Archives

Ref.TH.1470-CERN

MULTIPERIPHERAL DYNAMICS BEYOND MODELS

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ABSTRACT

We examine the physical content and the formal apparatus of multiperipherism, showing that their essence is an upper bound with suitable decrease and factorization properties. From this bound, we derive an exact factorized representation for the production amplitudes leading to a rigorous multiperipheral integral equation. We give the exact formulae for the total cross-section and the n particle inclusive distributions. The multiperipheral models studied so far appear as possible approximations of our equations.

Ref.TH.1470-CERN
8 March 1972

1. INTRODUCTION

The multiperipheral ideas concerning the hadron production processes at high energy have been extensively developed in the past decade, starting from a variety of attitudes, which range from field theory to pure phenomenology [1)-23)]. Analyzing this large amount of work, we find that it is based on two concepts: the decrease of the production amplitudes with increasing momentum transfers (defined with an appropriate ordering of the final particles) and a special form of decomposition of the amplitudes into factors, which we call Q factorization [*)].

From the use of these two concepts, often disguised by different motivations, a complex mathematical formalism has been developed: the absorptive elastic amplitude and the inclusive distributions are expressed by means of the solution of an integral equation [1),2),12)-14)], which can be partially diagonalized [24)-28)] using its symmetry with respect to the group $O(2,1)$ or $O(3,1)$. One gets in this way the singularities in the complex angular momentum plane, which control the asymptotic behaviour of the observed quantities mentioned above.

Many simple models, which have been studied in detail, produce, as leading singularity in the complex angular momentum plane, an isolated pole with factorized residue. From this special feature one gets some simple high energy properties, as the logarithmic increase of multiplicity [2),10)], the Feynman scaling law [2),20)-23)] and the asymptotic factorization properties of the inclusive distributions [20)-23)].

These properties are in agreement with the qualitative aspects of the experimental data. However, both theoretical considerations and precise experiments suggest that some of them are an over-simplification. In order to obtain the realistic more complicated structure of the leading singularities in the complex ℓ plane, one has to resort to more refined multiperipheral models.

At this point, we feel the need of a careful discussion of the most general assumptions, which are the basis of any form of multiperipherism

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*) We are using the terminology of Ref. 23). In Ref. 12) the term "short range correlation" is used. The relevant definitions will be given in the next Section.

and permit to set up a mathematical formalism of the kind summarised above. In particular, this discussion should clarify whether important high energy phenomena, as diffraction dissociation and long range correlations [18),29),30)], can reasonably find their place in a multiperipheral scheme. The clear statement of the general assumptions, which we shall propose, will also lead to general model independent consequences, which can be directly compared with experiments, providing a global test of multiperipheral dynamics.

We stress that we are speaking of conditions on the true amplitude and not on any model approximated amplitude.

In order to pursue this program, we consider again the two basic concepts mentioned at the beginning. The first concept, namely the decrease of the amplitudes in the momentum transfers, can be stated in a mathematically clear way as an upper bound on the modulus of the amplitude.

The second concept, namely the Q factorization [*)] is essential for deriving the integral equation, as it permits to express the amplitudes, for any multiplicity, in terms of a few functions of few variables. We remark, however, that one cannot reasonably assume that the amplitudes are exactly given by a product of numerical functions of this kind. Therefore, if we want to deal with the exact amplitude, we have to enlarge the concept of Q factorization. The natural extension consists of representing the amplitudes as matrix elements of products of operator valued functions in a Banach space. As in the usual Q factorization, each function depends on a few kinematical variables.

The concept of generalized Q factorization is connected with the concept of upper bound by a theorem, which says that a necessary and sufficient condition for the existence of a Q factorized representation (in the operator sense) for the amplitude is the existence of a Q factorized (in the numerical sense) upper bound.

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*) We precise that, as carefully explained in Section 3, the Q factorized representation holds only in a certain region of the phase space. The amplitude in the whole phase space can be reconstructed using its symmetry under permutations of the final particles.

We realize that both the general concepts we started from are contained in the requirement that the amplitudes have upper bounds which are both decreasing with the momentum transfers and numerically Q factorized. We assume this requirement as the general definition of multiperipherality. We shall show that this assumption is sufficient in order to develop the whole mathematical formalism of multiperipherism, without any approximation. This formalism can provide a useful framework for more restrictive and detailed assumptions.

Leaving aside the mathematical formalism, our assumption, which contains the most general features of multipheripherism, is interesting from two points of view. First, one can rigorously derive from the bound simple consequences to be compared with experiments. In the second place, the upper bound, being a clear mathematical statement, can provide the right instrument for connecting the multiperipheral ideas with more fundamental physical principles.

We think that it is important to stress that the upper bound we shall assume contains an arbitrary numerical factor and, therefore, the amplitude being continuous, it does not impose any restriction when we consider the amplitude in a compact region of the space of the kinematical variables (including the multiplicity). In other words, our condition has an asymptotic character.

It follows that all the general consequences of this condition on the measurable quantities have an asymptotic nature as well and their comparison with experiments implies necessarily some extrapolation procedure. This feature, as is well known, is common to most of high energy S matrix theory.

In Section 2, we define the numerical Q factorization and we introduce the relevant kinematical variables. The basic concept of decrease in the momentum transfers is discussed in Section 3, also in connection with the symmetry of the amplitude with respect to the permutations of the final particles. In Section 4, we discuss the basic concept of Q factorization and we generalize it in two different ways which are proved to be equivalent in Section 5. The Sections 6 and 7 are devoted to develop, starting from the concepts introduced before, the mathematical formalism in an exact form. In particular, we express the n particle inclusive distributions in terms of the solution of a multiperipheral integral equation.

2. KINEMATICS

We consider a process with two incoming and $n+2$ outgoing particles with $n \geq 0$.

We assume, for simplicity, that all the particles are identical and spinless, giving only some hint about the modifications required in the general case.

We choose a certain ordering for the outgoing particles and we indicate the various four-momenta as in Fig. 1, where the four-momentum transfers Q_i are also defined. At this stage, this figure has no dynamical meaning. We remark that the four-momenta $P_A, Q_0, \ldots, Q_n, P_B$ can be chosen as a complete set of kinematical variables. It is often useful to introduce the convention

$$Q_{-1} = P_A \quad , \quad Q_{n+1} = -P_B \quad . \tag{2.1}$$

Now we introduce the concept of Q factorization: [12),23)].

Definition: A sequence of Lorentz invariant functions F_n of the above mentioned four-vectors is called numerically Q factorized of order k if for $n \geq k-2$ we can write

$$\begin{aligned} & F_n\left(P_B, Q_n, \ldots, Q_0, P_A\right) = \\ & = B\left(P_B, Q_n, \ldots, Q_{n-k+2}\right) \prod_{i=k-1}^{n} K\left(Q_i, \ldots, Q_{i-k+1}\right) \cdot \\ & \cdot A\left(Q_{k-2}, \ldots, Q_0, P_A\right) \quad , \end{aligned} \tag{2.2}$$

where the functions A, B and K are Lorentz invariant and independent of n.

The motivation for this definition and the comparison with other kinds of factorization are discussed in Ref. 23).

It is convenient to express the functions F_n, A, B and K in terms of invariants which depend only on a few neighbouring four-momenta Q_i. We have to define $3n+2$ independent invariants of this kind. A simple choice is

$$t_i = Q_i^2 \quad , \quad i = 0,1,\cdots,n \quad , \tag{2.3}$$

$$s_i = \left(P_{i+1} + P_i\right)^2 = \left(Q_{i-1} - Q_{i+1}\right)^2 \quad , \quad i = 0,1,\cdots,n \quad , \tag{2.4}$$

$$\sigma_i = \left(P_{i+1} + P_{i-1}\right)^2 = \left(Q_{i-2} - Q_{i-1} + Q_i - Q_{i+1}\right)^2, \quad i = 1,2,\cdots,n \; . \tag{2.5}$$

We have used the convention (2.1).

The definition of the physical region in terms of these variables is rather complicated. For this and other related reasons, we introduce also the Bali-Chew-Pignotti (BCP) variables [8),14)].

As we are assuming that all the particles have the same mass m, all the variables t_i are necessarily negative [*)]. For each vertex in Fig. 1 we define a frame of reference. These frames are connected with a given arbitrary frame by means of the Lorentz transformations $a_0, a_1, \ldots, a_{n+1}$. It is more clear to define these group elements implicitly by means of the formulae

$$\left\{\begin{aligned} P_B &= L\left(a_{n+1}\right)\left(m,0,0,0\right) \; , \\ Q_n &= L\left(a_{n+1} a_z(\chi_{n+1})\right)\left(0,0,0,\sqrt{-t_n}\right) \; , \\ P_{n+1} &= L\left(a_{n+1} a_z(\hat{\chi}_{n+1})\right)\left(m,0,0,0\right) \; , \end{aligned}\right. \tag{2.6}$$

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*) In the general mass case, and for small values of i or of $(n-i)$, some of the variables t_i can be positive. In any case they are smaller than the square of a certain difference of masses.

$$\begin{cases} Q_i = L(a_i)(0,0,0,\sqrt{-t_i})\,, \\ Q_{i-1} = L(a_i a_z(\chi_i))(0,0,0,\sqrt{-t_{i-1}})\,, \\ P_i = L(a_i a_z(\hat{\chi}_i))(m,0,0,0)\,, \quad i=1,2,\dots,n\,, \end{cases} \tag{2.7}$$

$$\begin{cases} Q_0 = L(a_0)(0,0,0,\sqrt{-t_0})\,, \\ P_A = L(a_0 a_z(\chi_0))(m,0,0,0)\,, \\ P_0 = L(a_0 a_z(\hat{\chi}_0))(m,0,0,0)\,, \end{cases} \tag{2.8}$$

where $a_z(\chi)$ is a boost along the z axis with rapidity χ and

$$\begin{cases} \sinh \chi_{n+1} = (-t_n)^{\frac{1}{2}}\,(2m)^{-1}\,, \\ \cosh \hat{\chi}_{n+1} = (2m^2 - t_n)\,(2m^2)^{-1}\,, \qquad \hat{\chi}_{n+1} \geqslant 0\,, \end{cases} \tag{2.9}$$

$$\begin{cases} \cosh \chi_i = (m^2 - t_i - t_{i-1})(4t_i t_{i-1})^{-\frac{1}{2}}\,, \quad \chi_i > 0\,, \\ \sinh \hat{\chi}_i = (m^2 + t_i - t_{i-1})(-4t_i m^2)^{-\frac{1}{2}}\,, \quad i=1,2,\dots,n\,, \end{cases} \tag{2.10}$$

$$\begin{cases} \sinh \chi_0 = (-t_0)^{\frac{1}{2}}\,(2m)^{-1}\,, \\ \hat{\chi}_0 = -\chi_0\,, \end{cases} \tag{2.11}$$

From Eqs. (2.6)-(2.8) it is easy to show that the elements

$$g_i = a_z(-\chi_{i+1})\, a_{i+1}^{-1}\, a_i \quad , \qquad i = 0, 1, \cdots, n \quad , \tag{2.12}$$

do not act on the z components and, therefore, they belong to SU(1,1).

One has to remark that the four vectors (2.6)-(2.8) do not change if we perform the substitution

$$a_k \longrightarrow a_k\, u_z(\gamma) . \tag{2.13}$$

where $u_z(\gamma)$ represents a rotation of an angle γ around the z axis. In terms of the variables g_i, the transformation (2.13) takes the form

$$\begin{cases} g_k \longrightarrow g_k\, u_z(\gamma) \quad , \\ g_{k-1} \longrightarrow u_z(-\gamma)\, g_{k-1} \quad . \end{cases} \tag{2.14}$$

This means that if we parameterize the elements g_i in the usual way

$$g_i = u_z(\mu_i)\, a_x(\xi_i)\, u_z(\nu_i) , \tag{2.15}$$

only the following BCP variables are relevant.

$$\begin{cases} t_i \quad , \quad i = 0, 1, \cdots, n \quad , \qquad -\infty < t_i < 0 \quad , \\ \xi_i \quad , \quad i = 0, 1, \cdots, n \quad , \qquad 0 \leq \xi_i < \infty \quad , \\ \omega_i = \nu_i + \mu_{i-1} \quad , \quad i = 1, 2, \cdots, n . \qquad 0 \leq \omega_i < 2\pi \quad . \end{cases} \tag{2.16}$$

Nevertheless, we shall often consider the scattering amplitudes as functions of the variables t_i and g_i, always keeping in mind that the amplitude has to be invariant under transformations of the kind (2.14).

The connection with the variables s_i is

$$s_i = t_{i-1} + t_{i+1} - \\ - \frac{1}{2t_i}\left\{\left(T(m^2, t_{i-1}, t_i)\,T(m^2, t_i, t_{i+1})\right)^{\frac{1}{2}} \cosh \xi_i + \\ + (m^2 - t_{i-1} - t_i)(m^2 - t_i - t_{i+1})\right\} , \quad i = 0, 1, \dots, n \quad . \tag{2.17}$$

For $i = 0$ and $i = n$, we have to use the convention

$$t_{-1} = t_{n+1} = m^2 \quad . \tag{2.18}$$

The function T is given by

$$T(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \quad . \tag{2.19}$$

From Eq. (2.17) we get the inequality

$$-\frac{1}{2t_i}\left[T(m^2, t_{i-1}, t_i)\,T(m^2, t_i, t_{i+1})\right]^{\frac{1}{2}}(\cosh \xi_i - 1) \leq s_i \leq \\ \leq -\frac{1}{2t_i}\left[T(m^2, t_{i-1}, t_i)\,T(m^2, t_i, t_{i+1})\right]^{\frac{1}{2}}(\cosh \xi_i + 1) \quad . \tag{2.20}$$

3. DECREASE IN THE MOMENTUM TRANSFERS

We indicate by $M_n(x)$ the invariant amplitude for a process with $n+2$ particles in the final state. The symbol x stands for a set of $3n+2$ invariants and can be considered as a point of a manifold Ω_n. A permutation π of the final particles induces a mapping ρ_π of Ω_n onto itself *). Clearly, as all our particles are identical bosons, the function $M_n(x)$ has the symmetry property

$$M_n(\rho_\pi x) = M_n(x) \quad , \qquad x \in \Omega_n . \tag{3.1}$$

By decrease in the momentum transfers, we mean the following statement: the function $M_n(x)$ is very small unless x belongs to one of the $(n+2)!$ regions Δ_n^π defined, for each fixed permutation π, by the condition that the $(n+1)$ invariants

$$t_i^\pi = \Big(P_A - \sum_{j=0}^{i} P_{\pi(j)}\Big)^2 , \qquad i = 0, 1, \dots, n , \tag{3.2}$$

are all small in absolute value. This statement can be translated into the inequality, valid in the physical region,

$$\left| M_n(x) \right| \leqslant \sup_\pi \hat{f}_n(\rho_\pi x) \quad , \tag{3.3}$$

where the function $\hat{f}_n(x)$ is strongly decreasing in its arguments $|t_0|, \dots, |t_n|$. More stringent conditions on this function will be discussed in the next Section.

In order to justify this assumption experimentally, it is useful to decompose the various four-vectors into longitudinal and transverse parts. We consider first the region Δ_n corresponding to the identical permutation. We choose the z axis along the direction of the incoming particles and we write

$$-t_i = \left[(Q_i^z)^2 - (Q_i^0)^2 \right] + \left[(Q_i^x)^2 + (Q_i^y)^2 \right] , \tag{3.4}$$

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*) In the general case the symbol x contains also indices describing the internal quantum numbers of the particles and ρ_π acts also on them.

One can show that both the quantities in the brackets are non-negative and, therefore, they are both small if t_i is small *). We easily get also the inequality

$$p_i = \left[(P_i^x)^2 + (P_i^y)^2 \right]^{\frac{1}{2}} \leq (-t_i)^{\frac{1}{2}} + (-t_{i-1})^{\frac{1}{2}} \quad , \tag{3.5}$$

from which we see that the transverse momenta p_i are small. Of course, this conclusion holds in all the regions Δ_n^{π} and we obtain the experimentally very well established result that the amplitude decreases strongly with increasing transverse momenta.

However, the assumption we are considering is stronger than the decrease with transverse momenta at least for two reasons. First, we see that in order to have the quantities

$$Q_k^x = - \sum_{i=0}^{k} P_i^x \tag{3.6}$$

small, it is not sufficient to require that the quantities P_i^x are all small, but, if k is large, a negative correlation among these quantities has to be present.

Second, the requirement of small momentum transfers implies also conditions on the longitudinal momenta, as we see from Eq. (3.4). If we consider the longitudinal momentum space introduced by Van Hove [31] we can define the regions $\tilde{\Delta}_n^{\pi}$ in this space which are the projections of the regions Δ_n^{π}. Events outside these regions are expected to be rare. A detailed experimental analysis of these features seems still necessary in order to get a conclusion.

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*) In the general mass case, $(Q_i^o)^2 - (Q_i^z)^2$ can be positive, but smaller than the square of some difference of masses, so that our conclusion is still valid.

The condition (3.3) is rather complicated, due to the symmetry of the functions $M_n(x)$ with respect to the permutations of the final particles. It is, therefore, convenient to introduce different functions $F_n(x)$, which are not symmetric, but have simpler decrease properties in the variables $|t_i|$.

With regard to the connection between the functions M_n and F_n, one can adopt two different attitudes.

a) The function M_n is given by the sum of $(n+2)!$ terms obtained by permuting the final particles in the function F_n

$$M_n(x) = \sum_{\pi} F_n(\mathcal{P}_\pi x) \quad . \tag{3.7}$$

b) For a given choice of the ordering of the final particles, the function F_n coincides with the function M_n in a given region Γ_n of the space Ω_n of the kinematical invariants.

Then, if we indicate by Γ_n^π the region in Ω_n defined by

$$x \in \Gamma_n^\pi \quad \Longleftrightarrow \quad \mathcal{P}_\pi x \in \Gamma_n \quad , \tag{3.8}$$

using the symmetry property of M_n we may write

$$M_n(x) = F_n(\mathcal{P}_\pi x) \qquad \text{for} \quad x \in \Gamma_n^\pi \quad . \tag{3.9}$$

In order to get a complete information on M_n we have to assume that

$$\bigcup_{\pi} \Gamma_n^\pi = \Omega_n \quad . \tag{3.10}$$

If some of the regions Γ_n^π overlap, we have the constraints *)

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*) If we particularize the present scheme interpreting the function F_n as the result of multiple exchanges along the chain of Fig. 1, Eq. (3.11) is just a statement of duality. Different kinds of duality correspond to different choices of the region Γ_n.

$$F_n(x) = F_n(\rho_\pi x) \qquad \text{for} \qquad x \in \Gamma_n \cap \Gamma_n^\pi \quad . \tag{3.11}$$

The advantage of the point of view b) is clear when we compute $|M_n|^2$. In fact, from Eq. (3.9) we get immediately

$$|M_n(x)|^2 = |F_n(\rho_\pi x)|^2 , \qquad x \in \Gamma_n^\pi \quad . \tag{3.12}$$

If we start instead from Eq. (3.7), non-diagonal terms will in general appear.

We remark that if the region Γ_n is open and Eq. (3.10) holds, we can always find a continuous function $\eta_n(x)$ with support in Γ_n and with the property

$$\sum_\pi \eta_n(\rho_\pi x) = 1 \quad . \tag{3.13}$$

We omit the proof of this statement as we shall give later an explicit expression for $\eta_n(x)$. Then, from Eq. (3.9) we have

$$M_n(x) = \sum_\pi F_n(\rho_\pi x)\eta_n(\rho_\pi x) \quad , \tag{3.14}$$

$$|M_n(x)|^2 = \sum_\pi |F_n(\rho_\pi x)|^2 \eta_n(\rho_\pi x) \quad . \tag{3.15}$$

Comparing Eq. (3.7) with Eq. (3.14), we see that the good features of the point of view a) are also present in the approach b), which we shall develop in the following.

Now we show that, if we choose a continuous function $f_n(x)$ such that

$$f_n(x) > \hat{f}_n(x) \quad , \qquad x \in \Omega_n \quad , \tag{3.16}$$

it is always possible to choose the region Γ_n in such a way that it is open and satisfies the condition (3.10) and to define the function $F_n(x)$ in such a way that

$$|F_n(x)| \leq f_n(x) \quad . \tag{3.17}$$

For instance one can define Γ_n as the region where the inequality

$$f_n(x) > \sup_{\pi} \hat{f}_n(P_\pi x) \tag{3.18}$$

holds. It is clear that in this region

$$|F_n(x)| = |M_n(x)| < f_n(x) \quad . \tag{3.19}$$

Outside this region, $F_n(x)$ is arbitrary and it can be chosen in such a way that Eq. (3.17) is satisfied. In order to show that Eq. (3.10) holds, one has just to remark that

$$\sup_{\pi} f_n(P_\pi x) > \sup_{\pi} \hat{f}_n(P_\pi x) \tag{3.20}$$

and, therefore, for every $x \in \Omega$ one can find a permutation π such that $P_\pi x$ satisfies Eq. (3.18), namely $x \in \Gamma_n^\pi$.

In this situation we can also choose a simple expression for the functions $\eta_n(x)$ introduced above. We have just to put

$$\eta_n(x) = g_n(x) \cdot \left[\sum_{\pi} g_n(P_\pi x) \right]^{-1} \tag{3.21}$$

with

$$g_n(x) = \left(f_n(x) - \sup_{\pi} \hat{f}_n(P_\pi x)\right) \cdot \\ \cdot \theta\left(f_n(x) - \sup_{\pi} \hat{f}_n(P_\pi x)\right) \quad . \tag{3.22}$$

In conclusion, we have replaced the set of amplitudes $M_n(x)$, which satisfy the bound (3.3), with the set of functions $F_n(x)$, which satisfy the simpler condition (3.17).

4. Q FACTORIZATION

Now we discuss the second basic concept of multiperipheral dynamics, i.e., Q factorization. The simplest kind of Q factorization, which appears in most models, has been defined in Section 2. The decrease of the amplitude in the momentum transfers is taken into account by the Q factorized models assuming that the single factors are decreasing functions of the momentum transfers on which they depend. This remark, together with the discussion of the preceding Section should make clear that the functions to which Q factorization is applied are not the amplitudes $M_n(x)$, but rather the functions $F_n(x)$ characterized by Eqs. (3.9) and (3.17).

The numerical Q factorization of Section 2 cannot be imposed as an exact requirement on the amplitude. For instance a k particle threshold singularity cannot be contained in the expression (2.2). One could try to approximate more and more the amplitude increasing the number k of variables contained in each factor. Our attitude, however, is quite different, as our aim is to formulate statements on the exact amplitude. Therefore, we are forced to generalize the concept of Q factorization.

We discuss in this Section two rather different ways of achieving this generalization, which in the next Section will be shown to be equivalent. The first way is to assume that the functions $|F_n(x)|$ have upper bounds $f_n(x)$ [see Eq. (3.17)] which are numerically Q factorized. There is a large arbitrariness in the choice of the number and the nature of the variables which appear in each factor. If we want to take into account only very general features, as the decrease in the momentum transfers t_i and the polynomial boundedness in the subenergies s_i, it is sufficient to consider the simple Q factorized upper bound

$$f_n(x) = c \prod_{i=0}^{n} \left[d(t_i) \left(\frac{s_i}{4m^2} \right)^{\alpha} \right] , \tag{4.1}$$

where the function $d(t)$ is suitably decreasing for large negative t and α is an exponent of the order of one.

Q factorized bounds of a more complicated kind can be divided in two classes: those which, in turn, are majorized by a function of the form (4.1) and those for which such a majorization is not possible. In the first case, if one is not interested in the details of the amplitude, one can just start from the bound (4.1). Examples of bounds of the second kind can be devised, which still permit to develop the multiperipheral formalism. At present, we do not discuss these cases in detail, as we consider them rather artificial.

We stress that, besides the decrease in $|t_i|$ and the polynomial boundedness in s_i, a very restrictive condition is implied by our Q factorized bound, namely a limitation on the behaviour of the amplitudes for increasing multiplicity n. By integration on the final momenta we can obtain an upper bound on the cross-sections $\sigma_n(s)$. Introducing also a rather natural lower bound for the total cross-section $\sigma(s)$, one can prove [32] that the average multiplicity increases at most as $\log s$. This procedure can give rise to a first experimental test of the Q factorized bound we are assuming.

If we introduce the BCP variables and we use Eq. (2.20), we realize that the bound (4.1) implies a bound of the kind

$$|F_n(x)| \leq c \prod_{i=0}^{n} \left[d(t_i) \left(\cosh \xi_i + 1 \right)^{\alpha} \right] \cdot \left[\frac{\sqrt{4m^2 - t_0}\,\sqrt{4m^2 - t_n}}{8m^2} \right]^{\alpha} \cdot \prod_{i=0}^{n-1} \left[\frac{T(m^2, t_i, t_{i+1})}{8m^2 \sqrt{t_i\, t_{i+1}}} \right]^{\alpha} . \tag{4.2}$$

The second way of generalizing the Q factorization consists in assuming that the functions $F_n(x)$ can be expressed in the form (2.2), where, however, the quantities B, K and A are not numerical functions, but matrices of suitable dimension. The same argument given above for numerical Q factorization shows that an exact representation cannot be obtained by means of finite dimensional matrices. Therefore, we have to consider infinite matrices or, more exactly, operators and vectors in suitable infinite dimensional normed spaces. The opportunity of dealing with normed spaces is due to the possibility of taking into account the upper bounds discussed above, while the restriction to Hilbert spaces would be unnatural.

Also in this case, there is a large arbitrariness in the choice of the variables appearing in each factor. In the next Section we shall adopt a special choice, which is particularly suitable for further developments and we shall prove that the two ways of generalizing the Q factorization discussed above are equivalent. The same result can be obtained with other choices of the variables.

5. CONSTRUCTION OF AN EXACT Q FACTORIZED REPRESENTATION

The special kind of operator Q factorized representation, which we want to study in detail, is given by the following formulae *)

$$F_n(g_n, t_n, \cdots, g_0, t_0) =$$

$$= \left(B(g_n, t_n) \, , \, \prod_{i=0}^{n-1} K(t_{i+1}, g_i, t_i) \; A(t_0) \right) ,$$

$$t_i < 0 \, , \quad g_i \in SU(1,1) \, , \quad n = 0, 1, 2, \cdots \, , \tag{5.1}$$

where $A(t)$ is an element of a Banach space $\mathcal{B}_t$ (which depends on t), $B(g,t)$ is an element of the dual space $\mathcal{B}_t'$ and $K(t',g,t)$ is a bounded operator from $\mathcal{B}_t$ to $\mathcal{B}_{t'}$. The necessity of introducing several Banach spaces, namely one for each value of t, is a consequence of the redundant choice of the variables appearing in the various factors. We stress that the quantities A, B, K are independent of n.

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*) We define the product symbol $\prod$ in such a way that the factors are ordered with decreasing indices.

The invariance of F_n with respect to the transformations of the kind (2.14) is ensured by the existence of the operators $U(\gamma, t)$ in $\mathcal{B}_t$ with the properties

$$\begin{cases} U(\gamma, t_{i+1})\, K(t_{i+1}, g_i, t_i)\, U(\gamma', t_i) = K(t_{i+1}, u_2(\gamma) g_i u_2(\gamma'), t_i) , \\ U(\gamma, t_0)\, A(t_0) = A(t_0) , \\ U^T(\gamma, t_n)\, B(g_n, t_n) = B(g_n u_2(\gamma), t_n) . \end{cases} \tag{5.2}$$

From Eq. (5.1) it follows that the functions F_n satisfy bounds of the kind

$$|F_n| \leq b(\xi_n, t_n) \prod_{i=0}^{n-1} k(t_{i+1}, \xi_i, t_i)\, a(t_0) , \tag{5.3}$$

where

$$\begin{cases} b(\xi_n, t_n) \geq \| B(g_n, t_n) \| , \\ k(t_{i+1}, \xi_i, t_i) \geq \| K(t_{i+1}, g_i, t_i) \| , \\ a(t_0) \geq \| A(t_0) \| . \end{cases} \tag{5.4}$$

Conversely, we shall show that

Proposition: Given an arbitrary sequence of continuous functions

$$F_n(g_n, t_n, \cdots, g_0, t_0) , \quad n = 0, 1, 2, \cdots , \tag{5.5}$$

which are invariant with respect to the transformations (2.14) and satisfy bounds of the kind (5.3), we can always find the vector valued functions $B(g_n,t_n)$, $A(t_0)$ and the operator valued functions $K(t_{i+1},g_i,t_i)$, $U(\gamma,t)$ satisfying the conditions (5.2) and the (5.4) and such that the functions F_n can be represented as in Eq. (5.1).

Proof: We can assume, without loss of generality, that

$$b(\xi_n, t_n) = k(t_{i+1}, \xi_i, t_i) = a(t_0) = 1 . \tag{5.6}$$

In the general case, we have just to divide all the quantities by the respective upper bounds.

We introduce the auxiliary Banach space $\mathcal{A}$, whose elements are sequences

$$\varphi_0 , \ \varphi_1(g_0,t_0) , \ \varphi_2(g_1,t_1,g_0,t_0) \ \cdots \quad , \tag{5.7}$$

where φ_0 is a complex number and φ_1, φ_2,... are generalized functions which represent arbitrary bounded complex measures. The norm is defined by

$$\|\varphi\| = |\varphi_0| + \sum_{n=1}^{\infty} \int |\varphi_n(g_{n-1},\dots,t_0)| \, d^3g_{n-1} \, dt_{n-1} \dots d^3g_0 \, dt_0 \quad . \tag{5.8}$$

In a similar way, we consider the Banach space $\tilde{\mathcal{A}}$, whose elements are sequences of measures of the kind

$$\psi_1(g_0), \ \psi_2(g_0,t_0,g_1), \ \dots \quad , \tag{5.9}$$

with a definition of the norm similar to Eq. (5.8).

Then, we consider the bilinear functionals, depending on the parameter t defined by

$$\phi(t,\psi,\varphi) = \sum_{m=1}^{\infty} \int F_{m-1}(g'_0, t'_0, \dots, t'_{m-2}, g'_{m-1}, t) \cdot$$

$$\cdot \psi_m(g'_0, t'_0, \dots, t'_{m-2}, g'_{m-1}) \cdot \varphi_0 \cdot d^3g'_0 dt'_0 \dots dt'_{m-2} d^3g'_{m-1} \quad +$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int F_{m+n-1}(g'_0, t'_0, \dots, t'_{m-2}, g'_{m-1}, t, g_{n-1}, t_{n-1}, \dots, g_0, t_0) \cdot$$

$$\cdot \psi_m(g'_0, t'_0, \dots, t'_{m-2}, g'_{m-1}) \, \varphi_n(g_{n-1}, t_{n-1}, \dots, g_0, t_0) \cdot$$

$$\cdot d^3g'_0 dt'_0 \dots dt'_{m-2} d^3g'_{m-1} d^3g_{n-1} dt_{n-1} \dots d^3g_0 dt_0 \quad . \tag{5.10}$$

From the definition of the norms and from Eqs. (5.3) and (5.6), we get

$$\left| \phi(t,\psi,\varphi) \right| \leqslant \|\psi\| \cdot \|\varphi\| \quad , \tag{5.11}$$

Now we define the operator $H(g,t)$ in $\mathcal{A}$ and the operator $\widetilde{H}(t',g)$ in $\widetilde{\mathcal{A}}$ in the following way

$$\begin{cases} [H(g,t)\varphi]_0 = 0 \quad , \\ [H(g,t)\varphi]_{n+1}(g_n, t_n, \dots, g_0, t_0) = \delta^3(g_n g^{-1}) \cdot \\ \qquad \cdot \delta(t_n - t) \, \varphi_n(g_{n-1}, t_{n-1}, \dots, g_0, t_0) \, , \quad n \geqslant 0 \quad , \end{cases} \tag{5.12}$$

$$\begin{cases} \left[\tilde{H}(t',g)\Psi\right]_1 = 0 \quad , \\ \left[\tilde{H}(t',g)\Psi\right]_{m+1}(g_0,t_0,\dots,t_{m-1},g_m) = \\ = \Psi_m(g_0,t_0,\dots,t_{m-2},g_{m-1})\,\delta(t_{m-1}-t')\,\delta^3(g_m g^{-1}) \,, \quad m \geqslant 1 \quad , \end{cases}$$

(5.13)

where $\delta^3(g)$ is a normalized Dirac measure on SU(1,1) concentrated on the unit. It is easy to see that these operators have the properties

$$\|H(g,t)\| = \|\tilde{H}(t',g)\| = 1 \quad ,$$

(5.14)

$$\phi(t',\Psi,\,H(g,t)\varphi) = \phi(t,\,\tilde{H}(t',g)\Psi,\,\varphi) \;.$$

(5.15)

We define also the operators $V(\gamma)$ in $\mathfrak{A}$ and $\tilde{V}(\gamma)$ in $\tilde{\mathfrak{A}}$ as follows

$$\begin{cases} \left[V(\gamma)\varphi\right]_0 = \varphi_0 \quad , \\ \left[V(\gamma)\varphi\right]_n(g_{n-1},t_{n-1},\dots,g_0,t_0) = \varphi_n(u_z(-\gamma)g_{n-1},t_{n-1},\dots,g_0,t_0) \quad , \end{cases}$$

(5.16)

$$\left[\tilde{V}(\gamma)\Psi\right]_n(g_0,t_0,\dots,g_{n-1}) = \Psi_n(g_0,t_0,\dots,g_{n-1}u_z(-\gamma)) \quad .$$

(5.17)

They have the properties

$$V(\gamma)\,H(g,t) = H(u_2(\gamma)g, t) \quad ,$$
$$\tilde V(\gamma)\,\tilde H(t',g) = \tilde H(t', g\,u_2(\gamma)) \, , \tag{5.18}$$

$$\Phi(t,\psi, V(\gamma)\varphi) = \phi(t, \tilde V(\gamma)\psi, \varphi) \; . \tag{5.19}$$

(5.20)

The last equation is a consequence of the invariance of F_n with respect to the transformations (2.14).

Finally we define the vectors $\alpha \in \mathcal{G}$ and $\beta(g) \in \tilde{\mathcal{G}}$ in the following way

$$\alpha_0 = 1 \;, \qquad \alpha_n = 0 \;, \qquad n \geq 1 \, , \tag{5.21}$$

$$[\beta(g)]_1(g_0) = \delta^3(g_0 g^{-1}) \quad , \quad [\beta(g)]_m = 0 \quad , \quad m \geq 2 \, . \tag{5.22}$$

They have the properties

$$\|\alpha\| = \|\beta(g)\| = 1 \quad , \tag{5.23}$$

$$V(\gamma)\alpha = \alpha \quad , \tag{5.24}$$

$$\tilde V(\gamma)\beta(g) = \beta(g\,u_2(\gamma)) \quad . \tag{5.25}$$

From all these definitions, we get

$$F_n\left(g_n, t_n, \cdots, g_0, t_0\right) = \phi\Big(t_n, \beta(g_n), \prod_{i=0}^{n-1} H(g_i, t_i)\,\alpha\Big) \quad . \tag{5.26}$$

Notice that all the information about the functions F_n is contained in the functional ϕ . The representation (5.26) is in general not economical, as the space $\mathcal{A}$ can be much larger than necessary. In order to eliminate this redundancy, we introduce the closed subspaces $\mathcal{O}_t \subset \mathcal{A}$ containing the vectors φ such that

$$\phi(t, \psi, \varphi) = 0 \tag{5.27}$$

for any choice of $\psi \in \tilde{\mathcal{A}}$. Then we introduce the quotient spaces

$$\mathcal{B}_t = \mathcal{A}/\mathcal{O}_t \quad . \tag{5.28}$$

From Eq. (5.15) we see that for any t'

$$H(g,t)\varphi \in \mathcal{O}_{t'} \quad \text{if} \quad \varphi \in \mathcal{O}_t \quad . \tag{5.29}$$

It follows that for any choice of t', the operator $H(g,t)$ defines an operator $K(t',g,t)$ from $\mathcal{B}_t$ to $\mathcal{B}_{t'}$. We define also the vector $A(t) \in \mathcal{B}_t$, which corresponds to the element $\alpha \in \mathcal{A}$. From the definition of the norm of a quotient space, it is clear that

$$\begin{cases} \|K(t', g, t)\| \leqslant 1 \quad , \\ \|A(t)\| \leqslant 1 \quad . \end{cases} \tag{5.30}$$

For each choice of t and of $\psi \in \tilde{\mathfrak{A}}$, we consider the linear bounded functional C on $\mathfrak{B}_t$ (that is an element of the dual space $\mathfrak{B}_t'$), which acts on the element $D \in \mathfrak{B}_t$ as follows

$$(C, D) = \phi(t, \psi, \varphi) \quad , \qquad \varphi \in D \quad . \tag{5.31}$$

From Eq. (5.11) we see that

$$\| C \| \leqslant \| \psi \| \quad . \tag{5.32}$$

We call $B(g,t)$ the element of $\mathfrak{B}_t'$ which corresponds to the element $\beta(g) \in \tilde{\mathfrak{A}}$. Clearly we have

$$\| B(g,t) \| \leqslant \| \beta(g) \| = 1 \quad . \tag{5.33}$$

The result (5.1) follows from Eqs. (5.31) and (5.26), by successive passages to the quotient spaces. The equations (5.30) and (5.33) show that the norms satisfy the bounds (5.4) under our special assumption (5.6).

In order to build the operators $U(\gamma, t)$ we remark that from Eq. (5.20) it follows that all the spaces $\mathcal{O}_t$ are invariant under $V(\gamma)$. Therefore, for any t, $V(\gamma)$ defines an operator $U(\gamma, t)$ in $\mathfrak{B}_t$ which, as it is easy to show using Eqs. (5.15), (5.18)-(5.20), (5.24), (5.25), has the required properties (5.2).

We remark that the proof given above is based on an explicit construction of the operator K and of the vectors A and B. We do not exclude that other choices of the objects A, B and K could give an equivalent representation of the same amplitudes. At this point, one should show that the representation we have constructed is, in a given sense, the most economical one. Here we do not give a complete treatment, but we limit ourselves to the following result:

<u>Proposition</u>: If the functions F_n have the representation

$$F_n = \hat{B}(g_n, t_n) \prod_{i=0}^{n-1} \hat{K}(t_{i+1}, g_i, t_i) \hat{A}(t_0) \quad , \tag{5.34}$$

where $\hat{B}$, $\hat{K}$ and $\hat{A}$ are matrices of dimension respectively $(1 \times r)$, $(r \times r)$ and $(r \times 1)$, the spaces $\mathcal{B}_t$ constructed in the preceding proof have dimension not larger than r.

<u>Proof</u>: From Eq. (5.10) we see that the functional ϕ can be written in the form

$$\phi(t, \psi, \varphi) = \sum_{i=1}^{r} R_i(t, \psi) S_i(t, \varphi) \quad . \tag{5.35}$$

From the definition of the space $\mathcal{O}_t$ we see that it contains the vectors φ such that the quantities

$$S_i(t, \varphi) \quad , \quad i = 1, 2, \dots r \quad , \tag{5.36}$$

vanish. It follows that an element of the space $\mathcal{B}_t$ defined by Eq. (5.28) is uniquely determined by the quantities (5.36), so that $\mathcal{B}_t$ has dimension not larger than r.

6. <u>THE INCLUSIVE DISTRIBUTIONS</u>

In Section 3 we have discussed the decrease property of the amplitudes in the momentum transfers and we have seen that this property can more easily be stated in terms of the functions $F_n(x)$, which, however, are not symmetric under permutations of the final particles. In this Section we want to study some integrals containing the amplitude, expressing them in terms of the functions $F_n(x)$.

The most interesting integrals are the unitarity integrals, which express the absorptive part of the non-forward elastic amplitude, and the integrals which give the inclusive distributions and in particular the total cross-section. We shall treat in detail only the second case.

It will also be convenient, generalizing a procedure due to Chew and de Tar [14], to express these integrals in terms of BCP variables.

The inclusive distribution for r observed final particles can be written as

$$\frac{d^{3r}\sigma}{d^3\underline{P}_0 \cdots d^3\underline{P}_{r-1}} \cdot \prod_{i=0}^{r-1} 2P_i^0 = \frac{1}{2}\left[s(s-4m^2)\right]^{-\frac{1}{2}} \cdot$$

$$\cdot\, G_r\left(P_A, P_B, P_0, \cdots, P_{r-1}\right) \qquad , \tag{6.1}$$

where

$$G_r\left(P_A, P_B, P_0, \cdots, P_{r-1}\right) = \sum_{n=\bar{n}}^{\infty} (2\pi)^{-3n-2} \cdot \frac{1}{(n+2-r)!} \cdot$$

$$\int \left| M_n(P_A, P_B, P_0, \cdots, P_{n+1})\right|^2 \delta^4(P_A + P_B - \sum_{i=0}^{n+1} P_i) \prod_{i=r}^{n+1} \frac{d^3\underline{P}_i}{2P_i^0} \quad . \tag{6.2}$$

If we want to include the elastic scattering, $\bar{n}$ is equal to $r-2$ or to 0, if $r \leq 2$.

Now we want to introduce the functions F_n using Eq. (3.15). For simplicity we put

$$\mathcal{F}_n(x) = \left| F_n(x)\right|^2 \eta_n(x) \quad . \tag{6.3}$$

The result can be written as

$$G_r(P_A, P_B, P_0, \dots P_{r-1}) = \sum_{\pi} \mathcal{G}_r\left(P_A, P_B, P_{\pi(0)}, \dots, P_{\pi(r-1)}\right) , \tag{6.4}$$

where the sum is over the $r!$ permutations of the observed particles and

$$\mathcal{G}_r\left(P_A, P_B, P'_0, \dots, P'_{r-1}\right) = \sum_{n=\bar{n}}^{\infty} (2\pi)^{-3n-2} \sum_{0 \le \nu_0 < \dots < \nu_{r-1} \le n+1}$$
$$\cdot \int \mathcal{F}_n\left(P_A, P_B, P_0, \dots, P_{n+1}\right) \prod_{k=0}^{r-1} \left[\delta^4(P'_k - P_{\nu_k}) d^4 P_{\nu_k}\right] \cdot$$
$$\cdot \delta^4\left(P_A + P_B - \sum_{j=0}^{n+1} P_j\right) \prod_j{}' \left[\delta\left(P_j^2 - m^2\right)\theta\left(P_j^0\right) d^4 P_j\right] , \tag{6.5}$$

where $\prod_j{}'$ means the product for $0 \le j \le n+1$ and $j \ne \nu_0, \dots, \nu_{r-1}$. Note that all the factorials have disappeared from Eq. (6.5). Remark also that the functions G_r are symmetric in the observed four-momenta, while the functions $\mathcal{G}_r$ are not.

Chew and de Tar [14] have developed a technique for introducing the BCP variables as integration variables in integrals of the kind (6.5). Here we describe a treatment essentially equivalent, but more suitable for our purposes. It is also convenient to introduce group theoretical variables in the inclusive distributions by putting

$$\begin{cases} P_A = L(b_A)(m, 0, 0, 0) , \\ P_B = L(b_B)(m, 0, 0, 0) , \\ P'_i = L(b_i)(m, 0, 0, 0) , \quad i = 1, 2, \dots, r . \end{cases} \tag{6.6}$$

We rewrite Eqs. (2.6)-(2.8) in the form

$$\begin{cases} Q_i = L(a_i)\left(0,0,0,\sqrt{-t_i}\right), \\ Q_i' = L\left(a_{i+1}a_2(\chi_{i+1})\right)\left(0,0,0,\sqrt{-t_i'}\right), \quad i=0,\cdots,n \end{cases} , \tag{6.7}$$

$$\begin{cases} P_A' = L\left(a_0 a_2(\chi_0)\right)\left(m_A',0,0,0\right), \\ P_B' = L\left(a_{n+1}\right)\left(m_B',0,0,0\right). \end{cases} \tag{6.8}$$

It is clear that we must have $Q_i = Q_i'$, $P_A = P_A'$ and $P_B = P_B'$. We impose these constraints by means of appropriate δ functions and we write Eq. (6.5) in the form

$$\mathcal{G}_r\left(P_A, P_B, P_0', \cdots, P_{r-1}'\right) = \sum_{n=\bar{n}}^{\infty} (2\pi)^{-3n-2} \sum_{0\le \nu_0 < \cdots < \nu_{r-1} \le n+1}$$

$$\int \mathcal{F}_n\left(P_A', P_B', P_0, \cdots P_{n+1}\right) \prod_{k=0}^{r-1} \delta^4\left(P_k' - P_{\nu_k}\right) \prod_j{}' \left[\delta\left(P_j^2 - m^2\right)\theta\left(P_j^0\right)\right] \cdot$$

$$\prod_{i=0}^{n} \left[\delta^4\left(Q_i - Q_i'\right) d^4Q_i\, d^4Q_i'\right] \delta^4\left(P_A - P_A'\right) d^4P_A'\ \delta^4\left(P_B - P_B'\right) d^4P_B' . \tag{6.9}$$

From Eq. (6.7) and the Lemma of the Appendix A we get

$$\delta^4\left(Q_i - Q_i'\right) = \frac{2\pi^2}{-t_i}\,\delta\left(t_i - t_i'\right)\delta_-^3\left(g_i\right) , \tag{6.10}$$

where g_i is defined by Eq. (2.12). In a similar way we get

$$\delta^4\left(P'_k - P_{\nu_k}\right) = \frac{2\pi^2}{m^2}\,\delta\left(P^2_{\nu_k} - m^2\right)\delta^3_+\left(b_k^{-1} a_{\nu_k} a_z(\hat{\chi}_{\nu_k})\right) , \tag{6.11}$$

$$\delta^4\left(P_A - P'_A\right) = \frac{2\pi^2}{m^2}\,\delta\left(m'^2_A - m^2\right)\delta^3_+\left(b_A^{-1} a_0 a_z(\chi_0)\right) , \tag{6.12}$$

$$\delta^4\left(P_B - P'_B\right) = \frac{2\pi^2}{m^2}\,\delta\left(m'^2_B - m^2\right)\delta^3_+\left(b_B^{-1} a_{n+1}\right) . \tag{6.13}$$

The Lemma of the Appendix B permits the substitutions

$$\delta\left(P_i^2 - m^2\right)\theta\left(P_i^0\right) d^4Q_i\, d^4Q'_{i-1} \longrightarrow$$

$$\longrightarrow \frac{1}{4\pi}\left[T\left(m^2, t_i, t'_{i-1}\right)\right]^{\frac{1}{2}} d^6a_i\, dt_i\, dt'_{i-1} \quad , \quad i = 1,2,\dots,n \quad , \tag{6.14}$$

$$\delta\left(P_0^2 - m^2\right)\theta\left(P_0^0\right) d^4Q_0\, d^4P'_A \longrightarrow$$

$$\longrightarrow \frac{1}{4\pi}\left[T\left(m^2, m'^2_A, t_0\right)\right]^{\frac{1}{2}} d^6a_0\, dm'^2_A\, dt_0 \quad , \tag{6.15}$$

$$\delta\left(P_{n+1}^2 - m^2\right)\theta\left(P_{n+1}^0\right) d^4Q'_n\, d^4P'_B \longrightarrow$$

$$\longrightarrow \frac{1}{4\pi}\left[T\left(m^2, m'^2_B, t'_n\right)\right]^{\frac{1}{2}} d^6a_{n+1}\, dm'^2_B\, dt'_n \quad . \tag{6.16}$$

We perform all these substitutions in Eq. (6.9) and we perform, using the δ functions, the integrations over $t'_0, \ldots, t'_n$, m'^2_A and m'^2_B. In this way we get

$$\mathcal{G}_r(b_A, b_B, b_0, \ldots, b_{r-1}) = \sum_{n=\bar{n}}^{\infty} (2\pi)^{-3n-2} \sum_{0 \leq \nu_0 < \cdots < \nu_{r-1} \leq n+1}$$

$$\int \mathcal{F}_n(g_n, t_n, \ldots, g_0, t_0) \left(\frac{2\pi^2}{m^2}\right)^{r+2} \delta^3_+\left(b_A^{-1} a_0 a_z(\chi_0)\right) \delta^3_+\left(b_B^{-1} a_{n+1}\right)$$

$$\prod_{k=0}^{r-1} \delta^3_+\left(b_k^{-1} a_{\nu_k} a_z(\hat{\chi}_{\nu_k})\right) \cdot \prod_{i=0}^{n} \left[\frac{2\pi^2}{-t_i} \delta^3_-(g_i)\, dt_i\right] \cdot$$

$$\prod_{j=0}^{n+1} \left[\frac{1}{4\pi} \left[T(m^2, t_j, t_{j-1})\right]^{\frac{1}{2}} d^6 a_j\right] , \tag{6.17}$$

where the convention (2.18) has been used.

Equation (6.17) has the remarkable property that all the kinematical terms which appear in the integrand are Q factorized. More exactly, each factor contains at most two consecutive variables t_i, t_{i-1} and only one group element g_i. Notice that in the preceding Section we have chosen, for the functions F_n, a Q factorized representation which has just the same structure and is, therefore, especially suitable for being used in connection with Eq. (6.17), as we shall see in the next Section.

7. THE MULTIPERIPHERAL INTEGRAL EQUATION

In this Section, we apply the Q factorization procedures developed in Sections 4 and 5 to the integrals (6.17) of the preceding Section. In this way we shall naturally introduce the multiperipheral integral equation.

First of all, we remark that the procedures developed in Section 5 can be directly applied to the function $\mathcal{F}_n(x)$ defined in Eq. (6.3), leading to the operator Q factorized representation

$$\mathcal{F}_n(g_n, t_n, \dots, g_0, t_0) = \left(\mathcal{B}(g_n, t_n), \prod_{i=0}^{n-1} \mathcal{K}(t_{i+1}, g_i, t_i)\, \mathcal{A}(t_0)\right). \tag{7.1}$$

Alternatively, one can develop a Q factorized representation for the functions $\eta_n(x)$ and to substitute it into Eq. (6.3) together with Eq. (5.1). One gets easily in this way an equation of the kind (7.1). This procedure gives the possibility of connecting the quantities $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{K}$ with the analogous quantities A, B and K. Moreover, it can be generalized to the study of non-forward unitarity.

Starting from Eq. (7.1), we define the operator valued kernel

$$\underline{\mathcal{K}}\left(t_{i+1}, a_{i+1}^{-1} a_i, t_i\right) = \frac{1}{16\pi^2 |t_i|} \left[T(m^2, t_{i+1}, t_i)\right]^{\frac{1}{2}} \cdot \mathcal{K}(t_{i+1}, g_i, t_i)\, \delta_-^3(g_i) \quad , \qquad i = 0, \dots, n-1 \quad , \tag{7.2}$$

and the vector valued functions

$$\underline{\mathcal{B}}\left(a_{n+1}^{-1} a_n, t_n\right) = \frac{1}{8|t_n|\pi} \left[T(m^2, m^2, t_n)\right]^{\frac{1}{2}} \cdot \delta_-^3(g_n)\, \mathcal{B}(g_n, t_n) \quad , \tag{7.3}$$

$$\underline{\mathcal{A}}(t_0) = \frac{1}{4\pi} \left[T(m^2, t_0, m^2)\right]^{\frac{1}{2}} \mathcal{A}(t_0) \quad . \tag{7.4}$$

Then Eq. (6.17) takes the form

$$\mathcal{G}_r\left(b_A, b_B, b_0, \dots, b_{r-1}\right) = \sum_{n=\bar{m}}^{\infty} \sum_{0 \leq \nu_0 < \dots < \nu_{r-1} \leq n+1}$$

$$\int \frac{2\pi^2}{m^2} \delta_+^3\left(b_B^{-1} a_{n+1}\right) \left(\underline{\mathcal{B}}\left(a_{n+1}^{-1} a_n, t_n\right), \prod_{i=0}^{n-1} \underline{\mathcal{H}k}\left(t_{i+1}, a_{i+1}^{-1} a_i, t_i\right) \cdot \right.$$

$$\left. \underline{\mathcal{A}}(t_0) \right) \frac{2\pi^2}{m^2} \delta_+^3\left(a_z(-\chi_0) a_0^{-1} b_A\right) \cdot \prod_{k=0}^{r-1} \left[\frac{2\pi^2}{m^2} \delta_+^3\left(b_k^{-1} a_{\nu_k} a_z(\hat{\chi}_{\nu_k})\right)\right] \cdot$$

$$dt_0 \cdots dt_n \, d^6 a_0 \cdots d^6 a_{n+1} \quad .$$

(7.5)

This expression can be simplified introducing the operator valued kernel

$$\underline{\mathcal{R}}\left(t', a'^{-1} a, t\right) = \delta(t'-t)\, \delta^6\left(a'^{-1} a\right) +$$

$$+ \underline{\mathcal{H}k}\left(t', a'^{-1} a, t\right) + \int \underline{\mathcal{H}k}\left(t', a'^{-1} a'', t''\right) \cdot$$

$$\cdot \underline{\mathcal{H}k}\left(t'', a''^{-1} a, t\right) dt''\, d^6 a'' + \cdots \quad ,$$

(7.6)

which is a solution of the integral equation

$$\underline{\mathcal{R}}\left(t', a'^{-1} a, t\right) = \delta(t'-t)\, \delta^6\left(a'^{-1} a\right) +$$

$$+ \int \underline{\mathcal{H}k}\left(t', a'^{-1} a'', t''\right) \underline{\mathcal{R}}\left(t'', a''^{-1} a, t\right) dt''\, d^6 a'' \quad .$$

(7.7)

Then one can write the function $\mathcal{G}_r$ as a sum of at most four integrals containing the kernel $\underline{\mathcal{R}}$. For instance, we have

$$\mathcal{G}_0(b_A, b_B) = \int \frac{2\pi^2}{m^2} \delta_+^3(b_B^{-1} a') \Big(\underline{\mathcal{B}}(a'^{-1} a, t), \underline{\mathcal{R}}(t, a^{-1} a_0, t_0)\, \underline{\mathcal{A}}(t_0) \Big) \frac{2\pi^2}{m^2} \delta_+^3(a_z(-\chi_0) a_0^{-1} b_A)\, d^6 a_0\, d^6 a\, d^6 a'\, dt_0\, dt\,, \tag{7.8}$$

$$\begin{aligned} \mathcal{G}_1(b_A, b_B, b) = & \int \frac{2\pi^2}{m^2} \delta_+^3(b_B^{-1} a') \Big(\underline{\mathcal{B}}(a'^{-1} a, t), \\ & \underline{\mathcal{R}}(t, a^{-1} a_0, t_0)\, \underline{\mathcal{A}}(t_0) \Big) \frac{2\pi^2}{m^2} \delta_+^3(a_z(-\chi_0) a_0^{-1} b_A) \frac{2\pi^2}{m^2} \Big[\delta_+^3(b^{-1} a' a_z(\hat{\chi})) + \\ & + \delta_+^3(b^{-1} a_0 a_z(\hat{\chi}_0)) \Big] d^6 a_0\, d^6 a\, d^6 a'\, dt_0\, dt \qquad + \\ & + \int \frac{2\pi^2}{m^2} \delta_+^3(b_B^{-1} a''') \Big(\underline{\mathcal{B}}(a'''^{-1} a'', t''), \underline{\mathcal{R}}(t'', a''^{-1} a', t') \\ & \underline{\mathcal{H}}(t', a'^{-1} a, t)\, \underline{\mathcal{R}}(t, a^{-1} a_0, t_0)\, \underline{\mathcal{A}}(t_0) \Big) \frac{2\pi^2}{m^2} \delta_+^3(a_z(-\chi_0) a_0^{-1} b_A) \\ & \frac{2\pi^2}{m^2} \delta_+^3(b^{-1} a' a_z(\hat{\chi}'))\, d^6 a_0\, d^6 a\, d^6 a'\, d^6 a''\, d^6 a''' \\ & dt_0\, dt\, dt'\, dt'' \qquad , \end{aligned} \tag{7.9}$$

where

$$\begin{cases} \cosh\hat{\chi} = (2m^2-t)(2m^2)^{-1} & , \quad \hat{\chi} \geq 0 , \\ \sinh\hat{\chi}' = (m^2+t'-t)(-4t'm^2)^{-\frac{1}{2}} & . \end{cases} \tag{7.10}$$

The first term in the right-hand side of Eq. (7.9) refers to the case in which the observed particle is the first or the last produced particle (in our conventional ordering); the second term takes into account all the other cases. The generalization to two or more observed particles is conceptually simple and gives rise to four different integrals. In this case the distinction between the functions G_r and $\mathcal{G}_r$, which is irrelevant for $r=0, 1$, is important.

We remark that Eq. (7.7) has essentially the same structure as the multiperipheral equation derived by Chew and de Tar [14], apart from the fact that our kernels are infinite dimensional operators for fixed values of their arguments. It is just this last feature that permits to consider (7.7) as an exact equation.

The Lorentz invariance manifests itself in the fact that Eq. (7.7) contains a convolution over SL(2C). As a consequence, it can be diagonalized by projection on the irreducible representations of this group, according to the techniques developed in Refs. 26)-28). As it is well-known, the projected equation determines the singularities in the complex λ plane, which control the asymptotic behaviour of the total cross-sections and of the inclusive distributions [20)-23)].

Equation (7.7), complemented by the appropriate bounds, can be considered as an exact dynamical equation in the same sense as the Schroedinger equation: as the Schroedinger equation gives the scattering amplitude in terms of a potential which is unknown, but energy independent, in a similar way Eq. (7.7) gives the inclusive distributions in terms of a kernel $\underline{\mathfrak{K}}$ which is also unknown, but independent of the total energy and the multiplicity.

ACKNOWLEDGEMENTS

One of us (L.S.) is grateful for the hospitality of the CERN Theoretical Study Division.

APPENDIX A: MEASURES ON SL(2C) AND RELATED FORMULAE

If H is a subgroup of the group A, we define a measure on A, which we indicate by $\delta_H(a)$ and has the property

$$\int_A f(a)\,\delta_H(a)\,da = \int_H f(h)\,dh \quad , \tag{A.1}$$

where dh is the invariant measure over H with an appropriate normalization.

We are interested in the case in which $A = SL(2C)$ and we shall use the simpler notations

$$\begin{cases} \delta_{SU(2)}(a) = \delta_+^3(a) \quad , \\ \delta_{SU(1,1)}(a) = \delta_-^3(a) \quad . \end{cases} \tag{A.2}$$

We define the four-vectors

$$\begin{cases} Q_W = (\sqrt{W}, 0, 0, 0) \ , \quad W > 0 \ , \\ Q_W = (0, 0, 0, \sqrt{-W}) \ , \quad W < 0 \ , \end{cases} \tag{A.3}$$

and we consider the measure

$$\delta^4\left[L(a)\,Q_W - Q_{W'}\right] \ , \quad W' \neq 0 \quad . \tag{A.4}$$

Clearly, this measure is concentrated in the set defined by

$$W = W' \ , \qquad a \in H_{W'} \ , \tag{A.5}$$

where

$$\begin{cases} H_W = SU(2) & \text{for } W > 0 , \\ H_W = SU(1,1) & \text{for } W < 0 , \end{cases} \tag{A.6}$$

and is invariant with respect to the transformation

$$a \to a h \quad , \quad h \in H_{W'} . \tag{A.7}$$

Therefore, it must have the form

$$\begin{cases} \delta^4 \left[L(a) Q_W - Q_{W'} \right] = \gamma(W)\, \delta(W - W')\, \delta^3_{\pm}(a) , \\ \pm = \operatorname{sign} W' . \end{cases} \tag{A.8}$$

In order to determine the function $\gamma(W)$, we have to choose the normalizations of the various invariant measures. For $SL(2C)$ we use the parametrization,

$$a = u_z(\mu)\, u_y(\theta)\, u_z(\nu)\, a_z(\zeta)\, u_y(\theta')\, u_z(\mu') ,$$

$$0 \leq \mu < 4\pi ,\ 0 \leq \theta \leq \pi ,\ 0 \leq \nu < 2\pi ,\ \zeta \geq 0 ,\ 0 \leq \theta' \leq \pi ,\ 0 \leq \mu' < 2\pi , \tag{A.9}$$

and we write the invariant measure as

$$da = (16)^{-1} d\mu\, d\cos\theta\, d\nu\, (\sinh\zeta)^2 d\zeta\, d\cos\theta'\, d\mu' . \tag{A.10}$$

For infinitesimal elements it is simpler to use the parametrization

$$L_{ik}(a) = \delta_{ik} + \eta_{ik} ,\quad \eta_{ik} = - g_{ii}\, g_{kk}\, \eta_{ki} ,\ \eta_{ii} = 0 , \tag{A.11}$$

and the invariant measure in an infinitesimal neighbourhood of the unit can be written in the form

$$d^6 a = (16)^{-1} d\eta_{10}\, d\eta_{20}\, d\eta_{30}\, d\eta_{12}\, d\eta_{23}\, d\eta_{31} \quad . \tag{A.12}$$

The invariant measure on SU(2) is defined by

$$h = u_z(\mu)\, u_y(\vartheta)\, u_z(\nu), \quad 0 \le \mu < 4\pi, \; 0 \le \vartheta \le \pi, \; 0 \le \nu < 2\pi, \tag{A.13}$$

$$d^3 h = (16\pi^2)^{-1}\, d\mu\, d\nu\, d\cos\vartheta \quad , \tag{A.14}$$

and near the unit it takes the form

$$d^3 h = (16\pi^2)^{-1}\, d\eta_{12}\, d\eta_{23}\, d\eta_{31} \quad . \tag{A.15}$$

The analogous formulae for SU(1.1) are

$$h = u_z(\mu)\, a_x(\xi)\, u_z(\nu), \quad 0 \le \mu < 4\pi, \; 0 \le \xi, \; 0 \le \nu < 2\pi, \tag{A.16}$$

$$d^3 h = (16\pi^2)^{-1}\, d\mu\, d\nu\, d\cosh\xi \quad , \tag{A.17}$$

$$d^3 h = (16\pi^2)^{-1}\, d\eta_{12}\, d\eta_{01}\, d\eta_{02} \quad . \tag{A.18}$$

Comparing with Eq. (A.1) we see that near the unit we can write

$$\delta_+^3(a) = \pi^{-2}\, \delta(\eta_{01})\, \delta(\eta_{02})\, \delta(\eta_{03}) \quad , \tag{A.19}$$

$$\delta_-^3(a) = \pi^{-2}\,\delta(\eta_{13})\,\delta(\eta_{23})\,\delta(\eta_{03}) \quad . \tag{A.20}$$

In the same approximation the left-hand side of Eq. (A.8) takes the form

$$\delta\left(\sqrt{W}-\sqrt{W'}\right)\,\delta\left(\eta_{01}\sqrt{W}\right)\delta\left(\eta_{02}\sqrt{W}\right)\,\delta\left(\eta_{03}\sqrt{W}\right)\,, \quad W>0, \tag{A.21}$$

$$\delta\left(\sqrt{-W}-\sqrt{-W'}\right)\,\delta\left(\eta_{03}\sqrt{-W}\right)\,\delta\left(\eta_{13}\sqrt{-W}\right)\delta\left(\eta_{23}\sqrt{-W}\right)\,, \quad W<0. \tag{A.22}$$

Comparing Eqs. (A.8), (A.19)-(A.20) we have for both the signs of W

$$\gamma(W) = \frac{2\pi^2}{|W|} \quad . \tag{A.23}$$

In conclusion, we have

<u>Lemma 1</u> If Q_W is given by Eq. (A.3) and W' is different from zero, we have

$$\delta^4\left[L(a)Q_W - Q_{W'}\right] = \frac{2\pi^2}{|W|}\,\delta(W-W')\,\delta_\pm^3(a)\,, \quad \pm = \operatorname{sign} W' . \tag{A.24}$$

The numerical coefficient is correct only if the invariant measures are normalized as in Eqs. (A.9)-(A.18).

APPENDIX B: A TRANSFORMATION OF VARIABLES

We consider a pair of four-vectors Q_1, Q_2 and we want to write them in the form

$$\begin{cases} Q_1 = L(a)\,\hat{Q}_1 \,, \\ Q_2 = L(a)\,\hat{Q}_2 \,, \end{cases} \tag{B.1}$$

where $a \in A = SL(2C)$ and the pair $\hat{Q}_1$, $\hat{Q}_2$ is a representative element of an orbit in the space of the pairs of four-vectors. These orbits can be labelled by means of the parameters

$$W_1 = (Q_1)^2 , \quad W_2 = (Q_2)^2 , \quad W_3 = (Q_1 - Q_2)^2 , \tag{B.2}$$

and of some other index, as we shall see at once.

It is useful to consider the quantity

$$\Delta = 4\left[(Q_1, Q_2)^2 - W_1 W_2\right] = T(W_1, W_2, W_3) , \tag{B.3}$$

where

$$T(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc . \tag{B.4}$$

We consider only orbits of the following "general" kinds. The orbits we do not consider form a set of zero measure.

a) $W_1 > 0$, $\Delta > 0$. We choose the representative elements

$$\begin{cases} \hat{Q}_1 = \left(\epsilon\sqrt{W_1}, 0, 0, 0\right) , \quad \epsilon = \pm 1 , \\ \hat{Q}_2 = \left(\dfrac{W_1 + W_2 - W_3}{2\epsilon\sqrt{W_1}}, 0, 0, \sqrt{\dfrac{\Delta}{4W_1}}\right) . \end{cases} \tag{B.5}$$

b) $W_1 < 0$, $\Delta > 0$. We choose

$$\begin{cases} \hat{Q}_1 = \left(0,\ 0,\ 0,\ \sqrt{-W_1}\right), \quad \epsilon = \pm 1, \\ \hat{Q}_2 = \left(\epsilon \sqrt{\frac{-\Delta}{4W_1}},\ 0,\ 0,\ \frac{W_3 - W_1 - W_2}{2\sqrt{-W_1}}\right). \end{cases} \tag{B.6}$$

c) $W_1 < 0$, $\Delta < 0$. We take

$$\begin{cases} \hat{Q}_1 = \left(0,\ 0,\ 0,\ \sqrt{-W_1}\right), \\ \hat{Q}_2 = \left(0,\ 0,\ \sqrt{\frac{\Delta}{4W_1}},\ \frac{W_3 - W_1 - W_2}{2\sqrt{-W_1}}\right). \end{cases} \tag{B.7}$$

The case $W_1 > 0$, $\Delta < 0$ cannot appear if the four-vectors are real.

We call K the subgroup of A containing the transformations which leave $\hat{Q}_1$ and $\hat{Q}_2$ invariant. It contains the rotations around the z axis in the cases a) and b) and the boosts along the x axis in the case c). It is clear that the four-vectors Q_1 and Q_2 given in Eq. (B.1) do not change if we perform a transformation of the kind

$$a \to a k, \qquad k \in K, \tag{B.8}$$

and, therefore, we may consider them as functions of W_1, W_2, W_3, ϵ, and $[a]$, where $[a]$ is an element of the coset space A/K.

We consider now an invariant measure $d^5[a]$ on A/K, i.e., a measure with the property

$$\int_{A/K} f([a'a])\, d^5[a] = \int_{A/K} f([a])\, d^5[a], \quad a' \in A. \tag{B.9}$$

As A and K are unimodular, this measure exists and is unique up to a constant factor [33]. If we fix the normalizations of the invariant measures on A and K, we may choose the normalization of $d^5[a]$ in such a way that

$$\int_A f(a)\, d^6 a = \int_{A/K} d^5[a] \int_K f(a_o k)\, dk \quad . \tag{B.10}$$

We have indicated by a_o a representative element of the coset $[a]$.

Now we consider the function

$$f(Q_1, Q_2) = \tilde{f}(W_1, W_2, W_3, \epsilon, a) =$$

$$= f\,[L(a)\hat{Q}_1, L(a)\hat{Q}_2] \quad , \tag{B.11}$$

which, as we have seen, can be considered as a function of the coset $[a]$. We are interested in the following change of integration variables

$$I = \int f(Q_1, Q_2)\, d^4Q_1\, d^4Q_2 =$$

$$= \sum_\epsilon \int dW_1\, dW_2\, dW_3\; J(W_1, W_2, W_3) \int_{A/K} d^5[a]\, \tilde{f}(W_1, W_2, W_3, \epsilon, a) \; . \tag{B.12}$$

The sum over ϵ is not present for $\Delta < 0$. In the domain of integration of the last integral, we have to exclude the region $W_1 > 0$, $\Delta < 0$.

Our aim is to compute the Jacobian J. The fact that it does not depend on a is a consequence of the uniqueness of the invariant measure on A/K. From the invariance of the measure d^4Q with respect to time inversion, it follows that J does not depend on ϵ . It is clear that the general form of Eq. (B.12) does not depend on the special choice (B.5)-(B.7) of the representative four-vectors $\hat{Q}_1$ and $\hat{Q}_2$. Also the special form of the Jacobian J is independent of this choice.

Now, in order to compute the jacobian J we can use the choice (B.5)-(B.7) and the invariant measures given in Eqs. (A.9)-(A.18). In the cases a) and b) we use for the group K the parametrization and the invariant measure

$$\begin{cases} k = u_z(\mu) \quad , \quad 0 \leq \mu < 4\pi \ , \\ dk = (4\pi)^{-1}\, d\mu \underset{\mu\to 0}{\simeq} (4\pi)^{-1}\, d\eta_{12} \quad , \end{cases} \tag{B.13}$$

and therefore for infinitesimal transformations

$$d^5[a] = \frac{\pi}{4}\, d\eta_{10}\, d\eta_{20}\, d\eta_{30}\, d\eta_{23}\, d\eta_{31} \quad . \tag{B.14}$$

In the case c) we put

$$\begin{cases} k = a_x(\xi) \quad , \quad -\infty < \xi < +\infty \ , \\ dk = (4\pi)^{-1}\, d\xi \underset{\xi\to 0}{\simeq} (4\pi)^{-1}\, d\eta_{10} \quad , \end{cases} \tag{B.15}$$

and for infinitesimal transformations we have

$$d^5[a] = \frac{\pi}{4}\, d\eta_{20}\, d\eta_{30}\, d\eta_{12}\, d\eta_{23}\, d\eta_{31} \quad . \tag{B.16}$$

As the Jacobian J does not depend on a, we can compute it, using Eqs. (B.1), (B.12), (B.14) and (B.16). The calculation is simple and in all the cases a), b), c) we obtain

$$J(w_1, w_2, w_3) = (4\pi)^{-1}\, |\Delta|^{1/2} \quad . \tag{B.17}$$

It is useful to summarize the results as follows:

Lemma 2 We consider the transformation of variables

$$\tilde{f}(W_1, W_2, W_3, \epsilon, a) = f[L(a)\hat{Q}_1, L(a)\hat{Q}_2], \tag{B.18}$$

where the pair of four-vectors $\hat{Q}_1$, $\hat{Q}_2$ is a representative element of an orbit in the space of the pairs of four-vectors, and depends smoothly on the parameters W_1, W_2, W_3. If $\Delta = T(W_1, W_2, W_3)$ is positive, they depend also on an index $\epsilon = \pm 1$. We indicate by K the subgroup of $A = SL(2C)$ which leaves $\hat{Q}_1$ and $\hat{Q}_2$ invariant. Then, the function (B.18) can be considered as a function of the coset $[a]$ of A/K. If we indicate by $d^5[a]$ an invariant measure on A/K, we have

$$\int f(Q_1, Q_2)\, d^4Q_1\, d^4Q_2 =$$

$$= (4\pi)^{-1} \sum_{\epsilon} \int dW_1\, dW_2\, dW_3\, |T(W_1, W_2, W_3)|^{1/2} \cdot$$

$$\cdot \int d^5[a]\; \tilde{f}(W_1, W_2, W_3, \epsilon, a) \quad , \tag{B.19}$$

where the sum over ϵ disappears in the region $\Delta < 0$ and the region where Δ is negative and some of the parameters W_i are positive has to be excluded from the integration domain. In the region where Δ is positive, we can use the invariant measure d^6a on A instead of $d^5[a]$. The numerical factor is correct only if the measure $d^5[a]$ is normalized as in Eqs. (B.14), (B.16), or in the case $\Delta > 0$, if d^6a is normalized as in Eq. (A.10).

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FIGURE CAPTION

Notations for the four-momenta in a production process.

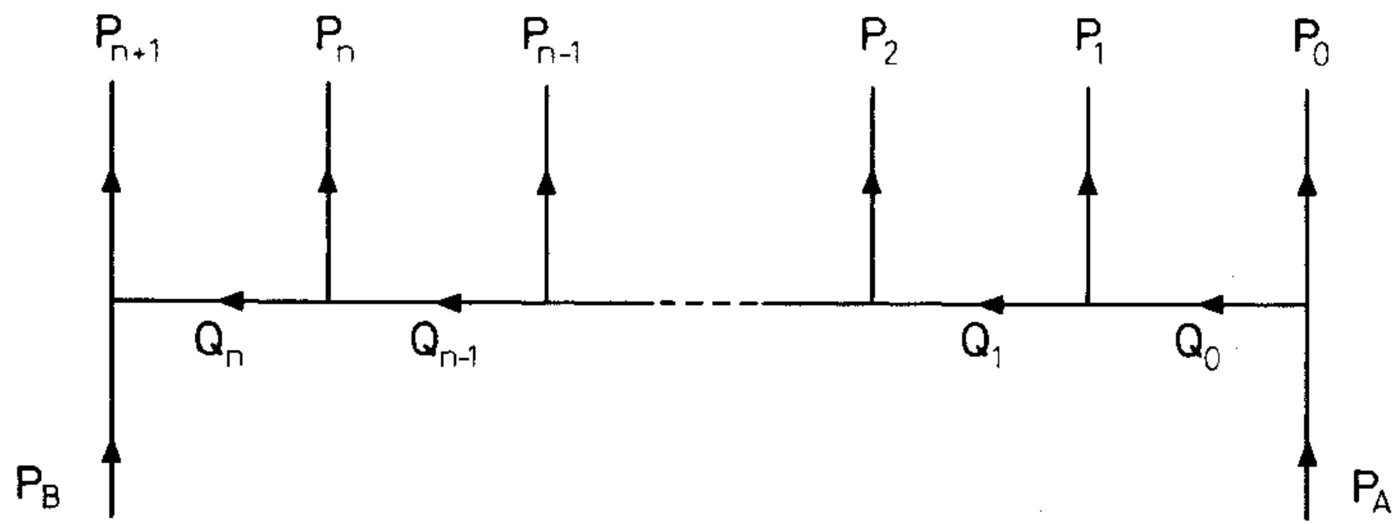

FIG.1