) in Figure 4 depicts the rebalancing region and the optimal portfolio policy for a two-asset example when $\Lambda = I$, while Panel (b) depicts the corresponding rebalancing region and optimal portfolio policy when $\Lambda = \Sigma$. The figure shows that, in both cases, the rebalancing region is a convex region containing the Markowitz portfolio. Moreover, it shows how the optimal trading strategy moves to the boundary of the rebalancing region.

4.2 The Multiperiod Case

The investor's decision for this case can be written as:

$$\max_{\{x_i\}_{i=1}^T} \sum_{t=1}^T \left[(1 - \rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right]. \quad (11)$$

As in the single-period case, it is not possible to provide closed-form expressions for the optimal portfolio policy, but the following theorem illustrates the analytical properties of the optimal portfolio policy.

Theorem 2. *Let Assumption 1 hold, then:*

1. *If the starting portfolio x_0 is equal to the target or Markowitz portfolio x^* , then the optimal policy is not to trade at any period.*

2. Otherwise it is optimal to trade at every period. Moreover, at the t th period it is optimal to trade to the boundary of the following rebalancing region:

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Lambda^{-1/p} \Sigma(x_s - x^*)\|_q}{p \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}, \quad (12)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2 shows that for the multiperiod case with market impact costs it is optimal to trade at every period (except in the trivial case where the starting portfolio coincides with the Markowitz portfolio). This is in contrast to the case with proportional transaction costs, where it is not optimal to trade at any period other than the first. The reason for this is that with strictly convex transaction costs, $p > 1$, the transaction cost associated with a trade can be reduced by breaking it into several smaller transactions. For the case with proportional transaction costs, on the other hand, the cost of a transaction is the same whether executed at once or broken into several smaller trades, and thus it makes economic sense to carry out all the trading in the first period to take advantage of the utility benefits from the beginning.

Note also that for the case with market impact costs at every period it is optimal to trade to the boundary of a *state-dependent* rebalancing region that depends not only on the starting portfolio, but also on the portfolio for every subsequent period. Finally, note that the size of the rebalancing region for period t , assuming the portfolios for the rest of the periods are fixed, increases with the transaction cost parameter κ and decreases with the discount factor ρ and the risk-aversion parameter γ .

The following proposition shows that the rebalancing region for period t contains the rebalancing region for every subsequent period. Moreover, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows, and thus the optimal portfolio x_T converges to the target portfolio x^* in the limit when T goes to infinity. The intuition behind this result is again that the strict convexity of the transaction cost function for the case with $p > 1$ implies that it is optimal to trade every period, but it is never optimal to trade all the way to the Markowitz portfolio, and thus the investor's portfolio converges to the Markowitz portfolio only as time goes to infinity.

Proposition 3. *Let Assumption 1 hold, then:*

1. The rebalancing region for the t -th period contains the rebalancing region for every subsequent period,
2. Every rebalancing region contains the Markowitz portfolio,
3. The rebalancing region converges to the Markowitz portfolio in the limit when the investment horizon goes to infinity.

The next corollary gives the rebalancing region for the two particular cases of transaction cost matrix we consider.

Corollary 3. *For the multiperiod investor defined in (11):*

1. When the transaction cost matrix is $\Lambda = I$, then the rebalancing region is

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Sigma (x_s - x^*)\|_q}{p \|x_t - x_{t-1}\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (13)$$

2. When the transaction cost matrix is $\Lambda = \Sigma$, then the rebalancing region is

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_s - x^*)\|_q}{p \|\Sigma^{1/p} (x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (14)$$

To gain intuition about the shape of the rebalancing regions characterized in (13) and (14), Panel (a) in Figure 5 shows the optimal portfolio policy and the rebalancing regions for the two commodity futures on gasoil and sugar with an investment horizon $T = 3$ when $\Lambda = I$, whereas Panel (b) depicts the corresponding optimal portfolio policy and rebalancing regions when $\Lambda = \Sigma$. The figure shows, in both cases, how the rebalancing region for each period contains the rebalancing region for subsequent periods. Moreover, every rebalancing region contains, but is not centered at, the Markowitz portfolio x^* . In particular, for each stage, any trade is to the boundary of the rebalancing region and the rebalancing is towards the Markowitz strategy x^* .

Finally, we study numerically the impact of the market impact cost growth rate p on the optimal portfolio policy. Figure 6 shows the rebalancing regions and trading trajectories for investors with different transaction growth rates $p = 1, 1.25, 1.5, 1.75, 2$. When the transaction cost matrix $\Lambda = I$, Panel (a) shows how the rebalancing region depends on p . In particular, for $p = 1$ we recover the case with proportional transaction costs, and

hence the rebalancing region becomes a parallelogram. For $p = 2$, the rebalancing region becomes an ellipse. And for values of p between 1 and 2, the shape of the rebalancing regions are similar to superellipses¹¹ but not centered at the target portfolio x^* . On the other hand, Panel (b) in Figure 6 shows how the trading trajectories depend on p for a particular investment horizon of $T = 10$ days. We observe that, as p grows, the trading trajectories become more curved and the investor converges towards the target portfolio at a slower rate. To conserve space, we do not provide the figure for the case $\Lambda = \Sigma$, but we find that for this case the trajectories are less curved as p grows, and becomes a straight line for $p = 2$.

5 Quadratic Transaction Costs

We now consider the case with quadratic transaction costs. The investor's decision is:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) - (1-\rho)^{t-1} \kappa \|\Lambda^{1/2} (x_t - x_{t-1})\|_2^2 \right]. \quad (15)$$

For the case with quadratic transaction costs, our framework differs from that in G&P in two respects only. First, G&P's work focuses on impact of predictability, whereas consistent with most of the existing literature on transaction costs we assume price changes are iid. Second, G&P consider an infinite horizon, whereas we allow for a finite investment horizon. The next theorem adapts the results of G&P to obtain an explicit characterization of the optimal portfolio policy.

Theorem 3. *Let Assumption 1 hold, then:*

1. *The optimal portfolio $x_t, x_{t+1}, \dots, x_{t+T-1}$ satisfies the following equations:*

$$x_t = A_1 x^* + A_2 x_{t-1} + A_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (16)$$

$$x_t = B_1 x^* + B_2 x_{t-1}, \quad \text{for } t = T. \quad (17)$$

where

$$A_1 = (1-\rho)\gamma [(1-\rho)\gamma\Sigma + 2\kappa\Lambda + 2(1-\rho)\kappa\Lambda]^{-1} \Sigma,$$

$$A_2 = 2\kappa [(1-\rho)\gamma\Sigma + 2\kappa\Lambda + 2(1-\rho)\kappa\Lambda]^{-1} \Lambda,$$

$$A_3 = 2(1-\rho)\kappa [(1-\rho)\gamma\Sigma + 2\kappa\Lambda + 2(1-\rho)\kappa\Lambda]^{-1} \Lambda,$$

¹¹The general expression for a superellipse is $|\frac{x}{a}|^m + |\frac{y}{b}|^n = 1$ with $m, n > 0$.

with $A_1 + A_2 + A_3 = I$, and

$$B_1 = (1 - \rho)\gamma[(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Sigma,$$

$$B_2 = 2\kappa[(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Lambda,$$

with $B_1 + B_2 = I$.

2. The optimal portfolio converges to the Markowitz portfolio as the investment horizon T goes to infinity.

Theorem 3 shows that the optimal portfolio for each stage is a combination of the Markowitz strategy (the target portfolio), the previous period portfolio, and the next period portfolio.

The next corollary shows the specific optimal portfolios for two particular cases of transaction cost matrix. We consider the case where the transaction costs matrix is proportional to the covariance matrix, which G&P argue is realistic.¹² In addition, we also consider the case where the transaction costs matrix is proportional to the identity matrix; that is $\Lambda = I$.

Corollary 4. For a multiperiod investor with objective function (15):

1. When the transaction cost matrix is $\Lambda = I$, then the optimal trading strategy satisfies

$$x_t = A_1x^* + A_2x_{t-1} + A_3x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (18)$$

$$x_t = B_1x^* + B_2x_{t-1}, \quad \text{for } t = T \quad (19)$$

where

$$A_1 = (1 - \rho)\gamma[(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1}\Sigma,$$

$$A_2 = 2\kappa[(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1},$$

$$A_3 = 2(1 - \rho)\kappa[(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1},$$

with $A_1 + A_2 + A_3 = I$, and

$$B_1 = (1 - \rho)\gamma[(1 - \rho)\gamma\Sigma + 2\kappa I]^{-1}\Sigma,$$

$$B_2 = 2\kappa[(1 - \rho)\gamma\Sigma + 2\kappa I]^{-1}.$$

with $B_1 + B_2 = I$.

¹²Note that although G&P argue that the case $\Lambda = \Sigma$ is realistic, they also solve explicitly the case with general transaction cost matrix Λ .

2. When the transaction cost matrix is $\Lambda = \Sigma$, then the optimal trading strategy satisfies

$$x_t = \alpha_1 x^* + \alpha_2 x_{t-1} + \alpha_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (20)$$

$$x_t = \beta_1 x^* + \beta_2 x_{t-1}, \quad \text{for } t = T. \quad (21)$$

where $\alpha_1 = (1 - \rho)\gamma / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$, $\alpha_2 = 2\kappa / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$, $\alpha_3 = 2(1 - \rho)\kappa / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and $\beta_1 = (1 - \rho)\gamma / ((1 - \rho)\gamma + 2\kappa)$, $\beta_2 = 2\kappa / ((1 - \rho)\gamma + 2\kappa)$ with $\beta_1 + \beta_2 = 1$.

3. When the transaction cost matrix is $\Lambda = \Sigma$, then the optimal portfolios for periods $t = 1, 2, \dots, T$ lay on a straight line.

Corollary 4 shows that, when $\Lambda = \Sigma$, the solution becomes simpler and easier to interpret than when $\Lambda = I$. Note that when $\Lambda = \Sigma$, matrices A and B in Theorem 3 become scalars α and β , respectively, and hence the optimal portfolio at period t can be expressed as a linear combination of the Markowitz portfolio, the previous period portfolio and the next period portfolio. For this reason, it is intuitive to observe that the optimal trading strategies for all the periods must lay on a straight line.

To conclude this section, Figure 7 provides a comparison of the optimal portfolio policy for the case with quadratic transaction costs (when $\Lambda = \Sigma$), with those for the cases with proportional and market impact costs (when $\Lambda = I$), for a multiperiod investor with $T = 3$. We have also considered other transaction cost matrices, but the insights are similar. The figure confirms that, for the case with quadratic transaction costs, the optimal portfolio policy is to trade at every period along a straight line that converges to the Markowitz portfolio. It can also be appreciated that the investor trades more aggressively at the first periods compared to the final periods. For the case with proportional transaction costs, it is optimal to trade to the boundary of the no-trade region shaped as a parallelogram in the first period and not to trade thereafter. Finally, for the case with market impact costs, the investor trades at every period to the boundary of the corresponding rebalancing region. The resulting trajectory is not a straight line.

6 Empirical Analysis

In this section, we study empirically the losses associated with ignoring transaction costs and investing myopically, as well as how these losses depend on the transaction cost parameter, the investment horizon, the risk-aversion parameter, and the discount factor. We first consider the case with proportional transaction costs, and then study how the monotonicity properties of the losses change when transaction costs are quadratic. We have also considered the case with market impact costs ($p = 1.5$), but the monotonicity properties for this case are in the middle of those for the cases with $p = 1$ and $p = 2$ and thus we do not report the results to conserve space.

For each type of transaction cost (proportional or quadratic), we consider three different portfolio policies. First, we consider the *target* portfolio policy, which consists of trading to the target or Markowitz portfolio in the first period and not trading thereafter. This is the optimal portfolio policy for an investor in the absence of transaction costs. Second, the *static* portfolio policy, which consists of trading at each period to the solution to the single-period problem subject to transaction costs. This is the optimal portfolio policy for a myopic investor who takes into account transaction costs. Third, we consider the *multiperiod* portfolio policy, which is the optimal portfolio policy for a multiperiod investor who takes into account transaction costs.

Finally, we evaluate the utility of each of the three portfolio policies using the appropriate multiperiod framework; that is, when considering proportional transaction costs, we evaluate the investor's utility from each portfolio with the objective function in equation (2); and when considering quadratic transaction costs, we evaluate the investor's utility using the objective function (15). These utilities are in units of wealth, and to facilitate the comparison of the different policies, we report the percentage utility loss of the suboptimal portfolio policies with respect to the utility of the optimal multiperiod portfolio policy. Note also that, as argued by (DeMiguel, Garlappi, and Uppal, 2009, Footnote 17), it can be shown that the mean-variance utility is approximately equal to the certainty equivalent of an investor with quadratic utility, and thus although we report utility losses in this section, they can be interpreted equivalently as certainty equivalent losses.

We consider an empirical dataset similar to the one used by Gârleanu and Pedersen (2013).¹³ In particular, the dataset is constructed with 15 commodity futures: Aluminum, Copper, Nickel, Zinc, Lead, and Tin from the London Metal Exchange (LME), Gasoil from the Intercontinental Exchange (ICE), WTI crude, RBOB Unleaded gasoline, and Natural Gas from the New York Mercantile Exchange (NYMEX), Gold and Silver from the New York Commodities Exchange (COMEX), and Coffee, Cocoa, and Sugar from the New York Broad of Trade (NYBOT). The dataset contains daily data from July 7th, 2004 until September 19th, 2012. For our evaluation, we replace the mean and covariance matrix of price changes with their sample estimators.

6.1 Proportional Transaction Costs

6.1.1 Base Case.

For our base case, we adapt the parameters used by G&P in their empirical analysis to the case with proportional transaction costs. We assume proportional transaction costs of 50 basis points ($\kappa = 0.005$), absolute risk-aversion parameter $\gamma = 10^{-6}$, which corresponds to a relative risk aversion of one for a small investor managing one million dollars¹⁴, annual discount factor $\rho = 2\%$, and an investment horizon of $T = 22$ days (one month). For all the cases, the investor's initial portfolio is the equally weighted portfolio; that is, the investor splits her one million dollars equally among the 15 assets.

For our base case, we observe that the utility loss associated with investing myopically (that is, the relative difference between the utility of the *multiperiod* portfolio policy and the *static* portfolio policy) is 60.46%. This utility loss is large because the no-trade region corresponding to the static portfolio policy contains the equally-weighted portfolio, and thus the static portfolio policy is to remain at the starting equally-weighted portfolio, which attains a much lower multiperiod mean-variance utility than the multiperiod portfolio policy. The utility loss associated with ignoring transaction costs altogether (that is, the relative difference between the utility of the *multiperiod* portfolio policy and the *target* portfolio

¹³We thank Alberto Martin-Utrera for making this dataset available to us.

¹⁴Gârleanu and Pedersen (2013) consider a smaller absolute risk aversion $\gamma = 10^{-9}$, which corresponds to a larger investor managing $M = 10^9$ dollars. It makes sense, however, to consider a smaller investor (and thus a larger absolute risk-aversion parameter) in the context of proportional transaction costs because these are usually associated with small trades.

policy) is 49.33%. This loss is large because the target portfolio policy requires a large amount of trading in the first period that results in large transaction costs. Summarizing, we find that the loss associated with either ignoring transaction costs or behaving myopically can be substantial. The following subsection confirms this is also true when we change relevant model parameters.

6.1.2 Comparative statics.

We study numerically how the utility losses associated with ignoring transaction costs (i.e., with the static portfolio), and investing myopically (i.e., with the target portfolio) depend on the transaction cost parameter, the investment horizon, the risk-aversion parameter, and the discount factor.

Panel (a) in Figure 8 depicts the utility loss associated with the target and static portfolios for values of the proportional transaction cost parameter κ ranging from 0 basis point to 460 basis points (which is the value of κ for which the optimal multiperiod policy is not to trade). As expected, the utility loss associated with ignoring transaction costs is zero in the absence of transaction costs and increases monotonically with transaction costs. Moreover, for large transaction costs parameters, the utility loss associated with ignoring transaction costs grows linearly with κ and can be very large.¹⁵ The utility losses associated with behaving myopically are unimodal (first increasing and then decreasing) in the transaction cost parameter, being zero for the case with zero transaction costs (because both the single-period and multiperiod portfolio policies coincide with the target or Markowitz portfolio), and for the case with large transaction costs (because both the single-period and multiperiod portfolio policies result in little or no trading). The utility loss of behaving myopically reaches a maximum of 80% for a level of transaction costs of around 5 basis points.

Panel (b) in Figure 8 depicts the utility loss associated with investing myopically and ignoring transaction costs for investment horizons ranging from $T = 5$ (one week) to $T = 260$ (over one year). Not surprisingly, the utility loss associated with behaving myopically grows with the investment horizon. Also, the utility loss associated with ignoring transaction costs

¹⁵In fact, the utility of the target portfolio policy is negative (and thus the utility loss is larger than 100%) for κ larger than 69 basis points because of the high transaction costs associated with this policy.

is very large for short-term investors, and decreases monotonically with the investment horizon. The reason for this is that the size of the no-trade region for the multiperiod portfolio policy decreases monotonically with the investment horizon, and thus the target and multiperiod policies become similar for long investment horizons. This makes sense intuitively: by adopting the Markowitz portfolio, a multiperiod investor incurs transaction cost losses at the first period, but makes mean-variance utility gains for the rest of the investment horizon. Hence, when the investment horizon is long, the transaction losses are negligible compared with the utility gains.

Finally, we find that the relative utility losses associated with investing myopically and ignoring transaction costs do not depend on the risk-aversion parameter as well as the discount factor ρ .

6.2 Quadratic Transaction Costs

In this section we study whether and how the presence of quadratic transaction costs (as opposed to proportional transaction costs) affects the utility losses of the static and target portfolios.

6.2.1 The Base Case.

Our base case parameters are similar to those adopted in Gârleanu and Pedersen (2013). We assume that the matrix $\Lambda = \Sigma$ and set the absolute risk aversion parameter $\gamma = 10^{-8}$, which corresponds to an investor with relative-risk aversion of one who manages 100 million dollars¹⁶, discount factor $\rho = 2\%$ annually, transaction costs parameter $\kappa = 1.5 \times 10^{-7}$ (which corresponds to $\lambda = 3 \times 10^{-7}$ in G&P's formulation), investment horizon $T = 22$ days (one month), and equal-weighted initial portfolio.

Similar to the case with proportional transaction costs, we find that the losses associated with either ignoring transaction costs or behaving myopically are substantial. For instance, for the base case with find that the utility loss associated with investing myopically

¹⁶Gârleanu and Pedersen (2013) choose a smaller absolute risk aversion parameter $\gamma = 10^{-9}$, which corresponds to an investor with relative-risk aversion of one who manages one billion dollars. The insights from our analysis are robust to the use of $\gamma = 10^{-9}$, but we choose $\gamma = 10^{-8}$ because this results in figures that are easier to interpret.

is 28.98%, whereas the utility loss associated with ignoring transaction costs is 109.14%. Moreover, we find that the utility losses associated with the target portfolio are relatively larger, compared to those of the static portfolio, for the case with quadratic transaction costs. The explanation for this is that the target portfolio requires large trades in the first period, which are penalized heavily in the context of *quadratic* transaction costs. The static portfolio, on the other hand, results in smaller trades over successive periods and this will result in overall smaller *quadratic* transaction costs.

6.2.2 Comparative Statics.

Panel (a) in Figure 9 depicts the utility loss associated with investing myopically and ignoring transaction costs for values of the quadratic transaction cost parameter κ ranging from 2.5×10^{-8} to 2.5×10^{-7} .¹⁷ Our findings are very similar to those for the case with proportional transaction costs. Not surprisingly, the utility losses associated with ignoring transaction costs are small for small transaction costs and grow monotonically as the transaction cost parameter grows. Also, the utility loss associated with the static portfolio policy is concave unimodal in the level of transaction costs. The intuition behind these results is similar to that provided for the case with proportional transaction costs.

Panel (b) in Figure 9 depicts the utility loss associated with investing myopically and ignoring transaction costs for values investment horizon T ranging from 5 days to 260 days. Our findings are similar to those for the case with proportional transaction costs. The target portfolio losses are monotonically decreasing with the investment horizon as these two portfolios become more similar for longer investment horizons, where the overall importance of transaction costs is smaller. The static portfolio losses are monotonically increasing with the investment horizon because the larger the investment horizon the faster the multiperiod portfolio converges to the target, whereas the rate at which the static portfolio converges to the target does not change with the investment horizon.

Finally, we find that the utility loss associated with investing myopically and ignoring transaction costs is monotonically decreasing in the absolute risk-aversion parameter. The explanation for this is that when the risk-aversion parameter is large, the mean-variance

¹⁷This corresponds to changing the value of λ in G&P from 5×10^{-8} to 5×10^{-7}

utility is relatively more important compared to the quadratic transaction costs, and thus the target and static portfolios are more similar to the multiperiod portfolio. This is in contrast to the case with proportional transaction costs, where the losses did not depend on the risk-aversion parameter. The reason for this difference is that with quadratic transaction costs, the mean-variance utility term and the transaction cost term are both quadratic, and thus the risk-aversion parameter does have an impact on the overall utility loss. Similar with the case with proportional transaction costs, the utility losses do not depend on the change in the value of discount factor as in the model with proportional transaction costs.

7 Model robustness

To gauge the robustness of our results to the use of a framework with iid returns and CRRA utility, we consider an investor who maximizes her CRRA utility (of either terminal wealth or intermediate consumption) by investing in a risk-free asset and a risky asset with iid returns, and who is subject to proportional transaction costs. We compute the investor's optimal portfolio policy using a numerical procedure similar to that used by DeMiguel and Uppal (2005), and find that the certainty equivalent loss from following the G&P-type portfolio policy given in Theorem 1 is typically smaller than 0.5%.

We consider a base case similar to that considered by DeMiguel and Uppal (2005): the investor has a risk-aversion parameter $\gamma = 3$, faces a risk-free asset with annual rate of return of 6% and a risky asset with mean return of 10% and return volatility of 20%. The investor has an initial endowment of \$1 invested in the risk-free asset, faces proportional transaction costs of 50 basis points, rebalances her portfolio once a year, and has an investment horizon of ten years.

For the base case where the investor maximizes her CRRA utility of terminal wealth, we find that the certainty equivalent loss from using the G&P portfolio policy is only 0.20%.¹⁸ To understand why this loss is so small, note that the main difference between the optimal portfolio policy and the G&P policy is that the optimal portfolio policy requires

¹⁸Note that the G&P-type portfolio policy given by Theorem 1 is a buy-and-hold policy. We compute the number of shares to be held by this policy by solving problem (4)–(5). Because we assume the starting price of the risky asset is \$1, the mean and covariance matrix of excess price changes is equal to the mean and covariance matrix of excess returns. It is straightforward to generalize this procedure to the case where the risky asset starting price is different from \$1, and it is easy to show that the certainty equivalent loss from using the G&P-type portfolio policy does not depend on the risky asset starting price.

rebalancing, whereas the G&P policy is a buy-and-hold policy. Nonetheless, we find that, because of the presence of transaction costs, even the optimal portfolio policy requires very little rebalancing (on average the investor trades only 2% of her wealth per year), and thus the optimal and G&P policies are quite similar.

Table 1 shows how the certainty equivalent loss of the G&P-type portfolio policy depends on the risk-aversion parameter, the stock return volatility, and the transaction cost parameter for the case where the investor maximizes her CRRA utility of terminal wealth. We find that the certainty equivalent loss decreases with risk aversion (because the amount invested in the risky asset, and thus the optimal amount of rebalancing, decreases), increases with the stock return volatility (because the optimal amount of rebalancing increases), and decreases with transaction costs (because the optimal amount of rebalancing decreases). For all values of these parameters that we try, we find that the certainty equivalent loss is below 0.50%.

We then consider an investor who maximizes her CRRA utility of intermediate consumption. To be able to make a sensible comparison, we augment the portfolio policy given by Theorem 1 by assuming that the investor's consumption to wealth ratio is equal to that of an investor who maximizes her CRRA utility of intermediate consumption, but ignores the presence of transaction costs.¹⁹ We find that the certainty equivalent loss for our base case with intermediate consumption is 0.14%, and thus even smaller than that for the case where the investor maximizes utility of terminal wealth. The reason for this is that the investor progressively liquidates her wealth to finance consumption, both for the optimal and G&P policies, and thus the amount of trading required to rebalance the risky asset position is smaller.

Table 2 shows how the certainty equivalent loss for the case with intermediate consumption depends on the risk-aversion, volatility, and transaction costs. As in the case without intermediate consumption, we find that the certainty equivalent loss decreases with risk aversion because the amount invested in the risky asset and thus the optimal amount of rebalancing decreases. We find that the certainty equivalent loss is not very sensitive to the

¹⁹In particular, we use the investor's consumption to wealth ratio given by (Ingersoll, 1987, p. 243), after fixing the the well-known typo in the book. This is a conservative choice because this consumption to wealth ratio is not optimal in the presence of transaction costs.

stock return volatility because volatility has two opposite effects on the amount of rebalancing. On the one hand, higher volatility implies smaller allocation to the risky asset, and thus less rebalancing. On the other hand, higher volatility implies higher fluctuations on portfolio weights, and thus more rebalancing. These two opposite effects seem to be of roughly similar strength and thus the certainty equivalent loss is not very sensitive to volatility. Finally, we find that the certainty equivalent loss increases with transaction costs (unlike in the case without intermediate consumption). The reason for this is that the particular procedure we have used to augment the G&P policy to finance intermediate consumption requires a large amount of trading on the risky asset, and thus large transaction costs. Nevertheless, we find that the certainty equivalent loss incurred from using the G&P-type policy is always below 0.50%.

Finally, we consider a case with two risky stocks with return correlation of 50%, and an investment horizon of 8 years, and find that the certainty equivalent loss of using the G&P policy is 0.11%. Overall, our results show that although the G&P-type policy relies on the assumptions of iid price changes and multiperiod mean-variance utility, it provides a reasonable approximation for the optimal portfolio policy for the case with iid returns and CRRA utility, and thus we think our analysis provides insights that are interesting beyond the G&P framework.

8 Conclusions

We study the optimal portfolio policy for a multiperiod mean-variance investor facing many risky assets subject to proportional, market impact, or quadratic transaction costs. We provide a closed-form expression for a no-trade region shaped as a parallelogram, and use these closed-form expressions to show how the no-trade region shrinks with the investment horizon and the risk-aversion parameter, and grows with the level of proportional transaction costs and the discount factor. Moreover, we show that the optimal portfolio policy can be conveniently computed by solving a single quadratic program for problems with up to thousands of risky assets. For the case with market impact costs, the optimal portfolio policy is to trade to the boundary of a state-dependent rebalancing region. In addition, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows

large. We also show numerically that the losses associated with ignoring transaction costs or investing myopically may be large, and study how they depend on the relevant parameters. Finally, we compute the optimal portfolio policy for a case with CRRA utility and iid returns, and show that the G&P-type policy that we study provides a good approximation to this case.

A Figures

Figure 1: No-trade region and level sets for proportional transaction costs.

This figure depicts the no-trade region and the level sets when the investment horizon $T = 5$ for an investor facing proportional transaction costs with $\kappa = 0.005$, annual discount factor $\rho = 2\%$, absolute risk-aversion parameter $\gamma = 10^{-4}$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

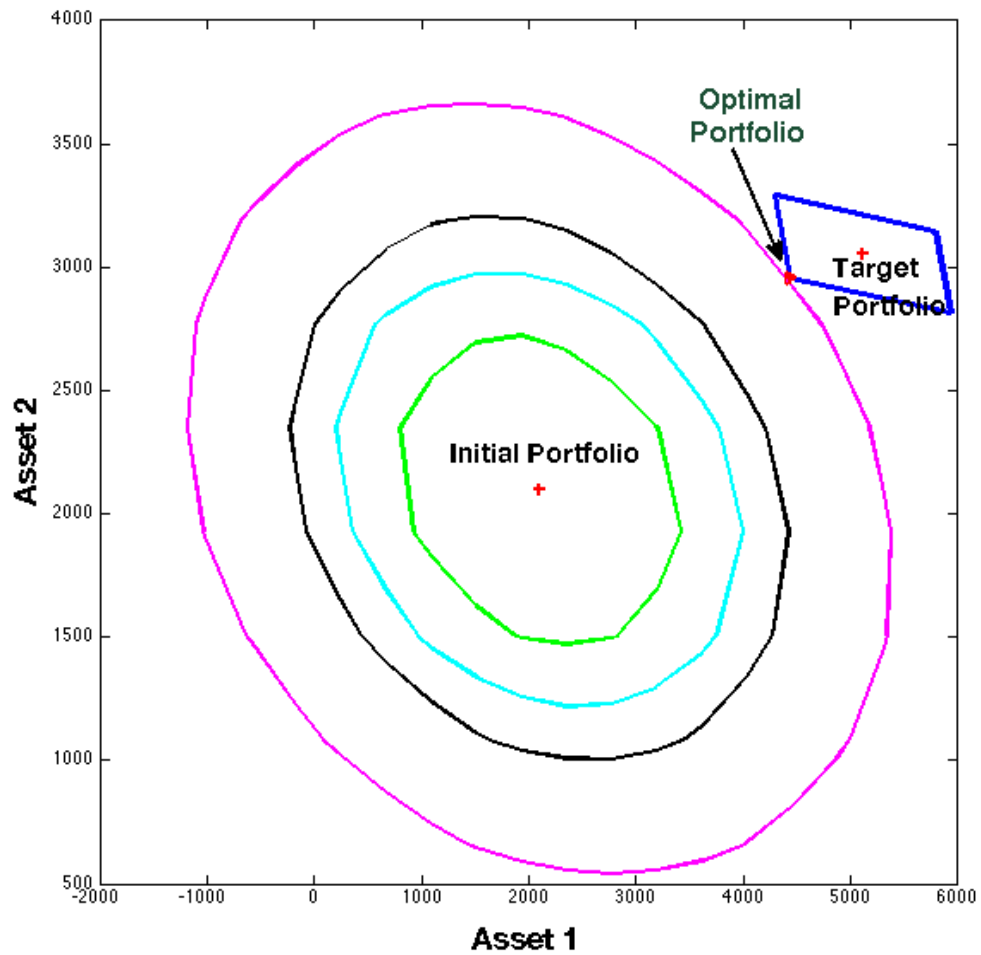
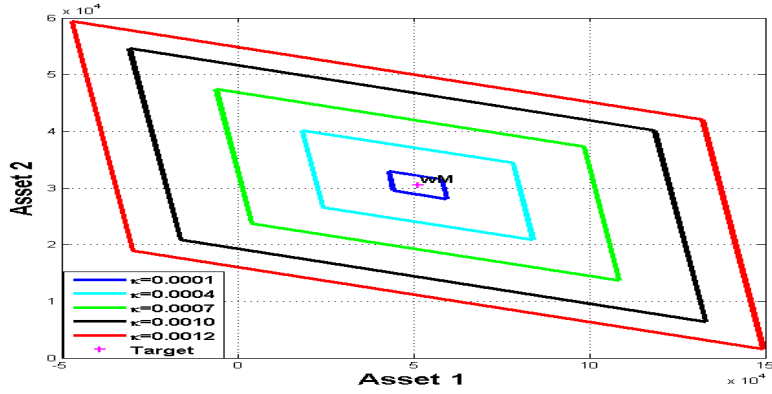


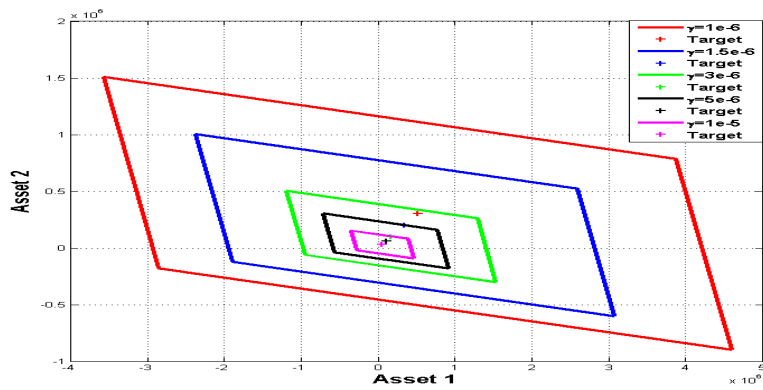
Figure 2: No-trade region: comparative statics.

This figure shows how the no-trade region for a multiperiod investor subject to proportional transaction costs depends on relevant parameters. For the base case, we consider a proportional transaction cost parameter $\kappa = 0.005$, annual discount factor $\rho = 2\%$, absolute risk-aversion parameter $\gamma = 10^{-6}$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

(a) No-trade regions for different κ



(b) No-trade regions for different γ



(c) No-trade regions for different ρ

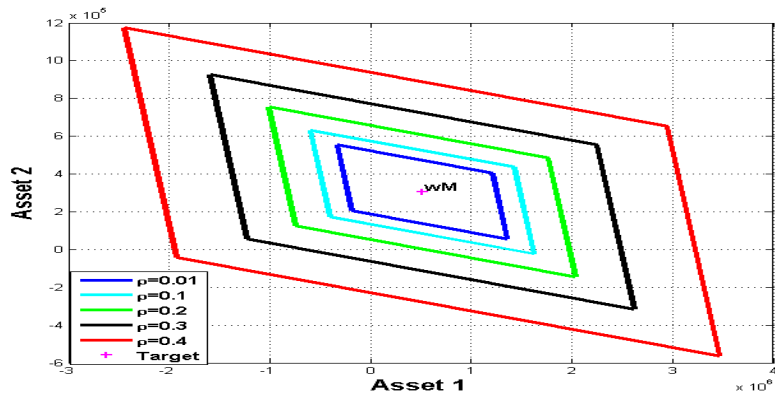
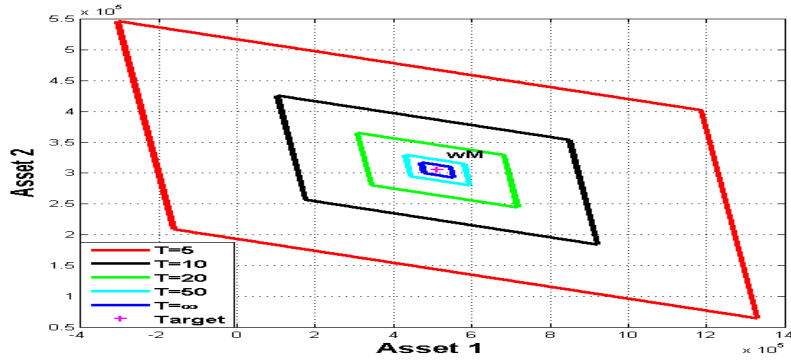


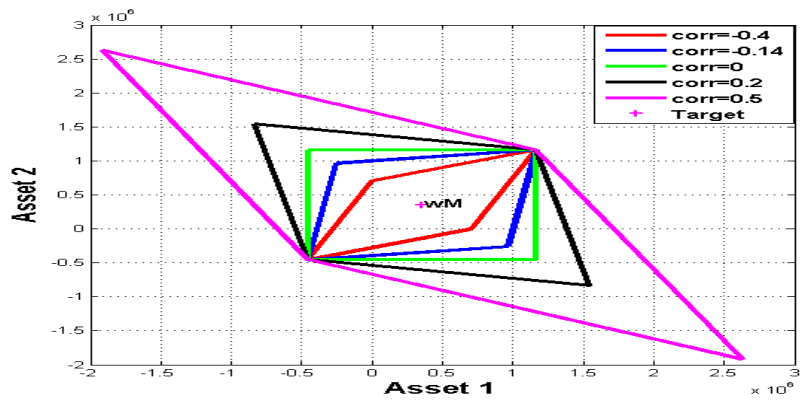
Figure 3: No-trade region: comparative statics.

This figure shows how the no-trade region for a multiperiod investor subject to proportional transaction costs depends on relevant parameters. For the base case, we consider a proportional transaction cost parameter $\kappa = 0.005$, annual discount factor $\rho = 2\%$, absolute risk-aversion parameter $\gamma = 10^{-6}$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

(a) No-trade regions for different T



(b) No-trade regions for different correlations



(c) No-trade regions for different σ^2

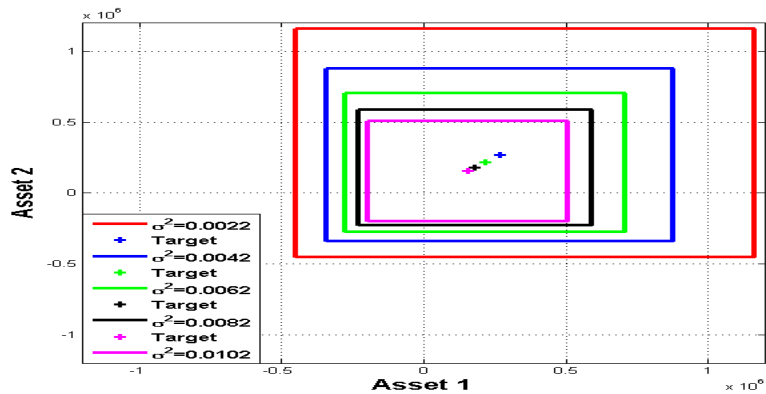


Figure 4: Rebalancing region for single-period investor

This figure depicts the rebalancing region for a single-period investor subject to market impact costs. We consider an absolute risk aversion parameter $\gamma = 10^{-7}$, annual discount factor $\rho = 2\%$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar. Panel (a) depicts the rebalancing region when the transaction cost matrix $\Lambda = I$, the exponent of the power function is $p = 1.5$, and the transaction cost parameter $\kappa = 1.5 \times 10^{-8}$, and Panel (b) when $\Lambda = \Sigma$, $p = 1.5$, and $\kappa = 5 \times 10^{-6}$.

(a)

Rebalancing region when $\Lambda = I$

(b) Rebalancing region when $\Lambda = \Sigma$

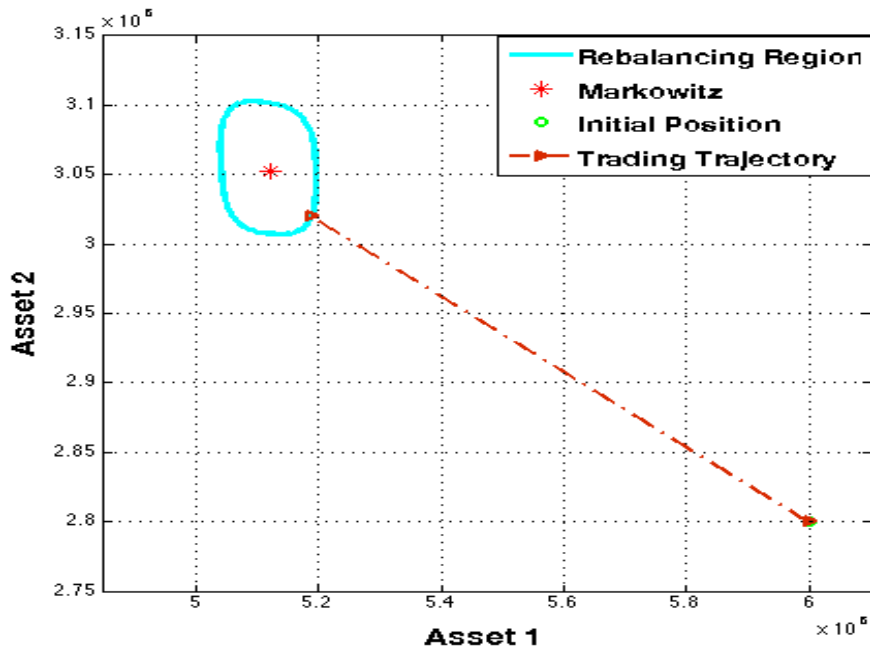
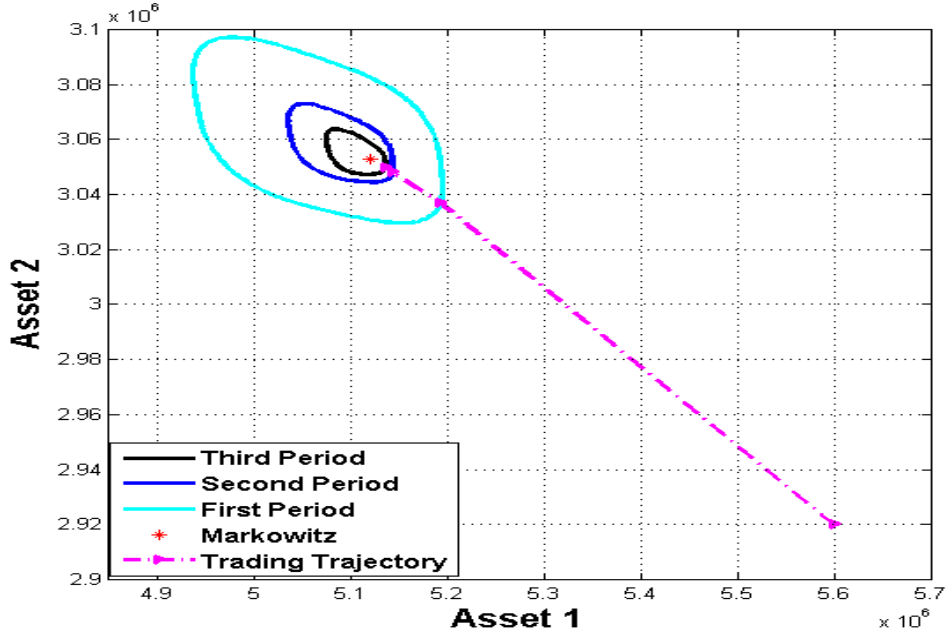


Figure 5: Rebalancing region for multiperiod investor

This figure depicts the rebalancing region for a multiperiod investor subject to market impact costs. We consider an absolute risk aversion parameter $\gamma = 10^{-7}$, annual discount factor $\rho = 2\%$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar. Panel (a) depicts the rebalancing region when the transaction cost matrix $\Lambda = I$, the exponent of the power function is $p = 1.5$, and the transaction cost parameter $\kappa = 1.5 \times 10^{-8}$, and Panel (b) when $\Lambda = \Sigma$, $p = 1.5$, and $\kappa = 5 \times 10^{-6}$.

(a) Rebalancing region when $\Lambda = I, T = 3$



(b) Rebalancing region when $\Lambda = \Sigma, T = 3$

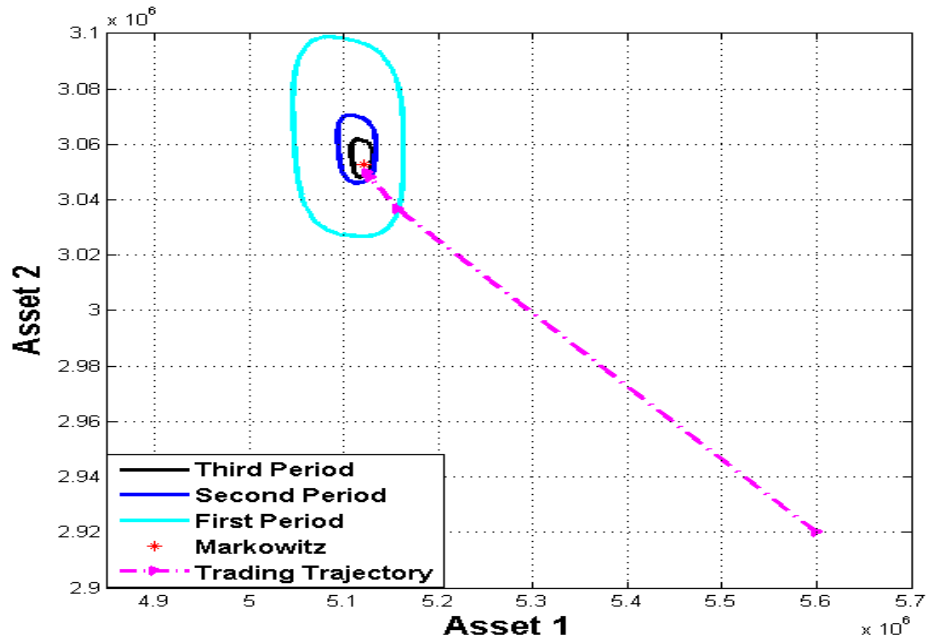
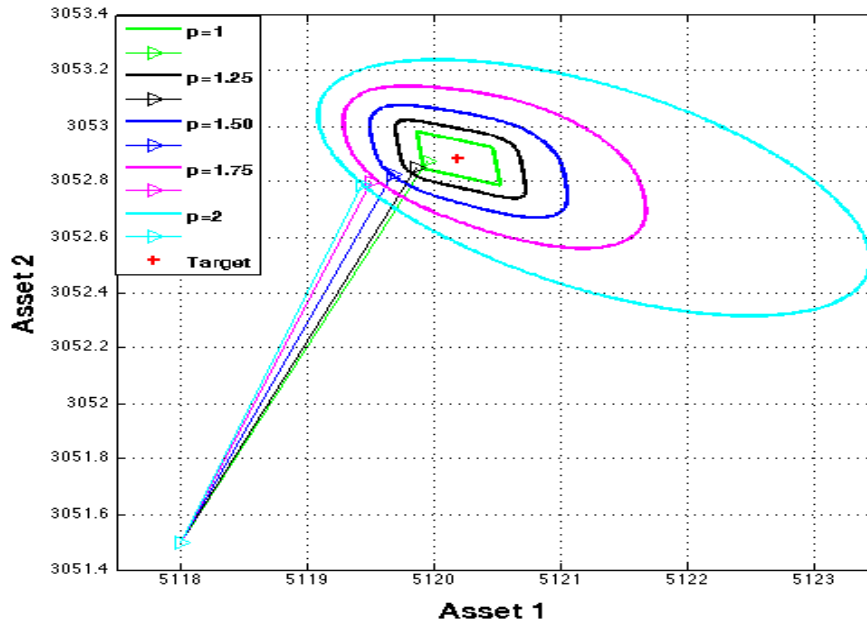


Figure 6: Rebalancing regions and trading trajectories for different exponents p .

This figure shows how the rebalancing regions and trading trajectories for the market impact costs model change with the exponent of the transaction cost function p . Panel (a) depicts the rebalancing regions for the single-period investor, with transaction cost parameter $\kappa = 1.5 \times 10^{-8}$, annual discount factor $\rho = 50\%$. Panel (b) depicts the multiperiod optimal trading trajectories when the investment horizon $T = 10$, with transaction cost parameter $\kappa = 5 \times 10^{-6}$, and annual discount factor $\rho = 5\%$. In both cases, we consider transaction costs matrix $\Lambda = I$, the risk-aversion parameter $\gamma = 10^{-4}$, and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

(a) Rebalancing regions depending on exponent p .



(b) Trading trajectories depending on exponent p .

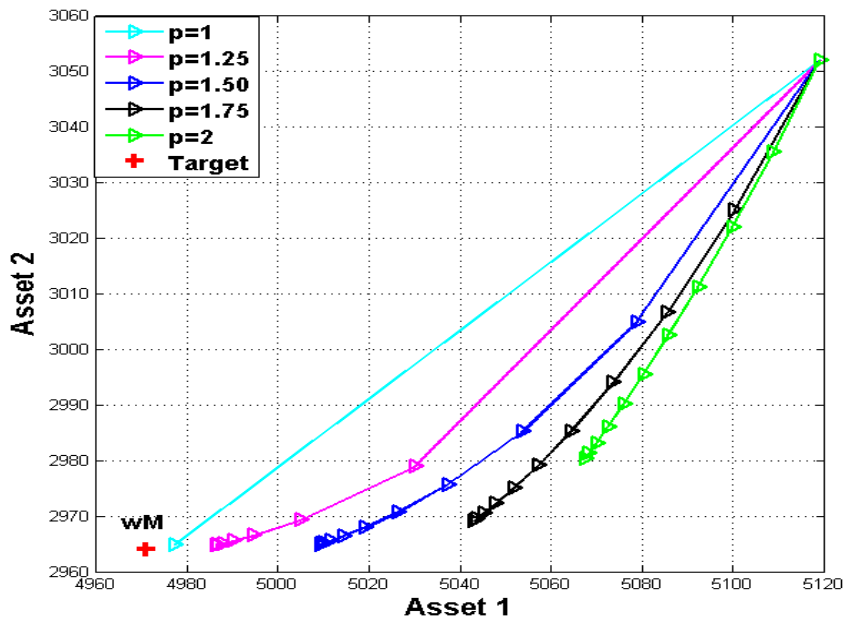


Figure 7: Trading trajectories for different transaction costs.

This figure depicts the trading trajectories for a multiperiod investor facing different types of transaction cost. We consider an investment horizon $T = 3$, a risk-aversion parameter $\gamma = 10^{-4}$, and annual discount factor $\rho = 20\%$. In the proportional transaction cost case, the transaction costs parameter $\kappa = 1.5 \times 10^{-8}$. In the market impact cost case, $\kappa = 2 \times 10^{-8}$, $\Lambda = I$, and $p = 1.5$. In the quadratic transaction cost case, $\kappa = 2 \times 10^{-4}$ and $\Lambda = \Sigma$.

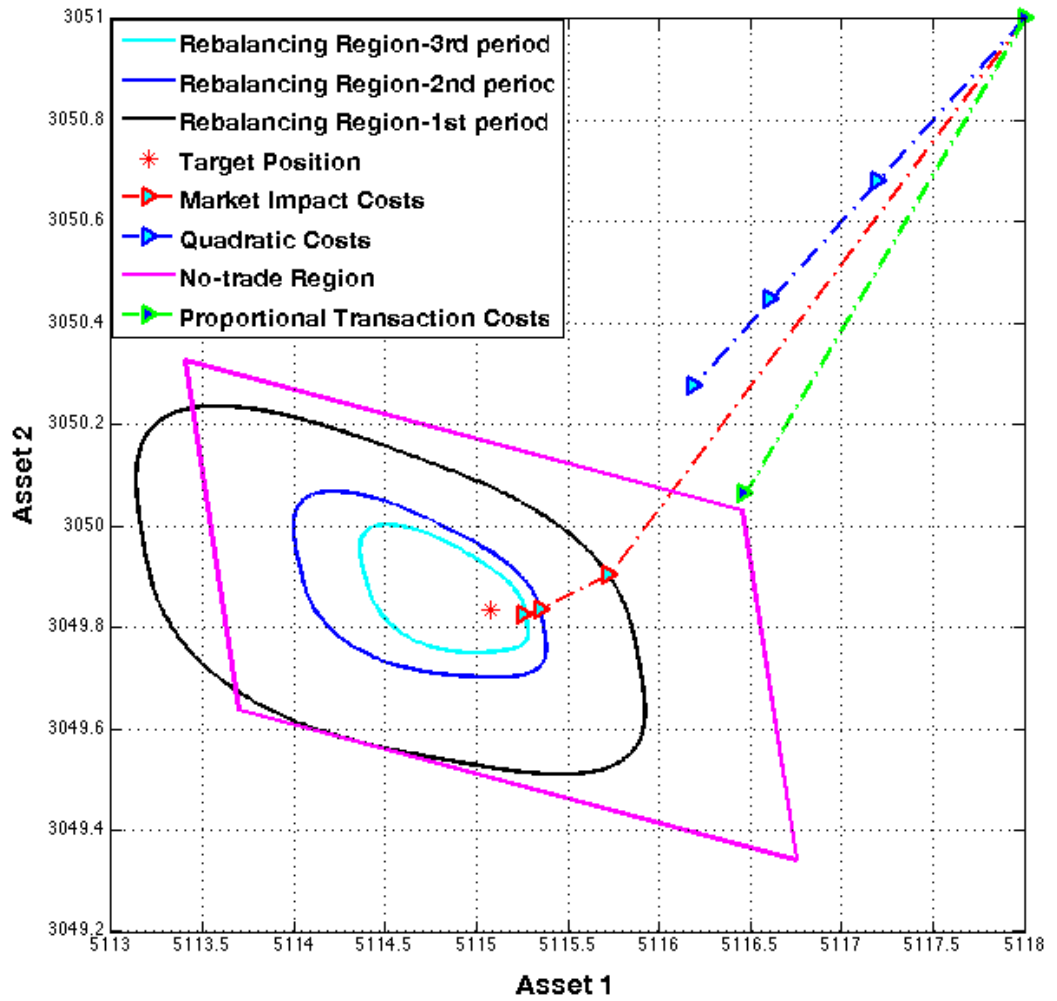
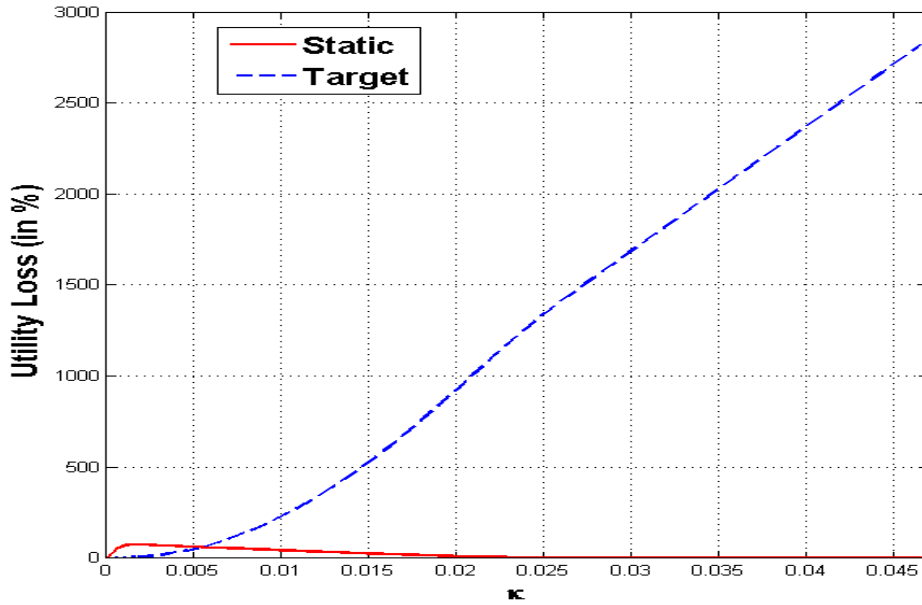


Figure 8: Utility losses with proportional transaction costs.

This figure depicts the utility loss of the static and target portfolios for the dataset with 15 commodity futures as a function of the transaction cost parameter κ (Panel (a)), and the investment horizon T (Panel (b)). In the base case, we consider proportional transaction costs parameter $\kappa = 0.0050$, risk-aversion parameter $\gamma = 1e - 6$, annual discount factor $\rho = 2\%$ and investment horizon $T = 22$. The price-change mean and covariance matrix are set equal to the sample estimators for the dataset that contains 15 commodity prices changes.

(a) Utility losses depending on κ .



(b) Utility losses depending on investment horizon T .

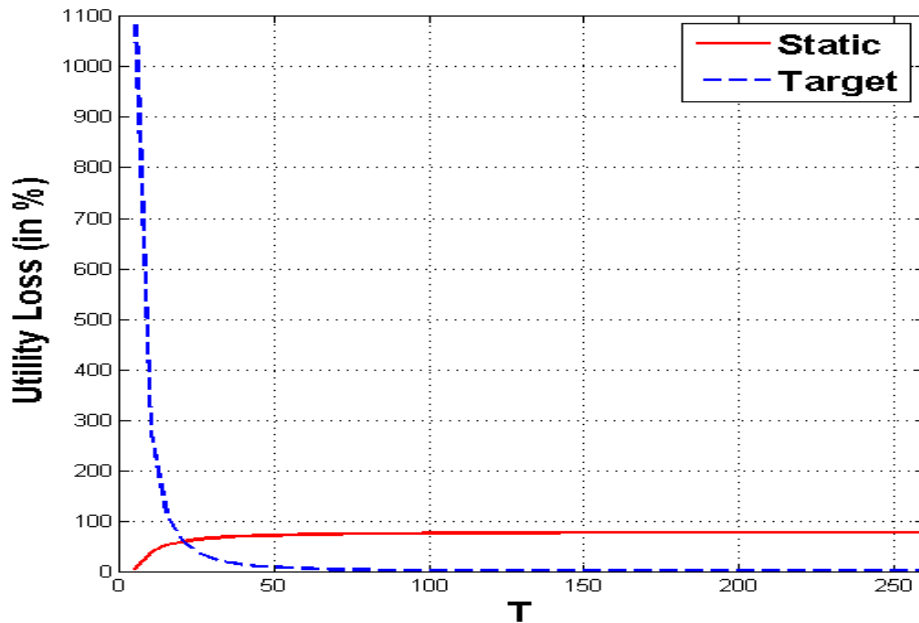
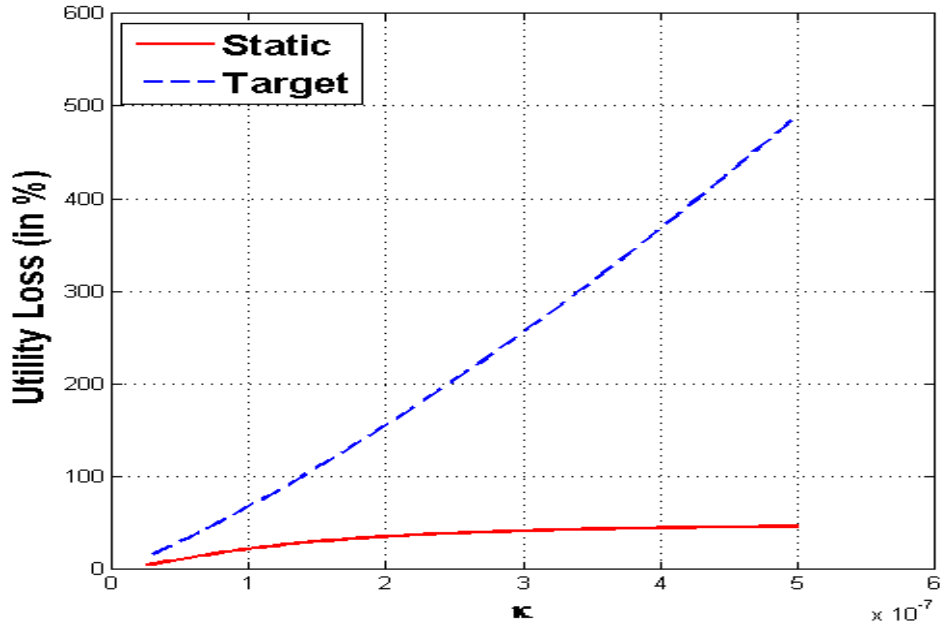


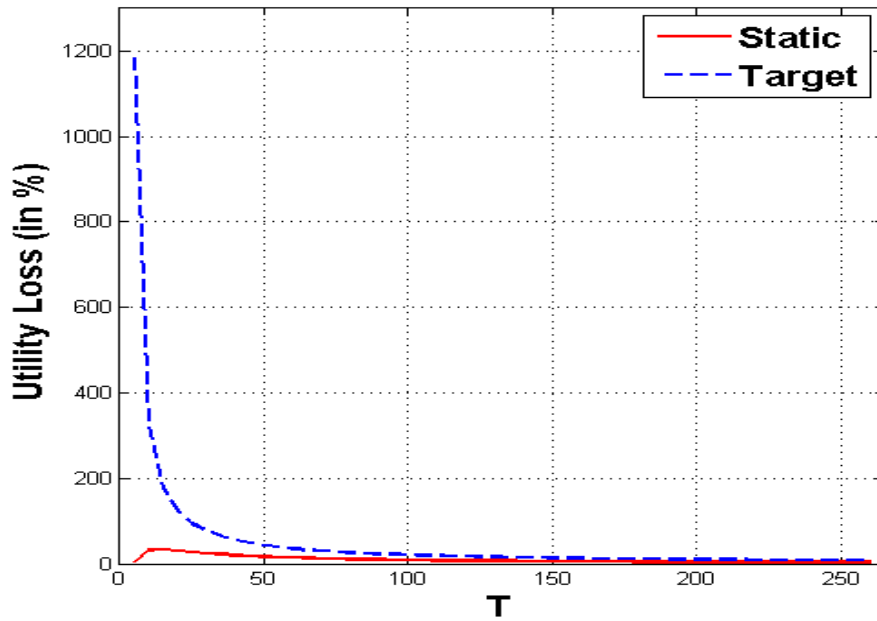
Figure 9: Utility losses with quadratic transaction costs.

This figure depicts the utility loss of the static and target portfolios for the dataset with 15 commodity futures as a function of the transaction cost parameter κ (Panel (a)), and the investment horizon T (Panel (b)). In the base case, we consider quadratic transaction costs parameter $\kappa = 1.5e - 7$, risk-aversion parameter $\gamma = 1e - 8$, annual discount factor $\rho = 0.02$ and investment horizon $T = 22$. The price-change mean and covariance matrix are set equal to the sample estimators for the dataset that contains 15 commodity prices changes.

(a) Utility losses depending on κ



(b) Utility losses depending on investment horizon T



B Tables

Table 1: Certainty equivalent loss: CRRA utility of terminal wealth

This table shows the certainty equivalent (CEQ) wealth for an investor with CRRA utility of terminal wealth and for different values of the investor's risk aversion parameter (γ), the stock return volatility (σ), and the proportional transaction cost rate, which are given in the first three columns. The fourth column gives the certainty equivalent of the optimal portfolio policy, and the last column gives the percentage loss in certainty equivalent incurred by using the G&P-type portfolio policy.

Parameters			CEQ	CEQ Loss
γ	σ	κ	\$	%
2	0.15	0.000	2.8756	0.19%
		0.005	2.8475	0.13%
		0.010	2.8206	0.08%
	0.20	0.000	2.6730	0.28%
		0.005	2.6565	0.18%
		0.010	2.6417	0.12%
	0.25	0.000	2.5857	0.45%
		0.005	2.5739	0.32%
		0.010	2.5640	0.25%
3	0.15	0.000	2.2643	0.19%
		0.005	2.2487	0.11%
		0.010	2.2342	0.07%
	0.20	0.000	2.1574	0.30%
		0.005	2.1479	0.20%
		0.010	2.1396	0.15%
	0.25	0.000	2.1103	0.40%
		0.005	2.1035	0.29%
		0.010	2.0980	0.24%
4	0.15	0.000	2.0720	0.19%
		0.005	2.0609	0.12%
		0.010	2.0508	0.07%
	0.20	0.000	1.9985	0.28%
		0.005	1.9917	0.19%
		0.010	1.9859	0.14%
	0.25	0.000	1.9658	0.34%
		0.005	1.9609	0.25%
		0.010	1.9570	0.21%

Table 2: Certainty equivalent loss: CRRA utility of intermediate consumption

This table shows the certainty equivalent (CEQ) consumption for an investor with CRRA utility of intermediate consumption and for different values of the investor's risk aversion parameter (γ), the stock return volatility (σ), and the proportional transaction cost rate, which are given in the first three columns. The fourth column gives the certainty equivalent consumption of the optimal portfolio policy, and the last column gives the percentage loss in certainty equivalent consumption incurred by using the G&P-type portfolio policy.

Parameters			CEQ	CEQ Loss
γ	σ	κ	\$	%
2	0.15	0.000	0.0165	0.08%
		0.005	0.0164	0.22%
		0.010	0.0163	0.45%
	0.20	0.000	0.0160	0.09%
		0.005	0.0159	0.17%
		0.010	0.0158	0.29%
	0.25	0.000	0.0157	0.14%
		0.005	0.0157	0.18%
		0.010	0.0156	0.25%
3	0.15	0.000	0.0465	0.07%
		0.005	0.0463	0.16%
		0.010	0.0461	0.32%
	0.20	0.000	0.0455	0.09%
		0.005	0.0454	0.14%
		0.010	0.0453	0.22%
	0.25	0.000	0.0451	0.12%
		0.005	0.0450	0.14%
		0.010	0.0449	0.19%
4	0.15	0.000	0.0655	0.07%
		0.005	0.0652	0.13%
		0.010	0.0650	0.25%
	0.20	0.000	0.0644	0.09%
		0.005	0.0643	0.12%
		0.010	0.0642	0.18%
	0.25	0.000	0.0640	0.10%
		0.005	0.0639	0.12%
		0.010	0.0638	0.15%

C Proofs of all results

Proof of Proposition 1

Part 1. When $\rho = 0$, the change in excess terminal wealth net of transaction costs for a multiperiod investor is

$$W_T = \sum_{t=1}^T \left[x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right]. \quad (\text{C1})$$

From Assumption 1, it is straightforward that the expected change in terminal wealth is

$$E_0(W_T) = \sum_{t=1}^T \left(x_t^\top \mu - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right). \quad (\text{C2})$$

Using the law of total variance, the variance of change in terminal wealth can be decomposed as

$$\text{var}_0(W_T) = E_0[\text{var}_s(W_T)] + \text{var}_0[E_s(W_T)]. \quad (\text{C3})$$

Taking into account that

$$\begin{aligned} E_0[\text{var}_s(W_T)] &= E_0 \left\{ \text{var}_s \left[\sum_{t=1}^T \left[x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right] \right] \right\} \\ &= E_0 \left[\sum_{t=s}^T x_t^\top \Sigma x_t \right] \\ &= \sum_{t=s}^T x_t^\top \Sigma x_t, \end{aligned} \quad (\text{C4})$$

and

$$\begin{aligned} \text{var}_0[E_s(W_T)] &= \text{var}_0 \left\{ E_s \left[\sum_{t=1}^T \left[x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right] \right] \right\} \\ &= \text{var}_0 \left\{ \sum_{t=1}^{s-1} \left[x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right] + \sum_{t=s}^T \left[x_t^\top \mu - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right] \right\} \\ &= \sum_{t=1}^{s-1} x_t^\top \Sigma x_t, \end{aligned} \quad (\text{C5})$$

the variance of the change in excess terminal wealth can be rewritten as

$$\text{var}_0(W_T) = \sum_{t=s}^T x_t^\top \Sigma x_t + \sum_{t=1}^{s-1} x_t^\top \Sigma x_t = \sum_{t=1}^T x_t^\top \Sigma x_t. \quad (\text{C6})$$

Consequently, the mean-variance objective of the change in excess terminal wealth for a multiperiod investor is

$$\begin{aligned}
& \max_{\{x_t\}_{t=1}^T} E(W_T) - \frac{\gamma}{2} \text{var}(W_T) \\
& \equiv \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left(x_t^\top \mu - \kappa \|\Lambda^{1/p} \|x_t - x_{t-1}\|_p^p \right) - \frac{\gamma}{2} \sum_{t=1}^T x_t^\top \Sigma x_t \\
& \equiv \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[\left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - \kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p \right]. \tag{C7}
\end{aligned}$$

Which is exactly objective function (1) when the value of discount factor $\rho = 0$.

Part 2. For the model with proportional transaction costs, the optimal policy is to trade at the first period to the boundary of the no-trade region given by (5), and not to trade for periods $t = 2, 3, \dots, T$. For an investor who is now sitting at period j with $j > 0$, the no-trade region for the remaining periods $t = j + 1, j + 2, \dots, T$ is given by

$$\|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma} \frac{\rho}{1 - (1 - \rho)^{T-j}}, \tag{C8}$$

which defines a region that contains the region defined by constraint (5). we could infer that the "initial position" for stage s , which is on the boundary of no-trade region defined by (5), is inside the no-trade region defined in (C8). Hence for any $\tau > j$, the optimal strategy for an investor who is sitting at j is to stay at the boundary of no-trade region defined in (5), which is consistent with the optimal policy obtained at time $t = 0$.

For the model with temporary market impact costs, for simplicity of exposition we consider $\Lambda = \Sigma$. At $t = 0$, the optimal trading strategy for period τ is on the boundary of the following rebalancing region

$$\frac{\|\sum_{s=\tau}^T (1 - \rho)^{s-\tau} \Sigma^{1/q} (x_{s|0} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{t|0} - x_{t-1|0})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \tag{C9}$$

For the investor who is now at $t = j$ for $j < \tau$, the optimal trading strategy is at the boundary of rebalancing region given by

$$\frac{\|\sum_{s=\tau}^T (1 - \rho)^{s-t} \Sigma^{1/q} (x_{s|j} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{t|j} - x_{t-1|j})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \tag{C10}$$

Because $x_{\tau|j} = x_{\tau|0}$ for $\tau \leq j$ and taking into account that $x_{j|j} = x_{j|0}$, we can infer that

$$\frac{\|\sum_{s=j}^T (1 - \rho)^{s-t} \Sigma^{1/q} (x_{s|j} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{j|j} - x_{j-1|j})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma} \tag{C11}$$

defines the same rebalancing region for period j as the following one

$$\frac{\|\sum_{s=j}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_{s|0} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{j|0} - x_{j-1|0})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (\text{C12})$$

That is, $x_{\tau|j} = x_{\tau|0}$ for $j < \tau$, and hence the optimal policies for the model with temporary market impact is time-consistent.

For the model with quadratic transaction costs, for simplicity of exposition we consider $\Lambda = \Sigma$. For an investor sitting at period $t = 0$, the optimal trading strategy for a future period $t = \tau$ is given by (20) (when $t = T$, simply let $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = 0$.), that is,

$$x_{\tau|0} = \alpha_1 x^* + \alpha_2 x_{\tau-1|0} + \alpha_3 x_{\tau+1|0}. \quad (\text{C13})$$

For the investor who is at $t = j$, for $j < \tau$, the optimal trading strategy is

$$x_{\tau|j} = \alpha_1 x^* + \alpha_2 x_{\tau-1|j} + \alpha_3 x_{\tau+1|j}. \quad (\text{C14})$$

Because $x_{\tau|j} = x_{\tau|0}$ for $\tau \leq j$ and taking into account that

$$\begin{aligned} x_{j|0} &= \alpha_1 x^* + \alpha_2 x_{j-1|0} + \alpha_3 x_{j+1|0} \\ &= x_{j|j} = \alpha_1 x^* + \alpha_2 x_{j-1|j} + \alpha_3 x_{j+1|j} \\ &= \alpha_1 x^* + \alpha_2 x_{j-1|0} + \alpha_3 x_{j+1|j}, \end{aligned} \quad (\text{C15})$$

which gives $x_{j+1|0} = x_{j+1|j}$. By using the relation recursively, we can show that for all $\tau > j$, it holds $x_{\tau|j} = x_{\tau|0}$. \square

Proof of Theorem 1

Part 1. Define Ω_t as the subdifferential of $\kappa \|x_t - x_{t-1}\|_1$

$$s_t \in \Omega_t = \left\{ u_t \mid u_t^\top (x_t - x_{t-1}) = \kappa \|x_t - x_{t-1}\|_1, \|u_t\|_\infty \leq \kappa \right\}, \quad (\text{C16})$$

where s_t denotes a subgradient of $\kappa \|x_t - x_{t-1}\|_1$, $t = 1, 2, \dots, T$. If we write $\kappa \|x_t - x_{t-1}\|_1 = \max_{\|s_t\|_\infty \leq \kappa} s_t^\top (x_t - x_{t-1})$, objective function (2) can be sequentially rewritten as

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \\ &= \max_{\{x_t\}_{t=1}^T} \min_{\|s_t\|_\infty \leq \kappa} \sum_{t=1}^T \left[(1-\rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top (x_t - x_{t-1}) \right] \\ &= \min_{\|s_t\|_\infty \leq \kappa} \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[(1-\rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top (x_t - x_{t-1}) \right]. \end{aligned} \quad (\text{C17})$$

The first order condition for the inner objective function of (C17) with respect to x_t is

$$0 = (1 - \rho)(\mu - \gamma \Sigma x_t) - s_t + (1 - \rho)s_{t+1}, \quad (\text{C18})$$

and hence

$$x_t = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_t, \quad \text{for } s_t \in \Omega_t, s_{t+1} \in \Omega_{t+1}. \quad (\text{C19})$$

Denote x_t^* as the optimal solution for stage t , there exists s_t^* and s_{t+1}^* such that

$$x_t^* = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_t^*, \quad \forall t. \quad (\text{C20})$$

We now let $s_t^* = \frac{1 - (1 - \rho)^{T - t + 2}}{\rho} s_T^*$, for $t = 1, 2, \dots, T - 1$ and $s_T^* = (1 - \rho)(\mu - \gamma \Sigma x_T^*)$. Rewrite x_t^* as

$$\begin{aligned} x_t^* &= \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_t^* \\ &= \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{r+1}^*) - \frac{1}{(1 - \rho)\gamma} \Sigma^{-1} s_r^* = x_r^*, \quad \forall t, r, \end{aligned} \quad (\text{C21})$$

where $\|s_t\|_\infty \leq \kappa$. By this means, we find the value of s_t^* such that $x_t^* = x_r^*$ for all $t \neq r$.

We conclude that $x_1 = x_2 = \dots = x_T$ satisfies the optimality conditions.

Part 2. Because $x_1 = x_2 = \dots = x_T$, one can rewrite the objective function (2) as

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[(1 - \rho)^t \left(x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\} \\ &= \max_{x_1} \left\{ \sum_{t=1}^T \left[(1 - \rho)^t \left(x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) \right] - \kappa \|x_1 - x_0\|_1 \right\} \\ &= \max_{x_1} \frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho} \left(x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - \kappa \|x_1 - x_0\|_1. \end{aligned} \quad (\text{C22})$$

Let s be the subgradient of $\kappa \|x_1 - x_0\|_1$ and let Ω be the subdifferential

$$s \in \Omega = \left\{ u \mid u^\top (x_1 - x_0) = \kappa \|x_1 - x_0\|_1, \|u\|_\infty \leq \kappa \right\}. \quad (\text{C23})$$

If we write $\kappa \|x - x_0\|_1 = \max_{\|s\|_\infty \leq \kappa} s^\top (x - x_0)$, objective function (C22) can be sequentially rewritten as:

$$\begin{aligned} & \max_{x_1} \frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho} \left(x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - \kappa \|x_1 - x_0\|_1 \\ &= \max_{x_1} \min_{\|s\|_\infty \leq \kappa} \frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho} \left(x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - s^\top (x_1 - x_0) \\ &= \min_{\|s\|_\infty \leq \kappa} \max_{x_1} \frac{(1 - \rho) - (1 - \rho)^{T+1}}{\rho} \left(x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - s^\top (x_1 - x_0). \end{aligned} \quad (\text{C24})$$

The first order condition for the inner objective function in (C24) is

$$0 = \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} (\mu - \gamma \Sigma x_1) - s, \quad (\text{C25})$$

and hence $x_1 = \frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s)$ for $s \in \Omega$. Then we plug x_1 into (C24),

$$\begin{aligned} \min_{\|s\|_\infty \leq \kappa} & \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left\{ \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s) \right]^\top \mu \right. \\ & \left. - \frac{\gamma}{2} \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s) \right]^\top \Sigma \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s) \right] \right\} \\ & - s^\top \left[\frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s) - x_0 \right] = \\ \min_{\|s\|_\infty \leq \kappa} & \frac{(1-\rho) - (1-\rho)^{T+1}}{2\rho\gamma} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s)^\top + \dots \\ & \Sigma^{-1} (\mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s) + s^\top x_0. \end{aligned} \quad (\text{C26})$$

Note that from (C25), we have $s = \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} (\mu - \gamma \Sigma x_1) = \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} [\gamma \Sigma (x^* - x_1)]$ as well as $\|s\|_\infty \leq \kappa$, where $x^* = \frac{1}{\gamma} \Sigma^{-1} \mu$ is the Markowitz portfolio. Replacing s in (C27), we conclude that problem (C27) is equivalent to the following

$$\min_{x_1} \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 \quad (\text{C28})$$

$$s.t. \quad \left\| \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \gamma \Sigma (x_1 - x^*) \right\|_\infty \leq \kappa. \quad (\text{C29})$$

Be aware of that the term $\frac{\gamma}{2} x_0^\top \Sigma x_0$ is a constant term for objective function (C28), it can be then rewritten as:

$$\begin{aligned} \max_{x_1} & \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 \\ \equiv \max_{x_1} & \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 + \frac{\gamma}{2} x_0^\top \Sigma x_0. \end{aligned} \quad (\text{C30})$$

It follows immediately that objective function (C30) together with constraint (C29) is equivalent to

$$\min_{x_1} \quad (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (\text{C31})$$

$$s.t. \quad \|\Sigma (x_1 - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T}. \quad (\text{C32})$$

Part 3. Note that constraint (5) is equivalent to

$$-\frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e \leq \Sigma (x_1 - x^*) \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e,$$

which is a parallelogram centered at x^* . To show that constraint (5) defines a no-trade region, note that when the starting portfolio x_0 satisfies constraint (5), $x_1 = x_0$ is a minimizer of objective function (4) and is feasible with respect to the constraint. On the other hand, when x_0 is not inside the region defined by (5), the optimal solution x_1 must be the point on the boundary of the feasible region that minimizes the objective. We then conclude that constraint (5) defines a no-trade region. \square

Proof of Proposition 2

Differentiating objective function (7) with respect to x_t gives

$$(1 - \rho)(\mu - \gamma \Sigma x) - \kappa p \Lambda^{1/p} |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)) = 0, \quad (\text{C33})$$

where $|a|^{p-1}$ denotes the absolute value to the power of $p - 1$ for each component

$$|a|^{p-1} = (|a_1|^{p-1}, |a_2|^{p-1}, \dots, |a_N|^{p-1}),$$

and $\text{sign}(\Lambda^{1/p}(x - x_0))$ is a vector containing the sign of each component for $\Lambda^{1/p}(x - x_0)$. Given that Λ is symmetric, rearranging (C33) we have

$$(1 - \rho)\Lambda^{-1/p}\gamma\Sigma(x^* - x) = \kappa p |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)). \quad (\text{C34})$$

Note that $x = x_0$ cannot be the optimal solution unless the initial position x_0 satisfies $x_0 = x^*$. Otherwise, take q -norm on both sides of (C34):

$$\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q = \frac{\kappa}{(1 - \rho)\gamma} p \|\Lambda^{1/p}(x - x_0)\|_q^{p-1} \cdot \|\text{sign}(\Lambda^{1/p}(x - x_0))\|_q, \quad (\text{C35})$$

where q is the value such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that $\|\Lambda^{1/p}(x - x_0)\|_q^{p-1} \cdot \|\text{sign}(\Lambda^{1/p}(x - x_0))\|_q = \|x - x_0\|_p^{p-1}$, we conclude that the optimal trading strategy satisfies

$$\frac{\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q}{p \|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} = \frac{\kappa}{(1 - \rho)\gamma}. \quad (\text{C36})$$

\square

Proof of Theorem 2

Differentiating objective function (11) with respect to x_t gives

$$\begin{aligned} (1 - \rho)^t (\mu - \gamma \Sigma x_t) - (1 - \rho)^{t-1} p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_t - x_{t-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_t - x_{t-1})) \\ + (1 - \rho)^t p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_{t+1} - x_t)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_{t+1} - x_t)) = 0. \end{aligned} \quad (\text{C37})$$

Specifically, given that Λ is symmetric, the optimality condition for stage T reduces to

$$(1 - \rho)\Lambda^{-1/p}(\mu - \gamma\Sigma x_T) = p\kappa|\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})). \quad (\text{C38})$$

Note that $x_T = x_{T-1}$ cannot be the optimal solution unless $x_{T-1} = x^*$. Otherwise, take q -norm on both sides of (C38) and rearrange terms,

$$\frac{\|\Lambda^{-1/p}\Sigma(x_T - x^*)\|_q}{p\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (\text{C39})$$

Summing up the optimal conditions recursively gives

$$p\kappa|x_t - x_{t-1}|^{p-1} \cdot \text{sign}(x_t - x_{t-1}) = \sum_{s=t}^T (1 - \rho)^{s-t+1} \gamma \Lambda^{-1/p}\Sigma(x^* - x_s). \quad (\text{C40})$$

Note that $x_t = x_{t-1}$ cannot be the optimal solution unless $x_{t-1} = x^*$. Otherwise, take q -norm on both sides, it follows straightforwardly that

$$\frac{\|\sum_{s=t}^T (1 - \rho)^{s-t} \Lambda^{-1/p}\Sigma(x_s - x^*)\|_q}{p\|\Lambda^{1/p}(x_t - x_{t-1})\|_p^{p-1}} = \frac{\kappa}{(1 - \rho)\gamma}. \quad (\text{C41})$$

We conclude that the optimal trading strategy for period t satisfies (C41) whenever the initial portfolio is not x^* . \square

Proof of Proposition 3

Part 1. We first define the function $g(x) = (1 - \rho)\Lambda^{-1/p}\Sigma(x - x^*)$. For period T it holds that $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$. Moreover, for the following last period it holds $\|g(x_{T-1}) + (1 - \rho)g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}$. Noting that $\|A + B\|_q \geq \|A\|_q - \|B\|_q$, it follows immediately that

$$\begin{aligned} p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} &\geq \|g(x_{T-1}) + (1 - \rho)g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1 - \rho)\|g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1 - \rho)p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds based on the fact $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$.

Rearranging terms,

$$\|g(x_{T-1})\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} + (1 - \rho)p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

which implies that

$$\frac{\|g(x_{T-1})\|_q}{p\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho)\frac{\kappa}{\gamma} \frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}},$$

which gives a wider area than the rebalancing region defined for x_T : $\frac{\|g(x_T)\|_q}{p\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$.

Similarly, $\|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \leq \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}$, it follows

$$\begin{aligned} \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} &\geq \|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \\ &\geq \|g(x_{T-2}) + (1 - \rho)g(x_{T-1})\|_p - (1 - \rho)^2\|g(x_T)\|_q, \\ &\geq \|g(x_{T-2}) + (1 - \rho)g(x_{T-1})\|_p - (1 - \rho)^2p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds because $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$.

Rearranging terms

$$\|g(x_{T-2}) + (1 - \rho)g(x_{T-1})\|_p \leq \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} + (1 - \rho)^2p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

which implies that,

$$\frac{\|g(x_{T-2}) + (1 - \rho)g(x_{T-1})\|_p}{p\|x_{T-2} - x_{T-3}\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho)^2\frac{\kappa}{\gamma}\frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}}.$$

The above inequality defines a region which is wider than the rebalancing region defined by

$$\frac{\|g(x_{T-1}) + (1 - \rho)g(x_T)\|_p}{p\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} \text{ for } x_{T-1}.$$

Recursively, we can deduce the rebalancing region corresponding to each period shrinks along t .

Part 2. Note that the rebalancing region for period t relates with the trading strategies thereafter. Moreover, the condition $x_1 = x_2 = \dots = x_T = x^*$ satisfies inequality (12), we then conclude that the rebalancing region for stage t contains Markowitz strategy x^* .

Part 3. The optimality condition for period T satisfies

$$(1 - \rho)(\mu - \gamma\Sigma x_T) - p\kappa\Lambda^{1/p}|\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})) = 0. \quad (\text{C42})$$

Let ω to be the vector such that $\lim_{T \rightarrow \infty} x_T = \omega$. Taking limit on both sides of (C42)

$$(1 - \rho)(\mu - \gamma\Sigma\omega) - p\kappa\Lambda^{1/p}|\Lambda^{1/p}(\omega - \omega)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(\omega - \omega)) = 0. \quad (\text{C43})$$

Noting that $\lim_{T \rightarrow \infty} x_T = \lim_{T \rightarrow \infty} x_{T-1} = \omega$, it follows

$$(1 - \rho)(\mu - \gamma\Sigma\omega) = 0,$$

which gives that $\omega = \frac{1}{\gamma}\Sigma^{-1}\mu = x^*$. We conclude that the investor will move to Markowitz strategy x^* in the limit case. \square

Proof of Theorem 3

Part 1. For $t = 1, 2, \dots, T - 1$, differentiating objective function (15) with respect to x_t gives

$$(1 - \rho)(\mu - \gamma \Sigma x_t) - \kappa(2\Lambda x_t - 2\Lambda x_{t-1}) - \kappa(1 - \rho)(2\Lambda x_t - 2\Lambda x_{t+1}) = 0, \quad (\text{C44})$$

rearranging terms

$$[(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda] x_t = (1 - \rho)\mu + 2\kappa\Lambda x_{t-1} + 2(1 - \rho)\kappa\Lambda x_{t+1}. \quad (\text{C45})$$

The solution can be written explicitly as following

$$\begin{aligned} x_t &= (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Sigma x^* \\ &\quad + 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Lambda x_{t-1} \\ &\quad + 2(1 - \rho)\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Lambda x_{t+1}. \end{aligned} \quad (\text{C46})$$

Define

$$\begin{aligned} A_1 &= (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Sigma, \\ A_2 &= 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda, \\ A_3 &= 2(1 - \rho)\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda, \end{aligned}$$

where $A_1 + A_2 + A_3 = I$.

For $t = T - 1$, the optimality condition is

$$(1 - \rho)(\mu - \gamma \Sigma x_T) - \kappa(2\Lambda x_T - 2\Lambda x_{T-1}) = 0, \quad (\text{C47})$$

the explicit solution is

$$x_T = (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Sigma x^* + 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Lambda x_{T-1}. \quad (\text{C48})$$

Define

$$\begin{aligned} B_1 &= (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Sigma, \\ B_2 &= 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Lambda, \end{aligned}$$

where $B_1 + B_2 = I$.

Part 2. As $T \rightarrow \infty$, taking limit on both sides of (C48) we then conclude that $\lim_{T \rightarrow \infty} x_T = x^*$. \square

Proof of Corollary 4

Substituting Λ with $\Lambda = I$ and $\Lambda = \Sigma$ respectively we obtain the optimal trading strategy.

To show that the trading trajectory of the case $\Lambda = \Sigma$ follows a straight line, noting that x_T is a linear combination of x_{T-1} and x^* , which indicates that x_T , x_{T-1} and x^* are on a straight line. On the contrary, if we assume that x_{T-2} is not on the same line, then x_{T-2} cannot be expressed as linear combination of x_T , x_{T-1} and x^* . Recall from equation (20) when $t = T - 1$ that

$$x_{T-1} = \alpha_1 x^* + \alpha_2 x_{T-2} + \alpha_3 x_T, \quad (\text{C49})$$

rearranging terms

$$x_{T-2} = \frac{1}{\alpha_2} x_{T-1} - \frac{\alpha_1}{\alpha_2} x^* - \frac{\alpha_3}{\alpha_2} x_T.$$

Noting that $\frac{1}{\alpha_2} - \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_2} = 1$, it is contradictory with the assumption that x_{T-2} is not a linear combination of x_T , x_{T-1} and x^* . By this means, we can show recursively that all the policies corresponding to the model when $\Lambda = \Sigma$ lay on the same straight line. \square

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